

Optimizing Wilson's momentum shell renormalization group

A. Tröster*

Faculty of Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria

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Previous attempts to accurately compute critical exponents from Wilson's momentum shell renormalization prescription suffered from the difficulties posed by the presence of an infinite number of irrelevant couplings. Taking the example of the one-dimensional long-ranged Ising model, we calculate the momentum shell renormalization flow in the plane spanned by the coupling constants (u_0, r_0) for different values of the momentum shell thickness parameter b by simulation using our recently developed Fourier Monte Carlo algorithm. We report strong anomalies in the b dependence of the fixed-point couplings and the resulting exponents y_τ and ω in the vicinity of a shell parameter $b^* < 1$ characterizing a thin but finite momentum shell. Evaluation of the exponents for this value of b yields a dramatic improvement of their numerical accuracy, indicating a strong damping of the influence of irrelevant couplings for $b=b^*$.

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I. INTRODUCTION

In a recent paper¹ we have presented a simulation method that in principle allows to calculate the renormalization-group (RG) flow and its fixed points (FPs) in the space \mathcal{K} of coupling constants $\mathbf{K}=(K_0, K_1, K_2, \dots)$ for Wilson's momentum shell (MS) RG scheme.^{2,3} These as well as the present simulations are based on our recently developed Fourier Monte Carlo (FMC) algorithm (see Refs. 4 and 5 and below for details) of the classical long-range ϕ^4 model⁶ with dimensionless Hamiltonian (we assume $\sigma > 0$ to ensure existence of the thermodynamic limit)

$$\mathcal{H}[s] = \frac{1}{2} \int d^d x \int d^d y \frac{s(\mathbf{x})s(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^{d+\sigma}} + \int d^d x \left[\frac{r_0}{2} s^2(\mathbf{x}) + \frac{u_0}{4} s^4(\mathbf{x}) \right], \quad (1)$$

whose critical behavior was first analyzed in Ref. 6. As we also discuss in more detail below, the accompanying critical exponents can be calculated from the eigenvalues of linearized RG transformations which are fitted to the actual RG flow patterns which we determine from a large number of FMC simulations. It is the nonperturbative nature of such a simulation-based approach that makes it particularly interesting, as the method in principle allows, e.g., to explore global features of the RG flow.

In the above context, the model defined by Eq. (1) is particularly convenient for two reasons. On the one hand, for noninteger σ the dispersion term k^σ is nonanalytic and therefore left invariant under the RG as a result of the general analytic nature of single RG transformations. The exponent η is thus found to obey $\eta=2-\sigma$ to all orders of perturbation theory⁶ and we arrive at the fact that here the RG field rescaling factor $z(b)=b^{(d+\sigma)/2}$ is known exactly. On the other hand, the model's upper critical dimension $d_c(\sigma)=2\sigma$ manifestly depends on σ . Thus, defining $\epsilon \equiv d_c(\sigma)-d=2\sigma-d$, it is possible to reinterpret the ϵ expansion as an expansion in the small parameter ϵ at variable σ and fixed integer dimension d . Moreover, for this model we are in the fortunate situation that both analytical results to order $O(\epsilon^2)$ as well as state-of-the-art numerical simulations using cluster algorithms in combination with finite-size scaling are available for

comparison.⁷ As was shown in Ref. 8, for $d > 1$ and $\sigma > 2 - \eta_{SR}$, the model exhibits a crossover to standard Ising critical behavior, where η_{SR} denotes the value of the critical exponent η corresponding the Ising universality class. We thus concentrate on the parameter range $d/2 < \sigma < 2 - \eta_{SR}$, for which a nontrivial FP \mathbf{K}^* was first predicted in Ref. 6.

