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Nonequilibrium quantum criticality in bilayer itinerant ferromagnets

So Takei,[1] William Witczak-Krempa,[2] and Yong Baek Kim[2,3]
[1]*Max-Planck-Institut für Festkörperforschung, D-70569 Stuttgart, Germany*
[2]*Department of Physics, University of Toronto, Toronto, Ontario, Canada M5S 1A7*
[3]*School of Physics, Korea Institute for Advanced Study, Seoul 130-722, Korea*
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We present a theory of nonequilibrium quantum criticality in a coupled bilayer system of itinerant electron magnets. The model studied consists of the first layer subjected to an in-plane current and open to an external substrate. The second layer is subject to no direct external drive, but couples to the first layer via short-ranged spin-exchange interaction. No particle exchange is assumed between the layers. Starting from a microscopic fermionic model, we derive an effective action in terms of two coupled bosonic fields which are related to the magnetization fluctuations of the two layers. When there is no interlayer coupling, the two bosonic modes possess different dynamical critical exponents z with $z=2$ ($z=3$) for the first (second) layer. This results in multiscale quantum criticality in the coupled system. It is shown that the linear coupling between the two fields leads to a low-energy fixed point characterized by the larger dynamical critical exponent $z=3$. We compute the correlation length in the quantum disordered and quantum critical regimes for both the nonequilibrium case and in thermal equilibrium where the whole system is held at a common temperature T. We identify an effective temperature scale T_{eff} with which we define a quantum-to-classical crossover that is in exact analogy with the thermal equilibrium case but with T replaced by T_{eff}. However, we find that the leading correction to the correlation length in the quantum critical regime scales differently with respect to T and T_{eff}. We also note that the current in the lower layer generates a drift of the magnetization fluctuations, manifesting itself as a parity-breaking contribution to the effective action of the bosonic modes. In this sense, the nonequilibrium drive in this system plays a role which is distinct from T in the thermal equilibrium case. We also derive the stochastic dynamics obeyed by the critical fluctuations in the quantum critical regime and find that they do not fall into the previously identified dynamical universality classes.

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I. INTRODUCTION

Understanding the role of nonequilibrium drives on systems tuned near a quantum critical point has received growing attention in recent years. One of the issues is to understand to what extent the concept of universality persists when systems depart far from equilibrium. Several works considered universal scaling behavior in nonlinear transport properties close to a superconductor-insulator quantum critical point.[1–3] Universal scaling of current in response to an external field was also studied for an itinerant electron system near a magnetic quantum critical point.[4] More recently, universal scaling functions of the nonlinear conductance and fluctuation dissipation ratios were obtained for a magnetic single-electron transistor near quantum criticality.[5,6]

Complimentary works have studied the effects of current flow on two-dimensional (2D) itinerant electron systems near a ferromagnetic-paramagnetic quantum critical point by generalizing the perturbative renormalization group to nonequilibrium situations.[7,8] These works developed scaling theories which showed that the leading effect of the nonequilibrium probe is to act as an effective temperature, T_{eff}. As in the corresponding equilibrium theory,[9,10] T_{eff} was found to be a relevant perturbation at the equilibrium Gaussian fixed point and to be responsible for inducing a quantum-to-classical crossover similar to the one triggered by thermal fluctuations. A large-T_{eff} quantum critical regime, analogous to the corresponding regime in equilibrium quantum criticality, was also identified. In this regime, the long-time, long-wavelength behavior of the order-parameter fluctuations was found to obey Langevin dynamics similar to one of the dynamical universality classes considered by Hohenberg and Halperin.[11] This has led to an important identification of a nonequilibrium dynamical universality class.

The form of the long-time, long-wavelength dynamics obeyed by order-parameter fluctuations depends intimately on the geometry of the system and on the manner in which the nonequilibrium perturbation is applied. For a 2D system, the orientation of the current flow with respect to the plane of the system influences the dynamics. When inversion symmetry is broken,[8] for instance, with the application of current parallel to the plane, the dynamics obeyed by the order-parameter fluctuations differs from the case where the current is applied normal to the plane. Introducing particle exchange by tunnel coupling the system to an external reservoir also affects the dynamical critical phenomena. A system, initially characterized by conserved order-parameter fluctuations in the closed case, must be described by nonconserved fluctuations once it is opened. For itinerant ferromagnets,[10] this corresponds to a change in the dynamical critical exponent from $z=3$ to $z=2$.[7] The sensitivity of order-parameter dynamics to system geometry and the form of nonequilibrium perturbation opens up the possibility to explore a variety of nonequilibrium dynamical universality classes and contribute to the general understanding of nonequilibrium quantum criticality.

In this work, we consider nonequilibrium quantum criticality in a coupled bilayer system of itinerant ferromagnets

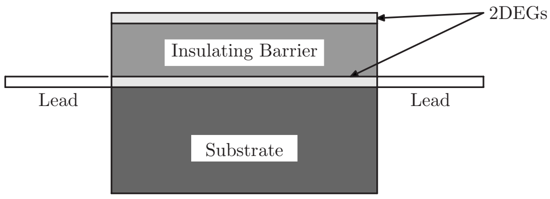

FIG. 1. Edge-on view of the model system considered in this work. The two itinerant electron films (2DEGs) are coupled via short-ranged spin-exchange interaction. The bottom 2DEG is tunnel coupled to a substrate and is driven out of equilibrium by a uniform electric field **E** applied in the in-plane direction. The top layer in turn is driven out of equilibrium by its interaction with the lower layer. There is no tunneling of electrons between the two 2DEGs.

tuned close to a spin-density wave instability. The two itinerant ferromagnets are separated by a three-dimensional (3D) insulating barrier (see Fig. 1) and are coupled via short-ranged spin-exchange interaction. We consider the case where there is no particle exchange between the layers. Departure from equilibrium is achieved by driving one of the layers with an in-plane electrical current. The resulting drift[8,12] of the spin fluctuations, in turn, causes the fluctuations in the other layer to also drift, ultimately driving the latter out of equilibrium. A steady state is established by tunnel coupling the driven layer to an external bath.

We show that the system can be characterized by two bosonic fields, which simply relate to the physical spin fluctuations of the two layers, and are coupled at the Gaussian level via the interlayer exchange interaction. The crucial complexity of the problem comes from the fact that the two fields possess different dynamics with different dynamical exponents, z, in the absence of interlayer coupling. This is a direct consequence of the fact that while one layer is open to an external substrate ($z=2$ mode), the other remains closed from any external bath ($z=3$ mode). As such, the bilayer system exhibits multiscale quantum criticality. We find that, because the fields couple linearly, the infrared properties of the system are governed by a $z=3$ fixed point.

We present a perturbative renormalization-group analysis of the system both in and out of equilibrium. In the equilibrium case the full system is held at a common temperature, T. Out of equilibrium, we introduce an effective temperature, T_{eff} (as in Ref. 8), which parametrizes the decoherence induced by the nonequilibrium drive. We study the flow of the system in the vicinity of the $z=3$ equilibrium Gaussian fixed point. We identify the line of crossover from the quantum disordered to the quantum critical regime in terms of the bare parameters of the system and compute both temperature and nonequilibrium corrections to the correlation length of the critical fluctuations in the quantum critical regime. We find that while the temperature corrections to the correlation length in thermal equilibrium gain two contributions that reflect the presence of the $z=2$ and 3 dynamics, the nonequilibrium contribution contains only one correction that reflects $z=2$ physics. This result signifies the fact that the nonequilibrium drive is applied only to the bottom layer whose corresponding fluctuations in the decoupled limit possess $z=2$ dynamics. This shows that the energy scale identified as effective temperature in a previous work (Ref. 8) does not strictly apply to this problem.

We also consider the Langevin dynamics obeyed by the critical fluctuations in the quantum critical regime where the system effectively becomes classical and the quantum fluctuations can be integrated out. Here, the analysis is carried out in the eigenbasis that diagonalizes the Gaussian effective action. Because one of the eigenmodes remains massive at the critical point, the long-time dynamics of the full system can be described by a single field. We solve the Langevin equation for this eigenmode both in and out of equilibrium. While $z=2$ physics seem to play no role in the long-time dynamics of the critical eigenmode in equilibrium, its effect is important in the nonequilibrium case. In the latter case, the Langevin dynamics display a hybrid effect of both $z=2$ and 3 physics. In either case, the obtained dynamics differ from any of the dynamical universality classes considered in Ref. 11.

The paper is organized as follows. In Sec. II, we begin by introducing the bilayer system, and present a theory to model the system. By integrating out all fermionic degrees of freedom, the effective two-field bosonic action is derived in Sec. III. We provide a brief mean-field analysis of the effective action and establish the fixed point in Sec. IV for the perturbative renormalization-group analysis which follows in Sec. V. The correlation length is calculated in Sec. VI, and the dynamics obeyed by the critical fluctuations in the quantum critical regime is discussed in Sec. VII. Finally, we conclude in Sec. VIII.

II. SYSTEM AND MODEL

The system of interest, shown in Fig. 1, consists of two 2D itinerant electron systems which are tuned close to a spin-density wave instability.[9,10] Throughout this work, we assume the systems to be paramagnetic but are nearly unstable toward an itinerant ferromagnetic state with an order-parameter symmetry of Ising nature. We assume no particle exchange between the two layers, but allow magnetic fluctuations in the layers to interact via short-ranged ferromagnetic spin exchange. The bottom layer is tunnel coupled to a substrate and is driven out of equilibrium by a uniform electric field **E** applied in the in-plane direction.

The total Hamiltonian of the system is given by

$$H = H_b + H_t + H_{b\text{-}t} + H_{sub} + H_{b\text{-}sub}, \tag{1}$$

where b (t) labels the bottom (top) layer and *sub* denotes the substrate. The Hamiltonian for each layer has a kinetic part and an onsite ferromagnetic exchange interaction term with a common interaction strength U. They can be written as

$$H_b = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}-e\mathbf{A}/c} c^{\dagger}_{b\mathbf{k},\sigma} c_{b\mathbf{k},\sigma} - \frac{U}{2}\sum_i (S^z_{b,i})^2, \tag{2}$$

$$H_t = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{t\mathbf{k},\sigma} c_{t\mathbf{k},\sigma} - \frac{U}{2}\sum_i (S^z_{t,i})^2, \tag{3}$$

where $c_{b(t)\mathbf{k},\sigma}$ is the annihilation operator for the bottom (top) fermions with in-plane momentum $\mathbf{k}$ and spin σ. The usual quadratic dispersion, $\epsilon_{\mathbf{k}} = |\mathbf{k}|^2/2m$, is assumed in both layers. $S^z_{\alpha,i} = \Sigma_\sigma \sigma c^{\dagger}_{\alpha i\sigma} c_{\alpha i\sigma}$ is the magnetization for layer α, where i

labels the in-plane position, and the usual factor of 1/2 was absorbed in the interaction U. e is the electron charge. We use a gauge in which the electromagnetic potential $\mathbf{A}=-c\mathbf{E}t$, and set $\hbar=1$.

Equations (2) and (3) have been written in a form which explicitly breaks SU(2) symmetry. Here, we are assuming that the itinerant magnets possess an easy axis so that their magnetization prefers to point up or down with respect to a certain crystal axis. Indeed, strong spin-orbit coupling in some itinerant magnets is known to give rise to pronounced magnetocrystalline anisotropy.[13,14] In this work, we also focus solely on the disordered side of the phase diagram where number of components for each of the order parameters are not expected to influence the results in an essential way.[9,10]

The Hamiltonian describing the interlayer interaction reads

$$H_{b\text{-}t} = -J\sum_i S^z_{b,i}S^z_{t,i}, \tag{4}$$

where $J>0$ is the strength of the ferromagnetic exchange interaction. The interaction may be mediated by magnetic fluctuations in the central paramagnetic insulator. We assume that these fluctuations possess long-range correlations in the out-of-plane direction (on the order of its thickness), but short-ranged correlations in the in-plane direction, which justifies the locality of the interaction in the in-plane direction. We refer the reader to Appendix A for further details on how the paramagnetic fluctuations of the central insulator can give rise to an effective spin-exchange interaction between the two layers. The substrate Hamiltonian and the tunneling Hamiltonian, which describes the coupling between the bottom layer and the substrate, are given by

$$H_{sub} = \sum_{\mathbf{k},k_z,\sigma} \epsilon_{\mathbf{k},k_z} a^\dagger_{\mathbf{k},k_z,\sigma} a_{\mathbf{k},k_z,\sigma}, \tag{5}$$

$$H_{b\text{-}sub} = \zeta \sum_{\mathbf{k},k_z,\sigma} a^\dagger_{\mathbf{k},k_z,\sigma} c_{b\mathbf{k},\sigma} + \text{H.c.}, \tag{6}$$

where $a_{\mathbf{k},k_z,\sigma}$ is the annihilation operator for the substrate fermions, and ζ parametrizes the tunneling strength between the layer and the substrate. The substrate is modeled as a Fermi-liquid bath with $\epsilon_{\mathbf{k},k_z}=\epsilon_{\mathbf{k}}+\epsilon(k_z)$, where k_z labels the out-of-plane momentum, and $\epsilon(k_z)$ is some dispersion not necessarily quadratic. The essential features of the bath are its density of states, ρ, which implies a broadening in the electronic levels associated with the bottom layer, $\Gamma=\pi\rho\zeta^2$ (the escape rate of electrons from the bottom layer to the substrate), and its resistivity, which we assume to be very high relative to that of the layer so we may couple the electric field only to the bottom layer.

