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Topological odd-parity superconductors

Masatoshi Sato

The Institute for Solid State Physics, The University of Tokyo, Kashiwanoha 5-1-5, Kashiwa, Chiba 277-8581, Japan

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We investigate topological phases of full-gapped odd-parity superconductors without or with time-reversal invariance. For odd-parity superconductors, a combination of the inversion and the $U(1)$ gauge symmetry is manifestly preserved, and the combined symmetry enables us to characterize the topological phases from the knowledge of the electron dispersion. Topologically protected gapless boundary states are predicted from the Fermi-surface topology. Simple criteria for topological odd-parity superconductors, in particular, that for a non-Abelian topological phase supporting a non-Abelian anyon are provided. Implications for nodal odd-parity superconductors are also discussed.

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Recently, there has been considerable interest in topological phases which are characterized by bulk topological invariants and the corresponding topologically protected gapless boundary states. The prototype of the topological phase is the integer quantum-Hall states, where the band Thouless-Kohmoto-Nightingale-den Nijs integer (Chern number) gives the integer quantum-Hall effects,[1] and it ensures the stability of the gapless edge states at the same time.[2] While the time-reversal breaking (TRB) is necessary to have a nontrivial Chern number, there exist another topological invariants called the Z_2 invariants which classify the topological phases of the time-reversal invariant (TRI) insulators.[3–6] When the Z_2 invariants are nontrivial, there exist an odd number of the Kramers pairs of gapless edge modes in two dimensions and an odd number of the Kramers degenerate band crossings (Dirac cones) on the surface in three dimensions, respectively.

The concept of topological phases is also applicable to superconducting states[7–12] because there is a direct analogy between superconductors and insulators: the Bogoliubov de Gennes (BdG) Hamiltonian for a quasiparticle of a superconductor is analogous to the Hamiltonian of a band insulator and the superconducting gap corresponds to the gap of the band insulator. Indeed, the TRB chiral p-wave superconductors have a nontrivial Chern number, and they support a topologically protected chiral gapless edge state, in analogy with the integer quantum-Hall states.[7] Topological phases of noncentrosymmetric superconductors and s-wave superfluids, which support non-Abelian anyons, were also investigated.[8,11–13]

In addition to the analogous properties, there are topological features inherent to superconductors. Superconductors possess the particle-hole symmetry (PHS) exchanging the quasiparticle with the antiquasiparticle, which provides additional topological characteristics. In particular, for general superconductors without spin-rotation symmetry, there arise extra one-dimensional (1D) Z_2 invariants in TRB and TRI systems, and a three-dimensional (3D) winding number in TRI one.[10] As a result, the topological superconductors are characterized by the 1D Z_2 invariant and the two-dimensional (2D) Chern number for the TRB case, and the 1D and 2D Z_2 invariants, and the 3D winding number for the TRI one, respectively.

In this Rapid Communication, assuming the inversion symmetry in the normal state, we present a theory of topological odd-parity superconductors. For TRI single-band odd-parity superconductors, it has been revealed that the topological properties are characterized by the Fermi-surface topology in the normal state.[14] Here we extend these results to general odd-parity superconductors without or with time-reversal invariance, by using the 1D Z_2 invariants obtained from the PHS. Making connections between the 1D Z_2 invariants and the other topological invariants mentioned above, we provide characterization of topological odd-parity superconductors in terms of the topology of the Fermi surface.

Our theory predicts topologically protected gapless boundary states from the Fermi-surface structures. In addition, we present a simple criterion for a non-Abelian topological phase in 2D TRB odd-parity superconductors, where a vortex obeys the non-Abelian statistics. The non-Abelian topological phase is of particular interest in connection with the realization of fault-tolerant quantum computation based on the manipulation of non-Abelian anyons.[15]