We now come to the central issue of the present paper. In Ref. 1 we observed that the obtained numerical values for the only relevant critical exponent $y_1 \equiv y_\tau = 1/\nu$ and the leading irrelevant Wegner correction exponent $y_2 \equiv -\omega$ were only of an accuracy comparable to those of an analytic $O(\epsilon)$ calculation. This unpleasant fact is, however, not specific to our method but represents a central problem one must overcome in any attempt to extract accurate numerical information from Wilson's MS prescription. In fact, to design a feasible calculation we had to project the full RG flow around the nontrivial FP \mathbf{K}^* of Wilson-Fisher type from the infinite-dimensional space \mathcal{K} onto the two-dimensional plane spanned by the variables (u_0, r_0) . In an ϵ -expansion analysis, the corresponding fixed-point values (u_0^*, r_0^*) both turn out to be of order $O(\epsilon)$ while all other components of the true infinite-dimensional FP \mathbf{K}^* are of order $O(\epsilon^2)$. This behavior is, of course, due to the fact that r_0 and u_0 actually correspond to the projection of the actual relevant and the least irrelevant direction in the infinite-dimensional coupling space \mathcal{K} . In the same way, the fixed-point values r_0^* and u_0^* also correspond to the projection of the "true" fixed point $\mathbf{K}^*=(r_0^*, u_0^*, \dots) \in \mathcal{K}$ onto this subplane. Unfortunately, however, attempts to enlarge the dimensionality of this subspace by inclusion of further irrelevant couplings generally result in prohibitively complicated calculation schemes. In fact, also analytically it was just the difficulty posed by controlling the influence of irrelevant couplings that historically led to the abandonment of the momentum shell RG for higher order calculations in favor of field-theoretic perturbation theory. In the present work we propose a solution to this intrinsic problem of Wilson's scheme.

Recall that a general RG transformation consists of three steps, namely, (1) coarse graining, (2) rescaling of lengths/momenta, and (3) renormalization of field variable(s). This procedure defines a mapping $\mathcal{R}(\cdot, b): \mathcal{K} \rightarrow \mathcal{K}$, where b mea-

ures the “shell thickness” for RG transformations defined for a momentum shell $\Lambda/b < |\mathbf{k}| < \Lambda$ at fixed cutoff Λ in wave-vector space. Nontrivial critical behavior is connected to those fixed points $\mathcal{R}(\mathbf{K}^*, b) \equiv \mathbf{K}^*$ of $\mathcal{R}(\cdot, b)$ that exhibit an infinite correlation length $\xi[\mathbf{K}^*] = \infty$. In the absence of so-called Griffiths-Pearce singularities⁹ [which is usually granted at least in a neighborhood of \mathbf{K}^* (Ref. 10)] $\mathcal{R}(\cdot, b)$ can be linearized near \mathbf{K}^* , leading to the matrix equation

$$\delta \mathbf{K}'_i = \sum_j \mathcal{M}_{ij}[\mathbf{K}^*] \delta \mathbf{K}_j + O(\delta \mathbf{K}^2), \quad (2)$$

where $\delta \mathbf{K} = \mathbf{K} - \mathbf{K}^*$ and the eigenvalues of the matrix $M_{ij}[\mathbf{K}^*] := \frac{\partial \mathcal{R}_i(\mathbf{K}, b)}{\partial K_j} \Big|_{\mathbf{K}=\mathbf{K}^*}$ which are expressed by a set of exponents y_i as

$$\lambda_i \equiv b^{y_i} \quad (3)$$

are related to the physical critical exponents. These eigenvalues are termed relevant, irrelevant, and marginal if $y_i > 0$, $y_i < 0$, and $y_i = 0$, respectively.

In analyzing and comparing results of calculations based on different RG schemes, it is necessary to distinguish between *nonuniversal* quantities (e.g., the location of the FP \mathbf{K}^*) and *universal* ones like the values of the critical exponents and amplitude ratios. Quite trivially, nonuniversal quantities may strongly depend on a large number of additional features of a system under investigation that reflect its microscopic details but are irrelevant for its universal long-distance behavior. Specifically, the type of underlying lattice or the exact value and geometry of the chosen cutoff Λ are usually free at our disposal in designing a convenient computational or analytic approach aimed at computing critical quantities. For instance, in a real-space RG calculation¹⁰ any particular choice of an underlying lattice structure limits the possible values of b to a discrete set. Nevertheless, in an ideal “exact” calculation any such choice of b should cancel from the results obtained for universal quantities, since, after all, the RG is just a mathematical device that allows a certain bookkeeping of the physics at different length scales. For instance, in their seminal review³ Wilson and Kogut found it convenient to choose $b=2$ for illustrating structural aspects of the momentum shell RG. In contrast, the exact location of the fixed point \mathbf{K}^* may strongly depend on the chosen scheme. Different RG schemes may thus exhibit different fixed points. Yet, these fixed points should be located on the same critical manifold of the corresponding universality class, as the divergence of the correlation length is—of course—not related to any choice of RG scheme.

