Excitations of the $\nu=\frac{5}{2}$ fractional quantum Hall state and the generalized composite fermion picture

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We present a generalization of the composite Fermion picture for a multicomponent quantum Hall plasma which contains particle with different effective charges. The model predicts very well the low-lying states of a $\nu=\frac{5}{2}$ quantum Hall state found in numerical diagonalization.

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I. INTRODUCTION

The energy spectrum of a pure two-dimensional electron gas in a strong perpendicular magnetic field is completely determined by the Coulomb interactions, making it the paradigm for all "strongly interacting" systems (for which standard many-body perturbation theory is inapplicable). Jain's composite Fermion (CF) picture[1,2] has emerged as a comprehensive approach to understanding the most prominent incompressible states of the lowest Landau level (LL0). The original CF picture made use of a mean-field Chern-Simons (CS) gauge-field theory. The CFs are electrons that have an even number CS fluxes attached. The remarkable feature of this model is that it maps complicated fractional quantum Hall (FQH) states with filling factor ν into simpler and better understood states of CFs with effective filling factor ν^*. A good example is provided by the well-known series of Laughlin-Jain states[3,4] occurring at filling factors $\nu=\nu^*/(2p\nu\pm 1)$, where ν^* and p are integers. The mean-field approach deals with two scales of energy: the cyclotron energy $\hbar\omega_c$ and the Coulomb interaction scale e^2/λ where the cyclotron frequency is $\omega_c=(eB)/(mc)$ and the magnetic length is $\lambda=\sqrt{(\hbar c)/(eB)}$. In numerical studies and in some realistic systems, the cyclotron energy is much larger than the average Coulomb energy; thus the former scale is irrelevant. The low-lying energy states are completely determined by Coulomb interactions. Adiabatic addition of flux[5] introduces Laughlin correlations without the need of a new mean-field cyclotron energy scale. Generalization of the CF picture to a plasma containing more than one type of particle was first introduced by Wójs *et al.*[6] It was used to predict the low-lying band of states of systems containing both electrons and valence-band holes. In this case, the constituents (electrons and negatively charged excitons) have the same charge. In this paper, we propose another generalization of CF model to include multicomponent plasmas in which particles with different charges are involved. We use this model to suggest an explanation of the low-lying bands of states of the $\nu=\frac{5}{2}$ FQH states.

The $\nu=\frac{5}{2}$ state has generated considerable recent interest as the result of the suggestion that its elementary excitations can be non-Abelian quasiparticles (QPs). Such excitations occur when the three-particle pseudopotential $V(\mathcal{R}_3)$ of Greiter *et al.*,[7] which forbids the formation of compact three particle clusters, is used to describe the interactions of electrons in the first-excited Landau level (LL1). Our simple generalized CF picture is used to interpret the ground state and low-lying bands of excitations of electrons in LL1 interacting through the standard Coulomb pseudopotential appropriate for such electrons confined to a very narrow quantum well. New experimental results of Choi *et al.*[8] have been used to interpreted incompressible states in terms of formation of pairs[9–11] (with $\ell_P=2\ell-1$) or large clusters at filling factors $\nu_1=\nu-2$ of LL1 with $2/3\geq\nu_1\geq 1/3$. The most prominent IQL state occurs at $2\ell=2N-3$ (or at $2\ell=2N+1$). Here we present numerical results for the energy spectrum for $N=8$, 10, 12, and 14. We demonstrate that in addition to the $L=0$ ground states, the two lowest bands of excitations can be interpreted using our simple generalization of Jain's CF picture to a two-component plasma of Fermion pairs (FPs) and unpaired electrons.

The paper is organized as follows. The next section is dedicated to the solution of a two-particle system in a magnetic field. Section III discusses CS gauge transformation and adiabatic addition of CS flux. In Sec. IV, we present the generalized CF model and its application to the $\nu=\frac{5}{2}$ FQH state. We draw conclusions in the last section.

