Hydrodynamic theory of supersolids: Variational principle, effective Lagrangian, and density-density correlation function

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We develop an effective low-energy, long-wavelength theory of a bulk supersolid—a putative phase of matter with simultaneous crystallinity and Bose condensation. Using conservation laws and general symmetry arguments we derive an effective action that correctly describes the coupling between the Bose condensation and the elasticity of the solid. We use our effective action to calculate the correlation and response functions for the supersolid, and we show that the onset of supersolidity produces peaks in the response function, corresponding to propagating second sound modes in the solid. With a further study on the dissipative hydrodynamics of supersolids we show that the Brillouin peaks of the second sound modes in the response function actually originate from the splitting of the central Rayleigh peak corresponding to the defect diffusion mode under the supersolid transition. Light scattering may provide a direct measure of this splitting.

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I. INTRODUCTION

In 1969 Andreev and Lifshitz proposed a novel phase of matter in quantum Bose crystals wherein a Bose condensate of point defects would coexist with the crystallinity of the solid.[1] This is perhaps the most conceptually clear picture of what we now call a "supersolid" although suggestions of the coexistence (or noncoexistence) of Bose condensation and crystallinity can be traced to the earlier work of Penrose and Onsager,[2] and Chester.[3] Andreev and Lifshitz provided an elegant (albeit incomplete) formulation of the hydrodynamics of supersolids and predicted propagating modes analogous to second (or fourth) sound in liquid ^{4}He. Their hydrodynamic formulation was further extended by Saslow[4] and by Liu.[5] Experimental searches for signatures of the supersolid phase proved fruitless[6] until recently, when Kim and Chan[7–9] observed rotational inertia anomalies in solid ^{4}He that they interpreted as evidence for supersolidity. Their work fueled extensive searches for further evidence of this elusive phase of matter[10–17] and there are now a number of comprehensive reviews of the experimental and theoretical progress in this area—see Refs. 18–21.

The present work is a detailed study of the hydrodynamics of bulk supersolids, of the type originally proposed by Andreev and Lifshitz, for which we derive an effective Lagrangian density. The advantages of calculating a Lagrangian density are twofold. First, the effective action constructed from the Lagrangian density is a powerful tool for calculating and elucidating the collective modes of the supersolid phase, and the correlation and response functions in the supersolid phase. Second, beyond linearized hydrodynamics, the effective action can also be use to study the dynamics and interaction of topological defects—vortices and dislocations—in the supersolid. In this paper we focus on the former point whereas the latter will be dealt with in subsequent publications.[22]

Recently there have been several works to derive Lagrangians for supersolids.[23–26] Son[23] expressed a Lagrangian for supersolids in terms of Galilean invariant hydrodynamic variables set by conservation laws and broken symmetries. Josserand *et al.*[24] derived another Lagrangian density by using the homogenization procedure to a nonlocal version of the Gross-Pitaevskii equation. Ye[25] proposed a phenomenological supersolid Lagrangian by introducing an arbitrary coupling constant between elasticity and superfluid. Peletminskii[26] used Poisson-bracket formalism with Galilean invariance, and derived the Andreev and Lifshitz hydrodynamics. However, their Lagrangian densities either are restricted to zero temperature behavior,[23] do not generate the Andreev and Lifshitz hydrodynamics,[24] violate Galilean invariance,[25] or incorporate additional auxiliary fields.[26] In contrast, our derivation of the Lagrangian density for a supersolid relies extensively on a variational principle used to obtain the dynamic equations in various continuum systems: normal fluids,[27–29] superfluids,[29–34] normal solids,[27,35] liquid crystals,[36] trapped superfluid gases,[37] and relativistic fluids.[38] This approach starts with a Lagrangian density of physically well-defined kinetic and potential-energy densities depending upon hydrodynamic variables set by conservation laws and broken symmetries. The hydrodynamic variables, in general, are not independent from each other but related by several constraints, e.g., conservation equations. These constraints are added to the Lagrangian density by using Lagrange multipliers which later are identified as Clebsch potentials for the velocities and also can be removed from the Lagrangian density by using some of the Euler-Lagrange equations.[27] Finally, the resulting Lagrangian density, which is manifestly Galilean invariant, becomes a function of all the physical hydrodynamic variables and describes nonzero temperature hydrodynamics.

Once the Lagrangian density of a system is constructed one can also investigate linear responses to external disturbances from equilibrium by calculating time-dependent correlation functions from the Lagrangian density. One of our important results is the calculation of the density-density correlation function of supersolids. Collective modes generated by slow disturbances appear as peaks in the correlation function. Analogous to superfluids under a supersolid transition, a new propagating second sound mode appears, and we calculate the contributions of the second sound modes of supersolids in the density-density correlation function. We also extend the analysis to the dissipative hydrodynamics of Andreev and Lifshitz, and show that light scattering, which measures fluctuations in the local dielectric constant,[39] may provide a direct measure of the density-density correlation function.

We should also state what this work is *not*—it is not an explanation of the recent experiments on possible supersolidity in ^{4}He as the prevailing wisdom suggests that structural disorder plays a key role in most of the experiments and our simplified model assumes an ordered solid.

This work is organized as follows. In Sec. II we derive the supersolid hydrodynamics and the effective Lagrangian density using the variational principle. We show that the Euler-Lagrange equations obtained from the Lagrangian density are equivalent to the hydrodynamic equations of motion derived by Andreev and Lifshitz[1] including a term nonlinear in the elastic strain which they omitted in their analysis. We also discuss the connection to the work of Saslow,[4] Liu,[5] Son,[23] and Josserand *et al.*[24] In Sec. III we use a quadratic version of the Lagrangian to investigate the linearized hydrodynamics of a supersolid. Finally, in Sec. IV the collective modes and the density-density correlation function of a model supersolid are calculated in detail. The appendices provide additional detail as an aid to the reader.

II. VARIATIONAL PRINCIPLE AND AN EFFECTIVE LAGRANGIAN OF SUPERSOLIDS

We start with a Lagrangian density for a supersolid in the Eulerian description (in which all quantities are depicted at fixed position $\mathbf{x}$ and time t)

$$\mathcal{L}_{\mathrm{SS}} = \frac{1}{2}\rho_{sij}v_{si}v_{sj} + \frac{1}{2}(\rho\delta_{ij} - \rho_{sij})v_{ni}v_{nj} - U_{\mathrm{SS}}(\rho,\rho_{sij},s,R_{ij}), \tag{1}$$

where ρ_{sij} is the superfluid density tensor, ρ is the total density, $\mathbf{v}_s$ is the velocity of the supercomponents, $\mathbf{v}_n$ is the velocity of the normal components, s is the entropy density, and

$$R_{ij} \equiv \partial_i R_j \tag{2}$$

is the deformation tensor, with $\mathbf{R}$ the coordinate affixed to material elements ($\partial_i \equiv \partial/\partial x_i$ and $\partial_t \equiv \partial/\partial_t$ in what follows). The first two terms in the Lagrangian density are the kinetic-energy densities of the supercomponent and the normal component, respectively, and the third term is the internal energy density which is a function of ρ, ρ_{sij}, s, and R_{ij}. In contrast to a superfluid, R_{ij} appears explicitly in U_{SS} for a supersolid, a reflection of the solid's broken translational symmetry. As shown in Appendix B, the internal energy density satisfies the thermodynamic relation

$$dU_{\mathrm{SS}} = Tds + \left[\mu + \frac{1}{2}(v_{ni} - v_{si})^2\right]d\rho - \lambda_{ik}dR_{ik} - \frac{1}{2}(v_{ni} - v_{si})(v_{nj} - v_{sj})d\rho_{sij}, \tag{3}$$

where μ is the chemical potential per unit mass and λ_{ij} the stress tensor. Given the Lagrangian density in Eq. (1), the action is

$$S_{\mathrm{SS}} = \int dt \int d^3x \mathcal{L}_{\mathrm{SS}}. \tag{4}$$

The equations of motion for a supersolid are obtained from variations of S_{SS} with respect to the dynamical variables. However, as illustrated in Appendix A, the dynamical variables are not independent and one must insure that conservation laws and broken symmetries are incorporated in the action through the use of auxiliary fields (Lagrange multipliers). For a three-dimensional supersolid there are five conserved quantities: the mass, the entropy, and the three components of the momentum. Among these constraints we impose only the mass and entropy conservation laws, and show below that the momentum conservation is the byproduct of the variational principle. Conservation of mass is expressed through the equation of continuity

$$\partial_t\rho + \partial_i j_i = 0, \tag{5}$$

where the mass current j_i is

$$j_i = \rho_{sij}v_{sj} + (\rho\delta_{ij} - \rho_{sij})v_{nj}. \tag{6}$$

The entropy conservation law is

$$\partial_t s + \partial_i(sv_{ni}) = 0 \tag{7}$$

in which only $\mathbf{v}_n$ is involved because the entropy is transported by the normal component. Finally, we account for the broken translational symmetry using *Lin's constraint*[29]

$$\frac{D_n R_i}{Dt} = 0, \tag{8}$$

where $D_n/Dt \equiv \partial_t + v_{ni}\partial_i$. This constraint states that the Lagrangian coordinates (i.e., the initial positions of particles) do not change along the paths of the normal component. Indeed, Lin's constraint was first introduced to generate vorticity in the Lagrangian description of an isentropic normal fluid.[29] We incorporate all of the constraints, Eqs. (5)–(8), into the Lagrangian density Eq. (1) by using the Lagrange multipliers α, ϕ, and β_i, with the result

$$\mathcal{L}_{\mathrm{SS}} = \frac{1}{2}\rho_{sij}v_{si}v_{sj} + \frac{1}{2}(\rho\delta_{ij} - \rho_{sij})v_{ni}v_{nj} - U_{\mathrm{SS}}(\rho,\rho_{sij},s,R_{ij}) + \alpha[\partial_t s + \partial_i(sv_{ni})] + \phi\{\partial_t\rho + \partial_i[\rho_{sij}v_{sj} + (\rho\delta_{ij} - \rho_{sij})v_{nj}]\} + \beta_i[\partial_t(sR_i) + \partial_j(sR_i v_{nj})]. \tag{9}$$

Note that in our formulation Lin's constraint is combined with the entropy conservation law.