In principle, as long as we aim for an ideal (analytical) calculation, we are thus free to work with any convenient choice of $b > 1$. In this respect, the advantages of choosing $b \equiv 1 + \mu$, $\mu \ll 1$, infinitesimally close to 1 are manifold, as this allows to replace discrete recursion relations by differential equations. Quite generally, following Ref. 11, if we define the Gell-Mann-Low functions $\beta_i[\mathbf{K}] := -\partial \mathcal{R}_i(\mathbf{K}; 1 + \mu) / \partial \mu \Big|_{\mu=0}$, a FP \mathbf{K}^* is determined by the simultaneous vanishing of all $\beta_i[\mathbf{K}^*] \equiv 0$, $i=1, 2, \dots$, and the exponents y_i directly arise as the eigenvalues of the matrix $\mathcal{B}_{ij}[\mathbf{K}] := -\partial \beta_i[\mathbf{K}] / \partial K_j \Big|_{\mathbf{K}=\mathbf{K}^*}$ without any reference to the physically

irrelevant auxiliary quantity $b = 1 + \mu$. Technically, the determination of most low-order contributions involving loop integrals in perturbative momentum shell RG calculations simplifies dramatically, as, e.g., $\int_{\Lambda/(1+\mu) < k < \Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + r_0)^n} = \frac{\hat{\Omega}_d \Lambda^d}{(\Lambda^2 + r_0)^n} \cdot \mu + O(\mu^2)$, where $\hat{\Omega}_d = \frac{\Omega_d}{(2\pi)^d}$ and Ω_d denotes the surface of the unit sphere in d dimensions. On a deeper level, the simplifications arising from such a choice of b allow to formulate the momentum shell RG completely in terms of an (alas complicated) *exact* RG differential equation,¹² exact meaning that in principle irrelevant couplings are fully taken into account. However, a discussion of the subsequent developments, which led to the at present most advanced formulation of the RG (see, e.g., Ref. 13 and references therein) will not be attempted here, for the simple reason that our simulations must necessarily be carried out at finite shell thickness.

However, in an actual (analytic or computer-based) computation, truncations and approximations are unavoidable, and thus one can anticipate that in practical calculations the exact invariance with respect to the choice of b will be broken. And indeed, contrary to the strategy of Wegner and Houghton,¹² in Ref. 14 Aharony undertook RG calculations to order $O(\epsilon^2)$ using a *large* b factor. The corresponding claim that the choice $b \gg 1$ would serve to suppress the unwanted influence of irrelevant couplings discussed above was subsequently worked out by Bruce *et al.*¹⁵ perturbatively to second order but did not lead to a widespread calculation scheme due to the difficulties posed by determining higher order loop integrals for finite shell thickness. Using FMC simulations, however, it becomes possible to study the case of arbitrary b nonperturbatively, as we now show.

II. SIMULATIONS

As opposed to analytical arguments, computer simulations are necessarily confined to finite systems. Using a simple-cubic lattice of L^d sites with periodic boundary conditions and lattice constant $a=1$, neighboring \mathbf{k} -vector components must at least differ by $2\pi/L$, such that both the limit $b \rightarrow 1$ as well as the limit $b \rightarrow \infty$ are unreachable for finite L . Thus the question arises naturally how the quality of the results obtained using a particular approximation scheme might be influenced by the concrete choice of b . Of course, from the computational point of view, finding a b^* value close to one in accordance with Ref. 12 would be definitely more attractive than the opposite case $b^* \gg 1$ predicted in Ref. 15, as FMC simulations using thin momentum shells involve only a small number of nonzero modes (see below). In trying to find out whether the choice of a large or small b value for the simulations will also improve their numerical quality, our strategy is thus to calculate RG flows and the accompanying critical exponents for a wide range of different values of b . As the emphasis is on covering a possibly large range of b values while at the same time minimizing finite-size effects, the present simulations were carried out for one-dimensional systems. For $d=1$ the σ values which qualify for exhibiting a Wilson-Fisher-type FP are confined to $1/2 < \sigma < 1$ since $\eta_{SR} = 1$.⁷ The case $d=1$ is also special insofar as there is no

phase transition in the short-range Ising model for $\sigma > 1$ and the crossover to short-range critical behavior observed for $\sigma = 2 - \eta_{SR}$ in $d > 1$ is replaced by a Kosterlitz-Thouless type of transition at $\sigma = 1$.¹⁶

