Nonequilibrium dynamic correlation-length scaling method

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The finite-size-scaling method in the equilibrium Monte Carlo (MC) simulations and the finite-time-scaling method in the nonequilibrium relaxation simulations are compromised. MC time data of various physical quantities are scaled by the MC time data of the dynamic correlation length. It corresponds to changing the system size in the finite-size-scaling method. This scaling method is tested in the three-dimensional ferromagnetic Ising model and in the three-dimensional $\pm J$ Ising spin-glass model. The transition temperature and the critical exponents are obtained. We also comment on the definition of the dynamic correlation length in the nonequilibrium relaxation process. The Ornstein-Zernike formula is not always appropriate.

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I. INTRODUCTION

Monte Carlo (MC) simulations and the finite-size-scaling analysis are very important tools in the study of phase transitions.[1] The thermodynamic limit is taken by the scaling analysis on the finite-size and equilibrium (infinite-time) data obtained by the equilibrium MC simulations. Then, the critical temperature and the critical exponents are estimated. The applications serve as a strong bridge between the experimental and the theoretical physics. We may estimate various physical parameters, predict unknown properties, and propose interesting experiments on real materials.

The nonequilibrium relaxation method[2–6] is an alternate version of the MC simulations. The method directly deals with the MC relaxation functions of physical quantities. In the standard equilibrium simulations, we take the infinite-time (equilibrium) limit first. Then, the infinite-size limit is taken by the finite-size-scaling analysis. This procedure is reversed in the nonequilibrium relaxation method. We take the infinite-size limit first by preparing a very large system and stopping the simulation before the finite-size effects appear. Then, the infinite-time (equilibrium) limit is taken by the finite-time-scaling analysis. The critical temperature and the critical exponents are estimated in the same manner as the standard finite-size-scaling analysis.

An advantage of the nonequilibrium relaxation method may be a better computational efficiency in the slow-dynamic systems. It is very important particularly when the computational resources are limited. In the equilibrium simulations, we discard the nonequilibrium relaxation MC steps and collect data only after the equilibrium states are realized. The discarding steps are very long in the slow systems. We must perform different MC simulations to obtain data on different-size systems. A linear size of system takes discrete and limited values due to the very long discarding steps. These disadvantages become advantages in the nonequilibrium relaxation method. We can use the long nonequilibrium MC data that have been discarded in the equilibrium simulations. Different time data (relaxation functions) can be obtained by one simulation. The finite-time-scaling fit is easier because the time data are almost continuous.

In general, there is a question whether the MC data are sufficient or not in the scaling analysis. In the finite-time-scaling analysis, the finite time, t, must be long enough to extract the $t\rightarrow\infty$ properties. In the finite-size-scaling analysis, the finite size, L, must be large enough to extract the $L\rightarrow\infty$ properties. Although it is not the definite answer to the question above, it is possible to check a size-time relation by observing the dynamic correlation length, $\xi(t)$, in the nonequilibrium relaxation method. This quantity directly exhibits how a typical size of correlated clusters grows with time. We may consider that the finite time is long enough if the dynamic correlation length satisfies various critical relations that must hold in the equilibrium limit.

The critical relations between the dynamic correlation length and other physical quantities lead us to find another scaling method to investigate the phase transitions. This is the nonequilibrium dynamic correlation-length scaling method we introduce in this paper. This scaling method works within the nonequilibrium relaxation scheme. Nonequilibrium relaxation functions of physical quantities are scaled by the dynamic correlation length. For example, we scale the magnetic susceptibility, χ, by plotting $\chi(t,T)/\xi(t,T)^{2-\eta}$ versus $\xi(t,T)/|T-T_c|^{-\nu}$, where T denotes the temperature, T_c denotes the critical temperature, and ν and η denote the critical exponents. This procedure is just a replacement of L with $\xi(t)$: we plot $\chi(L,T)/L^{2-\eta}$ versus $L/|T-T_c|^{-\nu}$ in the finite-size scaling.

