Parametric pumping and kinetics of magnons in dipolar ferromagnets

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The time evolution of magnons subject to a time-dependent microwave field is usually described within the so-called "S-theory," where kinetic equations for the distribution function are obtained within the time-dependent Hartree-Fock approximation. To explain the recent observation of "Bose-Einstein condensation of magnons" in an external microwave field [Demokritov *et al.*, Nature (London) **443**, 430 (2006)], we extend the S-theory to include the Gross-Pitaevskii equation for the time-dependent expectation values of the magnon creation and annihilation operators. We explicitly solve the resulting coupled equations within a simple approximation where only a single condensed mode is retained. We also re-examine the usual derivation of an effective boson model from a realistic spin model for yttrium-iron garnet films and argue that in the parallel pumping geometry (where both the static and the time-dependent magnetic field are parallel to the macroscopic magnetization) the time-dependent Zeemann energy cannot give rise to magnon condensation.

PACS number(s): 75.30.Ds, 76.20.+q, 03.75.Kk

I. INTRODUCTION

When ordered magnets are exposed to microwave radiation of sufficiently high power, one typically observes an exponential growth of the population of certain groups of spin-wave modes during some intermediate time interval. This is an example for a general phenomenon, which is usually referred to as *parametric resonance*. A particularly suitable system for observing parametric resonance are yttrium-iron garnet (YIG) crystals, because the spin waves in this system have a very low damping.[1] Early microscopic theories explaining parametric resonance in magnetic insulators have been developed by Suhl[2] and Schlömann *et al.*[3] In the 1970s Zakharov *et al.*[4] developed a comprehensive kinetic theory of parametric resonance in magnon gases which is sometimes called "S-theory." In this approach kinetic equations for the time-dependent distribution functions $n_k(t) = \langle a_k^\dagger(t) a_k(t) \rangle$ and $p_k(t) = \langle a_{-k}(t) a_k(t) \rangle$ are derived within the self-consistent time-dependent Hartree-Fock approximation. Here $a_k(t)$ and $a_k^\dagger(t)$ are the annihilation and creation operators of magnons with momentum k in the Heisenberg picture. Subsequently the nonlinear kinetic equations of the S-theory and extensions thereof have been studied by many authors.[5–10]

Quite recently Demokritov and co-workers[11,12] observed a new coherence effect of magnons in YIG under the influence of an external microwave field, which they interpreted as Bose-Einstein condensation (BEC) of magnons at room temperature. A similar phenomenon has been observed in superfluid ^{3}He, where NMR pumping can cause the magnetization to precess phase coherently.[13] The emergence of this coherent state can also be viewed as magnon BEC.[14,15] Whether or not the experiments by Demokritov and co-workers[11,12] can be considered to be an analog of BEC in atomic Bose gases (which nowadays is routinely realized using ultracold atoms in an optical trap) has been discussed controversially in the literature.[16,17] We argue below that the coherent state generated in these experiments[11,12] should perhaps not be called a Bose-Einstein condensate, because the condensation is not accompanied by spontaneous symmetry breaking in this case; instead, the microwave field gives rise to a term in the Hamiltonian, which explicitly breaks the $U(1)$ symmetry of the magnon Hamiltonian.

Unfortunately, the conventional S-theory is insufficient to describe the experimental situation, because the coherent magnon state generated in the experiments is characterized by finite expectation values of the magnon annihilation and creation operators $a_k(t)$ and $a_k^\dagger(t)$ for certain special values of k. In the condensed phase, the kinetic equations for the pair correlators $n_k(t)$ and $p_k(t)$ should therefore be augmented by equations of motion for the expectation values $\langle a_k(t) \rangle$ and $\langle a_k^\dagger(t) \rangle$. Recall that in the theory of the interacting Bose gas the corresponding equation of motion for the order parameter is called Gross-Pitaevskii equation;[18] this equation is missing in the conventional S-theory, which therefore does not completely describe the coherent magnon state in the regime of strong pumping. In this work we shall outline an extension of S-theory, which includes the order parameter dynamics on equal footing with the kinetic equations for the distribution functions. Since we would like to clarify conceptual points rather than performing explicit quantitative calculations, we shall derive our extended S-theory within the framework of a simple toy model, which we motivate in the following section.