The present simulations are based on our FMC algorithm which was introduced in Ref. 4 to study coarse-grained free energies. While we refer to Refs. 4 and 5 for full details of the algorithm, here it suffices to say that its central idea is to use the real and imaginary parts of the Fourier amplitudes of the underlying spin field configurations as the basic MC variables. The algorithm is thus advantageous in simulations where only modes corresponding to \mathbf{k} vectors from certain subspaces of the Brillouin zone are allowed to fluctuate while all other modes are kept fixed to zero. In particular, as explained in Refs. 1 and 4, this is the case for the coarse-graining step of our reproduction of Wilson's RG scheme. Starting from parameter values (u_0, r_0) , the coarse-graining step is performed by averaging out all "fast" fluctuations $\tilde{s}(\mathbf{k})$ for Fourier amplitudes with wave vectors $\Lambda/b < |\mathbf{k}| < \Lambda$, leading to an effective potential for the fluctuating homogeneous amplitude $\tilde{s}(\mathbf{0})$ which is proportional to the "bare" magnetization of the coarse-grained system. The fast modes $\tilde{s}(\mathbf{k})$ thus act like a "heat bath" for $\tilde{s}(\mathbf{0})$ while all other modes with wave vectors $0 < |\mathbf{k}| \leq \Lambda/b$ are kept zero during the simulation run. The resulting effective potential for $\tilde{s}(\mathbf{0})$ is fitted to a potential of the ϕ^4 type. Altogether, this procedure yields a mapping¹

$$(u_0, r_0) \mapsto (\tilde{u}_0, \tilde{r}_0) \quad (4)$$

of the original onto coarse-grained parameters $(\tilde{u}_0, \tilde{r}_0)$. To account for the rescaling step in Wilson's scheme necessary to restore the original momentum scale Λ , these parameters are rescaled to give

$$(u'_0, r'_0) = [b^{-3d} z^4(b) \tilde{u}_0, b^{-d} z^2(b) \tilde{r}_0]. \quad (5)$$

As mentioned above, it was shown in Ref. 6 that a fixed point of Wilson-Fisher type emerges for the choice $z(b) = b^{(d+\sigma)/2}$ of the field rescaling factor $z(b)$. In $d=1$ the recursion relations [Eq. (5)] then reduce to

$$(u'_0, r'_0) \equiv \mathcal{R}^{\text{FMC}}[(u_0, r_0), b] = (b^{2\sigma-1} \tilde{u}_0, b^\sigma \tilde{r}_0). \quad (6)$$

Specifically, we considered the parameter values $\sigma = 0.65, 0.75$, and 0.85 . In full similarity to the procedure used in Ref. 1, 30×30 initial parameter points (u_0, r_0) were used to determine each RG flow pattern and we chose to average the resulting flow diagram over six independent runs for improved statistics. After roughly locating the fixed point $(u_0^*, r_0^*) \equiv (u_0^*, r_0^*)$, we applied a two-dimensional version of Eq. (2) to the simulated flow pattern, fitting it with a linearized ansatz

$$\begin{pmatrix} u'_0 \\ r'_0 \end{pmatrix} = \begin{pmatrix} u_0^* \\ r_0^* \end{pmatrix} + \begin{pmatrix} \mathcal{M}_{uu} & \mathcal{M}_{ur} \\ \mathcal{M}_{ru} & \mathcal{M}_{rr} \end{pmatrix} \begin{pmatrix} u_0 - u_0^* \\ r_0 - r_0^* \end{pmatrix} \quad (7)$$

in a neighborhood of $\pm 8\%$ around this point in which the matrix entries \mathcal{M}_{ij} as well as the exact fixed-point values (u_0^*, r_0^*) are treated as fit parameters.