III. EFFECTIVE KELDYSH BOSONIC ACTION

To harness the nonequilibrium nature of the problem, we formulate our model using the Keldysh path-integral formalism.[15] We begin by writing the Keldysh partition function

$$\mathcal{Z}^K = \int \mathcal{D}[\bar{c}^\pm, c^\pm, \bar{a}^\pm, a^\pm] e^{i\mathcal{S}^+ - i\mathcal{S}^-}, \tag{7}$$

where + and − label the action for the forward and backward parts of the time-loop contour, respectively. The full action is given by

$$\mathcal{S}^\kappa = \mathcal{S}^\kappa_b + \mathcal{S}^\kappa_t + \mathcal{S}^\kappa_{b\text{-}t} + \mathcal{S}^\kappa_{sub} + \mathcal{S}^\kappa_{b\text{-}sub}, \tag{8}$$

where

$$\mathcal{S}^\kappa_b = \int d^3x \bar{c}^\kappa_{b\sigma}\left[i\partial_t - \frac{(i\partial_{\mathbf{r}} + e\mathbf{A}/c)^2}{2m} + \mu_b\right]c^\kappa_{b\sigma} + \frac{U}{2}\int d^3x (S^{z,\kappa}_b)^2, \tag{9}$$

$$\mathcal{S}^\kappa_t = \int d^3x \bar{c}^\kappa_{t\sigma}\left[i\partial_t + \frac{\partial^2_{\mathbf{r}}}{2m} + \mu_t\right]c^\kappa_{t\sigma} + \frac{U}{2}\int d^3x (S^{z,\kappa}_t)^2, \tag{10}$$

$$\mathcal{S}^\kappa_{b\text{-}t} = J\int d^3x S^{z,\kappa}_b S^{z,\kappa}_t, \tag{11}$$

$$\mathcal{S}^\kappa_{sub} = \int d^3x dz \bar{a}^\kappa_\sigma\left[i\partial_t + \frac{\partial^2_{\mathbf{r}}}{2m} - \epsilon(-i\partial_z) + \mu_s\right]a^\kappa_\sigma, \tag{12}$$

$$\mathcal{S}^\kappa_{b\text{-}sub} = -\zeta\int d^3x(\bar{a}^\kappa_{\sigma,z=0}c^\kappa_{b\sigma} + \bar{c}^\kappa_{b\sigma}a^\kappa_{\sigma,z=0}). \tag{13}$$

We are using $x=(t,\mathbf{r})$, $\mathbf{r}$ being the in-plane coordinate; z denotes the out-of-plane direction. All fermionic fields are now expressed in terms of Grassmann variables. We have omitted position and time dependences from all fields for brevity and assumed summation over the spin. $\kappa=\pm$ label a branch of the Keldysh contour on which the corresponding field resides. μ_b, μ_t, and μ_s are the chemical potentials of the bottom layer, top layer, and substrate, respectively.

A. Free fermionic action

The free fermionic action in Keldysh space can be obtained by setting $U=J=0$. After performing a standard change of basis,[15]

$$c^1 = \frac{c^+ + c^-}{\sqrt{2}}, \quad c^2 = \frac{c^+ - c^-}{\sqrt{2}},$$

$$\bar{c}^1 = \frac{\bar{c}^+ - \bar{c}^-}{\sqrt{2}}, \quad \bar{c}^2 = \frac{\bar{c}^+ + \bar{c}^-}{\sqrt{2}}, \tag{14}$$

the Keldysh action for layer α is simply given by

$$\mathcal{S}_{\alpha,0} = \int_{k,\sigma} (\bar{c}^1_{\alpha k\sigma} \quad \bar{c}^2_{\alpha k\sigma})\begin{pmatrix}[G^{-1}_{\alpha k}]^R & [G^{-1}_{\alpha k}]^K \\ 0 & [G^{-1}_{\alpha k}]^A\end{pmatrix}\begin{pmatrix}c^1_{\alpha k\sigma} \\ c^2_{\alpha k\sigma}\end{pmatrix}. \tag{15}$$

Here, $k=(\omega,\mathbf{k})$ labels the energy-momentum three vector and we have introduced the notation $\int_{k,\sigma}=\Sigma_\sigma\int\frac{d^2\mathbf{k}}{(2\pi)^2}\frac{d\omega}{2\pi}$. The top-layer electron Green's functions correspond to those of free equilibrium electrons,

$$G^{R(A)}_{tk} = \frac{1}{\omega - \xi^t_{\mathbf{k}} \pm i0^+},$$

$$G^K_{tk} = -2\pi i \,\mathrm{sgn}(\omega)\delta(\omega - \xi^t_{\mathbf{k}}), \tag{16}$$

where $\xi^t_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu_t$. We note that the Keldysh Green's function is given in its zero-temperature form.

The bottom layer is affected by both the electric field and the substrate, and this substantially modifies the Green's functions. Derivation of the Green's functions will be briefly outlined here. More detailed derivations can be found in Refs. 8 and 12. The first step is to integrate out the substrate degrees of freedom. For a frequency-independent tunneling amplitude [cf. Eq. (6)] the effects of the substrate can be parametrized by a constant level-broadening parameter $\Gamma = \pi\rho\zeta^2$, which measures the rate at which electrons in the bottom layer escape into the substrate. Then the retarded and Keldysh self-energies which renormalize the bottom Green's functions are given by

$$\Sigma^R_k = -i\Gamma = (\Sigma^A_k)^*, \tag{17}$$

$$\Sigma^K_k = -2i\Gamma[1 - 2g(\omega)]. \tag{18}$$

Here, $g(\omega)$ is the Fermi-Dirac distribution of the substrate, where ω is measured with respect to the chemical potential μ_s of the substrate. We note here that because of our particular system geometry, the scattering into the bath dominates over electron-paramagnon scattering, which leads to considerable analytical simplifications. For instance, in Ref. 4, the geometry is that of a thin strip which is coupled to a bath only at its boundary. In that case, electron-paramagnon scattering is crucial and one must calculate the electron and paramagnon distributions self-consistently. We also note that coupling the entire lower layer (and not only its boundaries) to an infinite bath allows one to assume that the Joule heat generated by the electric field is efficiently carried away into the bath.

We now consider the effect of the nonequilibrium drive. At this point we specify the magnitude of the electric field that will be considered in this work. We will be comparing the electric field E to Γ, the escape rate of electrons from the bottom layer to the substrate, or rather to the corresponding time scale $\tau = 1/2\Gamma$. We introduce a quantity which will be important in the remainder of this work: the effective temperature scale set by the nonequilibrium field, T_{eff}. This quantity parametrizes the consequences of the nonequilibrium-induced decoherence. In this system, where a uniform current flows in the in-plane direction, T_{eff} is given by[4,8]

$$T_{\mathrm{eff}} = eEv_F\tau. \tag{19}$$

We will limit ourselves to weak fields: $T_{\mathrm{eff}}/E_F \ll 1$ and $\tau T_{\mathrm{eff}} \ll 1$, where E_F is the Fermi energy of the bottom layer. Restricting ourselves to this regime and using the canonical momentum, the retarded and advanced Green's functions take their equilibrium form[8]

$$G^{R(A)}_{bk} = \frac{1}{\omega - \xi^b_{\mathbf{k}} \pm i\Gamma}. \tag{20}$$

Here, $\xi^b_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu_b$. The nonequilibrium Keldysh Green's function can be obtained using the linearized quantum Boltzmann equation[16,17] (QBE) for the lesser Green's function,

$$e\mathbf{E}\cdot(\nabla_{\mathbf{k}} + \mathbf{v}_{\mathbf{k}}\partial_\omega)G^<_{bk} = \Sigma^<_{bk}A_{bk} - 2\Gamma G^<_{bk}. \tag{21}$$

With the usual parametrization, the lesser Green's function can be written as

$$G^<_{bk} = if_{bk}A_{bk}, \tag{22}$$

where f_{bk} is the distribution function and A_{bk} the spectral function. With this parametrization one obtains a QBE for the distribution function,

$$e\mathbf{E}\cdot(\nabla_{\mathbf{k}} + \mathbf{v}_{\mathbf{k}}\partial_\omega)f_{bk} = 2\Gamma[-f_{bk} + g(\omega)]. \tag{23}$$

Following Ref. 8, we simplify Eq. (23) by dropping the gradient term in the left-hand side, which is justified in the weak-field limit. One then is required to solve the following QBE:

$$e\mathbf{E}\cdot\mathbf{v}_{\mathbf{k}}\partial_\omega f_{bk} = 2\Gamma[-f_{bk} + g(\omega)]. \tag{24}$$

The Keldysh Green's function is then obtained by solving Eq. (24) for the distribution function and substituting this into the standard formula,

$$G^K_{bk} = [1 - 2f_{bk}](G^R_{bk} - G^A_{bk}). \tag{25}$$

At zero temperature, the nonequilibrium electron distribution can be easily obtained and is given by

$$f_{bk} = \theta(-\omega) + \frac{1}{2}[\mathrm{sgn}(\omega) + \mathrm{sgn}(\mathbf{E}\cdot\mathbf{v}_{\mathbf{k}})]e^{-|\omega|/|e\mathbf{E}\cdot\mathbf{v}_{\mathbf{k}}\tau|}, \tag{26}$$

where we have used the fact that $g(\omega) = \theta(-\omega)$ at $T=0$ (θ being the Heaviside function). We see from Eqs. (24) and (26) that in equilibrium ($\mathbf{E}=0$), the bottom electron distribution reduces to the distribution of the substrate electrons signifying the state of thermal and chemical equilibrium between the two systems. We will hereafter assume for simplicity $\mu_t = \mu_b = \mu_s = \mu$. We therefore drop the superscript t and b from the top and bottom dispersions and assume that both dispersions and frequency ω are measured with respect to the common chemical potential μ.

We note here that the linear-response approach to obtaining the electron-distribution function is justified here as long as the shift of the Fermi surface in momentum space ($eE\tau$) is much less than the Fermi wave vector. Indeed, we will find, as in similar previous works,[7,8] that the nonequilibrium field is a relevant perturbation and grows under the renormalization-group flow. In this work, we restrict our flow up to the scale where $eE\tau \sim k_F$ so that Eqs. (25) and (26) remain valid.

B. Interactions

The interactions are given by

$$\mathcal{S}_{\rm int}^{\kappa}=\int d^3x\left[JS_b^{z,\kappa}S_t^{z,\kappa}+\frac{U}{2}(S_b^{z,\kappa})^2+\frac{U}{2}(S_t^{z,\kappa})^2\right]. \quad (27)$$

The interactions are decoupled by two one-component bosonic fields, $m_b^{\kappa}(x)$ and $m_t^{\kappa}(x)$, which physically represent the magnetization in the bottom and the top layers, respectively. The decoupling leads to

$$\mathcal{S}_{\rm int}^{\kappa}=-U\int d^3x\left[\frac{1}{2}(m_b^{\kappa})^2+\frac{j}{2}m_b^{\kappa}m_t^{\kappa}-(m_b^{\kappa}+jm_t^{\kappa})S_b^{z,\kappa}\right] +[b\leftrightarrow t], \quad (28)$$

where $j=J/U$ is the interlayer exchange coupling normalized by the intralayer exchange interaction. At this point, it is useful to introduce a new basis for the bosonic fields

$$\begin{pmatrix}\phi_b^{\kappa}\\ \phi_t^{\kappa}\end{pmatrix}=\begin{pmatrix}1 & j\\ j & 1\end{pmatrix}\begin{pmatrix}m_b^{\kappa}\\ m_t^{\kappa}\end{pmatrix}. \quad (29)$$

The transformation is well-defined given $j^2\neq 1$. The ϕ fields reduce to the physical layer magnetizations when the interlayer coupling $J\to 0$. Both bases give an equivalent description of the problem but the analysis is simplified in the ϕ basis because ϕ_b (ϕ_t) couples directly only to b (t) fermions, as can be seen from the third term in Eq. (28). Integrating out the fermions will not generate additional couplings between the bosons. In this new basis Eq. (28) takes the form

$$\mathcal{S}_{\rm int}^{\kappa}=-U\int d^3x\frac{1}{1-j^2}\left[\frac{1}{2}(\phi_b^{\kappa})^2+\frac{1}{2}(\phi_t^{\kappa})^2-j\phi_b^{\kappa}\phi_t^{\kappa}\right]-\phi_b^{\kappa}S_b^{z,\kappa} -\phi_t^{\kappa}S_t^{z,\kappa}. \quad (30)$$

The final Keldysh action for the interactions is obtained after performing a change of basis in Keldysh space. The corresponding transformation for the bosons reads

$$\phi^{\rm cl}=\frac{\phi^{+}+\phi^{-}}{\sqrt{2}},\quad \phi^{\rm q}=\frac{\phi^{+}-\phi^{-}}{\sqrt{2}} \quad (31)$$

with the analogous transformation for the conjugated fields. "cl" and "q" stand for classical and quantum Keldysh components, respectively. The resulting action then reads

$$\mathcal{S}_{\rm int}^{\kappa}=\int_{k,q}\sum_{\sigma,\alpha}\frac{\sigma}{\sqrt{2}}(\bar{c}^1_{\alpha k\sigma}\ \ \bar{c}^2_{\alpha k\sigma})\begin{pmatrix}\phi_{\alpha}^{\rm cl} & \phi_{\alpha}^{\rm q}\\ \phi_{\alpha}^{\rm q} & \phi_{\alpha}^{\rm cl}\end{pmatrix}_{k-q}\begin{pmatrix}c^1_{\alpha q\sigma}\\ c^2_{\alpha q\sigma}\end{pmatrix} -\frac{1}{2\tilde{U}}\int_q\sum_{\alpha,\beta}[\phi^{\rm cl}_{\alpha,-q}(\delta_{\alpha\beta}-j(\tau_x)_{\alpha\beta})\phi^{\rm q}_{\beta q}+{\rm c.c.}], \quad (32)$$

where $\tilde{U}=U(1-j^2)$. α and β here are summed over b and t and $q=(\Omega,\mathbf{q})$ labels bosonic frequency and momentum.