We are now in a position to derive the hydrodynamic equations of motion for supersolids. First of all, the variation in the action with respect to v_{si} produces

$$v_{si} = \partial_i\phi. \tag{10}$$

Therefore, the superfluid component of the velocity is a potential flow as expected [rotational flow can be obtained by introducing another constraint; see Ref. 32]. The remaining equations of motion are

(a) $\delta\rho$

$$\frac{1}{2}v_{\mathrm{n}}^2 - \frac{\partial U_{\mathrm{SS}}}{\partial \rho} - \partial_t \phi - v_{\mathrm{n}i} v_{\mathrm{s}i} = 0, \tag{11}$$

(b) $\delta\rho_{\mathrm{s}ij}$

$$\frac{\partial U_{\mathrm{SS}}}{\partial \rho_{\mathrm{s}ij}} = -\frac{1}{2}(v_{\mathrm{s}i} - v_{\mathrm{n}i})(v_{\mathrm{s}j} - v_{\mathrm{n}j}), \tag{12}$$

(c) δs

$$\frac{D_{\mathrm{n}}\alpha}{Dt} + R_i \frac{D_{\mathrm{n}}\beta_i}{Dt} + \frac{\partial U_{\mathrm{SS}}}{\partial s} = 0, \tag{13}$$

(d) $\delta v_{\mathrm{n}i}$

$$\partial_i \alpha + R_j \partial_i \beta_j = \frac{1}{s}(\rho\delta_{ij} - \rho_{\mathrm{s}ij})(v_{\mathrm{n}j} - v_{\mathrm{s}j}), \tag{14}$$

(e) δR_i

$$\frac{D_{\mathrm{n}}\beta_i}{Dt} - \frac{1}{s}\partial_j\left(\frac{\partial U_{\mathrm{SS}}}{\partial R_{ji}}\right) = 0. \tag{15}$$

In the above equations of motion we have eliminated the gradient of ϕ by using Eq. (10). In addition to the derived equations of motion, the variations with respect to the Lagrange multipliers reproduce the imposed constraints, Eqs. (5)–(8). Therefore, Eqs. (5)–(8) and (10)–(15) are the hydrodynamic equations for supersolids.

In the following we demonstrate that the equations of motion derived above are equivalent to the nondissipative supersolid hydrodynamics developed by Andreev and Lifshitz,[1] Saslow,[4] and Liu.[5] First, the taking gradient of Eq. (11) produces the Josephson equation

$$\partial_t v_{\mathrm{s}i} = -\partial_i \mu - \frac{1}{2}\partial_i v_{\mathrm{s}}^2, \tag{16}$$

where we have used the thermodynamic relation for $\partial U_{\mathrm{SS}}/\partial\rho$ given in Eq. (3). Second, we derive the momentum conservation equation; the following identity simplifies the derivation:

$$\frac{D(a\partial_i b)}{Dt} = \partial_i b \frac{Da}{Dt} + a\partial_i\left(\frac{Db}{Dt}\right) - a\partial_j b \partial_i v_j, \tag{17}$$

where $D/Dt \equiv \partial_t + v_i\partial_i$. Take D_{n}/Dt of Eq. (14), and eliminate the Lagrange multipliers by using Eqs. (7), (8), and (13)–(15). The result is

$$\begin{aligned}\frac{D_{\mathrm{n}}}{Dt}[(\rho\delta_{ij} - \rho_{\mathrm{s}ij})(v_{\mathrm{n}j} - v_{\mathrm{s}j})] = {}& -s\partial_i\left(\frac{\partial U_{\mathrm{SS}}}{\partial s}\right) - \partial_i R_j \partial_k\left(\frac{\partial U_{\mathrm{SS}}}{\partial R_{kj}}\right)\\ & - (\rho\delta_{ij} - \rho_{\mathrm{s}ij})(v_{\mathrm{n}j} - v_{\mathrm{s}j})\partial_k v_{\mathrm{n}_k}\\ & - (\rho\delta_{jk} - \rho_{\mathrm{s}jk})(v_{\mathrm{n}k} - v_{\mathrm{s}k})\partial_i v_{\mathrm{n}j}.\end{aligned} \tag{18}$$

Third, combine Eq. (18) with the thermodynamic relation, Eq. (3), the continuity equation, Eq. (5), and the Josephson equation, Eq. (16). After some algebra, we obtain the momentum-conservation law

$$\partial_t j_i + \partial_j \Pi_{ij} = 0, \tag{19}$$

where j_i is the mass current given in Eq. (6) and Π_{ij} is the (nondissipative) stress tensor

$$\begin{aligned}\Pi_{ij} = {}& \rho v_{\mathrm{s}i} v_{\mathrm{s}j} + v_{\mathrm{s}i} p_j + v_{\mathrm{n}j} p_i - R_{ik}\lambda_{jk}\\ & - [\epsilon - Ts - \mu\rho - (v_{\mathrm{n}j} - v_{\mathrm{s}j})p_j]\delta_{ij},\end{aligned} \tag{20}$$

where $p_i \equiv (\rho\delta_{ij} - \rho_{\mathrm{s}ij})(v_{\mathrm{n}j} - v_{\mathrm{s}j})$ and ϵ satisfies a thermodynamic relation given by Eq. (B3). Note that the Josephson equation, Eq. (16), and the momentum conservation equation, Eq. (19), are Eqs. (9) and (12) of Andreev and Lifshitz[1] [Andreev and Lifshitz neglected nonlinear strain terms, effectively replacing R_{ik} by δ_{ik} in the last term of Eq. (20) above]. Moreover, the momentum conservation equation is equivalent to Eq. (4.16) of Saslow[4] when $\mathbf{v}_{\mathrm{s}}$ is taken as a Galilean velocity, and Eq. (3.40) of Liu[5] in the case where the superthermal current vanishes.

The Lagrangian density used to derive the hydrodynamics of supersolids, Eq. (9), can be recast into a more familiar and compact form by using the equations of motion, as illustrated for an ideal fluid in Appendix A. To see this, we integrate the terms involving the Lagrange multipliers by parts (neglecting boundary terms) and use Eqs. (10) and (13) to eliminate α and β_i. We then obtain

$$\begin{aligned}\mathcal{L}_{\mathrm{SS}} = {}& -\rho\partial_t\phi - \frac{1}{2}\rho_{\mathrm{s}ij}\partial_i\phi\partial_j\phi + \frac{1}{2}(\rho\delta_{ij} - \rho_{\mathrm{s}ij})v_{\mathrm{n}i}v_{\mathrm{n}j}\\ & - (\rho\delta_{ij} - \rho_{\mathrm{s}ij})v_{\mathrm{n}j}\partial_i\phi - f(\rho, \rho_{\mathrm{s}ij}, T, R_{ij}),\end{aligned} \tag{21}$$

where $f \equiv U_{\mathrm{SS}} - Ts$ satisfies the thermodynamic relation

$$\begin{aligned}df = {}& -s dT + \left[\mu + \frac{1}{2}(v_{\mathrm{n}i} - v_{\mathrm{s}i})^2\right]d\rho - \lambda_{ik} dR_{ik}\\ & - \frac{1}{2}(v_{\mathrm{n}i} - v_{\mathrm{s}i})(v_{\mathrm{n}j} - v_{\mathrm{s}j})d\rho_{\mathrm{s}ij}.\end{aligned} \tag{22}$$

When cast in this form, we see that the coupling between the superfluid and the normal fluid [the fourth term in Eq. (21)] is $-(\rho\delta_{ij} - \rho_{\mathrm{s}ij})v_{\mathrm{n}i}v_{\mathrm{s}j}$; this is a "current-current" interaction, where the coupling constant is the normal fluid density. This coupling coefficient is *universal*—it is determined by conservation laws and Galilean invariance.