The present scaling method removes a shortcoming of the finite-time-scaling analysis in the nonequilibrium-relaxation method. It is an ambiguity of the dynamic exponent, z. This exponent connects the correlation time, τ, and the correlation length, ξ, through the dynamic-scaling relation as $\tau\sim\xi^z$. We can obtain the critical temperature and the critical exponents apart from a value of z in the present scaling method. On the other hand, the exponent ν is not solely obtained in the finite-time-scaling analysis.[6–12] It always appears as a form $z\nu$. It is necessary to estimate z independently in order to obtain ν. However, a numerical estimate of z has been a tough task in the slow-dynamic systems. It sometimes depends on the temperature in spin-glass models.[13] We will show that an estimate of z is easy in the present scaling method using the finite-time and finite-size numerical data.

We also propose a definition of the dynamic correlation length. The Ornstein-Zernike formula[14] is often used to define this value. We show that this definition is not always

appropriate particularly in the nonequilibrium relaxation regime.

Section II explains models and a definition of the dynamic correlation length. Section III explains the scaling procedure. Numerical results are presented in Sec. IV. The ferromagnetic Ising model in three dimensions and the $\pm J$ Ising spin-glass model in three dimensions are taken as examples. Section V is devoted to the summary and discussions.

II. MODEL AND DEFINITION OF THE DYNAMIC CORRELATION LENGTH

Let us consider the following spin model to demonstrate the present scaling method:

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j,$$

where $\langle i,j \rangle$ denotes the nearest-neighbor pairs in the cubic lattice, J_{ij} denotes the exchange interaction, and S_i denotes the Ising spin. We consider the uniform ferromagnetic model ($J_{ij}=1$) and the spin-glass model ($J_{ij}=\pm J$ with an equal probability) in this paper.

The dynamics of the present MC simulations is the Metropolis type. The linear system size is denoted by L. The skewed periodic boundary conditions are applied: the spin number is $L \times L \times (L+1)$.

The correlation length of an ordered domain in spin models is usually defined by the correlation function $C(r) = \langle S_i S_{i+r} \rangle$ as[14]

$$C(r) = \langle S_i S_{i+r} \rangle \propto \frac{\exp[-r/\xi]}{r}$$

in the equilibrium state near the transition temperature. Here, we sometimes use $r^{1+\eta}$ instead of r in the denominator. In the Monte Carlo simulations, it is easier to estimate the correlation length by the Fourier transform of the susceptibility, $\chi(\boldsymbol{k})$, as[15]

$$\xi = \frac{1}{k_{\min}} \sqrt{\frac{\chi(0)}{\chi(\boldsymbol{k}_{\min})} - 1}, \quad \boldsymbol{k}_{\min} = \frac{2\pi}{L} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{1}$$

Here, $\boldsymbol{k}_{\min}$ denotes the smallest wave vector along one direction in a finite-size lattice. The expression is exact in the limit of $\boldsymbol{k}_{\min} \to 0$.

The $1/r$ factor in the spin-correlation function is only guaranteed by the mean-field theory in the equilibrium state.[14] It is not trivial that we may use Eq. (1) in the nonequilibrium relaxation process, where the system size is much larger than the correlation length: $\xi \ll L$. In such case, the spin-correlation function may decay in a simple exponential form without the $1/r$ factor. In order to check the r dependence, we plot in Fig. 1 the correlation functions, $C(r)$ and $rC(r)$, against r in the nonequilibrium regime ($t=501$) and in the equilibrium regime ($t=3981$). The system is the uniform ferromagnetic model in three dimensions. The temperature, $T=4.515$, is above the transition temperature.

The linearity of $rC(r)$ is better than that of $C(r)$ when $t=3981$. The $1/r$ factor is necessary in estimating the correlation length in the equilibrium state. On the other hand, the linearity is poor and convex upward when $t=501$. The linearity of $C(r)$ seems to be better.