C. Integrating out fermions

We now integrate out all remaining fermionic modes to obtain an effective bosonic theory in the ϕ basis. Here, we follow the standard procedure and expand the resulting action up to quartic order in the bosonic fields. This then yields

$$\mathcal{S}_{\rm eff}=\mathcal{S}_{\rm eff}^{(2)}+\mathcal{S}_{\rm eff}^{(4)}, \quad (33)$$

where the Gaussian part reads

$$\mathcal{S}_{\rm eff}^{(2)}=-\int_q\{\Phi_{bq}^{\dagger}[(\tilde{U}\nu)^{-1}\hat{\tau}_x+\hat{\Pi}_{bq}]\Phi_{bq}-\Phi_{bq}^{\dagger}(\epsilon\hat{\tau}_x)\Phi_{tq} +(b\leftrightarrow t)\}, \quad (34)$$

where $\epsilon=j(\tilde{U}\nu)^{-1}$, $\nu=m/\pi$ is the two-dimensional density of states at the Fermi energy, and $\hat{\tau}_x$ is a Pauli matrix in Keldysh space. The two-component vector $\Phi_{\alpha}^T=(\phi_{\alpha}^{\rm cl}\ \ \phi_{\alpha}^{\rm q})$ denotes the structure of the fields in Keldysh space. The hat above the polarization function, $\hat{\Pi}_{\alpha q}$, denotes its matrix nature with the usual Keldysh causality structure,[15,17]

$$\hat{\Pi}_{\alpha q}=\begin{pmatrix}0 & \Pi_{\alpha q}^{A}\\ \Pi_{\alpha q}^{R} & \Pi_{\alpha q}^{K}\end{pmatrix}. \quad (35)$$

Here, the retarded and Keldysh polarization functions are defined by

$$\Pi_{\alpha q}^{R}=\frac{-i}{\nu}\int_k[G^R_{\alpha,k+q}G^K_{\alpha k}+G^K_{\alpha,k+q}G^A_{\alpha k}], \quad (36)$$

$$\Pi_{\alpha q}^{K}=\frac{-i}{\nu}\int_k[G^K_{\alpha,k+q}G^K_{\alpha k}+G^R_{\alpha,k+q}G^A_{\alpha k}+G^A_{\alpha,k+q}G^R_{\alpha k}], \quad (37)$$

and $\Pi_{\alpha q}^{A}=(\Pi_{\alpha q}^{R})^{*}$. The quartic terms read

$$\mathcal{S}_{\rm eff}^{(4)}=\int d^3x\sum_{\alpha,n=1}^{n=4}u_n^{\alpha}[\phi_{\alpha}^{\rm cl}(x)]^{4-n}[\phi_{\alpha}^{\rm q}(x)]^{n}. \quad (38)$$

Note that as advertised above, the quartic part of the action does not couple the bosonic fields. As a matter of fact, in the ϕ basis, the *only* mixing appears in the quadratic part of the action. We now turn to the evaluation of the polarization functions, which will reveal the dynamics of the bosonic fields as well as the effects of the nonequilibrium drive on the latter.

Integrating out soft fermionic modes is now known to yield nonanalytic corrections to the static spin susceptibility and invalidate the standard Landau-Ginzburg-Wilson description for itinerant ferromagnets in relevant dimensions.[18] However, these nonanalyticities are also known to cancel for the case of ordering field with Ising symmetry.[19]

D. Polarization functions

We begin with the evaluation of $\hat{\Pi}_{tq}$, which is computed using equilibrium electron propagators of the top layer. The retarded component can then be obtained by analytic continuation of the standard Matsubara polarization function used for a clean itinerant ferromagnet[9,10,20] yielding

$$\Pi_{tq}^{R}=-1+|\mathbf{q}|^2-i\frac{\Omega}{v_F|\mathbf{q}|}. \quad (39)$$

The Keldysh polarization function for the top layer can be straightforwardly obtained using the fluctuation-dissipation theorem. At $T=0$, it reads

$$\Pi_{tq}^{K}=-2i\frac{|\Omega|}{v_F|\mathbf{q}|}. \quad (40)$$

The polarization functions, $\hat{\Pi}_{bq}$, have been obtained elsewhere[8] and the detailed calculations here will be relegated to Appendix B. Here we state the results. The retarded polarization function for $|\mathbf{q}|<(v_F\tau)^{-1}$ and $T_{\text{eff}}\tau \ll 1$ is given by

$$\Pi^R_{bq} = -1 + |\mathbf{q}|^2 - i\tau(\Omega - \mathbf{v}_d \cdot \mathbf{q}). \tag{41}$$

We note here that the condition $|\mathbf{q}|<(v_F\tau)^{-1}$ was crucial in obtaining the $z=2$ dynamical term above with a constant Landau damping coefficient. In the limit where $|\mathbf{q}| > (v_F\tau)^{-1}$, an expansion with respect to $(|\mathbf{q}|v_F\tau)^{-1}$ becomes possible and the dynamical term for the bottom layer gets modified to $i\Omega/v_F|\mathbf{q}|$. We shall focus on excitation energies less than Γ and momenta less than Γ/v_F where nonconservation due to escape into the leads is dominant.

The parity-breaking drift term contains the drift velocity which reads

$$\mathbf{v}_d = \frac{e\mathbf{E}\tau}{m}. \tag{42}$$

The Keldysh polarization function contains information about decoherence arising from the noise in the current and is given by

$$\Pi^K_{bq} = -2i|\Omega|\tau\left[1 + I\left(\frac{T_{\text{eff}}}{|\Omega|}\right)\right], \tag{43}$$

where

$$I(x) \coloneqq x\int \frac{d\theta}{2\pi}|\cos\,\theta|e^{-1/x|\cos\,\theta|}. \tag{44}$$

Absence of drift in the Keldysh polarization indicates that the noise does not drift in the presence of the current in the bottom layer.

Once these polarization functions are inserted into Eq. (33) one obtains a theory of two coupled bosonic modes, one of which (Φ_b) is characterized by a dynamical critical exponent $z=2$ and the other (Φ_t) by $z=3$. In the decoupled limit ($\epsilon \to 0$), the former fluctuations are nonconserved in nature because the bottom layer is open to the external bath. On the other hand, the top layer does not couple to any bath and is thus characterized by conserved fluctuations.

IV. PRELIMINARY CONSIDERATIONS

A. Mean-field analysis

We begin with a mean-field analysis of the effective bosonic action [Eqs. (34) and (38)] in order to understand how a finite interlayer interaction and nonequilibrium drive affect the underlying Stoner condition. Assuming a mean-field ferromagnetic state, whose solution is both static and uniform (i.e., $\Omega \to 0$ and $\mathbf{q}\to\mathbf{0}$), we may write down the following Landau-Ginzburg theory for the bilayer system:

$$\mathcal{S}_{\text{LG}} = \delta(\phi_b^2 + \phi_t^2) - 2\epsilon\phi_b\phi_t + u_b\phi_b^4 + u_t\phi_t^4. \tag{45}$$

Here, $\delta = 1/(\tilde{U}\nu) - 1$; we have set the area to unity. It is clear from Eq. (45) that the interlayer coupling j gives rise to a renormalization of the Stoner criterion. However, the (parity-breaking) electric field does not give an additional renormalization as the relevant corrections vanish for $\Omega \to 0$ and $\mathbf{q}\to\mathbf{0}$.[21]

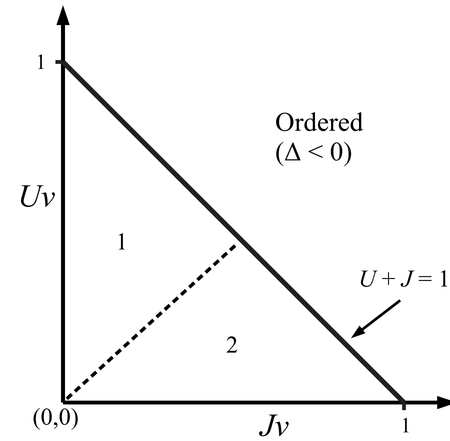

FIG. 2. Mean-field phase diagram. The solid line separates the ordered ($\Delta<0$) and disordered ($\Delta>0$) phases. The dashed line corresponds to $U=J$; in region 1 (2), $\epsilon>0$ (<0). In this work, we concentrate on region 1 of the disordered phase, near the transition line.

When $\epsilon=0$, i.e., $j=0$, the two fields decouple and the usual Stoner condition for the instability (i.e., $1/U_c=\nu$) is regained for each of the two layers. However, for $\epsilon>0$, the Stoner condition for a critical U is shifted. To see this, we begin with the saddle-point conditions for the theory,

$$\delta\phi_b - \epsilon\phi_t + 2u_b\phi_b^3 = 0, \tag{46}$$

$$\delta\phi_t - \epsilon\phi_b + 2u_t\phi_t^3 = 0. \tag{47}$$

In general, $u_b \neq u_t$, because these constants microscopically arise from particle-hole fluctuations, and the fermions in the bottom and top layers possess different dynamics. We see that in the mean-field equation for a given layer, the complimentary layer enters like a magnetic field that tends to align both order parameters. Decoupling the equations leads to $0 = (\delta^2/\epsilon - \epsilon)\phi_\alpha + \frac{\delta}{\epsilon}(u_\alpha + \frac{\delta^2}{\epsilon^2}u_{\bar{\alpha}})\phi_\alpha^3 + \frac{3\delta^2 u_\alpha u_{\bar{\alpha}}}{\epsilon^3}\phi_\alpha^5 + \frac{3\delta u_\alpha^2 u_{\bar{\alpha}}}{\epsilon^3}\phi_\alpha^7 + \frac{u_\alpha^3 u_{\bar{\alpha}}}{\epsilon^3}\phi_\alpha^9$, where $\bar{\alpha}$ is the label complimentary to α. We see that for $\delta > \epsilon$ the system is disordered and that the transition occurs at $\delta=\epsilon$ as one increases U. The modified Stoner condition for the instability is then given by

$$U_c = \frac{1}{\nu} - J. \tag{48}$$

We see that the critical U is reduced from the one corresponding to the single-layer Stoner criterion because the interlayer interaction promotes the ferromagnetic state. The considerations above lead us to introduce the tuning parameter for the transition $\Delta = \delta - \epsilon$. The mean-field phase diagram is shown in Fig. 2. Throughout this work, we will be considering properties of the system in the vicinity of the transition and in the disordered region of the phase diagram where $\Delta > 0$, i.e., $\delta > \epsilon$. In addition, we will be assuming $\delta \gtrsim \epsilon > 0$

and $\Delta \ll \delta, \epsilon$, which corresponds to a region close to the transition line in region 1 of Fig. 2.

B. Fixed point

The effective bosonic action [Eqs. (34) and (38)] consists of two coupled modes, ϕ_b and ϕ_t, that are characterized by different dynamical critical exponents. In the decoupled limit ($\epsilon \to 0$), the ϕ_b spin fluctuations are nonconserved in nature because the bottom layer is open to the substrate, and this gives rise to its $z=2$ dynamics. ϕ_t fluctuations, on the other hand, are conserved with $z=3$ dynamics because the top layer is not subject to particle exchange with an external bath. For finite ϵ, these modes are coupled at the Gaussian level. As such, the model exhibits multiscale quantum criticality. The phenomenon has been addressed in the context of the Pomeranchuk (or nematic) instability in two spatial dimensions,[22,23] and in the study of quantum phase transitions that spontaneously develop ferromagnetic order in a helical Fermi liquid,[24] the latter describing the low-energy physics of surface states of a three-dimensional topological insulator. In both situations, the low-energy bosonic theory consists of two modes, one longitudinal and the other transverse with different dynamical critical exponents. The lowest order coupling appears at the quadratic level, i.e., $\phi_\perp^2 \phi_\parallel^2$, in contrast with the linear coupling present in our situation. In both the Pomeranchuk and helical liquid scenarios, the longitudinal mode is Landau-damped with $z=3$ and the transverse one is ballistic and undamped with $z=2$. In Ref. 23, it was argued that the low-energy behavior of the system at $T=0$ is governed by the undamped $z=2$ mode since it possesses the smaller effective dimension. This claim is supported by the fact that renormalizations due to fluctuations to the mass and the vertex are dominated in the infrared by the $z=2$ mode.