In the following, we consider a general Hamiltonian H for full-gapped odd-parity superconducting states[16]

$$H = \frac{1}{2}\sum_{\boldsymbol{k}\alpha\alpha'} (c_{\boldsymbol{k}\alpha}^{\dagger}, c_{-\boldsymbol{k}\alpha}) H(\boldsymbol{k}) \begin{pmatrix} c_{\boldsymbol{k}\alpha'} \\ c_{-\boldsymbol{k}\alpha'}^{\dagger} \end{pmatrix},$$

$$H(\boldsymbol{k}) = \begin{pmatrix} \mathcal{E}(\boldsymbol{k})_{\alpha\alpha'} & \Delta(\boldsymbol{k})_{\alpha\alpha'} \\ \Delta^{\dagger}(\boldsymbol{k})_{\alpha\alpha'} & -\mathcal{E}^{T}(-\boldsymbol{k})_{\alpha\alpha'} \end{pmatrix}, \quad (1)$$

where $c_{\boldsymbol{k}\alpha}^{\dagger}(c_{\boldsymbol{k}\alpha})$ denotes the creation (annihilation) operator of electron with momentum $\boldsymbol{k}$. The suffix α labels other degrees of freedom for electron such as spin, orbital degrees of freedom, sublattice indices, and so on. $\mathcal{E}(\boldsymbol{k})$ is a Hermitian matrix describing the normal dispersion of the electron. The system in the normal state is symmetric under the inversion $c_{\boldsymbol{k}\alpha} \to \Sigma_{\alpha'} P_{\alpha\alpha'} c_{-\boldsymbol{k}\alpha'}$ with $P^2=1$, $P^{\dagger}\mathcal{E}(\boldsymbol{k})P=\mathcal{E}(-\boldsymbol{k})$. The parity of the gap function $\Delta(\boldsymbol{k})$ is odd, $P^{\dagger}\Delta(\boldsymbol{k})P^{*}=-\Delta(-\boldsymbol{k})$. For the time being, we do not suppose the time-reversal invariance in the system.

An important ingredient of our theory is the PHS of the BdG Hamiltonian (1)

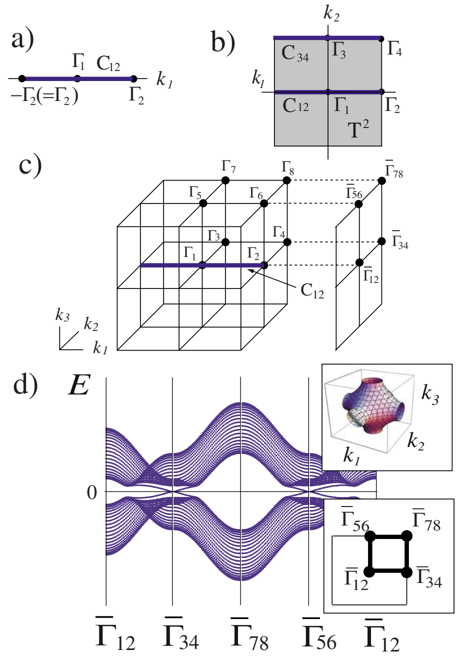

FIG. 1. (Color online) The TRI momenta Γ_i, and the TRI closed path C_{ij} connecting Γ_i and Γ_j in the BZ. (a) 1D BZ. The solid line denotes C_{12}. (b) 2D BZ T^2. (c) 3D BZ and the surface BZ of a 100 face. (d) 2D band structure for a slab with a 100 face for the 3D TRI odd parity superconductor that has the Fermi surface with $(-1)^{\bar{\nu}[C_{34}]}=(-1)^{\bar{\nu}[C_{56}]}=-1$ and the gap function $\Delta(\boldsymbol{k})=i\boldsymbol{d}(\boldsymbol{k})\cdot\boldsymbol{\sigma}\sigma_y$ with $d_i(\boldsymbol{k})=\Delta\sin k_i$. The insets show the Fermi surface (top) and the 2D surface BZ (bottom).

$$CH(\boldsymbol{k})C^{\dagger}=-H^{*}(-\boldsymbol{k}),\quad C=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{2}$$

