Oscillatory transient regime in the forced dynamics of a nonlinear auto oscillator

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We demonstrate that the forced transient dynamics of a nonlinear (nonisochronous) auto-oscillator is qualitatively different from the dynamics of a quasilinear oscillator described by the classical Adler's model. If the normalized amplitude μ of the driving force exceeds a certain critical value μ_{cr}, the transition to the synchronized regime becomes oscillatory with a frequency proportional to $\propto\sqrt{\mu-\mu_{cr}}$ and a synchronization time that is almost independent of μ. The discovered effect is illustrated on the example of a strongly nonlinear spin torque nano-oscillator (STNO) where the finite transient synchronization time can limit the possible range of STNO modulation frequencies.

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Nonlinear (or nonisochronous) auto-oscillators, i.e., auto-oscillators where the generated frequency depends on the oscillation amplitude, have several unique properties that make them qualitatively different from the common quasilinear (or isochronous) auto-oscillators.[1,2] Recently, nonisochronous auto-oscillating systems have become extremely important due to the large interest in nanosized magnetic spin torque nano-oscillators (STNO) (Refs. 3–5) that possess strong and nontrivial nonlinear properties.[6] STNOs are also interesting for microwave applications due to their unique combination of attractive properties, such as a wide range of generated frequencies,[7] fast modulation rates,[8,9] and easy integration into modern on-chip nanoelectronic circuits.

Recently, several experimental groups have performed studies of STNO synchronization (or injection locking) to an external microwave current.[10–13] These experimental results were analyzed, without exception, using the classical theory of injection locking of quasilinear isochronous auto-oscillators developed in 1946 by Adler.[14] In Adler's theory a simple dynamical equation for the phase difference ψ between the auto-oscillation and the injected driving signal is obtained in the form

$$\frac{d\psi}{dt} = -\Delta\omega - F\sin(\psi). \tag{1}$$

Here $\Delta\omega=\omega_e-\omega_0$ is the mismatch between the frequency of the injected signal ω_e and the frequency of free auto-oscillations ω_0 and F is proportional to the amplitude of the injected signal. Despite its simplicity, Eq. (1) correctly describes phase locking of quasilinear (or isochronous) auto-oscillators of different physical nature and allows one to find all the major characteristics of the phase-locking process. In particular, Eq. (1) predicts the stationary characteristics, such as the frequency interval of phase locking $|\Delta\omega|<F$ and the stationary phase relation $\psi_A=-\arcsin(\Delta\omega/F)$ between the driving signal and the locked oscillations. In addition, Eq. (1) allows one to analyze the transitional process in phase locking and predicts that the phase ψ approaches its locked value ψ_A *monotonically*—approximately exponentially with a time constant $\tau_A=1/(F\cos\psi_A)$ that is inversely proportional to the amplitude of the driving signal F.

In this Brief Report we show that for nonlinear auto-oscillators and sufficiently large periodic driving signals, $F>F_{cr}$, Adler's model breaks down and fails to describe both the stationary and transitional properties of the synchronization process. The most striking discrepancies are: (i) pronounced transient *oscillations* of the auto-oscillator phase difference ψ during its approach to synchronization and (ii) a synchronization time τ_s, which is *independent* of the driving amplitude F. Although the obtained results are general and equally applicable to any nonisochronous auto-oscillator, they are particularly important for STNOs, where the critical amplitude of the driving signal which represents the boundary between the regions of Adlerian and non-Adlerian transitional dynamics is surprisingly small.

The above-mentioned qualitative discrepancies between the transitional synchronization dynamics of a nonisochronous auto-oscillator and the Adler's model were discovered in our direct numerical simulations of STNO synchronization to an external microwave current. In these macrospin simulations (similar to the ones performed in Refs. 12 and 15) we assumed that the STNO has a standard spin-valve geometry consisting of a thin permalloy "free" layer separated by a Cu spacer from a synthetic antiferromagnet "fixed" layer placed in a bias magnetic field directed perpendicular to the layers and tilting the in-plane magnetization of the "fixed" magnetic layer at an angle γ_0 to the bias field. In such a case, the normalized (unit-length) uniform magnetization vector $\mathbf{m}$ of the STNO free layer obeys the Landau-Lifshitz-Gilbert-Slonczewski equation of the form[15–17]

$$\frac{d\mathbf{m}}{dt} = -|\gamma|\mathbf{m}\times\mathbf{H}_{eff} + \alpha\mathbf{m}\times\frac{d\mathbf{m}}{dt} + |\gamma|\alpha_J\mathbf{m}\times(\mathbf{m}\times\mathbf{p}). \tag{2}$$