To carefully determine the nature of the equilibrium fixed point at $T=0$, let us consider the correction to the mass in perturbation theory. We begin with our Gaussian action in the Matsubara form

$$\mathcal{S}_{\rm eq}^{(2)} = \int_q (\phi_{bq}^* \quad \phi_{tq}^*) \begin{pmatrix} \mathcal{L}_{bq}^{\rm eq} & -\epsilon \\ -\epsilon & \mathcal{L}_{tq}^{\rm eq} \end{pmatrix} \begin{pmatrix} \phi_{bq} \\ \phi_{tq} \end{pmatrix}, \tag{49}$$

where $\mathcal{L}_{bq}^{\rm eq} = \delta + |\mathbf{q}|^2 + |\Omega|\tau$ and $\mathcal{L}_{tq}^{\rm eq} = \delta + |\mathbf{q}|^2 + |\Omega|/v_F|\mathbf{q}|$ are the equilibrium inverse susceptibilities. The propagators for the modes, obtained by inverting the 2×2 inverse susceptibility matrix, are $\langle \phi_b(q)\phi_b^*(q)\rangle = \mathcal{L}_{tq}^{\rm eq}/D_q^{\rm eq}$, $\langle \phi_t(q)\phi_t^*(q)\rangle = \mathcal{L}_{bq}^{\rm eq}/D_q^{\rm eq}$, and $\langle \phi_b(q)\phi_t^*(q)\rangle = \langle \phi_t(q)\phi_b^*(q)\rangle = \epsilon/D_q^{\rm eq}$, where $D_q^{\rm eq} = \mathcal{L}_{bq}^{\rm eq}\mathcal{L}_{tq}^{\rm eq} - \epsilon^2$ is the determinant of the matrix. While the off-diagonal mass, ϵ, does not gain corrections from quartic fluctuations, the diagonal mass, δ, does gain renormalizations corresponding to the usual tadpole diagram of Fig. 3. For the ϕ_b sector, this is given by

$$\Delta\delta_b \propto \int \frac{d^2q}{(2\pi)^2}\frac{d\Omega}{2\pi} \frac{\delta + |\mathbf{q}|^2 + \dfrac{|\Omega|}{v_F|\mathbf{q}|}}{D_q^{\rm eq}}. \tag{50}$$

The contribution of the $z=3$ fluctuations near the critical point can be estimated by setting $\Omega \sim |\mathbf{q}|^3$. Then Eq. (50) essentially reduces to

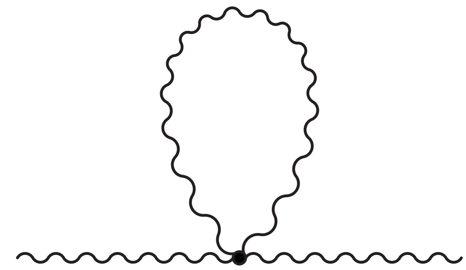

FIG. 3. Tadpole diagram for the mass renormalization. The curly lines correspond to magnetization fluctuations ϕ_α belonging to the same sector, b or t. As noted before, there's no mixing at the quartic level in the ϕ basis.

$$\Delta\delta_b|_{z=3} \sim \int \frac{d^5q}{2\Delta + |\mathbf{q}|^2} \sim \Delta^{3/2}, \tag{51}$$

where $\Delta = \delta - \epsilon$ is the distance to the critical point, and we have assumed $\delta \approx \epsilon$. For $z=2$ fluctuations, $\Omega \sim |\mathbf{q}|^2$, and the contribution can be estimated as

$$\Delta\delta_b|_{z=2} \sim \int \frac{d^4q}{2\Delta + |\mathbf{q}|} \sim \Delta^3. \tag{52}$$

In the vicinity of the critical point (as $\Delta \to 0$), the $z=3$ fluctuations are expected to dominate. A similar calculation can be carried out for the ϕ_t sector leading to the same conclusion.

The key difference here from the analysis in Ref. 23 is that the modes are coupled at the Gaussian level and that the two dynamical terms enter together in all of the propagators. It is then clear that close enough to the critical point (where the characteristic length scale of the bosonic fluctuations $\xi > v_F\tau$), the conserved dynamical term $(\Omega/v_F|\mathbf{q}|)$ will dominate over the nonconserved counterpart $(\Omega\tau)$, and the low-energy long-wavelength theory will be governed by $z=3$ dynamics. We now study the renormalization-group flow in the vicinity of this $z=3$ fixed point.

V. RENORMALIZATION-GROUP ANALYSIS

To aid with the analysis we begin with the effective Gaussian action [cf. Eq. (34)]

$$\mathcal{S}_{\rm eff}^{(2)} = -\int_q^\Lambda (\Phi_{bq}^\dagger \quad \Phi_{tq}^\dagger) \begin{pmatrix} \hat{\mathcal{L}}_b & -\epsilon\hat{\tau}_x \\ -\epsilon\hat{\tau}_x & \hat{\mathcal{L}}_t \end{pmatrix} \begin{pmatrix} \Phi_{bq} \\ \Phi_{tq} \end{pmatrix}, \tag{53}$$

where $\hat{\mathcal{L}}_\alpha = (\tilde{U}\nu)^{-1}\hat{\tau}_x + \hat{\Pi}_{\alpha q}$. The explicit expressions for the matrix elements read

$$\mathcal{L}_b^R = \delta + |\mathbf{q}|^2 + i\mathbf{v}_d\cdot\mathbf{q}\tau - i\Omega\tau, \tag{54}$$

$$\mathcal{L}_b^K = -2i|\Omega|\tau\left[1 + I\left(\frac{T_{\rm eff}}{|\Omega|}\right)\right] \tag{55}$$

and

$$\mathcal{L}_t^R = \delta + |\mathbf{q}|^2 - i\frac{\Omega}{v_F|\mathbf{q}|}, \tag{56}$$

$$\mathcal{L}_t^K = -2i\frac{|\Omega|}{v_F|\mathbf{q}|}. \tag{57}$$

The quartic action was given in Eq. (38).

A. Flow equations out of equilibrium

We study the flow in the vicinity of the $z=3$ Gaussian fixed point with $T_{\mathrm{eff}}=\Delta=u_n^\alpha=0$. Here, $T=0$ throughout. At this fixed point the most relevant dynamical term (i.e., $\Omega/v_F|\mathbf{q}|$) remains marginal. In Eq. (53), we have defined our momentum cutoff Λ, which in turn defines an associated cut-off energy scale $v_F\Lambda$. All energy quantities will be considered to be in units of this scale. The bare parameters are taken to be $\Delta, T_{\mathrm{eff}}, u_n^\alpha, \delta, \epsilon \ll 1$ with $\delta \sim \epsilon \gg \Delta$. We begin by integrating out fluctuations whose modes reside in the shell $\Lambda/b \le q \le \Lambda$. Here, $b>1$ is the scaling variable. In order to preserve Keldysh causality, we integrate over all frequencies (i.e., $-\infty<\Omega<\infty$) at every mode-elimination step. We next rescale momentum, $q\to q'/b$, to restore the cutoff back to Λ. At the $z=3$ fixed point, the frequency scales as $\Omega\to\Omega'/b^3$ and this keeps the coefficients of $|\mathbf{q}|^2$ and $\Omega/v_F|\mathbf{q}|$ terms invariant under the flow provided that the fields scale as $\phi_{\alpha q}^{\mathrm{cl,q}}\to b^{7/2}[\phi_{\alpha q}^{\mathrm{cl,q}}]'$. The $(T_{\mathrm{eff}}/\Omega)$ scaling present in the Keldysh components suggests that T_{eff} is relevant at the fixed point and scales with dimension of three, i.e., $T_{\mathrm{eff}}\to T'_{\mathrm{eff}}/b^3$. The nonconserved dynamical term is irrelevant at the fixed point, and this is reflected in the scaling $\tau\to\tau' b$. Then, the requirement to maintain the drift term invariant under the flow dictates the scaling of the drift velocity as $v_d\to v'_d/b^2$. Note that the drift term contains the combination τv_d, which has scaling dimension one, just as in Ref. 8. Both the diagonal (δ) and the off-diagonal (ϵ) mass terms scale with the usual dimension of two. Finally, at the $z=3$ fixed point, all quartic coupling constants are irrelevant with $[u_n^\alpha]=-1$. Therefore, in the standard lore,[9,10] the order-parameter self-interactions are not expected to play a significant role in the effective low-energy theory.

To summarize our scaling analysis, we write down the set of renormalization group (RG) equations to lowest (linear) order in the quartic coupling,

$$\frac{dT_{\mathrm{eff}}(b)}{d\ln b} = 3T_{\mathrm{eff}}(b), \tag{58}$$

$$\frac{d\epsilon(b)}{d\ln b} = 2\epsilon(b), \tag{59}$$

$$\frac{dv_d(b)}{d\ln b} = 2v_d(b), \tag{60}$$

$$\frac{d\tau(b)}{d\ln b} = -\tau(b), \tag{61}$$

$$\frac{du_n^\alpha(b)}{d\ln b} = -u_n^\alpha(b) + \mathcal{O}([u_n^\alpha]^2), \tag{62}$$

$$\frac{d\delta_\alpha(b)}{d\ln b} = 2\delta_\alpha(b) + 3u_1^\alpha(b)f_\alpha(T_{\mathrm{eff}}(b),\tau(b)). \tag{63}$$

We will distinguish renormalized parameters from the bare ones by explicitly writing the b dependence for the former. In Eq. (63), α labels b and t. Indeed, the bare values satisfy $\delta_b=\delta_t=\delta$; the label denotes the fact that these mass terms are subject to different renormalizations from fluctuations. The function, $f_\alpha(T_{\mathrm{eff}}(b),\tau(b))$ is given by

$$f_\alpha(T_{\mathrm{eff}}(b),\tau(b)) = iK_2\int\frac{d\Omega}{2\pi}D_\alpha^K(1,\Omega), \tag{64}$$

where $K_2=\int\frac{d^2\mathbf{q}}{(2\pi)^2}\delta(q-1)=1/2\pi$, we have set the momentum cutoff $\Lambda=1$, and

$$D_{\alpha p}^K = \frac{\mathcal{L}_{\alpha p}^K|\mathcal{L}_{\bar{\alpha}p}^R|^2 + \epsilon^2\mathcal{L}_{\bar{\alpha}p}^K}{|\mathcal{L}_{bp}^R\mathcal{L}_{tp}^R - \epsilon^2|^2} \tag{65}$$

is the Keldysh Green's function for the mode ϕ_α, it was obtained by inverting the inverse susceptibility matrix of Eq. (53). Solving these equations, we arrive at

$$T_{\mathrm{eff}}(b) = T_{\mathrm{eff}}b^3, \tag{66}$$

$$\epsilon(b) = \epsilon b^2, \tag{67}$$

$$v_d(b) = v_d b^2, \tag{68}$$

$$\tau(b) = \tau b^{-1}, \tag{69}$$

$$u_n^\alpha(b) = u_n^\alpha b^{-1}, \tag{70}$$

$$\delta_\alpha(b) = b^2\left[\delta + 3u_1^\alpha\int_0^{\ln b}dx e^{-3x}f_\alpha(T_{\mathrm{eff}}e^{3x},\tau e^{-x})\right]. \tag{71}$$

B. Flow equations in equilibrium

To elucidate the nonequilibrium problem at later stages, we also consider flow equations for an equilibrium situation in which the entire system is at temperature T. Here, the electric field is set to zero, thus $T_{\mathrm{eff}}=0$. In this case, the Gaussian action, Eq. (53), becomes

$$\mathcal{L}_b^R = \delta + |\mathbf{q}|^2 - i\Omega\tau, \tag{72}$$

$$\mathcal{L}_b^K = -2i\Omega\tau\coth\left(\frac{\Omega}{2T}\right) \tag{73}$$

and

$$\mathcal{L}_t^R = \delta + |\mathbf{q}|^2 - i\frac{\Omega}{v_F|\mathbf{q}|}, \tag{74}$$

$$\mathcal{L}_t^K = -2i\frac{\Omega}{v_F|\mathbf{q}|}\coth\left(\frac{\Omega}{2T}\right). \tag{75}$$

An analogous renormalization-group analysis as in the nonequilibrium case can be applied here, and the solutions to the corresponding flow equations become

$$T(b) = Tb^3, \tag{76}$$

$$\epsilon(b) = \epsilon b^2, \tag{77}$$

$$\tau(b) = \tau b^{-1}, \tag{78}$$

$$u_n^\alpha(b) = u_n^\alpha b^{-1}, \tag{79}$$

$$\delta_\alpha(b) = b^2\left[\delta + 3u_1^\alpha \int_0^{\ln b} dx e^{-3x} f_\alpha(Te^{3x}, \tau e^{-x})\right]. \tag{80}$$

The f_α functions are still given formally by Eqs. (64) and (65) with the replacement $T_{\text{eff}} \to T$, and the equilibrium inverse susceptibilities, Eqs. (72)–(75), must be used.

VI. CORRELATION LENGTH

In the current bosonic basis, there are three two-point correlation functions one can study, each with its own mass scale, which we define as

$$\Delta_{\alpha\beta}^{-1} = \langle \phi_{\alpha q} \phi_{\beta q}^* \rangle|_{q=0}, \tag{81}$$

where q is a frequency-momentum three vector. In the above equation we have omitted the Keldysh indices because one can use an equilibrium formalism to define the mass scales as they do not depend on the nonequilibrium drive. Note also that $\Delta_{bt} = \Delta_{tb}$. Within the Gaussian theory, the bare mass scales read

$$\Delta_{bb} = \Delta_{tt} = \frac{\delta^2 - \epsilon^2}{\delta}, \tag{82}$$

$$\Delta_{bt} = \frac{\delta^2 - \epsilon^2}{\epsilon}. \tag{83}$$

In the relevant limit, $\delta > 0$, $\epsilon > 0$, and $\delta \gtrsim \epsilon \gg \Delta$, Eqs. (82) and (83) become

$$\Delta_{bb} = \Delta_{tt} \approx \Delta_{bt} \approx 2\Delta. \tag{84}$$

Here, the mass scales were computed to lowest order in $\Delta/\delta \ll 1$ and $\Delta/\epsilon \ll 1$. One sees that they all vanish as the transition is approached ($\Delta \to 0$). This allows one to then define the physical correlation length of the system, ξ, via $\xi^{-2} := \Delta$. We now compute ξ both in and out of equilibrium focusing on the corrections arising from either T or T_{eff}.

A. In equilibrium

We begin by considering the correlation length in equilibrium. We define the quantum-to-classical crossover line for the system and compute the leading temperature correction to the correlation length in the quantum critical regime.