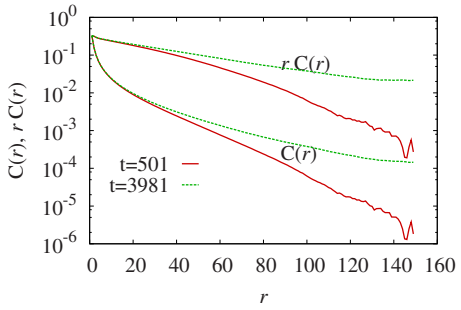

FIG. 1. (Color online) The r dependence of the spin-correlation function in the nonequilibrium regime ($t=501$) and in the equilibrium regime ($t=3981$). The system is the uniform ferromagnetic Ising model in three dimensions. The linear system size $L=299$ and the temperature $T=4.515$. The lower two lines are plots of $C(r)$ and the upper two lines are those of $rC(r)$.

If the correlation function exhibits a simple exponential decay as

$$C(r) = \langle S_i S_{i+r} \rangle \propto \exp[-r/\xi],$$

the correlation length should be estimated by the following expression:

$$\xi = \frac{1}{k_{\min}} \sqrt{\sqrt{\frac{\chi(0)}{\chi(\boldsymbol{k}_{\min})}} - 1}. \tag{2}$$

It is difficult to judge whether we should use Eq. (1) or Eq. (2) just by the linearity of $C(r)$ or $rC(r)$. The difference is small as shown in Fig. 1. There is also a possibility that another definition for $\xi(t)$ is appropriate.[16] In this paper, we only consider two expressions presented above and check a consistency with the dynamic-scaling relation. A definition that exhibits a better consistency is chosen.

The finite-t and finite-L data of the dynamic correlation length at the temperature, T, is denoted by $\xi(t,L,T)$. We expect that $\xi(t,L,T) \propto t^{1/z}$ from the dynamic-scaling relation. The correlation length divided by L becomes scale invariant at the transition temperature. Therefore, we plot $\xi(t,L,T)/L$ against $t^{1/z}/L$. The plotted data should ride on the single line with its slope unity at the critical temperature, if we choose a correct value of z. Let us call it a z scaling. In the paramagnetic phase, the larger-size data deviate downward. In the ordered phase, the larger-size data deviate upward. These are the finite-size effects that appear as the system approaches the equilibrium state.

Figure 2 shows the z scaling in the uniform ferromagnetic Ising model in three dimensions. Relaxation data at three temperatures are plotted in the same figure by shifting arbitrary in the vertical direction. The data estimated by Eq. (2) yield a better scaling and the slope unity. The dynamic exponent $z=2.03(2)$ is consistent with previous estimates: $z=2.04(3)$ from the scaling analysis,[17] $z=2.06(2)$,[3] $z=2.05(2)$,[18] and $z=2.055(10)$ (Refs. 19 and 20) from the nonequilibrium relaxation method, and $z=2.04(1)$ from the

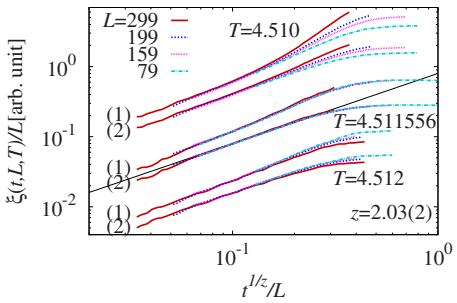

FIG. 2. (Color online) The z-scaling plot in the uniform ferromagnetic Ising model in three dimensions. Data labeled by (1) are estimated by Eq. (1) and those labeled by (2) are estimated by Eq. (2). Three temperatures are below (T=4.510), very close to (T=4.511556), and above (T=4.512) the transition temperature. A solid line is a guide for eyes with its slope unity.

damage-spreading analysis.[21] The data estimated by Eq. (1) are well scaled but the slope does not agree with unity. If we choose z so that it yields the slope unity, the relaxation data of different sizes do not ride on the same line. [The slope by Eq. (1) suggests that $\xi(t,L,T)/L \propto (t^{1/z}/L)^{1.13}$, but the scaling fails if we take $z=2.03/1.13=1.80$ for $t^{1/z}/L$.] Therefore, the dynamic-scaling relation contradicts itself even in the ferromagnetic model. It is an evidence that a use of Eq. (1) is not appropriate in the nonequilibrium relaxation scheme.