From Eq. (2), we can say that if $|u_n(\boldsymbol{k})\rangle$ is a quasiparticle state with positive energy $E_n(\boldsymbol{k})>0$ satisfying $H(\boldsymbol{k})|u_n(\boldsymbol{k})\rangle = E_n(\boldsymbol{k})|u_n(\boldsymbol{k})\rangle$, then $C|u_n^{*}(-\boldsymbol{k})\rangle$ is a quasiparticle state with negative energy $-E_n(-\boldsymbol{k})<0$. In the following, we use a positive (negative) n for $|u_n(\boldsymbol{k})\rangle$ to represent a positive (negative) energy quasiparticle state, and set:

$$|u_{-n}(\boldsymbol{k})\rangle = C|u_n^{*}(-\boldsymbol{k})\rangle. \tag{3}$$

To define the 1D Z_2 invariants, we introduce the gauge fields $A_i^{(\pm)}(\boldsymbol{k})=i\Sigma_{n\gtrless 0}\langle u_n(\boldsymbol{k})|\partial_{k_i}|u_n(\boldsymbol{k})\rangle$ and their sum $A_i(\boldsymbol{k}) = A_i^{(+)}(\boldsymbol{k})+A_i^{(-)}(\boldsymbol{k})$. From Eq. (2), we have

$$A_i^{(+)}(\boldsymbol{k}) = A_i^{(-)}(-\boldsymbol{k}) \tag{4}$$

and using this and the fact that $A_i(\boldsymbol{k})$ is the total derivative of a function, we can prove that the Wilson loop of $A_i^{(-)}(\boldsymbol{k})$ along a TRI closed path C in the Brillouin zone (BZ)

$$W[\mathrm{C}] = \frac{1}{2\pi}\oint_{\mathrm{C}} dk_i A_i^{(-)}(\boldsymbol{k}) \tag{5}$$

is quantized as $e^{2\pi i W[\mathrm{C}]}=\pm 1$.[17] Now consider the TRI closed path C_{ij} passing through the TRI momenta Γ_i and Γ_j. See Fig. 1. (The TRI momentum satisfies $\Gamma_i=-\Gamma_i+\boldsymbol{G}$ with a reciprocal lattice vector $\boldsymbol{G}$, and because of the periodicity of the BZ, C_{ij} forms a closed path.) By using these particular

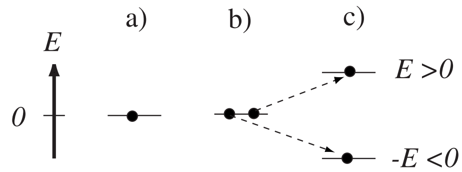

FIG. 2. Z_2 classification of edge state. (a) Topologically protected zero mode. [(b) and (c)] Topologically trivial edge modes.

TRI closed paths, the 1D Z_2 invariants $(-1)^{\nu[C_{ij}]}$ are defined as

$$(-1)^{\nu[C_{ij}]} = e^{2\pi i W[C_{ij}]}. \tag{6}$$

The physical meaning of the 1D Z_2 invariants is evident if we consider a full-gapped 1D odd-parity superconductor. In one dimension, we have a single 1D Z_2 invariant $(-1)^{\nu[C_{12}]}$ where C_{12} is the TRI closed path in Fig. 1(a). When $(-1)^{\nu[C_{12}]}=-1$ $(+1)$, the system is topologically nontrivial (trivial) and the bulk-edge correspondence implies that there exist an odd (even) number of the zero-energy states on the boundary. Without loss of generality, we consider the simplest nontrivial topological phase which supports a single-boundary zero mode. Then it is found that the PHS ensures the topological stability of the zero mode against small perturbation: from the PHS, a nonzero mode having energy E must be paired with a nonzero mode having the opposite one $-E$ so the single zero mode in Fig. 2(a) cannot acquire nonzero energy (unless the gap of the system closes). On the other hand, if we add another zero mode as depicted in Fig. 2(b), these zero modes are topologically unstable: in this case, we have a pair of zero modes so they can be nonzero modes smoothly as demonstrated in Fig. 2(c). This argument consistently indicates that the 1D superconductor has two topologically different phases and they are distinguished by the Z_2 number $(-1)^{\nu[C_{12}]}$.