1. Quantum disordered regime

In this regime, the f_α functions are computed at $T=0$. This gives a temperature-independent shift proportional to u_1^α to the two diagonal masses. We then have

$$\delta_\alpha(b) = b^2\left[\delta + 3u_1^\alpha \int_0^{\infty} dx e^{-3x} f_\alpha(0, \tau e^{-x})\right],$$

$$=: b^2(\delta + \Delta\delta_\alpha^0). \tag{85}$$

The correlation length in this regime is then given by

$$\xi^{-2} \approx \Delta + \Delta\delta_b^0 + \Delta\delta_t^0 =: r, \tag{86}$$

where we have computed to lowest order in $\Delta/\delta \ll 1$ and $\Delta\delta_\alpha^0/\delta \ll 1$. To find the condition on T for the occurrence of the regime, we impose that $\Delta(b') = 1$ while $T(b') < 1$. Recall that $\Delta(b) \approx \Delta_{bb}(b) \approx \Delta_{tt}(b) \approx \Delta_{bt}(b)$. This then translates to a condition on the bare quantities,

$$T < r^{3/2}. \tag{87}$$

2. Quantum critical regime

When the inequality in Eq. (87) is reversed, the characteristic energy scale of the fluctuations, $1/\xi^z$, becomes smaller than temperature and the fluctuations become classical. Here, we compute the leading temperature correction to the correlation length at scale b^*, where $T(b^*) = 1$. Since $\delta \sim \epsilon \gg r \approx T^{2/3}$, both δ and ϵ become much larger than 1 at some stage of the flow. We define the scale b_1 as $\delta(b_1) \sim \epsilon(b_1) = 1$. Then the integral in Eq. (80) must be split into two regions: $1 < b < b_1$ and the other $b_1 < b < b^*$. In the former region, the integral can be computed assuming $\delta \sim \epsilon \ll 1$ inside Eq. (65). In the latter region, the integral is computed assuming $\delta \sim \epsilon \gg 1$. The detailed derivation will be relegated to Appendix C; here we state the results. We define

$$\Delta\delta_\alpha := 3u_1^\alpha \int_0^{\ln b^*} dx e^{-3x}[f_\alpha(Te^{3x}, \tau e^{-x}) - f_\alpha(0, \tau e^{-x})]. \tag{88}$$

Here, one integrates up to b^* by splitting the integral as advertised. One then arrives at the following temperature correction to leading order:

$$\Delta\delta_\alpha = \frac{u_1^\alpha}{4\pi^2} T\left(1 + \frac{3\tau}{2} T^{1/3}\right). \tag{89}$$

The correlations length in the quantum critical regime is then given by

$$\xi_{\text{eq}}^{-2} \approx r + \frac{u_1^b + u_1^t}{4\pi^2} T\left(1 + \frac{3\tau}{2} T^{1/3}\right). \tag{90}$$

The linear-temperature correction in Eq. (90) is consistent with the corresponding correction obtained in Ref. 10 for the case $d=2$ and $z=3$. The $T^{4/3}$ correction, which is proportional to τ, is an additional correction that arises because of the $z=2$ dynamics present at the $z=3$ fixed point. Both the leading $z=3$ correction and a subleading $z=2$ correction enter into the correction because both modes are subject to a common temperature T.

B. Out of equilibrium

We now compute the leading nonequilibrium correction to the correlation length at scale b^*, where $T_{\text{eff}}(b^*) = 1$. We now assume $T=0$. Here, we only concentrate on the quantum critical regime because the correlation length in the quantum disordered regime is still given by Eq. (86). Of course, the condition for the occurrence of this regime now reads $T_{\mathrm{eff}} < r^{3/2}$.

The leading correction due to the electric field can be computed by splitting the scaling regimes into $1<b<b_1$ and $b_1<b<b^*$ as in the equilibrium case. We then obtain

$$\Delta\delta_\alpha \approx \frac{3u_1^\alpha}{4\pi^3}\tau T_{\mathrm{eff}}^{4/3}. \tag{91}$$

The correlations length in the quantum critical regime is then given by

$$\xi_{\mathrm{neq}}^{-2} \approx r + 3\frac{u_1^b + u_1^t}{4\pi^3}\tau T_{\mathrm{eff}}^{4/3}. \tag{92}$$

Once again, the detailed calculations are presented in Appendix C. In contrast to the equilibrium result, Eq. (90), the correlation length gains a correction of order $T_{\mathrm{eff}}^{4/3}$ only. The difference arises because in the decoupled limit the two layers are at different effective temperatures. This can be readily checked by comparing the two Keldysh polarization functions, Eqs. (40) and (43). One sees that the top layer is at zero effective temperature while the bottom possesses a finite T_{eff} because the field directly couples to the latter. Once the interlayer coupling is restored both ϕ_b and ϕ_t sectors feel the effect of the field as it is evident from Eq. (92). However, due to the absence of an effective temperature in the top Keldysh polarization function, a linear-T_{eff} correction does not arise in Eq. (92).

Results in this section suggest certain similarities between the role of T_{eff} in the nonequilibrium case and temperature T in the equilibrium case. Both T_{eff} and T induce a quantum-to-classical crossover, across which the behavior of critical fluctuations evolves from quantum to effectively classical. The crossover energy scale is defined in Eq. (87) where T can also be replaced by T_{eff}. Furthermore, Eq. (92) implies that T_{eff} can cutoff the divergence in the correlation length just as temperature T can as shown in Eq. (90). However, the lack of a linear-T_{eff} correction for the correlation length indicates that T_{eff} cannot be identified *strictly* as an effective temperature of the system.

We note that in equilibrium two corrections that reflect the presence of $z=2$ and 3 dynamics enter into the correlation length. The dominant contribution (i.e., linear-T contribution) arises essentially from the $z=3$ dynamical terms in the regime of interest. In systems with multiscale quantum criticality, modes with different dynamics often nontrivially contribute to various thermodynamic quantities in different bare parameter regimes. In thermal equilibrium, interesting crossover behavior in thermodynamic quantities in a system with multiscale quantum criticality has been addressed in the context of Pomeranchuk instability[23] and the Kondo-Heisenberg model.[25]

C. Eigenmode analysis

We have found that one must use caution in determining the mass scale which relates to the inverse square of the physical correlation length, for neither δ nor ϵ alone defines this scale. A sensible procedure would be to determine the eigenmodes of the system that diagonalize the Gaussian action Eq. (34) and directly read off the corresponding masses by observing the eigenvalues of the action. This analysis was carried out and is presented in Appendix D. We find that while one of the eigenmodes becomes critical at $\delta=\epsilon$, the other remains massive. We then integrate out the massive mode and construct an effective critical theory in terms of a single critical mode, λ_q. In the Appendix D, we carry out a renormalization-group procedure on this single-component critical theory and the corresponding T and T_{eff} corrections to the correlation length of this field is computed. We find that the correlation length obtained in the eigenbasis gives identical results to those presented in Eqs. (90) and (92).

VII. ORDER-PARAMETER DYNAMICS

We now consider the dynamics obeyed by the critical fluctuations in the quantum critical regime where the theory is effectively classical and the quantum fluctuations can be integrated out from the effective low-energy theory. The standard procedure to obtain the dynamics is outlined in Ref. 15 and has been applied in Refs. 7 and 8 in establishing nonequilibrium dynamical universality classes. In a completely different setting but in a related work, order-parameter dynamics near phase transitions in a driven interacting bilayer lattice gas has also been addressed in Refs. 26 and 27.

We begin by decoupling the terms quadratic in the quantum Keldysh fields by making use of a second Hubbard-Stratonovich transformation. The decoupling field plays the role of a noise source for the classical fluctuations. Finally, we integrate out the quantum fluctuations and obtain a stochastic equation, or Langevin equation, for the classical fluctuations. We hereafter drop the cl subscript from the fields. All couplings will also be assumed to take their renormalized values.

In the ϕ_α basis, the system of coupled, linear Langevin equations is

$$\begin{pmatrix} \mathcal{L}_{bq}^R & -\epsilon \\ -\epsilon & \mathcal{L}_{tq}^R \end{pmatrix}\begin{pmatrix} \phi_{bq} \\ \phi_{tq} \end{pmatrix} = \begin{pmatrix} \eta_{bq} \\ \eta_{tq} \end{pmatrix}, \tag{93}$$

where $\eta_{\alpha q}$ are the Fourier components of the noise sources; their correlators are given by the Keldysh inverse susceptibilities

$$\langle \eta_{\alpha q}\eta_{\beta q'}\rangle = \delta_{\alpha\beta}\delta(q+q')i\Pi_{\alpha q}^K. \tag{94}$$

One can obtain the noise-averaged correlators for the fluctuations by inverting the coefficient matrix

$$\langle \phi_{bq}\phi_{bq}^*\rangle = \frac{|\mathcal{L}_{tq}^R|^2}{|D_q|^2}i\Pi_{bq}^K + \frac{\epsilon^2}{|D_q|^2}i\Pi_{tq}^K, \tag{95}$$

$$\langle \phi_{tq}\phi_{tq}^*\rangle = \frac{|\mathcal{L}_{bq}^R|^2}{|D_q|^2}i\Pi_{tq}^K + \frac{\epsilon^2}{|D_q|^2}i\Pi_{bq}^K, \tag{96}$$

$$\langle \phi_{tq}\phi^*_{bq}\rangle = \frac{\epsilon}{|D_q|^2}[\mathcal{L}^R_{tq} i\Pi^K_{bq} + \mathcal{L}^A_{bq} i\Pi^K_{tq}], \tag{97}$$

where $D_q=\mathcal{L}^R_{bq}\mathcal{L}^R_{tq}-\epsilon^2$ is the determinant of the coefficient matrix. We mention here that for finite interlayer coupling all bosonic correlators feel the presence of the parity-breaking drift term because they all contain $\mathcal{L}^R_{bq}$. One can repeat the same procedure for the critical eigenmode (see Appendix D), whose Langevin equation reads

$$\chi_R^{-1}\lambda_q = \eta_q \tag{98}$$

with the noise correlator $\langle \eta_q \eta_{q'}\rangle = i\chi_K^{-1}\delta(q+q')$. From Appendix D, we have that

$$\chi_R^{-1} = \frac{1}{2}(\mathcal{L}^R_{bq} + \mathcal{L}^R_{tq}) - \epsilon, \tag{99}$$

$$\chi_K^{-1} = \frac{1}{2}(\Pi^K_{bq} + \Pi^K_{tq}). \tag{100}$$

The noise-averaged correlator for the eigenfield then reads

$$\langle \lambda_q \lambda^*_q\rangle = \frac{\frac{i}{2}(\Pi^K_{bq} + \Pi^K_{tq})}{\left|\frac{1}{2}(\mathcal{L}^R_{bq} + \mathcal{L}^R_{tq}) - \epsilon\right|^2}. \tag{101}$$

We now argue that the three correlators in the ϕ basis and the one for the critical eigenmode contain the same physics in the regime of interest. Indeed, we are considering $\Delta \approx \delta - \epsilon \ll \delta, \epsilon$, such that at the end of scaling $\delta, \epsilon > 1$. Thus we can write $|\mathcal{L}^R_{\alpha q}|^2 \approx \delta_\alpha^2$. Neglecting differences between δ_b and δ_t due to one-loop renormalizations, we have $\langle \phi_{\alpha q}\phi^*_{\beta q}\rangle \approx \langle \lambda_q \lambda^*_q\rangle/2$. Hence, we will study the fluctuation dynamics in terms of the critical eigenmode, λ.

In equilibrium, Eq. (98) in momentum-time space becomes

$$\frac{1}{v_F|\mathbf{q}|}\partial_t \lambda(\mathbf{q},t) = -(\Delta + |\mathbf{q}|^2)\lambda(\mathbf{q},t) + \eta(\mathbf{q},t), \tag{102}$$

where we have dropped the $z=2$ term from the time-derivative because we are in the regime of $|\mathbf{q}| < \Gamma/v_F$. In the quantum critical regime, the noise correlators become

$$\langle \eta(\mathbf{q},t)\eta(\mathbf{q}',t')\rangle \approx \frac{2T}{v_F|\mathbf{q}|}\delta(t-t')\delta(\mathbf{q}+\mathbf{q}'). \tag{103}$$

We then find the noise-averaged correlation function for the fluctuations

$$\langle \lambda(\mathbf{q},t)\lambda(-\mathbf{q},t')\rangle \approx \frac{T}{\Delta} e^{-v_F|\mathbf{q}|\Delta|t-t'|}. \tag{104}$$

In the out-of-equilibrium case, the Langevin equation becomes

$$\frac{1}{v_F|\mathbf{q}|}\partial_t \lambda(\mathbf{q},t) = -\left(\Delta + |\mathbf{q}|^2 + \frac{i}{2}\mathbf{v}_d\cdot\mathbf{q}\tau\right)\lambda(\mathbf{q},t) + \eta(\mathbf{q},t). \tag{105}$$

In the limit of large T_{eff}, we find

$$\langle \eta(\mathbf{q},t)\eta(\mathbf{q}',t')\rangle \approx \frac{2T_{\text{eff}}\tau}{\pi}\delta(t-t')\delta(\mathbf{q}+\mathbf{q}'). \tag{106}$$

The noise-averaged correlation function then reads

$$\langle \lambda(\mathbf{q},t)\lambda(-\mathbf{q},t')\rangle \approx \frac{T_{\text{eff}}\tau v_F|\mathbf{q}|}{\pi\Delta} e^{-v_F|\mathbf{q}|[\Delta+(i/2)\mathbf{v}_d\cdot\mathbf{q}\tau]|t-t'|}. \tag{107}$$

In equilibrium, the correlation function describes fluctuations which are conserved in nature with the decay rate vanishing in the long-wavelength limit as $\sim|\mathbf{q}|$. Because the effects of temperature are felt equally by both bosonic modes, the $z=2$ dynamics play no role in the long-wavelength theory of the order-parameter dynamics. The dynamics obeyed here is essentially identical to Model B in Ref. 11 but with the Landau damping parameter replaced by $v_F|\nabla|$. The situation is different in the nonequilibrium case. Here, the delta-correlated noise contribution only arises from Π_b^K that reflects $z=2$ dynamics since the driving field (the source of noise) is only coupled to the bottom electrons. The Langevin equation then displays a hybrid effect of both $z=2$ and 3 physics; while the white noise correlator reflects $z=2$ physics, the damping is governed by the $z=3$ dynamical term. As a result, the corresponding noise-averaged correlation function for the fluctuations yields a unique behavior where long-wavelength correlations vanish as $\sim|\mathbf{q}|$. In addition, the advertised drift of the fluctuations can be explicitly seen in Eq. (107).