Figure 3 shows the same z scaling in the $\pm J$ Ising spin-glass model in three dimensions. Relaxation data at three temperatures are plotted in the same figure by shifting arbitrary in the vertical direction. The temperatures are above (T=1.40), near (T=1.18), and below (T=1.10) the transition temperature. The dynamic exponent at each temperature is estimated so that the relaxation data fall onto the same line with the slope unity. The linearity is better if we use Eq. (2). The estimate of $\xi(t,L,T)$ using Eq. (1) bends upward as the system approaches the equilibrium limit while the slope approaches unity in the nonequilibrium limit ($t \to 0$). This bending behavior may mislead us to underestimate the value of z. It is also noted that the value of z depends on the temperature as $zT \simeq 7$.[13] The finite-size effects appear when the temperature is above the transition temperature (T=1.40). The size effects are very weak in the spin-glass phase (T=1.10).

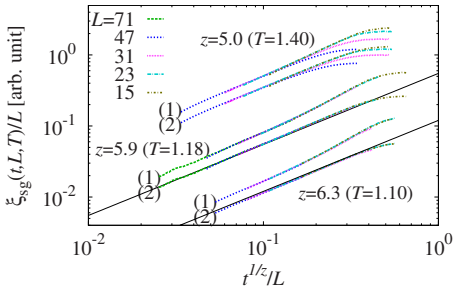

FIG. 3. (Color online) The z-scaling plot in the $\pm J$ Ising spin-glass model in three dimensions. Data labeled by (1) are estimated by Eq. (1) and those labeled by (2) are estimated by Eq. (2). Three temperatures, T=1.4, 1.18, and 1.10, are above, near, and below the transition temperature. A solid line is a guide for eyes with its slope unity.

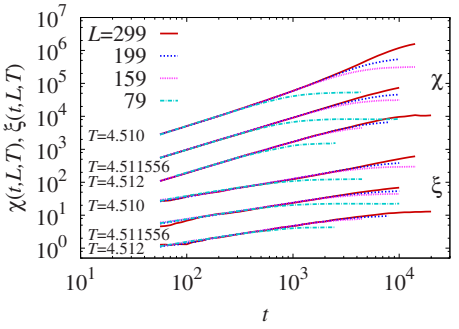

FIG. 4. (Color online) Size dependences of raw relaxation data of the dynamic correlation length, $\xi(t,L,T)$, and the magnetic susceptibility, $\chi(t,L,T)$ in the ferromagnetic model. The upper three data are the susceptibility and the lower three data are the dynamic correlation length.

It is shown that the dynamic correlation length in the nonequilibrium relaxation process should be estimated by using Eq. (2). The dynamic-scaling relation is satisfied in the nonequilibrium process if we take this definition. We perform our scaling analysis using Eq. (2) in this paper.

III. SCALING PROCEDURE

We explain our scaling procedures step by step in this section. The first two steps are the standard procedure of the nonequilibrium relaxation method. The relaxation functions of the physical quantity and the dynamic correlation length are obtained.

(1) We prepare a system with a linear scale L and perform a MC simulation at the temperature T. We start simulations from the paramagnetic state. The correlation length is zero in the initial state. A physical quantity, A, is measured at an MC step, t, and is stored in memory as $A(t,L,T)$. Changing a random number sequence, we perform independent simulations and take an average of $A(t,L,T)$ over these runs.

(2) We change the system size and perform simulations. Relaxation functions of $A(t,L,T)$ and $\xi(t,L,T)$ for different sizes are obtained and compared to check the finite-size effects. Data that are free from the size effects are denoted by $A(t,T)$ and $\xi(t,T)$.

Now, we will perform three scaling analyses. The first scaling analysis determines the dynamic exponent, z. This analysis is already presented in Figs. 2 and 3 as the z scaling.