Here $|\gamma|/2\pi=28$ GHz/T is the modulus of the gyromagnetic ratio for electron spin, $\mathbf{H}_{eff}=H_e\mathbf{z}-M_s(\mathbf{m}\cdot\mathbf{z})\mathbf{z}$ is the effective magnetic field, $\mu_0H_e=1.5$ T is the external magnetic field applied along the normal $\mathbf{z}$ to the STNO free layer,

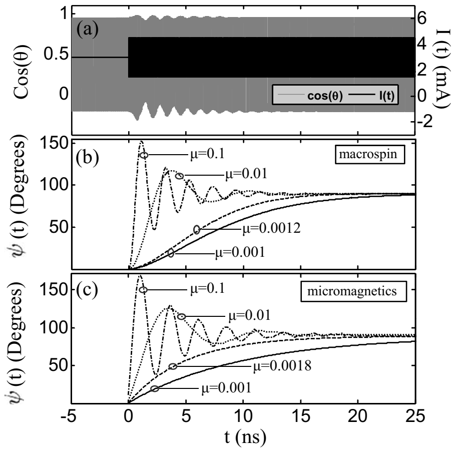

FIG. 1. (a) Time dependence of the STNO signal calculated in a macrospin approximation for $\mu = I_{rf}/I_{\mathrm{dc}} = 0.5$; I_{rf} was switched on at $t=0$ after allowing the STNO to settle into a stable free-running precession for 50 ns. (b) Macrospin and (c) micromagnetic simulations of the transient behavior of the phase difference between the STNO signal and the injected microwave current I_{rf} for different values of μ.

$\mu_0 M_s = 0.8$ T is the saturation magnetization of the free layer, $\alpha = 0.01$ is the Gilbert damping constant, and α_J is the spin torque magnitude defined as $\alpha_J = \frac{\hbar \epsilon I}{2\mu_0 M_s e V}$, where $\hbar$ is the Planck constant, $\varepsilon = 0.35$ is the dimensionless spin torque efficiency, I is the applied current, μ_0 is the magnetic permeability of free space, e is the electron charge, and $V = 3 \times 10^4$ nm^3 is the volume of the free layer. The unit vector $\mathbf{p} = \cos(\gamma_0)\mathbf{z} + \sin(\gamma_0)\mathbf{x}$ in Eq. (2), which defines the direction of spin polarization in the current I, coincides with the magnetization direction of the STNO fixed layer, which we set to $\gamma_0 = 60°$.

The critical dc bias current $I = I_{\mathrm{dc}} = I_c$ at which the stable auto-oscillations appeared in the model Eq. (2) was $I_c = 2.32$ mA. The total applied current in the nonautonomous (driven) regime contained the constant part $I_{\mathrm{dc}} = 3$ mA [supercriticality parameter $\zeta = I_{\mathrm{dc}}/I_c = 1.29$ resulting in a free-running STNO frequency of $\omega_0/(2\pi) = 25.3$ GHz] and the variable (microwave) part I_{rf}: $I(t) = I_{\mathrm{dc}} + I_{rf}\sin(\omega_e t)$, where ω_e is the driving frequency. The normalized amplitude of the microwave signal (or modulation depth) $\mu = I_{rf}/I_{\mathrm{dc}}$ was varied. In the phase-locked regime the generated STNO frequency becomes exactly equal to ω_e and a fixed (independent of the initial conditions) phase difference ψ_0 develops between the driving microwave current and the STNO oscillation.

The results of the macrospin numerical simulations demonstrating the STNO transition from the free running to phase-locked regime are shown in Fig. 1 for various amplitudes of the driving current I_{rf}. The STNO was first prepared in a free-running state ($I_{rf} = 0$) for 50 ns to achieve a stable free-running regime [only the last 5 ns of the free-running regime are shown in Fig. 1(a)]. At $t=0$ the microwave current I_{rf} was switched on with $\omega_e = \omega_0$. Since ω_e and ω_0 coincide, the phase locking manifests itself only by establishing a fixed phase relations between these oscillations. Figure 1(a) shows the time dependence of $\cos[\theta(t)] = \mathbf{m}(t) \cdot \mathbf{p}$ (which is proportional to the STNO output signal) for the normalized driving amplitude $\mu = I_{rf}/I_{\mathrm{dc}} = 0.5$. One can clearly see a transient beating of the envelope of the STNO signal, which indicates an *oscillatory* approach to the phase-locked state. This oscillatory transition is shown explicitly in Fig. 1(b), where we plot the time dependence of the phase difference $\psi(t)$ between the phasors representative of the STNO output signal and the external signal, respectively, for several values of μ.