VIII. CONCLUSION

In summary, we considered critical properties of a nonequilibrium bilayer system of itinerant electron magnets. Starting from a microscopic fermionic model subject to an external drive, we derived a coupled theory in terms of two bosonic fields which are related to the physical magnetization fluctuations of the two layers. In the limit of no interlayer coupling, the fields obey different dynamics with different dynamical critical exponents ($z=2$ and 3), leading to multiscale quantum criticality in the coupled system. We found that the applied current leads to both a drift of the magnetic fluctuations in the coupled bilayer system and to decoherence. The latter phenomenon is more subtle in our system compared with the analogous single-layer scenario because the nonequilibrium drive is applied to only one of the layers. This causes it to play a role distinct from temperature in the thermal equilibrium case where both layers are held at a common temperature. The differences are illustrated by comparing temperature and nonequilibrium effects on the correlation length and the dynamics obeyed by the critical fluctuations. A crucial feature of the work is that the two fields couple linearly. We found that the infrared properties of the system then are governed by the dynamics corresponding to the higher effective dimension. In light of Ref. 23, it would be interesting to consider a coupled order-parameter theory where the fields couple quadratically in the context of itinerant electron magnets. In this case, the effective theory possesses a discrete $Z_2 \times Z_2$ symmetry corresponding to independent transformations $m_\alpha \to -m_\alpha$ for $\alpha = b, t$ ensuring that no linear coupling can be generated during the RG transformation. If both fields become critical simultaneously, the low-energy properties are then expected to be governed by the field with the lower effective dimension.

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APPENDIX A: INTERLAYER SPIN EXCHANGE

We now consider how the central insulator can mediate an effective spin-exchange interaction between the top and the bottom layers. Imagine we have two 2D itinerant ferromagnets sandwiching a thin but 3D insulator with thickness L. For concreteness, we envisage the insulator as a quantum Ising paramagnet in the vicinity of a quantum phase transition to a long-ranged magnetically ordered phase. The continuum quantum field theory for this model can be expressed in terms of a coarse-grained order-parameter fluctuation field, ϕ, subject to a confining potential. For simplicity, we truncate this potential at quadratic order with some invertible dynamic correlation matrix for the fluctuations, $\chi(\mathbf{x}_i, \mathbf{x}_j, \tau)$, where $\mathbf{x}_i = (\mathbf{r}_i, z_i)$ labels the lattice sites in the 3D insulator. The QFT in the vicinity of the magnetic quantum critical point is then given by[28]

$$S_{\text{ins}} = \int d\tau \sum_{ij} \phi(\mathbf{x}_i, \tau)\chi^{-1}(\mathbf{x}_i, \mathbf{x}_j, \tau)\phi(\mathbf{x}_j, \tau). \tag{A1}$$

Since we are only interested in how this central insulator generates an effective spin exchange, we will work in the equilibrium formalism for simplicity.

We now consider the interaction terms between the insulator and the two itinerant magnetic layers. We assume that the plane of the bottom layer is $z=0$ and the top layer is at $z=L$. The simplest plausible interaction would be a ferromagnetic onsite exchange ($K>0$) between the layers and the insulator

$$S_{t\text{-ins}} = -K \int d\tau \sum_i \hat{S}_t^z(\mathbf{r}_i, \tau)\phi(\mathbf{x}_i, \tau)\delta_{z_i, L}, \tag{A2}$$

$$S_{b\text{-ins}} = -K \int d\tau \sum_i \hat{S}_b^z(\mathbf{r}_i, \tau)\phi(\mathbf{x}_i, \tau)\delta_{z_i, 0}. \tag{A3}$$

Here, $\hat{S}_t^z$ and $\hat{S}_b^z$ are the Ising spin fluctuations in the top and the bottom layers, respectively. We integrate out the central fluctuations ϕ assuming that their in-plane correlations are short ranged to obtain the following effective action:

$$S_{\text{eff}} = \int d\tau \sum_{\mathbf{r}_i} [-K^2\chi(L,L)S_t^z(\mathbf{r}_i,\tau)S_t^z(\mathbf{r}_i,\tau) - K^2\chi(0,0)S_b^z(\mathbf{r}_i,\tau)S_b^z(\mathbf{r}_i,\tau) - JS_t^z(\mathbf{r}_i,\tau)S_b^z(\mathbf{r}_i,\tau)], \tag{A4}$$

where we have assumed that the in-plane correlations of the insulator are short ranged and independent of τ: $\chi(\mathbf{x}_i, \mathbf{x}_j, \tau) = \delta_{\mathbf{r}_i \mathbf{r}_j}\chi(z_i, z_j)$. As advertised, an effective interlayer spin coupling $J = K^2[\chi(L,0) + \chi(0,L)]$ is generated. We see that $-K^2\chi(L,L)$ and $-K^2\chi(0,0)$ lead to finite renormalizations of the intralayer exchange term.

APPENDIX B: POLARIZATION FUNCTIONS FOR THE ϕ_b SECTOR

In equilibrium, the retarded polarization function in the ϕ_b sector is given by

$$\Pi_{bq}^R \approx -1 + |\mathbf{q}|^2 - i\Omega\tau, \tag{B1}$$

where we are assuming $\Omega\tau < 1$ and $q\tau v_F < 1$. In the regime where $q\tau v_F \gg 1$, the dynamical term becomes $\Omega/v_F q$, i.e., conserved in nature. This is because on sufficiently short-length scales the electrons are unaffected by the presence of the substrate. The equilibrium contribution to the Keldysh polarization is given simply through the fluctuation-dissipation theorem,

$$\Pi_{bq}^K = -2i|\Omega|\tau. \tag{B2}$$

To get the nonequilibrium corrections to the polarization functions we use the nonequilibrium Green's functions for the bottom electrons obtained in Sec. III A. From Eqs. (20) and (25), the nonequilibrium contribution to the retarded polarization reads

$$\delta\Pi_b^R(0,\mathbf{q}) \approx -i\frac{4\Gamma^2}{\nu\pi}\int_{\mathbf{k}} \frac{e\mathbf{E}\cdot\mathbf{v}_{\mathbf{k}}\tau}{(\xi_{\mathbf{q}+\mathbf{k}}^2 + \Gamma^2)(\xi_{\mathbf{k}}^2 + \Gamma^2)}. \tag{B3}$$

Expanding this result for small $\mathbf{q}$ one finds

$$\delta\Pi_b^R(0,\mathbf{q}) \approx i\frac{1}{4m\Gamma^2}e\mathbf{E}\cdot\mathbf{q}. \tag{B4}$$

The nonequilibrium contribution to the Keldysh polarization is given by

$$\delta\Pi_b^K(\Omega,\mathbf{0}) = \frac{-i}{\nu}\int_k \frac{-2i\Gamma}{(\omega+\Omega-\xi_{\mathbf{k}})^2+\Gamma^2}\frac{-2i\Gamma}{(\omega-\xi_{\mathbf{k}})^2+\Gamma^2}\{ -\text{sgn}(\omega+\Omega)\text{sgn}(\omega)e^{-|\omega|/|e\mathbf{E}\cdot\mathbf{v}_F\tau|} - \text{sgn}(\omega)\text{sgn}(\omega+\Omega)e^{-|\omega+\Omega|/|e\mathbf{E}\cdot\mathbf{v}_F\tau|} + [\text{sgn}(\omega)\text{sgn}(\omega+\Omega)+1]e^{-|\omega|/|e\mathbf{E}\cdot\mathbf{v}_F\tau|}e^{-|\omega+\Omega|/|e\mathbf{E}\cdot\mathbf{v}_F\tau|}\}. \tag{B5}$$

Here, we have linearized the fermion spectrum in the exponents. For a weak field (i.e., $T_{\text{eff}} \ll \tau^{-1}$) the functions in $\{\cdots\}$ in Eq. (B5) are strongly peaked at ω values where the numerators in the exponents vanish. Making use of this we obtain

$$\delta\Pi_b^K(\Omega,\mathbf{0}) \approx \frac{4i\Gamma^2}{\nu}\int_k \frac{1}{[\xi_{\mathbf{k}}^2+\Gamma^2]^2}\{\cdots\}, \tag{B6}$$

where $\{\cdots\}$ is unchanged from Eq. (B5). This makes the ω integral trivial, yielding the result

$$\delta\Pi_b^K(\Omega,\mathbf{0}) = -i\frac{4\Gamma^2}{\nu\pi}\int \frac{kdkd\theta}{(2\pi)^2}\frac{T_{\text{eff}}|\cos\theta|e^{-|\Omega|/T_{\text{eff}}|\cos\theta|}}{[\xi_{\mathbf{k}}^2+\Gamma^2]^2}, \tag{B7}$$

where θ is the angle subtended by the Fermi velocity and the electric field. Performing the momentum integral gives

$$\delta\Pi_b^K(\Omega,\mathbf{0}) \approx -2i\tau|\Omega|I\left(\frac{T_{\text{eff}}}{|\Omega|}\right). \tag{B8}$$

APPENDIX C: EVALUATION OF THE CORRELATION LENGTH

In this appendix, we present the derivation for the leading correction to the correlation length in the quantum critical regime. Here, the analysis is conducted in the $\phi_b-\phi_t$ basis. The results are summarized in Secs. VI A 2 and VI B.

Recall that $\delta_\alpha(b)$ in equilibrium is given by Eq. (80). The correction to the zero-temperature distance from criticality, r, is then given by

$$\Delta\delta_\alpha = 3u_1^\alpha\int_0^{\ln b^*} dxe^{-3x}[f_\alpha(Te^{3x},\tau e^{-x}) - f_\alpha(0,\tau e^{-x})]. \tag{C1}$$

If we define $\Delta f_\alpha := f_\alpha(Te^{3x},\tau e^{-x}) - f_\alpha(0,\tau e^{-x})$, we have

$$\Delta f_\alpha = iK_2\int\frac{d\Omega}{2\pi}\Delta D_\alpha^K(T,\tau), \tag{C2}$$

where $K_2=1/2\pi$ and

$$\Delta D_b^K(1,\Omega) = -2i\Omega\left[\coth\left(\frac{\Omega}{2T}\right)-\text{sgn}(\Omega)\right] \times\frac{\tau[(\delta+1)^2+\Omega^2]+\epsilon^2}{|(\delta+1-i\Omega\tau)(\delta+1-i\Omega)-\epsilon^2|^2}. \tag{C3}$$

$$\Delta D_t^K(1,\Omega) = -2i\Omega\left[\coth\left(\frac{\Omega}{2T}\right)-\text{sgn}(\Omega)\right] \times\frac{(\delta+1)^2+(\Omega\tau)^2+\tau\epsilon^2}{|(\delta+1-i\Omega\tau)(\delta+1-i\Omega)-\epsilon^2|^2}. \tag{C4}$$

The Ω integral gets most of the contribution from $|\Omega|\leq 2T$. So we can cutoff the integral and approximate $\coth(\Omega/2T)\approx 2T/\Omega$. Then we obtain

$$\Delta D_b^K(1,\Omega) \approx -2i(2T-|\Omega|)\frac{\tau[(\delta+1)^2+\Omega^2]+\epsilon^2}{|(\delta+1-i\Omega\tau)(\delta+1-i\Omega)-\epsilon^2|^2}, \tag{C5}$$

$$\Delta D_t^K(1,\Omega) \approx -2i(2T-|\Omega|)\frac{(\delta+1)^2+(\Omega\tau)^2+\tau\epsilon^2}{|(\delta+1-i\Omega\tau)(\delta+1-i\Omega)-\epsilon^2|^2}. \tag{C6}$$

To scale up to b^*, where $T(b^*)=1$, we split the integral into two regimes: $1<b<b_1$, and $b_1<b<b^*$, where $b_1=\delta^{-1/2}$. Up to scale b_1, $\delta,\epsilon\ll 1$ and $T\ll 1$. So in this regime, we can use the following expressions:

$$\Delta D_b^K \approx -2i(2T-|\Omega|)\frac{\tau}{1+(\Omega\tau)^2}, \tag{C7}$$

$$\Delta D_t^K \approx -2i(2T-|\Omega|)\frac{1}{1+\Omega^2}. \tag{C8}$$

With this approximation, the Ω integral can be done trivially. Expanding the result for small T, we get

$$\Delta f_b = \frac{4K_2}{\pi}\tau T^2 + \mathcal{O}(T^3), \tag{C9}$$

$$\Delta f_t = \frac{4K_2}{\pi}T^2 + \mathcal{O}(T^3). \tag{C10}$$

To evaluate the integral up to b_1, we first do a change of variables, $x\to T(b)=:y$, which makes the measure $dx=dy/3y$, and the lower and the upper limits T and $T/\delta^{3/2}$, respectively. Then the corrections are given by

$$\Delta\delta_\alpha^{(1)} = 3u_1^\alpha\int_T^{T/\delta^{3/2}}\frac{dy}{3y}\frac{T}{y}\Delta f_\alpha\left(y,\tau\left(\frac{T}{y}\right)^{1/3}\right). \tag{C11}$$