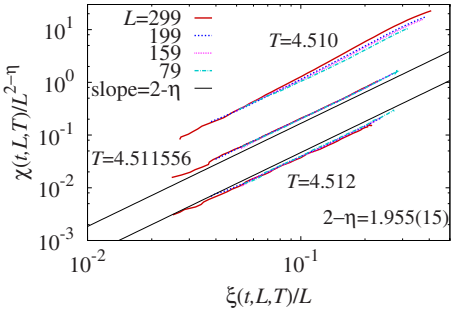

FIG. 5. (Color online) The η-scaling plot in the ferromagnetic model. The raw relaxation data of Fig. 4 are scaled.

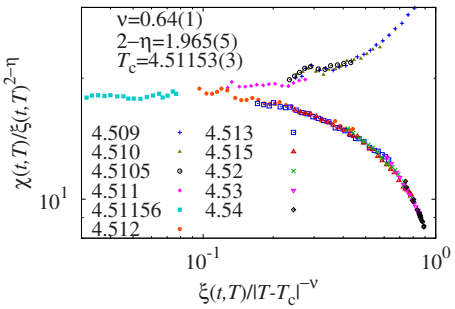

FIG. 6. (Color online) The dynamic correlation-length scaling plot of the magnetic susceptibility in the ferromagnetic model.

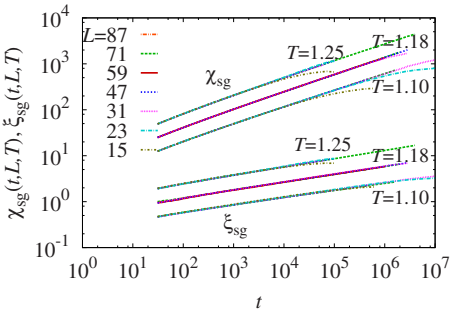

FIG. 7. (Color online) Size dependences of raw relaxation data of the dynamic correlation length, $\xi_{sg}(t,L,T)$, and the spin-glass susceptibility, $\chi_{sg}(t,L,T)$ in the $\pm J$ Ising spin-glass model in three dimensions. Three temperatures are above, near, and below the transition temperature, respectively. Data of $T=1.10$ are divided by 2 and those of $T=1.25$ are multiplied by 2 in order to separate from data of $T=1.18$.

This analysis is not always necessary. It is independent from the other two scaling analyses. We only perform this scaling when we need a value of z. This is a clear difference from (and is probably an advantage over) the standard nonequilibrium relaxation method with the finite-time-scaling analysis.

(3) *z scaling*. At the temperature near the critical temperature, we plot $\xi(t,L,T)/L$ versus $t^{1/z}/L$ in a log-log scale so that all the scaled data fall on a single scaling function with its slope unity. Only a value of z is a control parameter. Examples are Figs. 2 and 3.

The second scaling analysis determines the anomalous exponent η. Let us call it an η scaling. We perform the scaling analysis using the relaxation function of $\chi(t,L,T)$ and $\xi(t,L,T)$. Since the simulation starts from the paramagnetic state, both χ and ξ are expected to increase algebraically in time as $\chi \sim \xi^{2-\eta}$ when T is near T_c. Then, the nonequilibrium relaxation functions of $\chi(t,L,T)$ and $\xi(t,L,T)$ depend on L as

$$\frac{\chi(t,L,T)}{L^{2-\eta}} \propto \left(\frac{\xi(t,L,T)}{L}\right)^{2-\eta}. \tag{3}$$

(4) *η scaling*. We plot $\chi(t,L,T)/L^{2-\eta}$ against $\xi(t,L,T)/L$ in a log-log scale and determine η so that all relaxation functions fall on the same line with its slope $2-\eta$. In this scaling, η is an only scaling parameter. The determination of η is done first by this scaling plot. Examples are Figs. 5 and 8 in Sec. IV. Here, it is not necessary to know the critical temperature precisely. We only need a rough estimate. Since we work with the nonequilibrium relaxation method, the critical relaxation is observed near the critical temperature in the nonequilibrium process.