We conclude this section by discussing possible connections of Eq. (21) with the Lagrangian densities derived by Son[23] and by Josserand *et al.*[24] To connect to Son's results, we first invert Lin's constraint, Eq. (8), to obtain

$$v_{\mathrm{n}i} = -R_{ji}^{-1}\partial_t R_j, \tag{23}$$

where $R_{ji}^{-1} \equiv \partial x_i/\partial R_j$ and $R_{ij}R_{jk}^{-1} = \delta_{ik}$. Substituting this into Eq. (21), we obtain a Lagrangian similar in form to Eq. (23) of Son's paper (however, our energy density f depends upon ρ, ρ_{s}, and T in addition to R_{ij} extending to nonzero temperature hydrodynamics). On the other hand, as we pointed out earlier the hydrodynamics derived by Josserand *et al.* are not of Andreev and Lifshitz so that their Lagrangian density is different from ours. However we can compel our Eq. (21) to agree with their Eq. (4), once we identify their $\rho(n)$ with our normal fluid density $\rho\delta_{ij} - \rho_{\mathrm{s}ij}$ and replace their convective derivative $D\mathbf{u}/Dt$ with the velocity of normal components $\mathbf{v}_n$.

III. QUADRATIC LAGRANGIAN DENSITY AND THE LINEARIZED HYDRODYNAMICS OF SUPERSOLIDS

In this section we discuss the propagation of collective modes in supersolids by examining the response to small fluctuations away from equilibrium. The number of collective hydrodynamic modes of a system can be inferred by enumerating the system's conservation laws and broken symmetries.[40] Since we are more interested in the effect of density or defect fluctuations than thermal fluctuations, for simplicity we ignore thermal fluctuations in what follows. For a three-dimensional supersolid there are conservation laws for mass, three components of momentum, and energy; however, by ignoring thermal fluctuations we can omit the energy-conservation law. In addition the conservation laws, there are three broken translational symmetries (due to the crystallinity) and one broken gauge symmetry (due to the Bose-Einstein condensation). Thus, a three-dimensional supersolid without thermal fluctuations has eight conservation laws and broken symmetries (nine, if conservation of energy is included). Correspondingly, there are eight hydrodynamic modes: two pairs of the ordinary transverse propagating modes, a pair of longitudinal first sound modes, and a pair of longitudinal second sound modes (note that a solution of the hydrodynamic equations with dispersion $\omega=\pm ck$ would count as *two* modes—a pair of modes with one propagating and a second counterpropagating). The appearance of the longitudinal second sound modes is one of the key signatures of a supersolid.

We start by establishing the notation for the small fluctuations away from equilibrium. The equilibrium value of the densities will be denoted with a subscript of "0," and the density fluctuations will be denoted by $\delta\rho$ so that $\rho=\rho_0+\delta\rho$ and $\rho_{sij}=\rho_{s0ij}+\delta\rho_{sij}$. For the lattice fluctuations, let $\mathbf{u}$ denote the (small) deformation field away from the unstrained solid (i.e., the difference between the Lagrangian and the Eulerian coordinates)

$$\mathbf{x}=\mathbf{R}+\mathbf{u}. \tag{24}$$

Then the deformation tensor becomes

$$R_{ij}=\delta_{ij}-w_{ij}, \tag{25}$$

where $w_{ij}\equiv\partial_i u_j$. Finally, the velocity of the normal component is obtained by linearizing the inverted Lin's constraint, Eq. (23) so that to lowest order in the strain field the normal velocity is the time derivative of the displacement field

$$v_{ni}=\partial_t u_i. \tag{26}$$

Expanding the Lagrangian density, Eq. (21), up to second order in the small quantities $\delta\rho$, $\delta\rho_{sij}$, w_{ij}, $\partial_i\phi$, and $\partial_t\phi$, we obtain

$$\begin{aligned}\mathcal{L}_{SS}^{\text{quad}}=&-\rho_0\partial_t\phi-\lambda_{0ij}w_{ij}-\mu_0\delta\rho-\delta\rho\partial_t\phi\\&-\frac{1}{2}\rho_0(\partial_i\phi)^2-\left.\frac{\partial\mu}{\partial w_{ij}}\right|_\rho\delta\rho w_{ij}-\frac{1}{2}\left.\frac{\partial\mu}{\partial\rho}\right|_{w_{ij}}(\delta\rho)^2\\&+\frac{1}{2}\rho_{n0ij}(\partial_t u_i-\partial_i\phi)(\partial_t u_j-\partial_j\phi)-\frac{1}{2}\left.\frac{\partial\lambda_{ij}}{\partial w_{lk}}\right|_\rho w_{ij}w_{lk},\end{aligned} \tag{27}$$

where $\rho_{n0ij}\equiv\rho_0\delta_{ij}-\rho_{s0ij}$ and the thermodynamic relation for f, Eq. (22), is used. In the above expansion we have dropped constants which do not contribute to the equations of motion and have neglected terms proportional to $\partial f/\partial\rho_{sij}$ because they are of higher order [see Eq. (12)]; consequently, the quadratic Lagrangian turns out to be independent of fluctuations of the superfluid density. However, we have kept the first two terms in Eq. (27); although they are total derivatives and would seem to be irrelevant to the equations of motion, they are nontrivial for topological defects such as vortices or dislocations. In fact, one can show[22] that the first term produces the Magnus force on a vortex[41] and the second term generates the Peach-Koehler force on a dislocation.[42,43] We will defer the discussion of these effects to a subsequent publication.[22]

Now we are in a position to study the hydrodynamic modes of a supersolid. The quadratic Lagrangian density, Eq. (27), produces three linearized equations of motions which are equivalent to the continuity equation, the Josephson equation and the momentum-conservation equation. Before proceeding further, it is useful to rewrite the quadratic Lagrangian density in terms of the defect density fluctuation by using one of the equations of motion. By varying the action with respect to $\delta\rho$ we obtain

$$\delta\rho=\left.\frac{\partial\rho}{\partial w_{ij}}\right|_\mu w_{ij}+\left.\frac{\partial\rho}{\partial\mu}\right|_{w_{ij}}\delta\mu, \tag{28}$$

where we have used the linearized Josephson equation ($\partial_t\phi=-\mu_0-\delta\mu$) and the identity

$$\left.\frac{\partial x}{\partial y}\right|_z=-\left.\frac{\partial z}{\partial y}\right|_x\left.\frac{\partial x}{\partial z}\right|_y. \tag{29}$$

From Eq. (28) it is clear that the density fluctuation is an independent hydrodynamic variable—it is not slaved to the lattice deformation, as would be the case for a commensurate solid, where $\delta\rho=-\rho_0\boldsymbol{\nabla}\cdot\mathbf{u}$. Indeed, in a real (incommensurate) crystal a density fluctuation can be produced by lattice deformations or by point defects (vacancies and interstitials). To highlight the role of defects we will introduce the defect density fluctuation $\delta\rho_\Delta$ as our independent hydrodynamic variable, instead of $\delta\rho$. The local defect density is defined as the difference between the density of vacancies, ρ_V, and the density of interstitials, ρ_I

$$\rho_\Delta=\rho_I-\rho_V. \tag{30}$$

The minus sign is necessary so that the total defect density is conserved—in the bulk of the crystal vacancies and interstitials are created and destroyed in pairs (ignoring surface effects). Then we have

$$\delta\mu = \left.\frac{\partial\mu}{\partial w_{ij}}\right|_{\rho_\Delta} w_{ij} + \left.\frac{\partial\mu}{\partial\rho_\Delta}\right|_{w_{ij}} \delta\rho_\Delta \tag{31}$$

and from Eq. (28) we obtain

$$\delta\rho = \left.\frac{\partial\rho}{\partial w_{ij}}\right|_{\rho_\Delta} w_{ij} + \left.\frac{\partial\rho}{\partial\rho_\Delta}\right|_{w_{ij}} \delta\rho_\Delta, \tag{32}$$

where we have used Eq. (29) and the identity

$$\left.\frac{\partial x}{\partial y}\right|_0 = \left.\frac{\partial x}{\partial y}\right|_z + \left.\frac{\partial x}{\partial z}\right|_y \left.\frac{\partial z}{\partial y}\right|_0. \tag{33}$$

Equation (32) shows that a density fluctuation in an isothermal supersolid is caused either by a lattice deformation or by a defect density fluctuation $\delta\rho_\Delta$, just as in a normal solid.[44–46] Following Zippelius, Halperin, and Nelson (ZHN),[45] we can identify $(\partial\rho/\partial w_{ij})_{\rho_\Delta} = -\rho_0\delta_{ij}$ and $(\partial\rho/\partial\rho_\Delta)_{w_{ij}} = 1$. We finally obtain

$$\delta\rho = -\rho_0 w_{ii} + \delta\rho_\Delta, \tag{34}$$

which illustrates the roles of the lattice deformation $w_{ii} = \nabla\cdot\mathbf{u}$ and the defect density fluctuation in determining the total-density fluctuation. We note in passing that in a higher order expansion of the Lagrangian density the terms proportional to the superfluid density fluctuation must also be retained in Eq. (34). This would resemble the "three-fluid" scenario proposed by Saslow[47] in which the lattice density and velocity are introduced for the third fluid component.