First of all, we note that the stationary value of the phase difference ψ_0 is substantially different from zero ($\psi_0 \approx 90°$) in contrast with what one would expect from Eq. (1) for $\Delta\omega = \omega_e - \omega_0 = 0$. This significant *intrinsic* phase shift found for the first time in Ref. 12 is caused by the strong nonlinearity of the STNO generation frequency.[6,18,19] Second, one can see from Fig. 1(b) that the transient dependence of $\psi(t)$ is monotonic only for extremely small values of μ, whereas for all reasonable modulation depths, strong phase oscillations develop in the transient regime. The critical modulation depth, separating regions of monotonic (Adlerian) and oscillatory (non-Adlerian) transitional dynamics is as small as $\mu_{cr} = 0.0012$ [dashed curve in Fig. 1(b)].

To confirm that the observed transient oscillations are not an artifact of the macrospin approximation we have also performed full-scale micromagnetic simulations (similar to the simulations performed in Ref. 20) of injection locking of an STNO with the free layer in the form of circular nanopillar (diameter $D = 55$ nm, thickness $d = 3$ nm, exchange constant $A = 1.0 \times 10^{-11}$ J/m, and all other parameters the same as in the macrospin case).[21] Figure 1(c) shows the simulated transient phase dynamics for several values of modulation depth μ. It is clear from Fig. 1(c) that micromagnetic simulations also demonstrate the transition from an Adlerian (monotonic) to a non-Adlerian (oscillatory) regime. The critical modulation depth for the nanopillar STNO $\mu_{cr} = 0.0018$ is close to the one obtained in the macrospin case. We would like to note that since the values of μ_{cr} are so small, injection locking of STNOs almost always takes place in the non-Adlerian regime.

Another striking feature of the non-Adlerian transient dynamics is the very weak I_{rf} dependence of the synchronization time (time needed to reach the phase-locked state). For example, the envelopes of $\psi(t)$ for $\mu > \mu_{cr}$ in Figs. 1(b) and 1(c) have essentially the same time constant τ_s whereas the classical Adler's model Eq. (1) predicts that the time constant in these two cases should differ by a factor of 10. This is further illustrated in Fig. 2, where we show the dependence of the frequency of transient phase oscillations Ω on μ [panel (a)] and the decay constant $\Gamma_s = 1/\tau_s$ of these oscillations [panel (b)] both from macrospin simulations (filled dots) and micromagnetic simulations (empty squares). Oscillations appear for $\mu > \mu_{cr}$, and their frequency Ω increases with μ and reaches gigahertz values for accessible modulation depths $\mu \sim 0.5$. The decay rate Γ_s increases approximately linearly with μ [following the Adler's model Eq. (1)] only for $\mu < \mu_{cr}$, whereas in the non-Adlerian region $\mu > \mu_{cr}$ it remains virtually constant. This upper limit in Γ_s is expected to also have consequences for frequency modulation of STNOs since it may limit the maximum modulation frequency.

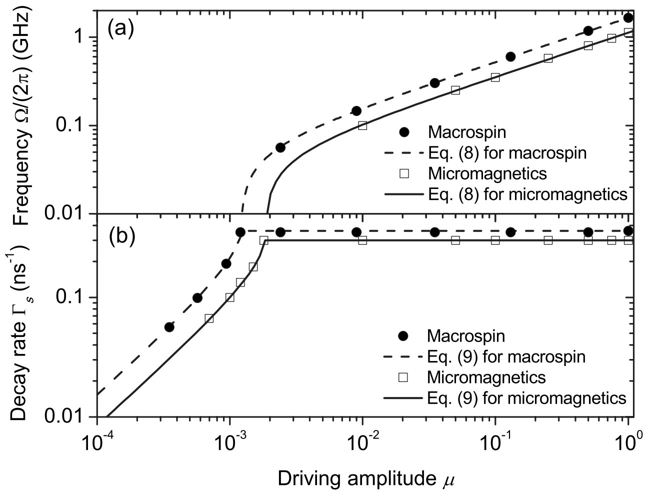

FIG. 2. Dependence of the frequency Ω of (a) the transient STNO phase oscillations and (b) its decay constant Γ_s on the normalized driving amplitude μ.