The last scaling analysis determines the critical temperature, T_c, and the exponent, ν. In this scaling, we only use data that are free from the size effect, $\xi(t,T)$ and $\chi(t,T)$. The equilibrium value of the correlation length diverges as $\xi \sim |T-T_c|^{-\nu}$. We suppose that this scaling relation is satisfied even in the nonequilibrium relaxation process. The susceptibility $\chi(t,T)$ should be scaled by $\xi(t,T)^{2-\eta}$ with η estimated above.

(5) *Dynamic correlation-length scaling*. We plot the relaxation functions $\chi(t,T)/\xi(t,T)^{2-\eta}$ versus $\xi(t,T)/|T-T_c|^{-\nu}$ for various temperatures and determine T_c and ν so that all the data ride on a single scaling function. Examples are Figs. 6 and 9 in Sec. IV. Here, a value of η is tuned to make the best scaling behavior.

It is noted that the finite-size L in the finite-size-scaling analysis is replaced by the dynamic correlation length $\xi(t,T)$ in the present analysis. We consider that the present scaling analysis is a natural extension from the finite-size-scaling analysis when we work within the nonequilibrium relaxation scheme.

IV. NUMERICAL RESULTS

A. Ferromagnetic model

We present numerical results in the ferromagnetic Ising model in three dimensions. Figure 4 shows raw relaxation

TABLE I. Previous estimates of T_c, ν, and η for the ferromagnetic Ising model in three dimensions.

Reference	T_c	ν	$2-\eta$
Ferrenberg and Landau (Ref. 22)	4.51142(5)	0.6289(8)	1.970(11)
Blöte *et al.* (Ref. 23)	4.51152(2)	0.6301(8)	1.963(5)
Salman and Adler (Ref. 24)	$4.51143(^{+10}_{-4})$		
Guida and Zinn-Justin (Ref. 25)		0.6304(13)	1.966(7)
Campostrini *et al.* (Ref. 26)		0.63002(23)	1.9636(4)
Ito *et al.* (Ref. 19)	4.511424(30)	0.635(5)	1.976(24)
Ito *et al.* (Ref. 20)	4.511526(30)	0.635(5)	1.976(24)
The present work	4.51153(3)	0.64(1)	1.965(5)

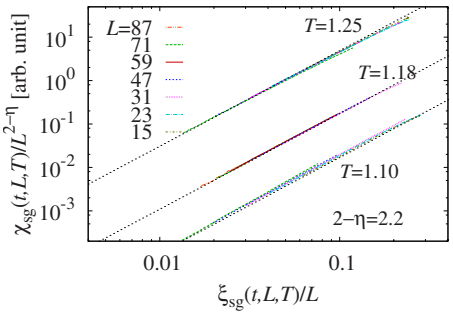

FIG. 8. (Color online) The η scaling in the $\pm J$ Ising spin-glass model in three dimensions. The raw relaxation data of Fig. 7 are scaled. Dotted lines are guide for eyes with its slope $2-\eta=2.2$.

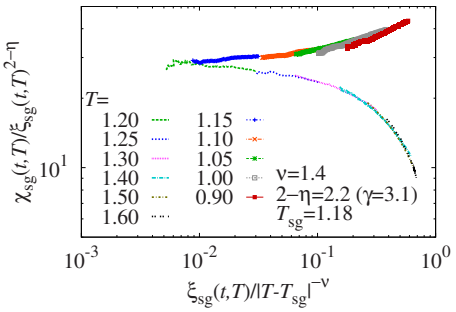

FIG. 9. (Color online) The dynamic correlation-length scaling plot of the spin-glass susceptibility.

data of the dynamic correlation length and the magnetic susceptibility. The dynamic correlation length is estimated by Eq. (2). The logarithmic slope of ξ at $T=4.511556$ is $1/z$ and that of χ is $(2-\eta)/z$. We plot in Fig. 5 a scaling result that determines η. The relaxation data are scaled with $2-\eta=1.955\pm0.015$, where the logarithmic slope of the scaled function agrees with $2-\eta$. If we use the data at $T=4.512$, the estimated value of $2-\eta$ becomes $2-\eta=1.90(5)$. The estimate improves as the temperature approaches the true critical temperature.