We can now use Eq. (34) to rewrite the quadratic Lagrangian density in terms of the defect density with the result

$$\begin{aligned}\mathcal{L}_{\rm SS}^{\rm quad} &= \rho_0 w_{ii}\partial_t\theta - \rho_{n0ij}\partial_t u_i \partial_j\theta - \frac{1}{2}\rho_0^2 \left.\frac{\partial\mu}{\partial\rho}\right|_{w_{ij}} w_{ii}^2 - \delta\rho_\Delta\partial_t\theta \\ &\quad - \frac{1}{2}\rho_{s0ij}\partial_i\theta\partial_j\theta - \left.\frac{\partial\mu}{\partial w_{ij}}\right|_{\rho} \delta\rho_\Delta w_{ij} + \rho_0 \left.\frac{\partial\mu}{\partial\rho_\Delta}\right|_{w_{ij}} w_{ii}\delta\rho_\Delta \\ &\quad - \frac{1}{2}\left.\frac{\partial\mu}{\partial\rho_\Delta}\right|_{w_{ij}} \delta\rho_\Delta^2 - \frac{1}{2}\left.\frac{\partial\lambda_{ji}}{\partial w_{lk}}\right|_{\rho} w_{ij}w_{lk} + \rho_0 \left.\frac{\partial\mu}{\partial w_{ij}}\right|_{\rho} w_{ij}w_{kk} \\ &\quad + \frac{1}{2}\rho_{n0ij}\partial_t u_i \partial_t u_j,\end{aligned} \tag{35}$$

where we introduced $\theta = \phi + \mu_0 t$. Next, we derive the linearized equations of motion from the Lagrangian density. First, note that the variation with respect to $\delta\rho_\Delta$ reproduces Eq. (31) because $\partial_t\theta$ is $-\delta\mu$. Second, taking the variation with respect to θ produces the linearized equation of continuity, expressed in terms of $\delta\rho_\Delta$

$$\partial_t\delta\rho_\Delta + \partial_i j_i^\Delta = 0, \tag{36}$$

where the defect current density is given by

$$j_i^\Delta = \rho_{s0ij}(\partial_j\theta - \partial_t u_j). \tag{37}$$

We see that the defect current arises from counterflow between the superfluid velocity $\nabla\theta$ and the normal fluid velocity $\partial_t\mathbf{u}$, and vanishes when $\rho_{s0ij} = 0$, in the normal state. In other words, $\partial_t\delta\rho_\Delta = 0$ in the normal state, in agreement with the nondissipative description of normal solids[45] in which defect currents are only produced through diffusion (i.e., the defect current is dissipative in the normal solid). The last equation of motion derived from the variation in u_i is

$$\begin{aligned}&\rho_{n0ij}\partial_t^2 u_j - \left(\left.\frac{\partial\mu}{\partial w_{ji}}\right|_{\rho} - \rho_{n0ij}\left.\frac{\partial\mu}{\partial\rho_\Delta}\right|_{w_{ij}}\right)\partial_j\delta\rho_\Delta \\ &\quad - \left(\left.\frac{\partial\lambda_{ji}}{\partial w_{lk}}\right|_{\rho} - \rho_{n0ij}\left.\frac{\partial\mu}{\partial w_{lk}}\right|_{\rho_\Delta} - \rho_0\left.\frac{\partial\mu}{\partial w_{ij}}\right|_{\rho}\delta_{lk}\right)\partial_j w_{lk} = 0.\end{aligned} \tag{38}$$

When the time derivative of Eq. (36) is combined with Eq. (31), we obtain

$$\begin{aligned}&\partial_t^2\delta\rho_\Delta - \rho_{s0ij}\left.\frac{\partial\mu}{\partial\rho_\Delta}\right|_{w_{ij}}\partial_i\partial_j\delta\rho_\Delta - \rho_{s0ij}\partial_i\partial_t^2 u_j \\ &\quad - \rho_{s0ij}\left.\frac{\partial\mu}{\partial w_{lk}}\right|_{\rho_\Delta}\partial_i\partial_j w_{lk} \\ &\quad = 0.\end{aligned} \tag{39}$$

Our linearized equations of motion, Eqs. (38) and (39), are equivalent to Eq. (19) of Andreev and Lifshitz.[1]

In the particular case in which the lattice sites are fixed (so that $\mathbf{u} = 0$) we recover from Eq. (39) the fourth sound modes obtained by Andreev and Lifshitz,[1] which have the dispersion relation

$$\omega^2 = \rho_{s0ij}\left.\frac{\partial\mu}{\partial\rho}\right|_{w_{ij}} q_i q_j, \tag{40}$$

where we have used $(\partial/\partial\rho_\Delta)_{w_{ij}} = (\partial/\partial\rho)_{w_{ij}}$. On the other hand, when there are no defect fluctuations ($\delta\rho_\Delta = 0$), Eqs. (38) and (39) are combined into

$$\rho_0\partial_t^2 u_i = \left.\frac{\partial\lambda_{ji}}{\partial w_{lk}}\right|_{\rho}\partial_j w_{lk} - \rho_0\left.\frac{\partial\mu}{\partial w_{lk}}\right|_{\rho_\Delta}\partial_i w_{lk} - \rho_0\left.\frac{\partial\mu}{\partial w_{ij}}\right|_{\rho}\partial_j w_{kk}. \tag{41}$$

A mode analysis of this equation would produce six sound modes of an anisotropic normal solid. We see that without defects there are no additional sound modes as expected.

IV. DENSITY-DENSITY CORRELATION FUNCTION OF A MODEL SUPERSOLID

In this section we will calculate the density-density correlation function of a model supersolid, a measurable quantity in a light scattering experiment. However, before delving into the calculations for a supersolid let us first review what is revealed by scattering light from a structureless, normal fluid [for example, see Ref. 48]. The mode counting for the fluid is simple—there are five collective modes due to conservation of mass, energy, and three components of momentum (in three dimensions). The five collective modes are a pair of transverse momentum diffusion modes and three longitudinal modes: a pair of propagating sound modes and a thermal diffusion mode. The density fluctuations important for light scattering only couple to the longitudinal modes so three modes are observed: the diffusion mode appears as the Rayleigh peak $\omega=0$ and the pair of sound modes as the Brillouin doublet at $\omega=\pm cq$ (with a sound speed c). In the absence of dissipation these peaks are δ functions; dissipation turns each δ function into a Lorentzian of width Dq^2 with D being an attenuation coefficient.

What happens in a superfluid? In addition to the five conserved densities that exist in a normal fluid there is a broken gauge symmetry so from mode counting we conclude there are six collective modes. Two of these are transverse momentum diffusion modes (just as for the normal fluid), leaving *four* longitudinal modes for the superfluid: a pair of propagating first sound modes and a new pair of propagating second sound modes. In essence, the central Rayleigh peak in the normal fluid splits into a new Brillouin doublet upon passing into the superfluid phase. This remarkable phenomenon has been observed in light scattering experiments on ^{4}He.[49,50] We show below that an analogous splitting occurs in a supersolid and should be observable in a light scattering experiment.

A. Dynamics of supersolid without dissipation

To facilitate the calculation of the density-density correlation function for a supersolid we will make two simplifying assumptions: the solid is isotropic and two dimensional. The isotropy causes the transverse and longitudinal modes to neatly decouple; the two dimensionality results in only one pair of propagating transverse modes, rather than two pair. Since we are interested in longitudinal fluctuations, the latter simplification is of little consequence to the main results of this section. With these assumptions, the thermodynamic relations are

$$\left.\frac{\partial \lambda_{ji}}{\partial w_{lk}}\right|_{\rho} = \tilde{\lambda}\delta_{ji}\delta_{lk} + \tilde{\mu}(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}), \tag{42}$$

$$\left.\frac{\partial \mu}{\partial w_{ij}}\right|_{\rho} = \gamma\delta_{ij}, \tag{43}$$

$$\left.\frac{\partial \mu}{\partial \rho_\Delta}\right|_{w_{ij}} = \left.\frac{\partial \mu}{\partial \rho}\right|_{w_{ij}} = \frac{1}{\rho_0^2\chi}, \tag{44}$$

where χ is the isothermal compressibility at constant strain, γ is a phenomenological coupling constant between the strain and the density, and $\tilde{\lambda}$ and $\tilde{\mu}$ are the bare Lamé coefficients at constant density. Then in Fourier space the Lagrangian density, Eq. (35), reduces to

$$\mathcal{L}_{\mathrm{SS}} = \frac{1}{2}[\,\delta\rho_\Delta(\mathbf{Q}) \quad \theta(\mathbf{Q}) \quad u_L(\mathbf{Q})\,]\mathbf{A}\begin{bmatrix}\delta\rho_\Delta(-\mathbf{Q})\\ \theta(-\mathbf{Q})\\ u_L(-\mathbf{Q})\end{bmatrix} + \frac{1}{2}(\rho_{n0}\omega_n^2 + \tilde{\mu}q^2)u_T(\mathbf{Q})u_T(-\mathbf{Q}), \tag{45}$$

where $\omega_n = i\omega$, $\mathbf{Q}=(\mathbf{q},\omega_n)$, $u_L=(\mathbf{q}\cdot\mathbf{u})/q$ with $q=|\mathbf{q}|$, $\mathbf{u}_T=\mathbf{u}-(u_L/q)\mathbf{q}$, and

$$\mathbf{A} = \begin{pmatrix} \dfrac{1}{\rho_0^2\chi} & -\omega_n & -iq\left(\gamma - \dfrac{1}{\rho_0\chi}\right) \\ \omega_n & q^2\rho_{s0} & i\omega_n q\rho_{s0} \\ iq\left(\gamma - \dfrac{1}{\rho_0\chi}\right) & i\omega_n q\rho_{s0} & \rho_{n0}\omega_n^2 + q^2\left(\lambda + \dfrac{1}{\chi} - 2\rho_0\gamma\right)\end{pmatrix}, \tag{46}$$

where $\lambda \equiv \tilde{\lambda} + 2\tilde{\mu}$. The collective modes are determined from the determinant $\Delta_{\mathbf{A}}$ of $\mathbf{A}$

$$\Delta_{\mathbf{A}} = \rho_{n0}\omega_n^4 + \left[\lambda + \rho_{n0}\left(\frac{1}{\rho_0\chi} - 2\gamma\right)\right]q^2\omega_n^2 - \rho_{s0}\left(\gamma^2 - \frac{\lambda}{\chi\rho_0^2}\right)q^4. \tag{47}$$