II. TWO CHARGED PARTICLES IN A MAGNETIC FIELD

A two-particle system is the simplest system that can be used to understand the physics of introducing the CS flux. The two particles have masses m_1 and m_2 and charges q_1 and q_2 and they are moving in the x-y plane. A dc uniform magnetic field $\vec{B}=B\hat{z}$ is applied perpendicular to the plane. We work in the symmetric gauge where the vector potential is $\vec{A}=\frac{1}{2}Br\hat{\phi}$, where $\hat{\phi}$ is the unit vector in the direction of increasing the angular coordinate ϕ. The Hamiltonian contains the kinetic terms and the Coulomb term

$$\hat{H}=\frac{1}{2m_1}\left[\vec{p}_1-\frac{q_1}{c}\vec{A}(\vec{r}_1)\right]^2+\frac{1}{2m_2}\left[\vec{p}_2-\frac{q_2}{c}\vec{A}(\vec{r}_2)\right]^2+\frac{q_1q_2}{|\vec{r}_1-\vec{r}_2|}. \tag{1}$$

The Hamiltonian can be separated into the center of mass (CM), relative (R) part, and a part describing the interaction between the CM and R motions (I)

$$H_{\rm CM}=\frac{1}{2M}\left[\vec{P}-\frac{Q}{c}\vec{A}(\vec{R})\right]^2+\frac{\vec{q}^2\vec{A}^2(\vec{R})}{2\mu c^2}, \tag{2}$$

$$H_R = \frac{1}{2\mu}\left[\vec{p} - \frac{q}{c}\vec{A}(\vec{r})\right]^2 + \frac{q_1 q_2}{r} + \frac{\tilde{q}^2\vec{A}^2(\vec{r})}{2Mc^2}, \qquad (3)$$

$$H_I = -\frac{\tilde{q}}{\mu c}[\vec{A}(\vec{R})\vec{p} + \vec{P}\vec{A}(\vec{r})] + \frac{\tilde{q}\bar{q}}{2\mu}\vec{A}(\vec{r})\vec{A}(\vec{R}). \qquad (4)$$

Here, relative coordinate and momentum are $\vec{r}=\vec{r}_1-\vec{r}_2$ and $\vec{p}=\frac{1}{2}(\vec{p}_1-\vec{p}_2)$, respectively. The reduced mass and charge are $\mu=(m_1m_2)/(m_1+m_2)$ and $q=(q_1m_2^2+q_2m_1^2)/(m_1+m_2)^2$. $R=\frac{1}{2}(\vec{r}_1+\vec{r}_2)$ and $\vec{P}=\vec{p}_1+\vec{p}_2$ represent the CM coordinate and momentum. Also, $\bar{q}=(q_1m_2+q_1m_2)/(m_1+m_2)$ and $\tilde{q}=(q_1m_2-q_2m_1)/(m_1+m_2)$. Throughout this paper, we consider that the particles (electrons, pairs of electrons) have the same charge-to-mass ratio (specific charge) $q_1/m_1=q_2/m_2$ and as a consequence, the $\tilde{q}=0$ and the CM and R motions decouple.

In the absence of electron-electron interaction, both CM and R motions are described by degenerate Landau levels. While the Coulomb interaction lifts the degeneracy, it does not alter fundamentally the shape of the wave function at least for small values of $\frac{q^2\mu}{\hbar}\sqrt{\frac{\hbar c}{qB}}$.

III. CHERN-SIMON FLUXES AND THEIR ADIABATIC ADDITION

The CF picture[1,2] results from the introduction of C-S fluxes. The vector potential $\vec{a}(\vec{r})$ that produces a flux of $\Phi=\alpha\Phi_0=\alpha hc/e$ is

$$\vec{a}(\vec{r}) = \Phi\int d^2r'\frac{\vec{z}\times(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^2}\rho(\vec{r}'), \qquad (5)$$

where $\rho(\vec{r})=\Psi^\dagger(\vec{r})\Psi(\vec{r})$ is the density operator. The magnetic field resulting from the CS vector potential is given by $\vec{b}(\vec{r})=\Phi\Sigma_i\delta(\vec{r}-\vec{r}_i)\hat{z}$, where $\vec{r}_i$ is the position of the ith electron.