For determining the b regime $0 \ll 1/b < 1$ we used a one-dimensional lattice with $L=N=8192$ sites while a smaller

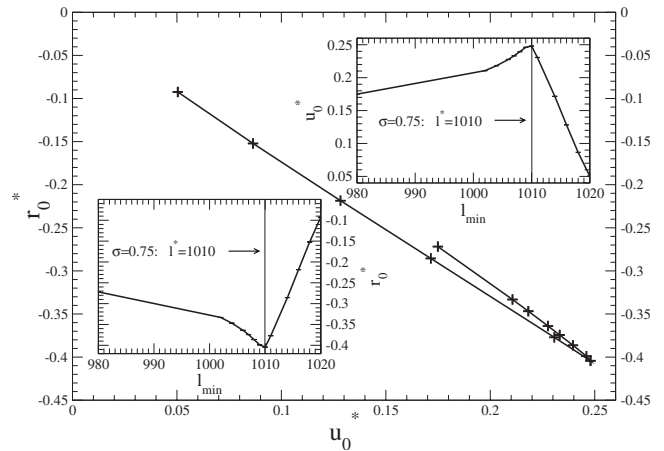


FIG. 1. b -dependent location of fixed points (u_0^*, r_0^*) at $\sigma = 0.75$. Insets: locations of u_0^* and r_0^* as functions of l_{\min} .

lattice with $L=N=1024$ was taken to explore the computationally much more expensive regime $0 < 1/b \ll 1$. Both types of simulations were performed at a fixed cutoff $\Lambda = \pi/4$ in the respective underlying one-dimensional Brillouin zones. In terms of the parametrization $k = \frac{2\pi}{L}m$, $m = -L/2 + 1, \dots, L/2$, this corresponds to the upper limit $|m| \leq l$ with a cutoff parameter $l = 1024$ and 128 , respectively. The allowed k values inside a momentum shell $\Lambda/b < |\mathbf{k}| \leq \Lambda$ are then parametrized by m values in the range $l_{\min} := l/b < |m| \leq l$. The allowed b values are thus given by $b = l/l_{\min}$, where $0 < l_{\min} < l$.

As shown in Fig. 1 representatively for $\sigma = 0.75$, for all considered values of σ we observe a pronounced anomaly in the location of the FP values u_0^*, r_0^* as functions of l_{\min} (cf. Fig. 1). Tentatively denoting the corresponding "critical" values of l_{\min} and b that simultaneously maximize both u_0^* and $-r_0^*$ by l^* and $b^* = l/l^*$, respectively, l^* is observed to be indeed quite close to but markedly smaller than $l-1$, which trivially would be the first possible l_{\min} value below l . Adopting the point of view of Ref. 12, one would be tempted to interpret the opening of such a finite gap between l and l^* as a finite-size effect. Indeed, l^* is observed to systematically shift to even smaller values for decreasing σ (cf. Table I), which would fit with this interpretation, as finite-size effects should of course develop stronger with increasing interaction range. However, a closer investigation for smaller system sizes reveals that, on the contrary, a critical value $b^* < 1$ ceases to exist for small system sizes while beyond a certain limiting system size, which is found to depend on σ , b^* asymptotically tends to a limiting value with growing L , as can be deduced from the simulation data shown in Fig. 2. In summary, the existence of a gap between the critical value b^*

TABLE I. l^* and $b^* = l/l^*$ as determined from our simulations for system size $L=8192$ and cutoff parameter $l=1024$.