We now evaluate the 1D Z_2 invariants by developing the method in Ref. 14. For an odd-parity superconductor, the combination of the inversion and the $U(1)$ gauge symmetry, $c_{\boldsymbol{k}\alpha}\to iP_{\alpha\alpha'}c_{\boldsymbol{k}\alpha'}$, is manifestly preserved, although each of them is spontaneously broken by the condensation $\Delta(\boldsymbol{k})$. Therefore, $H(\boldsymbol{k})$ has the following symmetry:

$$\Pi^{\dagger}H(\boldsymbol{k})\Pi = H(-\boldsymbol{k}),\quad \Pi=\begin{pmatrix}P & 0\\ 0 & -P^{*}\end{pmatrix}. \tag{7}$$

From this, we have $[H(\Gamma_i),\Pi]=0$ for the TRI momentum Γ_i. Thus, the quasiparticle state $|u_n(\Gamma_i)\rangle$ at Γ_i is simultaneously an eigenstate of Π, $\Pi|u_n(\Gamma_i)\rangle=\pi_n(\Gamma_i)|u_n(\Gamma_i)\rangle$. Evaluation of $(-1)^{\nu[C_{ij}]}$ is done by using the unitary matrices, $V_{mn}(\boldsymbol{k}) = \langle u_m(\boldsymbol{k})|\Pi C|u_n^{*}(\boldsymbol{k})\rangle$ and $W_{mn}(\boldsymbol{k})=\langle u_m(-\boldsymbol{k})|C|u_n^{*}(\boldsymbol{k})\rangle$. Since we have $\mathrm{tr}(V^{\dagger}\partial_{k_i}V)=2iA_i(\boldsymbol{k})$ from Eq. (3), $\nu[C_{ij}]$ is rewritten as

$$\nu[C_{ij}] = \frac{1}{\pi}\int_{\Gamma_i}^{\Gamma_j} dk_i A_i(\boldsymbol{k}) = \frac{1}{\pi i}\ln\left(\frac{\sqrt{\det V(\Gamma_i)}}{\sqrt{\det V(\Gamma_j)}}\right), \tag{8}$$

where we have used Eq. (4) and $\mathrm{tr}(V^{\dagger}\partial_{k_i}V)=\partial_{k_i}\ln\det V$. Furthermore, because $V_{mn}(\Gamma_i)$ is recast into $V_{mn}(\Gamma_i) = \pi_m(\Gamma_i)W_{mn}(\Gamma_i)$, we obtain

$$\det V(\Gamma_i) = \prod_n \pi_n(\Gamma_i)\det W, \tag{9}$$

where $\det W$ is independent of Γ_i because $\partial_{k_i} \ln \det W = 0$. Due to the PHS, $|u_n(\Gamma_i)\rangle$ and $|u_{-n}(\Gamma_i)\rangle$ share the same eigenvalue of Π and each eigenvalue appears twice in the product in Eq. (9). Therefore, taking the square root, we find $\sqrt{\det V(\Gamma_i)} = \Pi_{n<0}\pi_n(\Gamma_i)\sqrt{\det W}$. As a result, Eq. (8) reduces to

$$(-1)^{\nu[C_{ij}]} = \prod_{n<0} \pi_n(\Gamma_i)\pi_n(\Gamma_j). \tag{10}$$

In order to attribute the Fermi-surface properties to the 1D Z_2 invariant, we make the weak-paring assumption.[18] For ordinary superconductors, the superconducting gap is much smaller than the Fermi energy. Therefore, we reasonably assume that the typical energy scale of the gap function $\Delta(\Gamma_i)$ at the TRI momentum is much smaller than that of $\mathcal{E}(\Gamma_i)$. Under this assumption, we can take $\Delta(\Gamma_i) \to 0$ without gap closing. Because of the topological nature of $(-1)^{\nu[C_{ij}]}$, this adiabatic process does not change the value of $(-1)^{\nu[C_{ij}]}$.