Setting $\Delta_{\mathbf{A}}=0$, we find the longitudinal first sound speed c_L and second sound speed c_2

$$c_L^2 = \frac{\lambda}{2\rho_{n0}} + \frac{1}{2\rho_0\chi} - \gamma + \frac{1}{2}\sqrt{\left(\frac{\lambda}{\rho_{n0}} + \frac{1}{\rho_0\chi} - 2\gamma\right)^2 - \frac{4\rho_{s0}}{\rho_{n0}}\left[\frac{\lambda}{\chi\rho_0^2} - \gamma^2\right]}, \tag{48}$$

$$c_2^2 = \frac{\lambda}{2\rho_{n0}} + \frac{1}{2\rho_0\chi} - \gamma - \frac{1}{2}\sqrt{\left(\frac{\lambda}{\rho_{n0}} + \frac{1}{\rho_0\chi} - 2\gamma\right)^2 - \frac{4\rho_{s0}}{\rho_{n0}}\left[\frac{\lambda}{\chi\rho_0^2} - \gamma^2\right]}. \tag{49}$$

In particular, when $\rho_{s0}=0$ (normal solids), c_2 vanishes, and we only have

$$c_{\mathrm{NS}}^2 = (\tilde{\lambda} + 2\tilde{\mu} + 1/\chi)/\rho_0 - 2\gamma, \tag{50}$$

which agrees with the longitudinal sound speed obtained by Zippelius *et al.*[45] once we identify $\tilde{\lambda}=\lambda^{\mathrm{ZHN}}+2\gamma^{\mathrm{ZHN}}+1/\chi$ and $\gamma=(\gamma^{\mathrm{ZHN}}+1/\chi)/\rho_0$. Moreover, when the Lamé coefficients and the coupling constant γ vanish we recover the sound speed of a normal fluid. As discussed earlier, there is one pair of transverse sound modes with speed

$$c_T = \sqrt{\frac{\tilde{\mu}}{\rho_{n0}}}. \tag{51}$$

Finally, we can calculate the correlation functions from Eq. (45)

$$\langle \delta\rho_\Delta(\mathbf{Q})\delta\rho_\Delta(-\mathbf{Q})\rangle = \rho_{s0}q^2\frac{\rho_0\omega_n^2 + (\lambda - 2\rho_0\gamma + 1/\chi)q^2}{\Delta_\mathbf{A}}, \tag{52}$$

$$\langle \delta\rho_\Delta(\mathbf{Q})u_L(-\mathbf{Q})\rangle = iq\frac{\rho_{s0}}{\rho_0}\frac{\rho_0\omega_n^2 - (\rho_0\gamma - 1/\chi)q^2}{\Delta_\mathbf{A}} \tag{53}$$

and

$$\langle u_L(\mathbf{Q})u_L(-\mathbf{Q})\rangle = \frac{1}{\rho_0^2\chi}\frac{\rho_0^2\chi\omega_n^2 + \rho_{s0}q^2}{\Delta_\mathbf{A}}. \tag{54}$$

Since the density fluctuation is related to the defect density fluctuation and the strain tensor by Eq. (34), the density-density correlation function becomes

$$\langle \delta\rho(\mathbf{Q})\delta\rho(-\mathbf{Q})\rangle = A\left(\frac{1}{i\omega - c_Lq} - \frac{1}{i\omega + c_Lq}\right) + B\left(\frac{1}{i\omega - c_2q} - \frac{1}{i\omega + c_2q}\right), \tag{55}$$

where

$$A = -q\frac{\rho_0\rho_{n0}c_L^2 - \rho_{s0}\lambda}{2c_L\rho_{n0}(c_L^2 - c_2^2)}, \tag{56}$$

$$B = -q\frac{\rho_0\rho_{n0}c_2^2 - \rho_{s0}\lambda}{2c_2\rho_{n0}(c_2^2 - c_L^2)}. \tag{57}$$

Then, by performing the analytic continuation $i\omega_n = \omega + i\delta$, the density-density response function can be obtained from the imaginary part of the density-density correlation function

$$\chi''_{\rho\rho}(\mathbf{q},\omega) = -\pi A[\delta(\omega - c_Lq) - \delta(\omega + c_Lq)] - \pi B[\delta(\omega - c_2q) - \delta(\omega + c_2q)], \tag{58}$$

where we have used the identity

$$\frac{1}{\omega' - \omega - i\epsilon} = P\frac{1}{\omega' - \omega} + i\pi\delta(\omega - \omega'). \tag{59}$$

It is easy to show that the response function satisfies the thermodynamic sum rule (for the derivation of the static correlation function see Appendix B)

$$\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\frac{\chi''_{\rho\rho}(\mathbf{q},\omega)}{\omega} = \frac{\rho_0^2\chi\lambda}{\lambda - \rho_0^2\gamma^2\chi} \tag{60}$$

and the f-sum rule

$$\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\omega\chi''_{\rho\rho}(\mathbf{q},\omega) = \rho_0q^2. \tag{61}$$

B. Dynamics of supersolid with dissipation

We continue our discussion of the density correlation and response functions by including dissipative terms in the equations of motion. As mentioned above, the dissipative terms will broaden the δ-function peaks in the density response function. In addition, as noted by Martin *et al.*,[40] the dissipation is necessary to identify the "missing" defect diffusion mode in normal solids. The dissipative hydrodynamic equations of motion for a supersolid were first obtained by Andreev and Lifshitz,[1] who used standard entropy-production arguments to generate the dissipative terms. For an isotropic supersolid (including the nonlinear term neglected by Andreev and Lifshitz) we have (with the new dissipative terms on the right-hand side)

$$\partial_t\rho + \partial_i j_i = 0, \tag{62}$$

$$\partial_t j_i + \partial_j\Pi_{ij} = \zeta\partial_i\partial_k v_{nk} + \eta\partial^2 v_{ni} - \Sigma\partial_i\partial_k[\rho_s(v_{nk} - v_{sk})], \tag{63}$$

$$\partial_t u_i - v_{ni} + v_{nk}\partial_k u_i + u_i\partial_k v_{nk} = \Gamma\partial_k\lambda_{ki}, \tag{64}$$

$$\partial_t v_{si} + \partial_i\left(\mu + \frac{1}{2}v_s^2\right) = -\Lambda\partial_i\partial_k[\rho_s(v_{nk} - v_{sk})] + \Sigma\partial_i\partial_k v_{nk}, \tag{65}$$

where $\mathbf{j} = \rho_n\mathbf{v}_n + \rho_s\mathbf{v}_s$ is the total mass current, Σ and Λ coefficients of viscosity, ζ the bulk viscosity coefficient, η the shear viscosity coefficient, and Γ the diffusion coefficient for defects.

We next linearize the dissipative hydrodynamic equations by considering small fluctuations from the equilibrium values. Writing $\delta\mu$ and λ_{ij} in terms of $\delta\rho$ and δw_{ij}

$$\delta\mu = \frac{1}{\rho_0^2\chi}\delta\rho + \gamma w_{ii}, \tag{66}$$

$$\delta\lambda_{ij} = \gamma\delta_{ij}\delta\rho + \tilde{\lambda}\delta_{ij}w_{kk} + \tilde{\mu}(w_{ij} + w_{ji}). \tag{67}$$