σ	l^*	b^*
0.65	997	1.02708
0.75	1010	1.01386
0.85	1015	1.00887

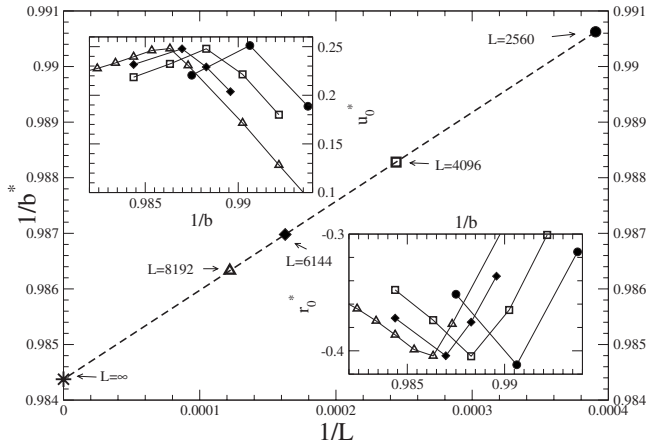


FIG. 2. L dependence of b^* for system sizes $L=2560$ (filled circle), 4096 (open square), 6144 (filled diamond), and 8192 (open triangle) at $\sigma=0.75$. A fit with the heuristic function $1/b^*(L) = 1/b^*(\infty) + c/L$, which is indicated by the dotted line, gives excellent agreement and thus indicates a saturation of $1/b^*(L)$ at a limiting value $1/b^*(\infty) \approx 0.9843$, clearly demonstrating that $b^*(\infty) = 1.01587$ is larger than 1 in the thermodynamic limit. Insets: locations of u_0^* and r_0^* as functions of $1/b$ for the same system sizes at $\sigma=0.75$. For sizes $L=1024$ and smaller no extrema of u_0^* and r_0^* as functions of b can be found owing to the comparatively large spacing $2\pi/L$ of the underlying discrete Brillouin zone.

and 1 is clearly *not* a finite-size effect.

From the fitted matrix elements \mathcal{M}_{ij} it is straightforward to compute the only relevant eigenvalue $\lambda_\tau = b^{y_\tau} > 1$ and the least irrelevant eigenvalue $\lambda_\omega = b^{-\omega} < 1$, and thus the exponents y_τ and ω . The resulting behavior of the obtained values of these exponents, which is summarized in Table II and Fig. 3, is also quite interesting. For all values of σ they are found to strongly depend on the chosen value of l_{\min} , with pronounced anomalies around the respective values of l^* . In detail, $y_\tau = y_\tau(l_{\min})$ shows a minimum near b^* and the value of $y_\tau^* := y_\tau(l^*)$ is observed to be located somewhat near the lower bounds but still in good agreement with the results of Ref. 7. On the other hand, the exponent $\omega = \omega(l_{\min})$ has a sharp anomaly in the vicinity of l^* , rising from an almost constant value to a value markedly above the corresponding $O(\epsilon)$ prediction. Interestingly, in doing so it is observed to pass through the values $\omega^* := \omega(l^*)$, which is again in good agreement with the corresponding results of Ref. 7 for the exponent ω . We also observe strong variations in the components of the two eigenvectors of the 2×2 -matrix $\mathcal{M}_{ij}[u_0^*(l), r_0^*(l)]$

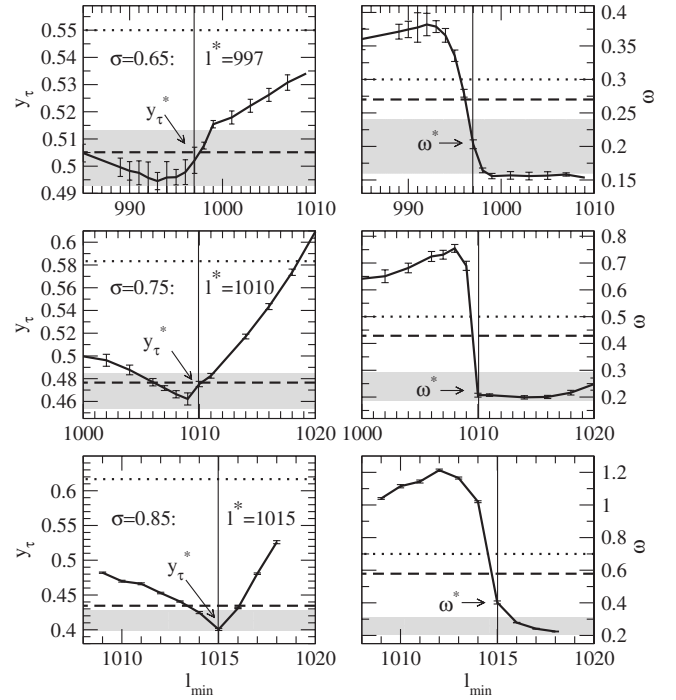


FIG. 3. l_{\min} dependence of exponents y_τ and ω (full lines) as obtained from our simulations compared to results from $O(\epsilon)$ (dotted lines), $O(\epsilon^2)$ (dashed lines), and simulation results of Ref. 7 (grey areas including error bars as published in Ref. 7).

accompanying the exponents $y_\tau(l)$ and $\omega(l)$ as functions of l around l^* .