To understand the origin of the observed non-Adlerian transient synchronization dynamics, we consider a generic model of a nonlinear (or nonisochronous) auto-oscillator, where the generation frequency $\omega(p)$ is a function of the generated power p.[6] In such an oscillator, even small power fluctuations, $\delta p = p - p_0$, from the free-running power p_0 may result in significant deviations of the generated frequency, $\omega(p) - \omega(p_0) \approx N\delta p$. Since the nonlinear frequency shift coefficient $N = d\omega(p)/dp$ adds the term $N\delta p$ to Eq. (1), which couples fluctuations in frequency and power, the original single Adler's equation must be replaced with a set of equations where the fluctuations of the generated power are also accounted for

$$\frac{d\psi}{dt} = -\Delta\omega - F \sin\psi + N\delta p, \tag{3a}$$

$$\frac{d\delta p}{dt} = -2\Gamma_p \delta p + 2p_0 F \cos\psi. \tag{3b}$$

Here $\psi(t) = \phi(t) - \omega_e t$ is the phase difference between the STNO signal and the external signal, F is the normalized external signal amplitude, and Γ_p is the damping rate of power fluctuations. For a linear oscillator ($N=0$) one retrieves the original Adler's model, Eq. (1).

For the macrospin STNO model, all parameters in Eq. (3) can be calculated analytically[6] to give: $p_0 = (\zeta-1)\omega_H/(2\omega_M)$, $N = 2\omega_M$, $\Gamma_p = \alpha\omega_H(\zeta-1)$, $F = \mu\cdot\alpha\omega_H \tan(\gamma_0)/(4\sqrt{p_0})$, $\omega_H = |\gamma|(H_a - M_s)$ is the ferromagnetic resonance frequency, and $\omega_M = |\gamma| M_s$, and these expressions are valid for moderate supercriticalities $\zeta \leq 1.5$. In the micromagnetic case, the parameters can be estimated using the same expressions.

The only stable stationary solution of Eq. (3) has the form

$$\psi_0 = \arctan(\nu) - \arcsin(\Delta\omega/\Delta\omega_0), \tag{4a}$$

$$\delta p_0 = p_0 \frac{\nu\Delta\omega + \sqrt{\Delta\omega_0^2 - \Delta\omega^2}}{(1+\nu^2)\Gamma_p}, \tag{4b}$$

where $\nu = Np_0/\Gamma_p$ is the dimensionless nonlinearity parameter (in our case $\nu \approx 1/\alpha = 100$) and $\Delta\omega_0 = \sqrt{1+\nu^2}F$ is the nonlinearity-enhanced frequency interval of phase locking.[6] The first term in Eq. (4a) describes the above-mentioned intrinsic phase shift of a strongly nonlinear STNO.

Linearizing Eq. (3) near the stable solution in Eq. (4), one can study the *transient* synchronization regime and find the decay rate λ of phase and power deviations from the stationary phase-locked state

$$\lambda = \Gamma_p + \frac{1}{2}F\cos\psi_0 \pm \sqrt{\left(\Gamma_p - \frac{1}{2}F\cos\psi_0\right)^2 - 2\nu\Gamma_p F \sin\psi_0}. \tag{5}$$

For a quasilinear ($\nu=0$), or Adlerian, auto-oscillator, Eq. (5) gives $\lambda_1 = 2\Gamma_p$ and $\lambda_2 = F\cos\psi_0$. λ_1 describes the damping rate of the power deviations δp and λ_2 the decay rate of the pure phase deviations $\psi - \psi_0$. The synchronization time $\tau = 1/\lambda$ quantifies the overall time needed to reach a phase-locked state. Since $\Gamma_p \gg F$ for realistic parameters, the synchronization time of an Adlerian oscillator is given by $\tau_A = 1/(F\cos\psi_0) \propto 1/F$.

To analyze the case of a strongly nonlinear ($|\nu| \gg 1$) auto-oscillator, we note that in this case one can neglect $F\cos\psi_0$ and simplify Eq. (5) to

$$\lambda = \Gamma_p\left(1 \pm \sqrt{1 - F/F_{cr}}\right) = \Gamma_p\left(1 \pm \sqrt{1 - \mu/\mu_{cr}}\right), \tag{6}$$

where the critical signal amplitude is $F_{cr} = \Gamma_p/(2\nu\sin\psi_0)$ or, in terms of the critical modulation depth,

$$\mu_{cr} \approx \frac{\alpha}{\tan\gamma_0}(\zeta-1)^{3/2}\sqrt{\frac{2\omega_H}{\omega_M}}. \tag{7}$$

For the material parameters used in our macrospin simulations we get the analytical value $\mu_{cr} = 0.0012$, which is in excellent agreement with the simulated result.