A charged particle is moving on a circular path in x-y having the wave function $\Psi(\vec{r})=e^{im\phi}u_m(r)$. When the flux tube is added via a gauge transformation, the eigenfunction is changed to

$$\bar{\Psi}(\vec{r}) = \exp\left[-\frac{iq}{\hbar c}\int \vec{a}(\vec{r})d\vec{r}\right]\Psi(\vec{r}). \qquad (6)$$

Considering $\vec{a}(\vec{r})=\hat{\phi}\Phi/(2\pi r)$, the new wave function is $\bar{\Psi}(\vec{r})=\exp[i(m-\alpha)\phi]u_m(r)$. The CS flux does not influence the radial-wave function, but it introduces a phase of $-i\alpha\phi$. For relative motion of the pair of charges, an odd integer α generates the famous change of statistics from bosons to fermions and vice versa. Introducing the CS flux makes the many-body Schrödinger equation extremely complicated. A simplification occurs when the mean-field approximation is made by replacing the density operator by its ground-state average density $n=\langle\Psi^\dagger\Psi\rangle$ in the expression of the CS vector potential. The particles move in an effective magnetic field $B^*=B+\alpha\phi_0 n$ and the mean-field Coulomb interaction vanishes. The mean-field approximation introduces an effective cyclotron energy scale $\hbar\omega_c^*=(\hbar qB^*)/(\mu c)$, which is irrelevant since $q^2/\lambda_r \ll \hbar\omega_c^*$.

Instead of making a gauge transformation, we can introduce the CS fluxes adiabatically.[5] We start with initial electron pair state and slowly increase the value of CS flux. Such technique leaves the phase of the wave function unchanged, but the radial-wave function is modified $u_m(r)\to u_{m+\alpha}(r)$ (in the absence of Coulomb interaction).

In the case of the system made of two particles with different charges, the fluxes attached to the two particles are increased from zero to $\alpha\phi_0$ and $\beta\phi_0$, respectively. The Hamiltonian is

$$\hat{H} = \frac{1}{2m_1}\left\{\vec{p}_1 - \frac{q_1}{c}[\vec{A}(\vec{r}_1) + \alpha\vec{a}(\vec{r}_1-\vec{r}_2)]\right\}^2 + \frac{1}{2m_2}\left\{\vec{p}_2 - \frac{q_2}{c}[\vec{A}(\vec{r}_2) - \beta\vec{a}(\vec{r}_1-\vec{r}_2)]\right\}^2 + \frac{q_1q_2}{|\vec{r}_1-\vec{r}_2|}, \qquad (7)$$

where $a(\vec{r})=\hat{\phi}(\phi_0/2\pi r)$. We suppose again that the particles have the same specific charges. The Hamiltonian can be decomposed into a CM part, an R part, and interaction between the CM and R motions

$$H_{\rm CM} = \frac{1}{2M}\left[\vec{P} - \frac{Q}{c}\vec{A}(\vec{R})\right]^2, \qquad (8)$$

$$H_R = \frac{1}{2\mu}\left\{\vec{p} - \frac{q}{c}[\vec{A}(\vec{r}) + (\alpha+\beta)\vec{a}(\vec{r})]\right\}^2 + \frac{q_1q_2}{r} + \frac{q_\alpha^2\vec{a}^2(\vec{r})}{2Mc^2}, \qquad (9)$$

$$H_I = -\frac{q_\alpha}{Mc}\vec{P}\vec{a}(\vec{r}) + \frac{q_\alpha q\vec{A}(\vec{R})\vec{a}(\vec{r})}{\mu c^2}, \qquad (10)$$

where $q_\alpha=q_1\alpha-q_2\beta$.

The CM and relative motions decouple completely if $q_\alpha=0$, i.e., $q_1\alpha=q_2\beta$. The result is independent of the Landau level in which motion takes place. This condition can be used to develop the generalized CF picture for a plasma containing particles with different charges. Section IV is a direct application of this result to introduction of correlation between unpaired electrons and FPs.

The relative motion Hamiltonian is very similar to Eq. (3). The effective magnetic field is modified by the CS flux. The new wave function will not change its phase, but the orbital part is modified. Our description of correlations treats the excitations as Fermions. They can be transformed into anyons by making a gauge transformation to eliminate the CS flux that was attached adiabatically to produce the desired correlations.[5]

If we think of electrons in the planar geometry, the CM and relative coordinates separate, although the CM and relative motions can belong to different Landau levels.[12] The condition for canceling the part of the Hamiltonian that describes the interaction between CM and relative motions is independent of the Landau levels to which they belong.