We replace $\delta\mu$ and $\delta\lambda_{ij}$ into Eqs. (62)–(65), and divide them into transverse and longitudinal parts. The equations of motion for transverse parts are

$$\rho_{n0}\partial_t v_n^T - \tilde{\mu}\partial^2 u^T - \eta\partial^2 v_n^T = 0, \tag{68}$$

where $\partial \equiv \sqrt{\partial_i^2}$ and

$$\partial_t u^T - v_n^T - \tilde{\mu}\Gamma\partial^2 u^T = 0. \tag{69}$$

These equations support a propagating transverse sound mode with the transverse sound speed $c_T = \sqrt{\tilde{\mu}/\rho_{n0}}$, as obtained in the previous section, and an attenuation constant $\Gamma_T = \eta + \rho_{n0}\tilde{\mu}\Gamma$. Next, the longitudinal equations of motion are

$$\partial_t\delta\rho + \rho_{s0}\,\partial v_s^L + \rho_{n0}\,\partial v_n^L = 0, \tag{70}$$

$$\rho_{n0}\partial_t v_n^L + \left(\frac{\rho_{n0}}{\rho_0^2\chi} - \gamma\right)\delta\rho - (\lambda - \rho_{n0}\gamma)\partial^2 u^L - (\tilde{\zeta} - 2\rho_{s0}\sigma - \rho_{s0}^2\Lambda)\partial^2 v_n^L - \rho_{s0}\sigma\partial^2 v_s^L = 0, \tag{71}$$

$$\partial_t u^L - v_n^L - \gamma\Gamma\,\partial\,\delta\rho - \lambda\Gamma\partial^2 u^L = 0, \tag{72}$$

$$\partial_t v_s^L + \frac{1}{\rho_0^2\chi}\partial\,\delta\rho + \gamma\partial^2 u^L - \sigma\partial^2 v_n^L - \rho_{s0}\Lambda\partial^2 v_s^L = 0, \quad (73)$$

where $\sigma \equiv \Sigma - \rho_{s0}\Lambda$ and $\tilde{\zeta} \equiv \zeta + \eta$. The Laplace-Fourier transform of Eqs. (70)–(73) yields

$$\mathbf{C}\begin{pmatrix} \delta\rho(\mathbf{q},z) \\ v_n^L(\mathbf{q},z) \\ u^L(\mathbf{q},z) \\ v_s^L(\mathbf{q},z) \end{pmatrix} = \begin{pmatrix} \delta\rho(\mathbf{q}) \\ v_n^L(\mathbf{q}) \\ u^L(\mathbf{q}) \\ v_s^L(\mathbf{q}), \end{pmatrix}, \quad (74)$$

where

$$\mathbf{C} = \begin{pmatrix} -iz & iq\rho_{n0} & 0 & iq\rho_{s0} \\ iq\left(\frac{1}{\rho_0^2\chi} - \frac{\gamma}{\rho_{n0}}\right) & -iz + q^2\frac{1}{\rho_{n0}}(\tilde{\zeta} - 2\rho_{s0}\sigma - \rho_{s0}^2\Lambda) & q^2\frac{1}{\rho_{n0}}(\lambda - \rho_{n0}\gamma) & q^2\frac{\rho_{s0}}{\rho_{n0}}\sigma \\ -iq\gamma\Gamma & -1 & -iz + q^2\lambda\Gamma & 0 \\ iq\frac{1}{\rho_0^2\chi} & q^2\sigma & -q^2\gamma & -iz + q^2\rho_{s0}\Lambda \end{pmatrix}. \quad (75)$$

From Eq. (75) we calculate two sound speeds c_L, Eq. (48), and c_2, Eq. (49), with two attenuation constants

$$D_L = -\frac{1}{\rho_{n0}(c_L^2 - c_2^2)}(c_L^2 n_1 + n_2), \quad (76)$$

$$D_2 = \frac{1}{\rho_{n0}(c_L^2 - c_2^2)}(c_2^2 n_1 + n_2), \quad (77)$$

where

$$n_1 \equiv \tilde{\zeta} - 2\rho_{s0}\sigma + \rho_{n0}\Gamma\lambda + \rho_{s0}(\rho_{n0} - \rho_{s0})\Lambda, \quad (78)$$

$$n_2 \equiv \frac{1}{\rho_0^2\chi}\{2\rho_0\rho_{s0}(\rho_0\chi\gamma - 1)\sigma + \rho_0\rho_{n0}(\lambda - \rho_0^2\chi\gamma^2)\Gamma + \rho_{s0}\tilde{\zeta} + \rho_0\rho_{s0}[\rho_{n0} - \rho_{s0} + \rho_0\chi(\lambda - 2\gamma\rho_{n0})]\Lambda\}. \quad (79)$$

Now we can see that when $\rho_s = 0$ (a normal solid), the second sound modes disappear but there remains the defect diffusion mode with the diffusion constant $D_2 = (\lambda - \rho_0^2\chi\gamma^2)\Gamma/\rho_0\chi c_L^2$.

We also calculate the density-density Kubo function from Eq. (75) (Ref. 51)

$$K_{\rho\rho}(\mathbf{q},z) = \frac{\chi_{\rho\rho}(\mathbf{q})}{k_BT}\frac{iz^3 + b_{\rho\rho}z^2 + d_{\rho\rho}q^2 z + e_{\rho\rho}q^2}{Z} + \frac{\chi_{u_L\rho}(\mathbf{q})}{k_BT}\frac{d_{\rho u_L}q^2 z + e_{\rho u_L}q^2}{Z}, \quad (80)$$

where the static susceptibilities $\chi_{\rho\rho}$ and $\chi_{u_L\rho}$ in Eq. (80) are given in Appendix B, and

$$Z \equiv (z^2 - c_L^2 q^2 + izq^2 D_L)(z^2 - c_2^2 q^2 + izq^2 D_2), \quad (81)$$

$$b_{\rho\rho} = -\Gamma\gamma q^2 - \frac{\rho_{s0}(\rho_{n0} - \rho_{s0})}{\rho_{n0}}\Lambda q^2 - \frac{\tilde{\zeta} - 2\rho_{s0}\sigma}{\rho_{n0}}q^2, \quad (82)$$

$$d_{\rho\rho} = -i\frac{\lambda}{\rho_{n0}} + i\gamma, \quad (83)$$

$$e_{\rho\rho} = \frac{\rho_{s0}\Lambda\lambda}{\rho_{n0}}q^2 - \rho_{s0}\Lambda\gamma q^2 + \frac{\rho_{s0}\sigma\gamma}{\rho_{n0}}q^2, \quad (84)$$

$$d_{\rho u_L} = (\lambda - \rho_0\gamma)q, \quad (85)$$

$$e_{\rho u_L} = i\rho_{s0}\left(\Lambda - \frac{\sigma}{\rho_{n0}}\right)\lambda q^3 + i\frac{\rho_{s0}}{\rho_{n0}}[2\sigma\rho_0 - \tilde{\zeta} - \rho_0\Lambda(\rho_0 - 2\rho_{s0})]\gamma q^3. \quad (86)$$

Then the susceptibility $\chi''_{\rho\rho}(\mathbf{q},\omega)$ can be obtained from the real part of Eq. (80) (Refs. 44 and 51)

$$\frac{\chi''_{\rho\rho}(\mathbf{q},\omega)}{\omega} = -\frac{iq^4 c_L^2 D_L I_1(q)}{(\omega^2 - c_L^2 q^2)^2 + (\omega q^2 D_L)^2} - \frac{iq^4 c_2^2 D_2 I_2(q)}{(\omega^2 - c_2^2 q^2)^2 + (\omega q^2 D_2)^2} + \frac{(\omega^2 - c_L^2 q^2) I_3(q)}{(\omega^2 - c_L^2 q^2)^2 + (\omega q^2 D_L)^2} + \frac{(\omega^2 - c_2^2 q^2) I_4(q)}{(\omega^2 - c_2^2 q^2)^2 + (\omega q^2 D_2)^2}, \quad (87)$$

where $I_1(q)$, $I_2(q)$, $I_3(q)$, and $I_4(q)$ are given in Appendix C.

Equation (87) is one of the central results of this paper, it is worth exploring some of its features and limits. First, one can show that Eq. (87) satisfies both the thermodynamic sum rule, Eq. (60), and f-sum rule, Eq. (61). Second, the first and second terms in Eq. (87) produce two Brillouin doublets centered at $\omega = \pm c_L q$ and $\omega = \pm c_2 q$ with widths $D_L q^2$ and $D_2 q^2$, respectively. The third and fourth terms in Eq. (87) are neg-

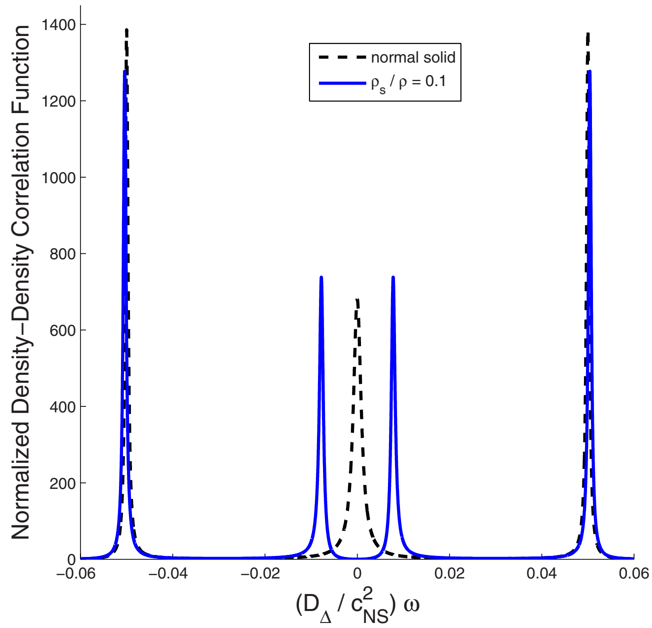

FIG. 1. (Color online) Density-density correlation functions of a normal solid [(black) dashed line] and a supersolid [(blue) solid line]. The supersolid fraction is assumed to be 10%.