The final dynamic correlation-length scaling plot of the susceptibility is shown in Fig. 6. Here, only the nonequilibrium-relaxation data that are free from the finite-size effects are plotted. For an estimated value of $(2-\eta)$ between 1.94 and 1.97, we search for values of T_c and ν so that the scaled data exhibit the best scaling behavior. The obtained values are $T_c=4.51153(3)$, $\nu=0.64(1)$, and $2-\eta=1.965(5)$. They are consistent with the previous estimates listed in Table I.

B. Spin-glass model

In this section, we present the numerical results of the $\pm J$ Ising spin-glass model in three dimensions. The spin-glass correlation length is estimated from the Fourier transform of the spin-glass susceptibility, χ_{sg}, defined by

$$\chi_{sg} = \frac{1}{N}\left[\sum_{i,j}\langle S_i S_j\rangle^2\right]_c,$$

where the bracket $[\cdots]_c$ denotes the configurational average and the bracket $\langle\cdots\rangle$ denotes the thermal average. The thermal average is estimated by the average over different real replicas

$$\langle S_i S_j\rangle = \frac{1}{m}\sum_{A=1}^{m} S_i^{(A)} S_j^{(A)}.$$

A replica number is denoted by m and the superscript (A) denotes a replica index. We prepare m real replicas for each random bond realization with different initial spin configurations. Spin states of each replica are updated using different random number sequences. The thermal average is taken only by this replica average in our nonequilibrium relaxation scheme. A replica number controls an accuracy of the thermal average. It is set to 256 in this paper. The dynamic spin-glass correlation length is estimated by Eq. (2), where χ is replaced by χ_{sg}. It has been defined by Eq. (1).[27,28]

Figure 7 shows the raw relaxation functions of the spin-glass susceptibility and the dynamic spin-glass correlation length. Here, the logarithmic slope of ξ_{sg} is $1/z(T)$ and that of χ_{sg} is $(2-\eta)/z(T)$. Figure 8 shows the scaling plot determining a value of η. The straight-line scaling is possible near $(T=1.18)$ and above $(T=1.25)$ the transition temperature by using the same value of $2-\eta=2.2$. The scaling behavior is

TABLE II. Previous estimates of T_{sg}, ν, and η for the $\pm J$ Ising spin glass model in three dimensions.

Reference	T_{sg}	ν	$2-\eta$
Kawashima and Young (Ref. 29)	1.11(4)	1.7(3)	2.35(5)
Palassini and Caracciolo (Ref. 27)	1.156(15)	1.8(2)	2.26(4)
Ballesteros *et al.* (Ref. 28)	1.138(10)	2.15(15)	2.337(15)
Campbell *et al.* (Ref. 30)	1.11	2.72(8)	2.40(4)
Hasenbusch *et al.*(Ref. 31)	1.109(10)	2.45(15)	2.375(10)
Bhatt and Young (Ref. 32)	$1.2(^{+0.1}_{-0.2})$	1.3(3)	2.3(2)
Ogielski and Morgenstern (Ref. 33)	1.20(5)	1.2(1)	
Mari and Campbell (Ref. 34)	1.195(15)	1.35(10)	2.225(25)
Nakamura *et al.* (Ref. 9)	1.17(4)	1.5(3)	2.4(1)
The present work	1.18(1)	1.40(5)	2.20(1)

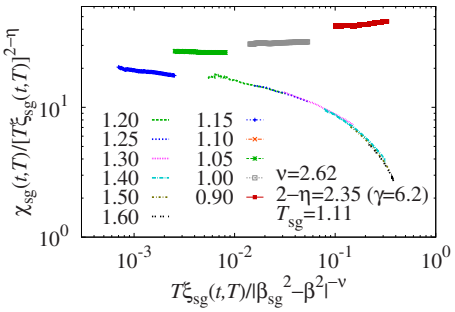

FIG. 10. (Color online) The dynamic correlation-length scaling plot of the spin-glass susceptibility using $|\beta_c^2-\beta^2|$ as the scaling variable.