For $\mu > \mu_{cr}$ the decay rates λ become complex [see Eq. (6)] and describe an oscillatory approach to the phase-locked state. The frequency of these transient oscillations is given by

$$\Omega = \Gamma_p\sqrt{\mu/\mu_{cr} - 1}, \tag{8}$$

where both $\Gamma_p = 0.36\ \text{ns}^{-1}$ and $\mu_{cr} = 0.0012$ can be calculated using the material parameters in our macrospin simulation. The calculated $\Omega(\mu)$ is shown as a dashed line in Fig. 2(a) and describes the simulated result perfectly. An equally good agreement [solid black line in Fig. 2(a)] is found in the micromagnetic case where the value $\Gamma_p = 0.3\ \text{ns}^{-1}$ was found numerically from the decay constant Γ_s in the non-Adlerian regime $\mu > \mu_{cr}$ (see below).

The decay constant Γ_s of the phase oscillation is given by the smallest of λ's in Eq. (6) when both of them are real (Adlerian regime), and by their real part when they are complex (non-Adlerian regime)

$$\Gamma_s = \begin{cases} \Gamma_p(1 - \sqrt{1 - \mu/\mu_{cr}}) & \mu < \mu_{cr} \\ \Gamma_p & \mu > \mu_{cr}. \end{cases} \quad (9)$$

In Fig. 2(b) we plot the analytically calculated Γ_s using the parameters of our macrospin and micromagnetic simulations. The agreement between the analytical calculation and the simulated results is again remarkable. Note, that in the non-Adlerian regime $\mu > \mu_{cr}$ the decay rate $\Gamma_s = \Gamma_p$ is constant, and measurements of Γ_s in this regime can be used to find the intrinsic damping rate Γ_p of the STNO.

Thus, the above presented analytical analysis shows that the transient non-Adlerian synchronization dynamics of STNOs observed in our numerical simulations is a *general* effect, which is present in *any nonisochronous* auto-oscillator. Our analysis also provides quantitative analytic expressions for both the frequency Eq. (8) and damping rate Eq. (9) of the transient phase oscillations in a nonisochronous auto-oscillator.

We can suggest a way to observe the transient phase oscillations of an STNO. If the injected current is pulsed with the repetition rate on the order of $1/\Gamma_p$, large sidebands at the frequencies $\omega_0 \pm \Omega$ should appear in the spectrum of the STNO oscillations. Since the transient frequency Ω may be significantly larger than the STNO generation linewidth, both the position and the shape of these sidebands can be measured experimentally, providing important information about such intrinsic STNO parameters as the nonlinear frequency shift N and the damping rate Γ_p of power fluctuations. In Fig. 3 we show the results of numerical simulations of an STNO phase locking to a *pulsed* microwave driving signal with a repetition period of 16 ns. One can clearly see the sidebands caused by the intrinsic transient STNO phase oscillations and their expected dependence on the modulation depth.

In conclusion, we have shown that the transient nonautonomous dynamics of an STNO for a sufficiently strong external signal cannot be described by the classical Adler's model. The reason for this non-Adlerian behavior is the strong nonlinearity of the auto-oscillator generation frequency, which couples power and phase fluctuations. While directly applicable to STNOs, this is a general property of all nonisochronous auto-oscillators.

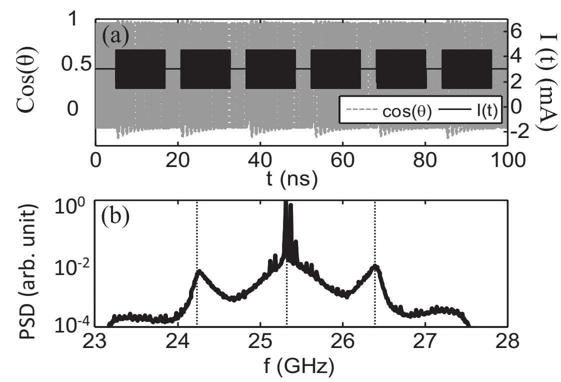

FIG. 3. (a) Time dependence of the STNO signal when subject to a pulsed microwave current with $\mu = 0.5$. (b) Spectrum of the STNO oscillations. Vertical lines in (b) indicate the free-running frequency ω_0 and expected positions of the sidebands $\omega_0 \pm \Omega$. (Other narrow sidebands are due to the direct frequency modulation at the pulse repetition rate.)

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