For the opposite limit of large $b \gg 1$ we report that within the system sizes and parameter regions accessible to our present simulations we were unable to observe a similar behavior. Indeed, contrary to our prior expectations, which were based on the calculations of Ref. 15, the value of y_τ was even found to monotonically increase beyond the $O(\epsilon)$ result with decreasing $1/b \ll 1$, with no definite sign of any onset of an opposite trend. Likewise, we found no anomalies in the corresponding fixed-point values u_0^* and r_0^* . However, this does not necessarily imply a contradiction to the results of Ref. 15, as these were derived for case of a short-ranged standard Landau-Ginzburg model and confined to second-order perturbation theory. In contrast, the approximation made in our present simulations, namely, the projection to the (u_0, r_0) plane, is of a related but nevertheless somewhat different nature.

TABLE II. Comparison of numerical values for exponents y_τ, ω as obtained from analytical first- and second-order ϵ expansion calculations and the simulations of Ref. 7 to the values obtained from the present simulations at the critical b -value b^* .

σ	y_τ				ω			
	$O(\epsilon)$	$O(\epsilon^2)$	Ref. 7	y_τ^*	$O(\epsilon)$	$O(\epsilon^2)$	Ref. 7	ω^*
0.65	0.55	0.505113	0.503(10)	0.5021(48)	0.3	0.270075	0.20(4)	0.2032(66)
0.75	0.583333	0.47653	0.469(15)	0.475(2)	0.5	0.428798	0.24(5)	0.206(1)
0.85	0.616667	0.434522	0.413(14)	0.400(2)	0.7	0.578571	0.26(5)	0.402(1)

III. DISCUSSION

From the point of view of the abstract RG theory, the observed b -dependent anomalies described above may be regarded as extremely unusual, to say the least. On one hand, they are in conflict with one of the key properties of the renormalization group, namely, the *semigroup property*

$$\mathcal{R}(\cdot, b_1) \circ \mathcal{R}(\cdot, b_2) = \mathcal{R}(\cdot, b_1 \cdot b_2) = \mathcal{R}(\cdot, b_2) \circ \mathcal{R}(\cdot, b_1), \quad (8)$$

which implies that the fixed point $\mathbf{K}^*(b)$ of a RG transformation $\mathcal{R}(\cdot, b)$ should actually be b independent. Moreover, in contrast to what follows from this relation, the observed strong anomalies in the components of the eigenvectors of $\mathcal{M}_{ij}[\mathbf{K}^*(b)]$ around b^* imply that $\mathcal{M}_{ij}[\mathbf{K}^*(b_1)]$ and $\mathcal{M}_{ij}[\mathbf{K}^*(b_2)]$ [and thus the operations $\mathcal{R}^{\text{FMC}}(\cdot, b_1)$ and $\mathcal{R}^{\text{FMC}}(\cdot, b_2)$] do not commute. Equally disturbing, as is obvious from Fig. 1, the above b -dependent fixed points show a kind of “hysteresis loop,” contradicting our expectation that they should correspond to critical systems and consequently their locations should collapse onto a single critical line.