In the process $\Delta(\Gamma_i) \to 0$, the BdG Hamiltonian at Γ_i reduces to $H(\Gamma_i) \to \text{diag}[\mathcal{E}(\Gamma_i), -\mathcal{E}^T(\Gamma_i)]$. By using an eigenstate $|\varphi_\alpha(\Gamma_i)\rangle$ of $\mathcal{E}(\Gamma_i)$ satisfying $\mathcal{E}(\Gamma_i)|\varphi_\alpha(\Gamma_i)\rangle = \varepsilon_\alpha(\Gamma_i)|\varphi_\alpha(\Gamma_i)\rangle$, an occupied state of $H(\Gamma_i)$ is given by $(|\varphi_\alpha(\Gamma_i)\rangle, 0)^t$ for $\varepsilon_\alpha(\Gamma_i) < 0$, and $(0, |\varphi_\alpha^*(\Gamma_i)\rangle)^t$ for $\varepsilon_\alpha(\Gamma_i) > 0$. Therefore, denoting the parity of $|\varphi_\alpha(\Gamma_i)\rangle$ as $P|\varphi_\alpha(\Gamma_i)\rangle = \xi_\alpha(\Gamma_i)|\varphi_\alpha(\Gamma_i)\rangle$, we find

$$\prod_{n<0} \pi_n(\Gamma_i) = \prod_\alpha \xi_\alpha(\Gamma_i)\prod_\alpha \text{sgn}\, \varepsilon_\alpha(\Gamma_i), \tag{11}$$

where the sum of α is taken for all eigenstates of $\mathcal{E}(\Gamma_i)$. We notice here that the product of the parity, $\Pi_\alpha\xi_\alpha(\Gamma_i)$, is independent of Γ_i since it is determined solely from $\det P$ and the dimensionality of the matrix $\mathcal{E}(\boldsymbol{k})$. Thus if we substitute Eq. (11) into Eq. (10), the contributions from the parity cancel, then we obtain the final expression

$$(-1)^{\nu[C_{ij}]} = \prod_\alpha \text{sgn}\, \varepsilon_\alpha(\Gamma_i)\text{sgn}\, \varepsilon_\alpha(\Gamma_j). \tag{12}$$

This formula is one of the main results in this Rapid Communication: from the bulk-edge correspondence, the gapless boundary states are predicted by the Fermi-surface structure determined by the electron dispersion $\varepsilon_\alpha(\Gamma_i)$. In addition, as is shown in the below, it connects the Chern number to the Fermi-surface structure. This connection provides a simple criterion for a non-Abelian topological phase in 2D TRB odd-parity superconductors.

The Chern number ν_{Ch} (Ref. 1) is defined by

$$\nu_{\text{Ch}} = \frac{1}{2\pi}\int_{T^2} \mathcal{F}^{(-)}(\boldsymbol{k}), \tag{13}$$

where $\mathcal{F}^{(-)}(\boldsymbol{k})$ is the field strength of $A_i^{(-)}(\boldsymbol{k})$ and T^2 the 2D BZ. By using Eq. (4), it can be linked to the 1D Z_2 invariants as

$$\nu_{\text{Ch}} = \frac{1}{2\pi}\int_{T^2} \mathcal{F}^{(-)}(\boldsymbol{k}) = \frac{1}{\pi}\int_{T^2_+} \mathcal{F}^{(-)}(\boldsymbol{k}) = \frac{1}{\pi}\oint_{\partial T^2_+} dk_i A_i^{(-)}(\boldsymbol{k})$$
$$= \nu[C_{12}] - \nu[C_{34}], \tag{14}$$

where T^2_+ is the upper half of T^2 and $\nu[C_{ij}]$ the 1D Z_2 invariants for C_{ij} in Fig. 1(b). Therefore, Eq. (12) yields the following connection between ν_{Ch} and the Fermi surface:

$$(-1)^{\nu_{\text{Ch}}} = \prod_{\alpha,i=1,2,3,4} \text{sgn}\, \varepsilon_\alpha(\Gamma_i) = (-1)^{p_0(S_{\text{F}})}, \tag{15}$$

where $p_0(S_{\text{F}})$ is the number of the connected components of the Fermi surface in the 2D BZ T^2.[19]