We then obtain

$$\Delta\delta_b^{(1)} = \frac{3}{\pi^2}u_1^b\tau T^{4/3}\frac{T^{2/3}}{\delta}, \tag{C12}$$

$$\Delta\delta_t^{(1)} = \frac{2}{\pi^2}u_1^t T\frac{T^{2/3}}{\delta}. \tag{C13}$$

To scale between b_1 and b^*, we use Eqs. (C5) and (C6) in the limit of $\delta\sim\epsilon\gg 1$. Since $\Delta\ll 1$ throughout the flow, we note that $\delta^2-\epsilon^2\sim\delta\Delta\leq 1$ during the flow, and we drop this term from the denominator. Then Eqs. (C5) and (C6) can be approximated by

$$\Delta D_b^K(1,\Omega) \approx -2i(2T-|\Omega|)\frac{\tau[\delta^2+\Omega^2]+\epsilon^2}{4\delta^2+[\delta\Omega(1+\tau)]^2}, \tag{C14}$$

$$\Delta D_t^K(1,\Omega) \approx -2i(2T-|\Omega|)\frac{\delta^2+(\Omega\tau)^2+\tau\epsilon^2}{4\delta^2+[\delta\Omega(1+\tau)]^2}. \tag{C15}$$

We then find that

$$\Delta f_b \approx \frac{\tau\delta^2+\epsilon^2}{4\pi^2\delta^2}T^2 \approx \frac{1+\tau}{4\pi^2}T^2, \tag{C16}$$

$$\Delta f_t \approx \frac{\tau\epsilon^2+\delta^2}{4\pi^2\delta^2}T^2 \approx \frac{1+\tau}{4\pi^2}T^2, \quad \text{(C17)}$$

where we have used the fact that $\delta \approx \epsilon$. The corrections from this region of the integral is then given by

$$\Delta\delta_\alpha^{(2)} = 3u_1^\alpha \int_{T/\delta^{3/2}}^1 \frac{dy}{3y}\frac{T}{y}\Delta f_\alpha\left(y, \tau\left(\frac{T}{y}\right)^{1/3}\right). \quad \text{(C18)}$$

We then obtain

$$\Delta\delta_\alpha^{(2)} = \frac{u_1^\alpha T}{4\pi^2}\left[1+\frac{3\tau}{2}T^{1/3}\right]. \quad \text{(C19)}$$

Since $T^{2/3} \ll \delta$ in this regime, Eqs. (C12) and (C13) are subleading to Eq. (C19). We therefore obtain to leading order

$$\Delta\delta_\alpha \approx \frac{u_1^\alpha T}{4\pi^2}\left[1+\frac{3\tau}{2}T^{1/3}\right]. \quad \text{(C20)}$$

In the nonequilibrium case, recall that $\delta_\alpha(b)$ is given by Eq. (71). The correction to the zero-temperature distance from criticality, r, is then given by

$$\Delta\delta_\alpha = 3u_1^\alpha \int_0^{\ln b^*} dx e^{-3x}[f_\alpha(T_{\text{eff}}e^{3x}, \tau e^{-x}) - f_\alpha(0, \tau e^{-x})]. \quad \text{(C21)}$$

In this nonequilibrium case, we obtain

$$\Delta D_b^K(1,\Omega) = -2i|\Omega| I\left(\frac{T_{\text{eff}}}{|\Omega|}\right)\frac{\tau[(\delta+1)^2+\Omega^2]}{|(\delta+1-i\Omega\tau)(\delta+1-i\Omega)-\epsilon^2|^2}, \quad \text{(C22)}$$

$$\Delta D_t^K(1,\Omega) = -2i|\Omega| I\left(\frac{T_{\text{eff}}}{|\Omega|}\right)\frac{\tau\epsilon^2}{|(\delta+1-i\Omega\tau)(\delta+1-i\Omega)-\epsilon^2|^2}. \quad \text{(C23)}$$

In the above, the drift term was dropped since it does not reach the cut-off scale at b^*.[8]

We first scale up to b_1. As in the equilibrium case, we have $\delta \ll 1$ and $\epsilon \ll 1$ in this region, and Eqs. (C22) and (C23) can be approximated by

$$\Delta D_b^K \approx \frac{-4i\tau T_{\text{eff}}}{\pi}\frac{1}{1+(\Omega\tau)^2}, \quad \text{(C24)}$$

$$\Delta D_t^K \approx \frac{-4i\tau T_{\text{eff}}}{\pi}\frac{\epsilon^2}{[1+(\Omega\tau)^2](1+\Omega^2)}. \quad \text{(C25)}$$

Once again, we have used the fact that ΔD_α^K is appreciable only for $|\Omega| \leq 2T_{\text{eff}}$ and that in this region

$$I\left(\frac{T_{\text{eff}}}{|\Omega|}\right) \approx \frac{T_{\text{eff}}}{|\Omega|}\frac{2}{\pi}. \quad \text{(C26)}$$

Performing the Ω integral up to $2T_{\text{eff}}$, we then obtain to lowest order in T_{eff}

$$\Delta f_b = \frac{4K_2}{\pi}\tau T_{\text{eff}}^2 + \mathcal{O}(T_{\text{eff}}^3), \quad \text{(C27)}$$

$$\Delta f_t = \frac{4K_2}{\pi}\tau\epsilon^2 T_{\text{eff}}^2 + \mathcal{O}(T_{\text{eff}}^3). \quad \text{(C28)}$$

The corrections up to b_1 can then be computed, giving

$$\Delta\delta_b^{(1)} = \frac{3}{\pi^2}u_1^b\tau T_{\text{eff}}^{4/3}\frac{T_{\text{eff}}^{2/3}}{\delta}, \quad \text{(C29)}$$

$$\Delta\delta_t^{(1)} = \frac{3}{2\pi^2}u_1^t\tau\epsilon^2 T_{\text{eff}}^{2/3}\frac{T_{\text{eff}}^{4/3}}{\delta^2} \approx \frac{3}{2\pi^2}u_1^t\tau T_{\text{eff}}^2. \quad \text{(C30)}$$

To scale in the region $b_1 < b < b^*$, we use

$$\Delta D_b^K(1,\Omega) \approx \frac{-4i\tau T_{\text{eff}}}{\pi}\frac{\tau\delta^2}{4\delta^2+[\delta\Omega(1+\tau)]^2}, \quad \text{(C31)}$$

$$\Delta D_t^K(1,\Omega) \approx \frac{-4i\tau T_{\text{eff}}}{\pi}\frac{\tau\epsilon^2}{4\delta^2+[\delta\Omega(1+\tau)]^2}. \quad \text{(C32)}$$

Performing the Ω integral and expanding for small T_{eff}, we obtain

$$\Delta f_b \approx \Delta f_t \approx \frac{\tau T_{\text{eff}}^2}{2\pi^3}, \quad \text{(C33)}$$

where we have again used the fact that $\delta \approx \epsilon$. The corrections are then given by

$$\Delta\delta_\alpha^{(2)} \approx \frac{3\tau u_1^\alpha}{4\pi^3}T_{\text{eff}}^{4/3}. \quad \text{(C34)}$$

As in the equilibrium case, the contributions from the first region of the integral [Eqs. (C29) and (C30)] are subleading to those from the second region [Eq. (C34)]. Therefore, to leading order, we obtain the following correction:

$$\Delta\delta_\alpha \approx \frac{3\tau u_1^\alpha}{4\pi^3}T_{\text{eff}}^{4/3}. \quad \text{(C35)}$$

APPENDIX D: EIGENMODE ANALYSIS

1. Determining the eigenmodes

In this appendix, we develop a critical theory in the basis of the eigenmodes. The renormalization-group analysis will be carried out in this basis, and the correlation length calculation will be reconsidered. We determine the eigenmodes of the system by diagonalizing the Gaussian action Eq. (34). What we find in the following is that the diagonalization procedure gives rise to two new bosonic modes which possess different masses. While one becomes critical at $\delta = \epsilon$, the other mode remains gapped. This allows one to integrate out the latter and arrive at an effective low-energy theory in terms of one critical mode. This is in stark contrast with Ref. 23 where both bosonic modes simultaneously become critical.

To aid with the eventual block diagonalization of the Keldysh action, we first diagonalize the equilibrium action in the Matsubara formalism, then map the theory back to Keldysh space. The Matsubara Gaussian action reads

$$\mathcal{S}^{(2)}_{\rm eq} = \int_q (\phi^*_{bq} \quad \phi^*_{tq}) \begin{pmatrix} \mathcal{L}^{\rm eq}_{bq} & -\epsilon \\ -\epsilon & \mathcal{L}^{\rm eq}_{tq} \end{pmatrix} \begin{pmatrix} \phi_{bq} \\ \phi_{tq} \end{pmatrix}, \tag{D1}$$

where $\mathcal{L}^{\rm eq}_{bq} = \delta + |\mathbf{q}|^2 + |\Omega|\tau$ and $\mathcal{L}^{\rm eq}_{tq} = \delta + |\mathbf{q}|^2 + |\Omega|/v_F|\mathbf{q}|$ are the equilibrium inverse susceptibilities. Expanding to lowest order with respect to the fluctuating parts of the polarization functions, the eigenvalues of the action read $\frac{1}{2}(\mathcal{L}^{\rm eq}_{bq} + \mathcal{L}^{\rm eq}_{tq}) \pm \epsilon$. The expansion is well defined in the interlayer coupling regime considered here (i.e., $\delta \gtrsim \epsilon > 0$). If we now denote the eigenmodes by λ_{1q} and λ_{2q}, the Gaussian action then reduces to

$$\mathcal{S}^{(2)}_{\rm eq} = \int \frac{d^2q}{(2\pi)^2} \frac{d\Omega}{2\pi} (\lambda^*_{1q} \quad \lambda^*_{2q}) \times \begin{pmatrix} (\mathcal{L}^{\rm eq}_{bq} + \mathcal{L}^{\rm eq}_{tq})/2 - \epsilon & 0 \\ 0 & (\mathcal{L}^{\rm eq}_{bq} + \mathcal{L}^{\rm eq}_{tq})/2 + \epsilon \end{pmatrix} \begin{pmatrix} \lambda_{1q} \\ \lambda_{2q} \end{pmatrix}. \tag{D2}$$

We can now map this Matsubara action back into the Keldysh form by simply replacing both fields by two-component Keldysh fields, i.e., $\lambda_{iq} \to \Lambda_{iq} = (\lambda^{\rm cl}_{iq} \ \lambda^{\rm q}_{iq})^T$, and replacing each polarization function by its corresponding matrix. The Keldysh Gaussian action is then given by

$$i\mathcal{S}^{(2)}_{\rm eff} = -i \int \frac{d^2q}{(2\pi)^2} \frac{d\Omega}{2\pi} (\Lambda^\dagger_{1q} \quad \Lambda^\dagger_{2q}) \times \begin{pmatrix} (\hat{\mathcal{L}}_{bq} + \hat{\mathcal{L}}_{tq})/2 - \epsilon\hat{\tau}_x & 0 \\ 0 & (\hat{\mathcal{L}}_{bq} + \hat{\mathcal{L}}_{tq})/2 + \epsilon\hat{\tau}_x \end{pmatrix} \begin{pmatrix} \Lambda_{1q} \\ \Lambda_{2q} \end{pmatrix}, \tag{D3}$$

where $\hat{\mathcal{L}}_{\alpha q} = (\widetilde{U}\nu)^{-1}\hat{\tau}_x + \hat{\Pi}_{\alpha q}$. We now consider the quartic terms. Recall that in the ϕ basis, the quartic interactions were given by

$$i\mathcal{S}^{(4)}_{\rm eff} = \int d^3x \{ -i[u^b_1(\phi^{\rm cl}_b)^3\phi^{\rm q}_b + u^b_3\phi^{\rm cl}_b(\phi^{\rm q}_b)^3] + [u^b_2(\phi^{\rm cl}_b)^2(\phi^{\rm q}_b)^2 + u^b_4(\phi^{\rm q}_b)^4]\} + (b \leftrightarrow t), \tag{D4}$$

where we have written terms in the form in which u^α_i are all real. To transform the quartic terms to the eigenbasis, we use the transformation matrix in the limit of $q \to 0$ since momentum- and frequency-dependent parts generate terms that are (RG) irrelevant compared with the leading constant coefficients. We then obtain

$$\begin{pmatrix} \phi^{\rm cl,q}_b \\ \phi^{\rm cl,q}_t \end{pmatrix} \approx \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \lambda^{\rm cl,q}_1 \\ \lambda^{\rm cl,q}_2 \end{pmatrix}. \tag{D5}$$

Inserting this transformation into Eq. (D4) gives the quartic terms in the eigenbasis.