IV. GENERALIZED COMPOSITE FERMION PICTURE AND THE LOW-LYING EXCITATIONS OF THE $\nu=\frac{5}{2}$ FRACTIONAL QUANTUM HALL STATE

We introduce a generalized CF picture for a multicomponent plasma. We think of every species of charged particle present in the plasma as a having different color (red, blue, etc.). We attach to each particle a flux tube carrying an integral number of different flux quanta of different colors. Each charge will see only the flux tubes having the same color and no particles will see its own flux. For example, a "red" charge will see the red flux tubes attached to all other particles. The correlations between the particles having the same color are identical to the one introduced in the original CF picture. The generalized CF model also introduces the correlations between different types of particles. We need to obtain the same correlations when "blue" charges interacts with blue fluxes on red charges as when red charges interact with red fluxes attached to blue charges. Adding pq_{red}/e blue fluxes to the a red charge causes the blue charge to have exactly the same blue-red correlations as adding pq_{blue}/e red fluxes to blue charge. The CS charge times the CS flux must be the same.

In this section, we use the Haldane's spherical geometry,[13] which maps the infinite planar surface onto a spherical one, magnetic field being produced by a monopole placed in the center of the sphere. The monopole strength is $2Q\Phi_0$ (where $2Q$ is an integer) gives rise to a magnetic field $B=2Q\Phi_0/(4\pi R^2)$ which is perpendicular to the spherical surface. The single-particle eigenstates are called monopole harmonics and denoted by $|Q,\ell,m\rangle$.[14,15] They are eigenfunctions of the square of the angular-momentum operator $\hat{\ell}^2$ and its z projection $\hat{\ell}_z$ with eigenvalues $\ell(\ell+1)$ and m, respectively, and $|m|\leq\ell$. Landau levels are replaced by angular-momentum shells $\ell=Q+n$, where n plays the role of the LL index. The energy of the state $|Q,\ell,m\rangle$ is $E_\ell=(\hbar\omega_C/2Q)[\ell(\ell+1)-Q^2]$. In order to obtain the energy spectrum of an N-particle system, the Coulomb interaction e^2/r_{ij} is diagonalized in the noninteracting basis set. In this geometry, the effective monopole strength seen by CFs of "color" a will be

$$2Q_a^* = 2Q - \sum_b (m_{ab} - \delta_{ab})(N_b - \delta_{ab}), \tag{11}$$

with $q_a m_{ab}=q_b m_{ba}$.

This model is applied to understand the lowest bands of states in the case of $\nu=\frac{5}{2}$ state. It is clear that the correlations and the elementary excitations are better understood for LL0 than for LL1 and higher Landau levels. The Coulomb interaction is described using the pseudopotentials $V(\mathcal{R})$, where $\mathcal{R}$ is the relative angular momentum $\mathcal{R}=2\ell-L'$, L' being the pair angular momentum. It is well known that Laughlin correlations (the avoidance of pair states with small values of $\mathcal{R}$) occur only when the pseudopotential $V_n(\mathcal{R})$ describing the interaction energy of a electron pair with angular momentum $L'=2\ell-\mathcal{R}$ in LLn is "superharmonic," i.e., rises with increasing L' faster than $L'(L'+1)$ as the avoided value of L' is approached.[16–18] A pseudopotential that is not superharmonic does not induce Laughlin correlations[19] and instead results in formation of pairs. In LL0, the pseudopotential is superharmonic for all values of $\mathcal{R}$. The CF picture applied for electrons in LL0 introduces $\ell^*=|\ell-p(N-1)|$, where p is an integer and explains that the lowest band of states will contain the minimum number of QP excitations required by the values of N and 2ℓ.[20] The quasiholes (QHs) reside in the angular-momentum shell $\ell_{\text{QH}}=\ell^*$; the quasielectrons (QEs) are in the shell $\ell_{\text{QE}}=\ell_{\text{QH}}+1$.