To understand this seemingly paradoxical behavior, let us go back to the very definition of the mapping $(u_0, r_0) \mapsto \mathcal{R}^{\text{FMC}}[(u_0, r_0), b]$ through steps (4) and (6). In fact, integrating out fluctuations between scales Λ and Λ/b_1 , it is well known that one not only reproduces ϕ^4 type of couplings but in principle may generate all other possible coupling types compatible with the given symmetry (i.e., sixth- and higher-order polynomial terms and higher-order dispersion terms) with certain nonzero prefactors (cf. Ref. 4). Our original justification for ignoring these terms was that in the sense of the RG they correspond to projections of irrelevant directions in \mathcal{K} . Indeed, fitting the resulting potential to a fourth-order polynomial actually amounts to projecting out these additional terms. With the coarse graining thus completed, we could move on to rescale the resulting coefficients $(\tilde{u}_0, \tilde{r}_0)$ according to Eq. (6) to obtain (u'_0, r'_0) . However, for the sake of the argument, let us postpone this step and instead perform a second coarse-graining step $(\tilde{u}_0, \tilde{r}_0) \mapsto (\tilde{u}_0, \tilde{r}_0)$ by now integrating out all modes between Λ/b_1 and $\Lambda/(b_1 \cdot b_2)$ for parameters $(\tilde{u}_0, \tilde{r}_0)$. Due to the intermediate discard of all generated higher irrelevant couplings as described, it is then perfectly clear that the result of this procedure is not equivalent to integrating out all modes between Λ and $\Lambda/(b_1 \cdot b_2)$ in a single step, where then irrelevant terms generated on the way from $\Lambda/(b_1 \cdot b_2)$ to Λ are automatically fully taken into account. It is this simple fact which is the reason for the violation of the semigroup property [Eq. (8)].

The inevitable conclusion is that the operation $\mathcal{R}^{\text{FMC}}(\cdot, b)$ does not resemble a meaningful RG transformation for arbitrary b due to the unavoidable implicit influence of higher

irrelevant couplings. It goes without saying that this fact also explains all other “paradoxical” observations we encountered for transformations at $b \neq b^*$. Nevertheless, for varying b our results strongly suggest that around $b=b^*$ the influence of irrelevant terms seems to be *minimized*, as the location $[u_0^*(b), r_0^*(b)]$ of the fixed point shows a stationary behavior around b^* and the exponents $y_\tau(b)$ and $\omega(b^*)$ are found to approach their true values at $b=b^*$. In other words, our simulations indicate that only the transformation $\mathcal{R}^{\text{FMC}}(\cdot, b^*)$ qualifies as a true RG transformation. Thus we propose to *define*

$$\mathcal{R}[(u_0, r_0), (b^*)^n] := \{\mathcal{R}^{\text{FMC}}[(u_0, r_0), b^*]\}^n, \quad n \in \mathbb{N} \quad (9)$$

for all integer powers of b^* , regarding $\mathcal{R}^{\text{FMC}}[(u_0, r_0), b^*]$ as our “basic” RG transformation. Moreover, since in our simulations b^* was observed to be close to 1 for the considered parameters, it is possible to even extend $\mathcal{R}[(u_0, r_0), b] := \mathcal{R}[(u_0, r_0), (b^*)^{\ln b / \ln b^*}]$ to arbitrary $b > 1$ by interpolation. In a forthcoming paper we intend to explore the exciting possibility of extrapolating the transformation to the limiting case $b \rightarrow 1$ and thus to study infinitesimal RG transformations by numerically calculating the Gell-Mann-Low functions $\beta_u(u_0, r_0)$ and $\beta_r(u_0, r_0)$.¹¹

In conclusion, our present investigation reveals that our original simulations of Wilson's method,¹ which had allowed to obtain a qualitative picture of the momentum RG flow in the (u_0, r_0) plane but had merely produced numerical results for the values of the critical exponents of order $O(\epsilon)$, is found to numerically represent a fairly accurate but still efficient algorithm for computing critical exponents once it is combined with an analysis of the b dependence of the discussed type. In fact, for the model investigated our results indicate that Wilson's scheme is “optimized” for a b -value b^* quite close to 1, which is highly welcome, as it implies that only a small fraction of all modes $\tilde{\mathfrak{f}}(\mathbf{k})$ enter in calculating the basic RG transformation $\mathcal{R}^{\text{FMC}}(\cdot, b^*)$, which makes for a computationally cheap algorithm at large system sizes.

Of course, to which extent our observations remain valid for other models and dimensions remains yet to be tested. To complete the picture, we intend to investigate the case $d > 1$ and possible applications to short-ranged models. The critical behavior of tethered or hexatic elastic membranes constitutes another highly interesting test ground for our approach.¹⁷

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*andreas.troester@univie.ac.at

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