From Eq. (15), we can derive a simple criterion for a non-Abelian topological phase: *If a full-gapped 2D TRB odd-parity superconductor has the Fermi surface with* $(-1)^{p_0(S_{\text{F}})} = -1$, *then its vortex is a non-Abelian anyon*. To derive this, suppose that $(-1)^{p_0(S_{\text{F}})} = -1$, i.e., $p_0(S_{\text{F}})$ is odd. Then Eq. (15) implies that the Chern number ν_{Ch} of the system is also odd, thus there exist an odd number of topologically protected chiral Majorana modes on an edge.[2] Now the key observation is that a vortex can be considered as a small circular edge.[7] With a proper consideration about the boundary condition around a vortex, we can say that the vortex also supports an odd number of Majorana zero-energy modes, which implies that the vortex is a non-Abelian anyon.[7,20]

Now consider TRI odd-parity superconductors. Because of the time-reversal invariance Θ with $\Theta^2 = -1$, the quasiparticle states form Kramers pairs, $|u_n^s(\mathbf{k})\rangle (s = \text{I}, \text{II})$ and $|u_n^{\text{I}}(\boldsymbol{k})\rangle = \Theta|u_n^{\text{II}}(-\boldsymbol{k})\rangle$. Since the Kramers pair share the same eigenvalue $\pi_n(\Gamma_i)$ of Π, Eq. (10) yields $(-1)^{\nu[C_{ij}]} = 1$. Thus the 1D Z_2 invariants $(-1)^{\nu[C_{ij}]}$ are always trivial for TRI odd-parity superconductors. However, as is shown in the following, use of the time-reversal invariance as well makes it possible to define nontrivial 1D Z_2 invariants.

To define the nontrivial Z_2 invariants, we consider the gauge field for the "half" of the Kramers doublets, say, $A_i^{\text{I}(-)}(\boldsymbol{k}) = i\Sigma_{n<0}\langle u_n^{\text{I}}(\boldsymbol{k})|\partial_{k_i}|u_n^{\text{I}}(\boldsymbol{k})\rangle$, and its Wilson loop $W^{\text{I}}[C_{ij}]$. Then the 1D Z_2 invariants $(-1)^{\tilde{\nu}[C_{ij}]}$ for TRI superconductors are defined as

$$(-1)^{\tilde{\nu}[C_{ij}]} = e^{2\pi i W^{\text{I}}[C_{ij}]}. \tag{16}$$

These Z_1 invariants $(-1)^{\tilde{\nu}[C_{ij}]}$ are the square roots of the original ones, $(-1)^{\tilde{\nu}[C_{ij}]} = (-1)^{\nu[C_{ij}]/2}$, and their quantization, $(-1)^{\tilde{\nu}[C_{ij}]} = \pm 1$, follows from $(-1)^{\nu[C_{ij}]} = 1$.

In a manner parallel to the TRB case, for odd-parity superconductors, the 1D Z_2 invariants $(-1)^{\tilde{\nu}[C_{ij}]}$ are determined from the Fermi-surface structure. Under the same weak-paring assumption, it is found that $(-1)^{\tilde{\nu}[C_{ij}]}$ satisfies the "square root" of Eq. (12)

$$(-1)^{\tilde{\nu}[C_{ij}]} = \prod_\alpha \text{sgn}\, \varepsilon_{2\alpha}(\Gamma_i)\text{sgn}\, \varepsilon_{2\alpha}(\Gamma_j), \tag{17}$$

where $\varepsilon_\alpha(\Gamma_i)$ is an eigenvalue of $\mathcal{E}(\Gamma_i)$ and we have set $\varepsilon_{2\alpha}(\Gamma_i) = \varepsilon_{2\alpha+1}(\Gamma_i)$ by using the Kramers degeneracy. Furthermore, other topological invariants can be linked to the Fermi-surface topology. As was mentioned above, besides the 1D Z_2 invariants $(-1)^{\tilde{\nu}[C]_{ij}}$, topological phases of