In LL1, the pseudopotential is only "weakly" superharmonic[21,22] for $R=1$ and as a consequence, it does not support Laughlin correlations at $1/3\leq\nu_1\leq 2/3$.[23] The correlations can be described in terms of the formation of $N_P=N/2$ pairs[23] when N is even. The electrons tend to form pairs with $\ell_P=2\ell-1$. To avoid violating the exclusion principle, we cannot allow FPs to be too close to one another. We do this by restricting the angular momentum of two pairs to values less than or equal to[24–27]

$$2\ell_{\text{FP}} = 2\ell_P - 3(N_P - 1), \tag{12}$$

implying that the FP filling factor satisfies the relation $\nu_{\text{FP}}^{-1}=4\nu_1^{-1}-3$. The factor of 4 is a reflection of N_P being half of N and the LL degeneracy g_P of the pairs being twice g for electrons. Correlations are introduced through a standard CF transformation applied to FPs

$$2\ell_{\text{FP}}^* = 2\ell_{\text{FP}} - 2p_P(N_P - 1). \tag{13}$$

Selecting $2p_P=4$ results in $2\ell_{\text{FP}}^*+1=N_P$ and pairs forming a $L=0$ incompressible quantum liquid (IQL) ground state. This occurs if the number of electrons and the angular-momentum shell satisfy the relation $2\ell=2N-3$ (or the electron-hole conjugate $2\ell=2N+1$). These IQL states are marked by circles in Fig. 1.

In order to understand the lowest bands of states, we will assume two types of elementary excitations. The first type will consist of an empty FP state in the lowest FP level [a quasihole FP (QHFP) with angular momentum ℓ_{FP}^*] plus one filled FP state in the first-excited FP level [a quasiparticle FP (QPFP) with angular momentum ℓ_{FP}^*+1]. Since $2\ell_{\text{FP}}^*=(N_P-1)$, this process gives rise to a "magnetoroton" state of a QHFP with $\ell_{\text{QHFP}}=(N_P-1)/2$ and a QPFP with $\ell_{\text{QPFP}}=(N_P+1)/2$. The resulting "magnetoroton" band has $L=1\oplus 2\oplus\cdots\oplus N_P$. This band is marked by squares in Fig. 1. Up to an overall constant, this band represents the interaction pseudopotential of a QHFP and a QPFP as a function of the total angular momentum. Unfortunately, the width of the band is not small compared to the minimum gap required to produce a magnetoroton of FPs so it is not as useful as the QE and QH pseudopotentials obtained from Laughlin correlated states in LL0 which contain a pair of QEs or a pair of QHs.[20]

Another possible band of low-lying excitations could result from breaking one of the CF pairs into two constituent unpaired electrons, each with charge $-e$ and angular momentum ℓ. We propose to treat the system of $N_P'=N_P-1$ FPs and the $N_e=2$ unpaired electrons by our generalized CF picture. Let the $2\ell_{\text{FP}}'=2\ell_P-3(N_P'-1)$. Then the following equations describe correlations of the N_P' FPs and $N_e=2$ unpaired electrons:

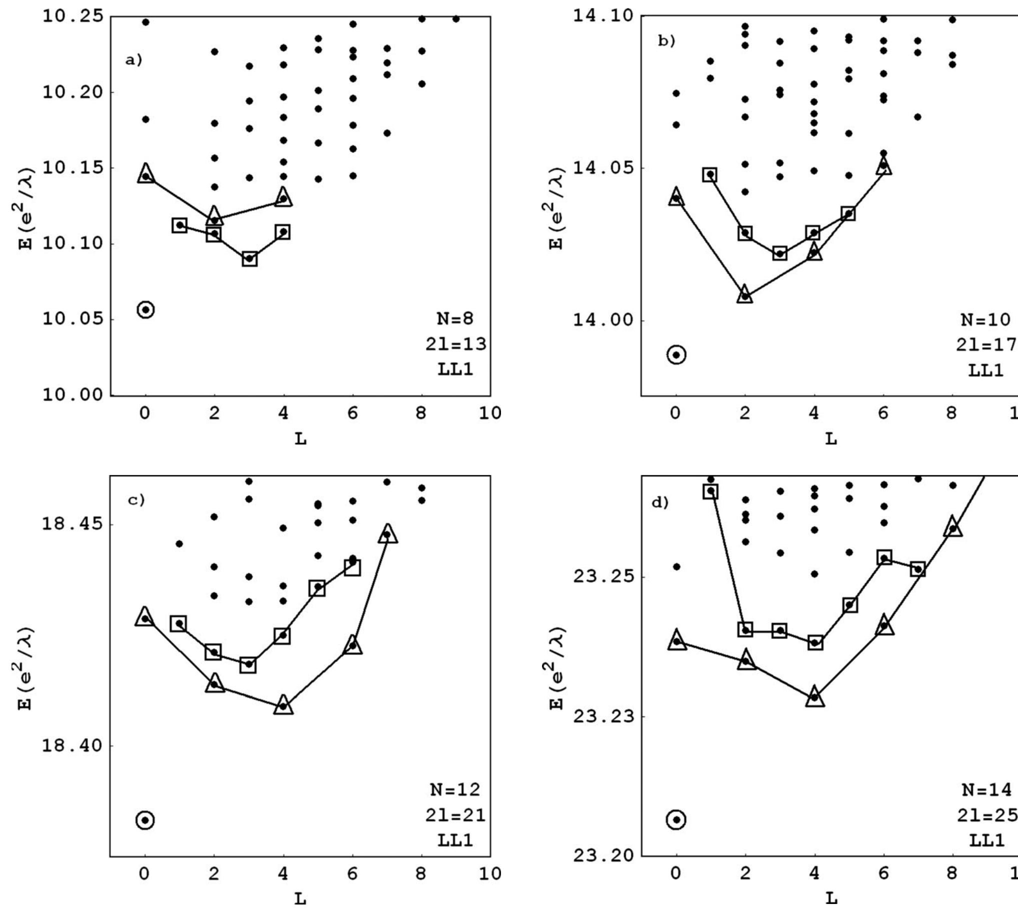

FIG. 1. Energy spectra of electrons in LL1 for $2\ell=2N-3$ generating incompressible ground states. Circles represent the IQL states, squares represent the pair magnetoroton states, and triangles represent the broken-pair states.

$$2\ell_{\mathrm{FP}}'^{*}=2\ell_{\mathrm{FP}}'-2p_P(N_P'-1)-2\gamma N_e, \tag{14}$$

$$2\ell_e^{*}=2\ell-2p_e(N_e-1)-\gamma N_P', \tag{15}$$

where γ is an integer representing the number of flux quanta attached to each unpaired electron in the CS transformation that is sensed by the Fermion pair. The fact that 2γ fluxes sensed by unpaired electrons must be attached to each FP and γ fluxes sensed by each FP sensed by each unpaired electron is a direct result of Sec. III. This gives the correlations that produce the $L=0$ ground state and the low-lying excitations.

Equation (14) tells us that the effective angular momentum of one FP is decreased from ℓ_{FP} by p_P times the number of other FPs and by γ times the number of unpaired electrons. Equation (15) tells us that the effective angular momentum of one unpaired electron is decreased by p_e times the number of other unpaired electrons and by $\gamma/2$ times the number of CF pairs. Note that $2p_{\mathrm{FP}}$ and $2p_e$ are even and that γ can be odd or even. Equations (14) and (15) define the generalized CF picture in which different types of Fermions, distinguishable from one another, experience correlations which leave them as Fermions (since $2p$ is even) and give the same correlations between members of two different species since the product of CS charge and the CS flux added is the same (i.e., $-e2\gamma=-2e\gamma$).

When using $p_P=2$, $p_e=1$, $\gamma=2$, $N_P'=N_P-1$, and $N_e=2$, and $2\ell_{\mathrm{FP}}'=2\ell_P-3(N_P'-1)$, Eq. (14) gives the FP's effective angular momentum as $\ell_{\mathrm{FP}}'^{*}=N_P-2$ and the FPs will form an IQL $L=0$ state. The two electrons become CFs occupying the level $2\ell_e^{*}=N-3$. They generate the "broken pair" band of states at $L=0\oplus 2\oplus\cdots\oplus N-4$. This band of states is represented by triangles in Fig. 1. In the planar geometry, CM and the relative motion in the absence of electron-electron interaction are constants. The calculated energy spectra give an $L=0$ incompressible liquid ground state plus many excited states. The excited states result from the electron-electron interaction appropriate for a pair of electron in LL1 and they depend on the total angular momentum L. The excitation energies relative to the $L=0$ ground state produce the bands of states shown in Fig. 1. Some of these states were discussed by Greiter *et al.*[7,28] but not in terms of a generalized CF picture capable of predicting the allowed values of L in the lowest band of energy levels.