The mass terms of the two eigenmodes in Eq. (D2) are $\delta - \epsilon$ and $\delta + \epsilon$ for Λ_1 and Λ_2, respectively. This implies that while Λ_1 becomes critical at $\delta = \epsilon$, Λ_2 remains gapped. We may therefore integrate out the gapped mode from the theory and obtain a single-component effective action only in terms of the critical eigenfield. In the process of integrating out Λ_2, we only retain terms that are quadratic in the latter. Terms linear and quadratic in Λ_2 emerge from the quartic terms, however, the former will generate contributions beyond quartic order in the critical field. After these simplifications, the quartic action becomes

$$i\mathcal{S}^{(4)}_{\rm eff} = -i \int d^3x [\bar{u}_1(\lambda_1)^3\lambda^{\rm q}_1 + \bar{u}_3\lambda_1(\lambda^{\rm q}_1)^3] + \int d^3x [\bar{u}_2(\lambda^{\rm cl}_1)^2(\lambda^{\rm q}_1)^2 + \bar{u}_4(\lambda^{\rm q}_1)^4] - i \int_{q,q',k} (\lambda^{\rm cl*}_{2q} \quad \lambda^{\rm q*}_{2q}) \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \begin{pmatrix} \lambda^{\rm cl}_{2q'} \\ \lambda^{\rm q}_{2q'} \end{pmatrix} \tag{D6}$$

with $Q_{11} = 3\bar{u}_1\lambda^{\rm cl}_{1,k+q}\lambda^{\rm q}_{1,-k-q'} - i\bar{u}_2\lambda^{\rm q}_{1,k+q}\lambda^{\rm q}_{1,-k-q'}$. Here, $\bar{u}_i = (u^b_i + u^t_i)/4$. Q_{12}, Q_{21}, and Q_{22} are also terms quadratic in λ_1, however, we do not write them explicitly here because they will not be necessary in the following discussion. Now performing the Gaussian integral over Λ_2, and expanding the resulting Tr ln to linear order in $\bar{u}_i$, we obtain

$$i\mathcal{S}^{(2)}_{\rm eff} = -i \int_q \Lambda^\dagger_{1q}[(\hat{\mathcal{L}}_{bq} + \hat{\mathcal{L}}_{tq})/2 - \epsilon\hat{\tau}_x]\Lambda_{1q} - i \int_q \Lambda^\dagger_{1q} \begin{pmatrix} 0 & \eta_1 \\ \eta_1 & i\eta_2 \end{pmatrix} \Lambda_{1q}, \tag{D7}$$

where

$$\eta_1 = \frac{3\bar{u}_1}{2} \int_k iD^K_{2k}, \quad \eta_2 = \bar{u}_2 \int_k iD^K_{2k}, \tag{D8}$$

and

$$\begin{pmatrix} D^K_{2q} & D^R_{2q} \\ D^A_{2q} & 0 \end{pmatrix} = -[(\hat{\mathcal{L}}_{bq} + \hat{\mathcal{L}}_{tq})/2 + \epsilon\hat{\tau}_x]^{-1}. \tag{D9}$$

Both η_1 and η_2 are real quantities, and we have used the fact that $D^R_{2\mathbf{q}}(t,t) = D^A_{2\mathbf{q}}(t,t) = 0$. We find that $\bar{u}_2 \to 0$ as $T \to 0$ and $T_{\rm eff} \to 0$ so η_2 does not change the form of the Keldysh term for Λ_1. We will omit these terms. The $\bar{u}_1$ term gives a small renormalization to the mass of the critical mode. We absorb the η_1 into the new mass: $\Delta = \delta - \epsilon + \eta_1$. The full effective action for the critical mode is now given by

$$i\mathcal{S}_{\rm eff} = -i \int_q \Lambda^\dagger_{1q} \begin{pmatrix} 0 & \chi^{-1}_A \\ \chi^{-1}_R & \chi^{-1}_K \end{pmatrix} \Lambda_{1q} - i \int d^3x [\bar{u}_1(\lambda^{\rm cl}_1)^3\lambda^{\rm q}_1 + \bar{u}_3\lambda^{\rm cl}_1(\lambda^{\rm q}_1)^3] + \int d^3x [\bar{u}_2(\lambda^{\rm cl}_1)^2(\lambda^{\rm q}_1)^2 + \bar{u}_4(\lambda^{\rm q}_1)^4]. \tag{D10}$$

In equilibrium,

$$\chi^{-1}_R = \Delta + |\mathbf{q}|^2 - \frac{i}{2}\left(\Omega\tau + \frac{\Omega}{v_F|\mathbf{q}|}\right), \tag{D11}$$

$$\chi^{-1}_K = -i\coth\left(\frac{\Omega}{2T}\right)\left(\Omega\tau + \frac{\Omega}{v_F|\mathbf{q}|}\right), \tag{D12}$$

and out of equilibrium,

$$\chi_R^{-1} = \Delta + |\mathbf{q}|^2 + \frac{i}{2}\mathbf{v}_d \cdot \mathbf{q}\tau - \frac{i}{2}\left(\Omega\tau + \frac{\Omega}{v_F|\mathbf{q}|}\right), \quad \text{(D13)}$$

$$\chi_K^{-1} = -i\frac{|\Omega|}{v_F|\mathbf{q}|} - i|\Omega|\tau\left[1 + I\left(\frac{T_{\text{eff}}}{|\Omega|}\right)\right]. \quad \text{(D14)}$$

Note that $I(T_{\text{eff}}/|\Omega|) \sim T_{\text{eff}}/|\Omega|$ for $T_{\text{eff}} > |\Omega|$ and $\coth(\Omega/2T) \sim T/\Omega$ for $T > |\Omega|$. We then see that, similar to previous works,[7,8] both T and T_{eff} act as a mass for the quantum fluctuations. At $T=0$ and $T_{\text{eff}}=0$, χ_K^{-1} vanishes at low energies as $\sim|\Omega|$. However, for $T \neq 0$ or $T_{\text{eff}} \neq 0$, $\chi_K^{-1} \neq 0$ as $\Omega \to 0$, and the theory effectively becomes classical.

2. RG analysis in eigenmode basis

We now perform a renormalization-group analysis of the effective action, Eq. (D10). Following a similar analysis as in the main text, we find the following set of RG equations to lowest (linear) order in the quartic coupling:

$$\frac{dT_{\text{eff}}(b)}{d\ln b} = 3T_{\text{eff}}(b), \quad \text{(D15)}$$

$$\frac{dv_d(b)}{d\ln b} = 2v_d(b), \quad \text{(D16)}$$

$$\frac{d\tau(b)}{d\ln b} = -\tau(b), \quad \text{(D17)}$$

$$\frac{d\bar{u}_i(b)}{d\ln b} = -\bar{u}_i(b) + \mathcal{O}([\bar{u}_j]^2), \quad \text{(D18)}$$

$$\frac{d\Delta(b)}{d\ln b} = 2\Delta(b) + 3\bar{u}_1(b) f(T_{\text{eff}}(b), \tau(b)). \quad \text{(D19)}$$

The function $f(T_{\text{eff}}(b), \tau(b))$ is given by

$$f(T_{\text{eff}}(b), \tau(b)) = iK_2 \int \frac{d\Omega}{2\pi} \chi_K(1, \Omega), \quad \text{(D20)}$$

where $\chi_K(p) = -\chi_K^{-1}(p)/|\chi_R^{-1}(p)|^2$. Solving these equations, we arrive at

$$T_{\text{eff}}(b) = T_{\text{eff}} b^3, \quad \text{(D21)}$$

$$v_d(b) = v_d b^2, \quad \text{(D22)}$$

$$\tau(b) = \tau b^{-1}, \quad \text{(D23)}$$

$$\bar{u}_i(b) = \bar{u}_i b^{-1}, \quad \text{(D24)}$$

$$\Delta(b) = b^2\left[\Delta + 3\bar{u}_1 \int_0^{\ln b} dx e^{-3x} f(T_{\text{eff}} e^{3x}, \tau e^{-x})\right]. \quad \text{(D25)}$$

An analogous renormalization-group analysis in the equilibrium case gives

$$T(b) = Tb^3, \quad \text{(D26)}$$

$$\tau(b) = \tau b^{-1}, \quad \text{(D27)}$$

$$\bar{u}_i(b) = \bar{u}_i b^{-1}, \quad \text{(D28)}$$

$$\Delta(b) = b^2\left[\Delta + 3\bar{u}_1 \int_0^{\ln b} dx e^{-3x} f(Te^{3x}, \tau e^{-x})\right]. \quad \text{(D29)}$$

We now recompute the correlation length in the eigenbasis. The scaling stops when $\Delta(b_1) \sim 1$. In the quantum disordered regime, T and T_{eff} is small enough so that $T(b_1) \ll 1$ and $T_{\text{eff}}(b_1) \ll 1$. In this case, the correlation length of the system can be obtained by setting $T=0$ and $T_{\text{eff}}=0$. One then obtains

$$\xi^{-2} = \Delta + 3\bar{u}_1 \int_0^{\infty} dx e^{-3x} f(0, \tau e^{-x}) =: r. \quad \text{(D30)}$$

The condition for the occurrence of the regime is obtained by requiring $T(b_1) < 1$ where $b_1 = r^{-1/2}$. Then the condition reads

$$T < r^{3/2}, \quad T_{\text{eff}} < r^{3/2}. \quad \text{(D31)}$$

The nonuniversal shift to the bare mass in Eq. (D30) indicates that we are above the upper critical dimension and that $\bar{u}_1$ is a dangerously irrelevant operator.

When the inequality in Eq. (D31) is violated, the critical fluctuations become classical in nature and the system enters the quantum critical regime. Here we compute the leading corrections to the correlation length in this regime. We scale up to either $T(b^*)=1$ or $T_{\text{eff}}(b^*)=1$. Recall that the solution for $\Delta(b)$ in equilibrium is given by Eq. (D29). The correction to the zero-temperature distance from criticality, r, is then given by

$$\delta r = 3\bar{u}_1 \int_0^{\ln b^*} dx e^{-3x}[f(Te^{3x}, \tau e^{-x}) - f(0, \tau e^{-x})]. \quad \text{(D32)}$$

If we define $\Delta f := f(Te^{3x}, \tau e^{-x}) - f(0, \tau e^{-x})$, we have

$$\Delta f = iK_2 \int \frac{d\Omega}{2\pi} \Delta\chi_K(T, \tau), \quad \text{(D33)}$$

and

$$\Delta\chi_K(1, \Omega) = -i\Omega\left[\coth\left(\frac{\Omega}{2T}\right) - \text{sgn}(\Omega)\right] \times \frac{\tau + 1}{(\Delta + 1)^2 + (\Omega\tau + \Omega)^2/4}. \quad \text{(D34)}$$

The Ω integral gets most of the contribution from $-2T \leq \Omega \leq 2T$. So we can cutoff the integral there and use $\coth(\Omega/2T) \approx 2T/\Omega$. Then we obtain

$$\Delta\chi_K(1, \Omega) \approx -i(2T - |\Omega|)\frac{1 + \tau}{1 + (\Omega\tau + \Omega)^2/4}, \quad \text{(D35)}$$

where we dropped Δ from the denominator since it remains small during the flow. The Ω integral can now be done trivially. Expanding the result for small T, we get

$$\Delta f = \frac{1}{\pi^2}(1+\tau)T^2 + \mathcal{O}(T^3). \tag{D36}$$

To evaluate the integral up to b^*, we first do a change of variables, $x \to T(b) =: y$, which makes the measure $dx = dy/3y$. The upper limit of the integral is 1, and we extend the lower limit down to 0. Then the correction is given by

$$\delta r = 3\bar{u}_1 \int_0^1 \frac{dy}{3y}\frac{T}{y}\Delta f\left(y, \tau\left(\frac{T}{y}\right)^{1/3}\right). \tag{D37}$$

We then obtain

$$\delta r = \frac{\bar{u}_1}{\pi^2}T\left(1 + \frac{3\tau}{2}T^{1/3}\right). \tag{D38}$$

We now move on to the nonequilibrium case. Recall that the solution $\Delta(b)$ for the nonequilibrium case is given by Eq. (D25). The correction to the zero-temperature distance from criticality, r, is then given by

$$\delta r = 3\bar{u}_1 \int_0^{\ln b^*} dx e^{-3x}[f(T_{\mathrm{eff}}e^{3x}, \tau e^{-x}) - f(0, \tau e^{-x})]. \tag{D39}$$

We then obtain

$$\Delta\chi_K(1,\Omega) = -i|\Omega|\tau \frac{I\left(\frac{T_{\mathrm{eff}}}{|\Omega|}\right)}{1 + (\Omega\tau + \Omega)^2/4}, \tag{D40}$$

where we have dropped the drift and the mass terms from the denominator since they remain small during the flow.[8] Once again, the Ω integral is appreciable only for $|\Omega| \le 2T_{\mathrm{eff}}$. In this region, we have

$$I\left(\frac{T_{\mathrm{eff}}}{|\Omega|}\right) \approx \frac{T_{\mathrm{eff}}}{|\Omega|}\frac{2}{\pi}. \tag{D41}$$

Performing the Ω integral up to $2T_{\mathrm{eff}}$, we then obtain to lowest order in T_{eff},

$$\Delta f = \frac{2}{\pi^3}\tau T_{\mathrm{eff}}^2 + \mathcal{O}(T_{\mathrm{eff}}^3). \tag{D42}$$

The mass correction is then given by

$$\delta r = 3\bar{u}_1 \int_0^1 \frac{dy}{3y}\frac{T_{\mathrm{eff}}}{y}\Delta f\left(y, \tau\left(\frac{T_{\mathrm{eff}}}{y}\right)^{1/3}\right). \tag{D43}$$

Performing the integral, we obtain

$$\delta r = \frac{3\bar{u}_1}{\pi^3}\tau T_{\mathrm{eff}}^{4/3}. \tag{D44}$$

To summarize the results, the correlation length in equilibrium reads

$$\xi_{\mathrm{eq}}^{-2} = r + \frac{\bar{u}_1}{\pi^2}T\left(1 + \frac{3\tau}{2}T^{1/3}\right) \tag{D45}$$

while out of equilibrium, one obtains

$$\xi_{\mathrm{neq}}^{-2} = r + \frac{3\bar{u}_1\tau}{\pi^3}T_{\mathrm{eff}}^{4/3}. \tag{D46}$$

This is in exact agreement with the results obtained in the main text [cf. Eqs. (90) and (92)].

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