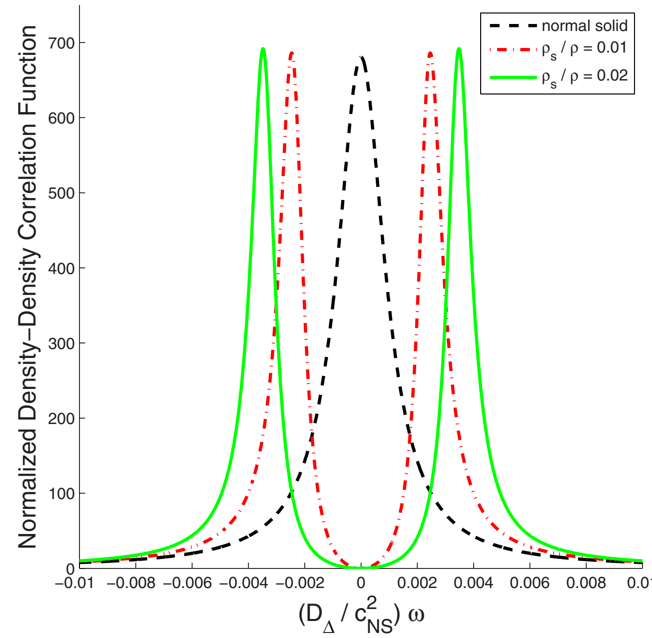

FIG. 2. (Color online) Splitting of the Rayleigh peak [(black) dashed line] due to the defect diffusion mode in the normal solid phase into the Brillouin doublet of the second sound modes in the supersolid phase. The (red) dash-dot line is for $\rho_s/\rho=1\%$ and the (green) solid line $\rho_s/\rho=2\%$.

ligible near the Brillouin doublets and in fact these terms vanish in the limit of zero dissipation. Therefore the nondissipative density-density correlation function, Eq. (58), is obtained from the first two terms in the limit $D_L, D_2 \to 0$. Finally, for normal solids ($\rho_s=0$), the second term in Eq. (87) vanishes and there is only one Brillouin doublet due to the longitudinal first sound modes. In this case the fourth term in Eq. (87) becomes the Rayleigh peak of the defect diffusion mode centered at $\omega=0$. Therefore, we see that in analogy with a superfluid,[49] *the defect diffusion peak that exists in a normal solid will split into a Brillouin doublet of second sound modes upon entering the supersolid phase*.

To get a sense of the size of this effect, let us substitute some physically realistic numbers into the correlation function. Assuming $\rho_s \ll \rho_0$ and $\gamma=\Lambda=\Sigma=0$, we have

$$I_1(q) = -I_2(q) + i\alpha\frac{\rho_0}{c_{\text{NS}}^2} = i\frac{\rho_0}{c_{\text{NS}}^2} - 2i\frac{(\alpha-1)^2}{\alpha^2}\frac{\rho_{s0}}{c_{\text{NS}}^2} + \mathcal{O}\left(\frac{\rho_{s0}^2}{\rho_0^2}\right), \tag{88}$$

$$\begin{aligned} I_3(q) &= -I_4(q) \\ &= -\frac{(\alpha-1)^2}{\alpha}\frac{\rho_0}{c_{\text{NS}}^2}D_\Delta q^2 \\ &\quad + 2q^2\frac{\alpha-1}{\alpha^2}\left[\frac{(\alpha-1)^2(\alpha-3)}{\alpha}D_\Delta + D_l\right]\frac{\rho_{s0}}{c_{\text{NS}}^2} + \mathcal{O}\left(\frac{\rho_{s0}^2}{\rho_0^2}\right), \end{aligned} \tag{89}$$

where the longitudinal sound speed of normal solid c_{NS} is given in Eq. (50), the defect diffusion constant $D_\Delta \equiv \Gamma/\chi$, and $\alpha \equiv \rho_0\chi c_{\text{NS}}^2$. We show in Fig. 1 the normalized density-density correlation functions of a normal solid and a supersolid. We have used the first sound speed $c_{\text{NS}}=550$ m/s, the density $\rho=0.19048$ g/cm^3, the isothermal compressibility $\chi=0.29615\times10^{-8}$ cm s^2/g for ^{4}He solid[52,53] at the molar volume 21 cm^3/mole, the viscosity of ^{4}He *fluid* of 2×10^{-5} g/cm s, the typical wave number involved in light scattering $q^{-1}=100$ nm, and $\Gamma=8\times10^{-11}$ cm^3 s/g. In Fig. 2 we show the splitting of the Rayleigh peak due to defect diffusion in a normal solid into a Brillouin doublet of second sound modes, for two values of the supersolid fraction.

V. CONCLUSION

Starting from general conservation laws and symmetry principles we derived the effective action for a supersolid—a state of matter with simultaneous broken translational symmetry and Bose condensation. The resulting effective action in Eq. (21) is one of the two important results of this work, and will be further developed in subsequent work on vortex and dislocation dynamics in supersolids. In this work, however, we used the linearized version of this action to calculate the second of our important results—the density-density correlation and response functions for a model supersolid (with isotropic elastic properties), see Eq. (87). In complete analogy with a superfluid, we showed that the onset of supersolidity causes the zero-frequency defect diffusion mode to split into propagating second sound modes and from our calculation we can determine the spectral weight in these modes as well as the weight in the "normal" longitudinal sound waves in a solid.

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APPENDIX A: VARIATIONAL PRINCIPLE FOR AN IDEAL FLUID AND THE GROSS-PITAEVSKII ACTION

This appendix demonstrates the simplicity of the variational principle for deriving the hydrodynamic equations of motion and the Lagrangian density for continuum systems. Consider the simplest case of an ideal fluid (IF) which is irrotational, inviscid and incompressible. The Lagrangian density for the ideal fluid is

$$\mathcal{L}_{\text{IF}} = \frac{1}{2}\rho v^2 - U_{\text{IF}}(\rho), \tag{A1}$$

where ρ is the mass density, $\mathbf{v}$ the velocity field, and U_{IF} the internal energy density which satisfies $dU_{\text{IF}}=\mu d\rho$. From the Lagrangian density we construct the action $S=\int dt \int d^3x \mathcal{L}_{\text{IF}}$. The variational principle states that the equations of motion are derived from variations in the action with respect to all the dynamical variables. The naive application of this principle to the ideal fluid leads to the trivial equation of motion $\mathbf{v}=0$. The difficulty is that the dynamical variables ρ and $\mathbf{v}$ are not independent but constrained by the conservation of mass

$$\partial_t \rho + \partial_i(\rho v_i) = 0. \tag{A2}$$

This *constraint* is incorporated into the Lagrangian density by introducing a Lagrange multiplier ϕ

$$\mathcal{L}_{\text{IF}} = \frac{1}{2}\rho v^2 - U_{\text{IF}}(\rho) + \phi[\partial_t \rho + \partial_i(\rho v_i)]. \tag{A3}$$

Then the equations of motion are obtained from variations in the action $S[\rho, \mathbf{v}, \phi]$ with respect to ρ, $\mathbf{v}$, and ϕ

$$\frac{\delta S}{\delta \rho} = \frac{1}{2}v^2 - \frac{\partial U_{\text{IF}}}{\partial \rho} - \frac{D\phi}{Dt} = 0, \tag{A4}$$

$$\frac{\delta S}{\delta v_i} = \rho v_i - \rho \partial_i \phi = 0, \tag{A5}$$

$$\frac{\delta S}{\delta \phi} = \partial_t \rho + \partial_i(\rho v_i) = 0. \tag{A6}$$

From Eq. (A5) we obtain the velocity field

$$v_i = \partial_i \phi, \tag{A7}$$

which implies that there is no vorticity as expected for an ideal fluid. We can derive the Euler equation from Eq. (A4) by taking its gradient, using Eq. (A7), and then the Gibbs-Duhem relation to obtain

$$\rho \frac{Dv_i}{Dt} = -\partial_i P, \tag{A8}$$

where P is the pressure. The Lagrangian density may be cast into an equivalent form by substituting $\mathbf{v}=\nabla\phi$ into Eq. (A3) and integrating by parts with the result

$$\mathcal{L}_{\text{IF}} = -\rho \partial_t \phi - \frac{1}{2}\rho(\partial_i \phi)^2 - U_{\text{IF}}(\rho). \tag{A9}$$

For comparison, consider a system of weakly interacting bosons with a condensate wave function $\psi(\mathbf{r},t)$ that satisfies Gross-Pitaevskii equation[54]

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2 \psi + g|\psi|^2 \psi. \tag{A10}$$

This equation of motion can be derived from the Lagrangian density

$$\mathcal{L}_{\text{GP}} = \frac{i\hbar}{2}[\psi^* \partial_t \psi - \psi \partial_t \psi^*] - \frac{\hbar^2}{2m}(\nabla \psi^*) \cdot (\nabla \psi) - \frac{g}{2}(\psi^* \psi)^2. \tag{A11}$$

Taking $\psi=\sqrt{n}e^{i\tilde{\phi}}$ with the number density n, the Gross-Pitaevskii Lagrangian density becomes

$$\mathcal{L}_{\text{GP}} = \frac{i\hbar}{2}\partial_t n - \hbar n \partial_t \tilde{\phi} - \frac{\hbar^2}{8mn}(\partial_i n)^2 - \frac{\hbar^2}{2m}n(\partial_i \tilde{\phi})^2 - \frac{g}{2}n^2. \tag{A12}$$