We have evaluated similar energy spectra for $2\ell=2N-3+j$, for $j=1,2,\ldots$, using the Coulomb potential appropriate for LL1 and zero well width. Degenerate QH states with the same angular momentum L (a requirement for non-Abelian statistics) are not observed.

V. CONCLUSIONS

We developed a generalized CF model in which a plasma contains particles with different charges. The adiabatic introduction of CS fluxes was used. It is worth noting that for the generalized CF picture, the correlations between a pair of particles can be thought of as resulting from adiabatic addition of fictitious CS flux quanta to one particle that is sensed by fictitious charge on the other. We applied our model to the system with $\nu=\frac{5}{2}$ filling factor. Our interpretation is an attempt to understand some of the low-lying excitations of the $\nu_1=1/2$ state in a simple CF-type picture. The numerical data confirm our model. Our results do not contain the degenerate states required by non-Abelian statistics.

[1] J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989).
[2] J. K. Jain, Phys. Rev. B **41**, 7653 (1990).
[3] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[4] J. K. Jain, Phys. Rev. B **40**, 8079 (1989).
[5] J. J. Quinn and J. J. Quinn, Phys. Rev. B **68**, 153310 (2003).
[6] A. Wójs, I. Szlufarska, K.-S. Yi, and J. J. Quinn, Phys. Rev. B **60**, R11273 (1999).
[7] M. Greiter, X.-G. Wen, and F. Wilczek, Nucl. Phys. B **374**, 567 (1992).
[8] H. C. Choi, W. Kang, S. Das Sarma, L. N. Pfeiffer, and K. W. West, Phys. Rev. B **77**, 081301(R) (2008).
[9] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[10] V. Scarola, K. Park, and J. K. Jain, Nature (London) **406**, 863 (2000).
[11] E. H. Rezayi and F. D. M. Haldane, Phys. Rev. Lett. **84**, 4685 (2000).
[12] N. d'Ambrumenil and A. M. Reynolds, J. Phys. C **21**, 119 (1988).
[13] F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
[14] T. T. Wu and C. N. Yang, Nucl. Phys. B **107**, 365 (1976).
[15] T. T. Wu and C. N. Yang, Phys. Rev. D **16**, 1018 (1977).
[16] A. Wójs and J. J. Quinn, Solid State Commun. **108**, 493 (1998).
[17] A. Wójs and J. J. Quinn, Solid State Commun. **110**, 45 (1999).
[18] J. J. Quinn and A. Wójs, Physica E (Amsterdam) **6**, 1 (2000).
[19] P. Sitko, K.-S. Yi, and J. J. Quinn, Phys. Rev. B **56**, 12417 (1997).
[20] X. M. Chen and J. J. Quinn, Phys. Rev. B **47**, 3999 (1993).
[21] A. Wójs, Phys. Rev. B **63**, 125312 (2001).
[22] A. Wójs and J. J. Quinn, Phys. Rev. B **71**, 045324 (2005).
[23] G. E. Simion and J. J. Quinn, Physica E (Amsterdam) **41**, 1 (2008).
[24] J. J. Quinn, A. Wójs, and K.-S. Yi, Phys. Lett. A **318**, 152 (2003).
[25] A. Wójs, D. Wodziński, and J. J. Quinn, Phys. Rev. B **71**, 245331 (2005).
[26] A. Wójs, D. Wodziński, and J. J. Quinn, Phys. Rev. B **74**, 035315 (2006).
[27] J. J. Quinn and J. J. Quinn, Solid State Commun. **140**, 52 (2006).
[28] M. Greiter, X.-G. Wen, and F. Wilczek, Phys. Rev. Lett. **66**, 3205 (1991).