TRI superconductors are characterized by the 2D Z_2 invariant $(-1)^{\nu_{2dTI}}$ (Ref. 3) and the 3D winding number ν_w.[10] It is found that $(-1)^{\nu_{2dTI}}$ and $(-1)^{\nu_w}$ are written as products of $(-1)^{\tilde{\nu}[C]_{ij}}$, and Eq. (17) leads to

$$(-1)^{\nu_{2dTI}} = \prod_{\alpha,i=1,2,3,4} \text{sgn}\, \varepsilon_{2\alpha}(\Gamma_i) = (-1)^{p_0(S_F)/2}, \quad (18)$$

$$(-1)^{\nu_w} = \prod_{\alpha,i=1,\cdots,8} \text{sgn}\, \varepsilon_{2\alpha}(\Gamma_i) = (-1)^{\chi(S_F)/4}. \quad (19)$$

Here $\nu_{2dTI}(\nu_w)$ is calculated for a full-gapped 2D (3D) TRI odd-parity superconductor, and Γ_i's are illustrated in Fig. 1(b) (Fig. 1(c)). $\chi(S_F)$ is the Euler characteristics of the Fermi surface.[14] From Eq. (18), the 2D Z_2 invariant is completely determined. On the other hand, Eq. (19) partially determines the 3D winding number, that is, $(-1)^{\nu_w}$. Nevertheless, it gives a sufficient condition for a topological superconductor: *If a full-gapped 3D TRI odd-parity superconductor has the electron dispersion* $\varepsilon_{2\alpha}(\boldsymbol{k})$ *with* $\prod_{\alpha,i=1,\ldots,8}\text{sgn}\, \varepsilon_{2\alpha}(\Gamma_i)=-1$, *then it is a topological superconductor with nonzero (odd)* ν_w.

The formulas (18) and (19) have already been reported for TRI single-band spin-triplet superconductors and multiband odd-parity superconductors with $P=1$.[14] Here they are extended to general TRI odd-parity superconductors. Furthermore, we have additional formulas (12), (15), and (17), which were not known before. From them, detailed information on topologically protected gapless boundary states is obtained. In Fig. 1(d), we give an example of gapless surface modes for a 3D TRI topological odd-parity superconductor. The 3D winding number ν_w is 2 in this model, so there exist two gapless modes on a surface, say a 100 face. Formula (17) consistently determines the position of the band crossings of the gapless surface modes: from the Fermi-surface structure in the inset, Eq. (17) leads to $(-1)^{\tilde{\nu}[C_{34}]}=(-1)^{\tilde{\nu}[C_{56}]}=-1$. Correspondingly, the position of the band crossings points is determined at $\bar{\Gamma}_{34}$ and $\bar{\Gamma}_{56}$ on the surface BZ. The predicted relation between the band crossings of the surface modes and the Fermi surface is one of experimental signatures of topological odd-parity superconductors.

In this Rapid Communication, we have mainly considered full-gapped odd-parity superconductors but our formulas (12) and (17) are applicable to a nodal odd-parity superconductor as well if the TRI path C_{ij} does not intersect a node of the superconducting gap. When the 1D Z_2 invariant $(-1)^{\nu[C_{ij}]}$ or $(-1)^{\tilde{\nu}[C_{ij}]}$ is nontrivial, a gapless surface state is predicted at $\bar{\Gamma}_{ij}$ in the surface BZ.

To conclude, we present a description of topological odd-parity superconductors in terms of the Fermi-surface structures in the normal state. All the topological invariants characterizing topological odd-parity superconductors are directly related to the topology of the Fermi surface by Eqs. (12) and (15) for the TRB case, and Eqs. (17)–(19) for the TRI one, respectively. These relations provide simple criteria for topological phases including a non-Abelian one in odd-parity superconductors.

Recently, I became aware that Ref. 21 also discussed a part of the generalization [Eq. (19)] independently in a different manner.

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