The first term contributes $i\hbar N/2$ to the action (with N the number of particles) and does not contribute to the dynamics. Identifying $\rho=mn$ and $\phi=(\hbar/m)\tilde{\phi}$, we see that the Gross-Pitaevskii Lagrangian density, Eq. (A12), is identical to the ideal fluid Lagrangian density, Eq. (A9), with

$$U_{\text{IF}}(\rho) = \frac{\hbar^2}{2m^2}(\nabla\sqrt{\rho})^2 + \frac{g}{2m^2}\rho^2. \tag{A13}$$

APPENDIX B: THERMODYNAMIC RELATIONS AND THE STATIC CORRELATION FUNCTIONS OF SUPERSOLIDS

In this appendix we calculate the thermodynamic relation for the potential energy density in Eq. (1). Given the Lagrangian density, Eq. (1), the total-energy density for a supersolid is defined as the sum of the kinetic-energy densities and the internal-energy density

$$E_{\text{SS}} = \frac{1}{2}\rho_{sij}v_{si}v_{sj} + \frac{1}{2}(\rho\delta_{ij} - \rho_{sij})v_{\text{n}i}v_{\text{n}j} + U_{\text{SS}}(\rho, \rho_{sij}, s, R_{ij}). \tag{B1}$$

Following Andreev and Lifshitz,[1] this total-energy density can be related to the energy density ϵ measured in the frame where the supercomponent is at rest as

$$E_{\text{SS}} = \frac{1}{2}\rho v_{\text{s}}^2 + (\rho\delta_{ij} - \rho_{sij})(v_{\text{n}j} - v_{\text{s}j})v_{\text{s}i} + \epsilon, \tag{B2}$$

where ϵ has a thermodynamic relation

$$d\epsilon = Tds + \mu d\rho - \lambda_{ik}dR_{ik} + (v_{\mathrm{n}i} - v_{\mathrm{s}i})d[(\rho\delta_{ij} - \rho_{\mathrm{s}ij})(v_{\mathrm{n}j} - v_{\mathrm{s}j})]. \tag{B3}$$

We can obtain the thermodynamic relation for the total energy E_{SS} by differentiating Eq. (B2) and using Eq. (B3) for $d\epsilon$ with the result

$$dE_{\mathrm{SS}} = Tds - \lambda_{ik}dR_{ik} - (v_{\mathrm{n}i} - v_{\mathrm{s}i})v_{\mathrm{n}j}d\rho_{\mathrm{s}ij} + \left[\mu + \frac{1}{2}(2v_{\mathrm{n}i}^2 - 2v_{\mathrm{n}i}v_{\mathrm{s}i} + v_{\mathrm{s}i}^2)\right]d\rho + \rho_{\mathrm{s}ij}v_{\mathrm{s}j}dv_{\mathrm{s}i} + (\rho\delta_{ij} - \rho_{\mathrm{s}ij})v_{\mathrm{n}i}dv_{\mathrm{n}j}. \tag{B4}$$

This thermodynamic relation agrees with Eq. (2.18) of Saslow[4] and Eq. (2.1) of Liu[5] after identifying $\epsilon^{\mathrm{Saslow,Liu}} = E_{\mathrm{SS}}$ and $\mu^{\mathrm{Saslow,Liu}} = \mu - v_{\mathrm{n}i}v_{\mathrm{s}i} + v_{\mathrm{s}i}^2/2$. Then the differentiation of Eq. (B1) and the use of Eq. (B4) give the thermodynamic relation for U_{SS}, Eq. (3).

For a supersolid at rest we can expand the free energy $F_{\mathrm{SS}} = E_{\mathrm{SS}} - TS$ up to the second order in the density fluctuations $\delta\rho$ and the strains $w_{ij} = \partial_i u_j$

$$F_{\mathrm{SS}} = \frac{1}{2}\left.\frac{\partial\mu}{\partial\rho}\right|_{w_{ij}}(\delta\rho)^2 + \left.\frac{\partial\mu}{\partial w_{ij}}\right|_{\rho}\delta\rho w_{ij} + \frac{1}{2}\left.\frac{\partial\lambda_{ij}}{\partial w_{lk}}\right|_{\rho} w_{ij}w_{lk}. \tag{B5}$$

Using Eqs. (42)–(44) for an isotropic supersolid, the free energy (in Fourier space) can be written as

$$F_{\mathrm{SS}} = \frac{1}{2}\tilde{\mu}q^2u_T^2 + \frac{1}{2}[\delta\rho(\mathbf{q}) \quad u_L(\mathbf{q})]\mathbf{B}\begin{bmatrix}\delta\rho(-\mathbf{q}) \\ u_L(-\mathbf{q})\end{bmatrix}, \tag{B6}$$

where

$$\mathbf{B} = \begin{pmatrix} \dfrac{1}{\rho_0^2\chi} & -iq\gamma \\ iq\gamma & q^2\lambda \end{pmatrix}. \tag{B7}$$

Then the static correlation functions can be easily read off from Eq. (B6)

$$\chi_{\rho\rho}(\mathbf{q}) = \beta\langle\delta\rho(\mathbf{q})\delta\rho(-\mathbf{q})\rangle = \frac{\rho_0^2\chi\lambda}{\lambda - \rho_0^2\gamma^2\chi}, \tag{B8}$$

$$\chi_{u_L\rho}(\mathbf{q}) = \beta\langle u_L(\mathbf{q})\delta\rho(-\mathbf{q})\rangle = \frac{i\rho_0^2\chi\gamma}{q(\lambda - \rho_0^2\gamma^2\chi)}. \tag{B9}$$

APPENDIX C: CALCULATION OF THE DENSITY-DENSITY CORRELATION FUNCTION

Each term in the Kubo function given in Eq. (80) can be separated into the first sound part and the second sound part by performing a partial-fraction expansion

$$\frac{a_{jk}z^3 + b_{jk}z^2 + d_{jk}q^2z + q^2e_{jk}}{(z^2 - c_L^2q^2 + izD_Lq^2)(z^2 - c_2^2q^2 + izD_2q^2)} = \frac{\tilde{A}_{jk}z + \tilde{B}_{jk}}{z^2 - c_L^2q^2 + izD_Lq^2} + \frac{\tilde{C}_{jk}z + \tilde{D}_{jk}}{z^2 - c_2^2q^2 + izD_2q^2}, \tag{C1}$$

where $j,k = (\rho, u_L)$. Then, $\tilde{A}_{jk}$, $\tilde{B}_{jk}$, $\tilde{C}_{jk}$, and $\tilde{D}_{jk}$ can be written in terms of a_{jk}, b_{jk}, d_{jk}, and e_{jk} along with the sound velocities (c_L and c_2) and the attenuation coefficients (D_L and D_2)

$$\tilde{A}_{jk} = \frac{a_{jk}[c_L^4 - c_2^2c_L^2 + q^2D_L(D_Lc_2^2 - D_2c_L^2)]}{(c_L^2 - c_2^2)^2 + q^2(D_L - D_2)(D_Lc_2^2 - D_2c_L^2)} + \frac{ib_{jk}(c_2^2D_L - D_2c_L^2) + d_{jk}(c_L^2 - c_2^2) + ie_{jk}(D_L - D_2)}{(c_L^2 - c_2^2)^2 + q^2(D_L - D_2)(D_Lc_2^2 - D_2c_L^2)}, \tag{C2}$$

$$\tilde{B}_{jk} = \frac{ia_{jk}c_L^2q^2(D_Lc_2^2 - D_2c_L^2) + b_{jk}c_L^2(c_L^2 - c_2^2)}{(c_L^2 - c_2^2)^2 + q^2(D_L - D_2)(D_Lc_2^2 - D_2c_L^2)} + \frac{id_{jk}c_L^2q^2(D_L - D_2) + e_{jk}[c_L^2 - c_2^2 + q^2D_L(D_2 - D_L)]}{(c_L^2 - c_2^2)^2 + q^2(D_L - D_2)(D_Lc_2^2 - D_2c_L^2)}. \tag{C3}$$

The coefficients $\tilde{C}_{jk}$ and $\tilde{D}_{jk}$ are the same as $\tilde{A}_{jk}$ and $\tilde{B}_{jk}$, respectively, but two sound velocities and two attenuation coefficients must be interchanged. Then the functions defined in the density-density correlation function, Eq. (87), are given by

$$I_1(\mathbf{q}) = \chi_{\rho\rho}(\mathbf{q})\tilde{A}_{\rho\rho} + \chi_{u_L\rho}(\mathbf{q})\tilde{A}_{\rho u_L}, \tag{C4}$$

$$I_2(\mathbf{q}) = \chi_{\rho\rho}(\mathbf{q})\tilde{C}_{\rho\rho} + \chi_{u_L\rho}(\mathbf{q})\tilde{C}_{\rho u_L}, \tag{C5}$$

$$I_3(\mathbf{q}) = \chi_{\rho\rho}(\mathbf{q})\tilde{B}_{\rho\rho} + \chi_{u_L\rho}(\mathbf{q})\tilde{B}_{\rho u_L} - iq^2D_LI_1(\mathbf{q}), \tag{C6}$$

$$I_4(\mathbf{q}) = \chi_{\rho\rho}(\mathbf{q})\tilde{D}_{\rho\rho} + \chi_{u_L\rho}(\mathbf{q})\tilde{D}_{\rho u_L} - iq^2D_2I_2(\mathbf{q}). \tag{C7}$$

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