poor in the low-temperature phase ($T=1.10$). It is noted that the scaling at $T=1.10$ becomes good if we use a value $2-\eta=2.3$. This value agrees with the previous estimate that gives $T_{sg} \simeq 1.1$.[27–31]

Using the estimated value of $2-\eta$, we plot the dynamic correlation-length scaling results in Fig. 9. The scaling behavior is good when the temperature is above the estimated spin-glass transition temperature, $T_{sg}=1.18$. The scaling behavior is fairly good below T_{sg}.

The previous estimates[9,27–34] for T_{sg} and critical exponents are summarized in Table II. They are roughly categorized into two groups. One[27–31] gives T_{sg} close to 1.1 and ν close to 2. The other[9,32–34] gives T_{sg} close to 1.2 and ν close to 1.3. The present result is consistent with the latter group. The latter group mostly takes the dynamic approach to the phase transition. Recently, Hukushima and Campbell[35] discussed that this discrepancy can be understood by the strong corrections to scaling.

Campbell *et al.*[30] proposed the β-scaling analysis, which uses $(\beta^2-\beta_{sg}^2)$ as the scaling variable. They estimated the transition temperature and the critical exponents as $T_{sg}=1.11$, $\nu=2.72(8)$, and $2-\eta=2.40(4)$. We also try this β-scaling analysis in Fig. 10. The high-temperature data are well scaled but the low-temperature data cannot be scaled. Our estimates are $T_c=1.11$, $\nu=2.62$, and $2-\eta=2.35$, which are consistent with their estimates but disagree with our present estimates using $(T-T_{sg})$. This discrepancy suggests that the present numerical simulations are not sufficient to extract the true critical phenomena both in a size scale and a time scale in the spin-glass model.[35] We checked that the β scaling is possible using the same transition temperature and critical exponents in the ferromagnetic model (figure not shown). The present size and time scales are sufficient in the ferromagnetic model.

V. SUMMARY AND DISCUSSION

The dynamic correlation-length scaling method is introduced. The basic idea of this method is that we investigate the phase transition through the correlation length. It is found that the scaling relations among physical quantities hold even in the nonequilibrium relaxation process. We can use finite-time and finite-size data in the scaling analysis as shown in Figs. 2, 3, 5, and 8. Although the raw relaxation data (Figs. 4 and 7) exhibit the finite-size effects, we can scale them to one scaling line. The critical divergence is observed from very early stage of the nonequilibrium relaxation process, if we scale the data by the dynamic correlation length. These finite time-size data have been discarded in previous scaling methods. We can utilize them in the present scaling method and improve the total computational efficiency.

The present dynamic correlation-length scaling analysis is regarded as an extension of the finite-size-scaling analysis replacing the size L with the dynamic correlation length $\xi(t)$. We may consider that the finite-time relaxation data at a time t corresponds to the equilibrium data of the size L with $L=\xi(t)$.

Since the present scaling method is entirely based on the dynamic correlation length, the definition is very important. We found that a use of the definition based on the Ornstein-Zernike formula is not appropriate in our scheme. Since the formula is based on the mean-field approximation, the present nonequilibrium process may be out of the applicable range of the approximation. The relaxation function of the dynamic correlation length estimated by Eq. (1) exhibits an extra increase before reaching an equilibrium value, as shown in Figs. 2 and 3. The logarithmic slope, which is $1/z$, is then overestimated. It affects a value of ν in the conventional finite-time-scaling analysis of the nonequilibrium relaxation method.[36] The introduced definition, Eq. (2), is based on a simple exponential decay of the correlation function, which may be valid when $L \gg \xi$. We also comment that the definition Eq. (2) may be used even in the equilibrium state where the mean-field approximation is not valid.

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