II. TOY MODEL FOR PARAMETRIC RESONANCE IN YIG

In order to understand a complex physical phenomenon, it is sometimes useful to study a simplified "toy model," which still contains some essential features of the phenomenon of interest. For our purpose, it is sufficient to consider a single anharmonic oscillator with an additional time-dependent term describing the creation and annihilation of pairs of particles. The Hamiltonian is

$$\hat{H}(t) = \epsilon_0 a^\dagger a + \frac{\gamma_0}{2} e^{-i\omega_0 t} a^\dagger a^\dagger + \frac{\gamma_0^*}{2} e^{i\omega_0 t} aa + \frac{u}{2} a^\dagger a^\dagger aa. \quad (1)$$

Here a and $a^\dagger$ are bosonic annihilation and creation operators, $\epsilon_0 > 0$ is some energy scale, and $u > 0$ is the interaction

energy. The second and third terms on the right-hand side of Eq. (1) describe the effect of an external microwave field, which oscillates with frequency $\omega_0 > 0$ and couples with strength γ_0 to the magnon gas. Below we shall show that this model contains the essential physics of parametric resonance and BEC of magnons; in particular, in the regime of strong pumping $|\gamma_0| > |\epsilon_0 - \omega_0/2|$ the model has a stationary non-equilibrium state, which corresponds to the coherent magnon state observed in the experiments by Demokritov and co-workers.[11,12]

Our toy model Eq. (1) involves only a single boson operator representing the magnon at the minimum of the dispersion, which is expected to condense. Of course, for experimentally relevant macroscopic samples of YIG a more realistic model should describe infinitely many magnon operators a_k labeled by crystal momentum k, so that the following bosonic "resonance Hamiltonian" should give a better description of the experimental situation,

$$\hat{H}_{\mathrm{res}}(t) = \sum_k \epsilon_k a_k^\dagger a_k + \frac{1}{2}\sum_k \left[\gamma_k e^{-i\omega_0 t} a_k^\dagger a_{-k}^\dagger + \gamma_k^* e^{i\omega_0 t} a_{-k} a_k\right]$$
$$+ \frac{1}{2}\sum_{k,k',q} u(k,k',q) a_{k+q}^\dagger a_{k'-q}^\dagger a_{k'} a_k. \quad (2)$$

If we assume that the $k=0$ boson condenses and retain only this degree of freedom on the right-hand side of Eq. (2), we arrive at our toy model Eq. (1). In the theory of superfluidity a similar reduced description involving only the order parameter is provided by the Gross-Pitaevskii equation.[18] Of course, the minimum of the dispersion in experimentally relevant samples of YIG occurs at certain nonzero wave vectors $\pm k_*$, so that it would be more accurate to retain the two modes a_{k_*} and a_{-k_*} and their mutual interactions in Eq. (2). Moreover, the fact that in the experiments[11,12] the wave vectors of the condensed magnons are different from the wave vectors of the magnons, which are initially generated by microwave pumping cannot be described within the framework of our toy model. Nevertheless, below we shall show that our simple model allows us to understand some conceptual points related to the nature of the coherent state observed in the experiments.[11,12]

The bosonic resonance Hamiltonian (2) has been the starting point of several theoretical investigations of parametric resonance in magnon gases.[4–10] This model is believed to be a realistic model for YIG in the parallel pumping geometry, where the static and the time-dependent components of the external magnetic fields are both parallel to the direction of the macroscopic magnetization. In the appendix we shall critically re-examine the usual derivation of Eq. (2) from an effective spin Hamiltonian for YIG and show that in spin language the time-dependent resonance term in the second line of Eq. (2) involves also the combinations $\cos(\omega_0 t)[S_i^x S_i^x - S_i^y S_i^y]$ and $\sin(\omega_0 t)[S_i^x S_i^y + S_i^y S_i^x]$, where S_i^α are the components of the spin operators at lattice site i. Terms of this type cannot be related to the Zeemann energy associated with a time-dependent magnetic field parallel to the magnetization. This is obvious for a ferromagnet with only exchange interactions, because in this case the magnon operators a_k and $a_k^\dagger$ can be identified with the Fourier components of the Holstein-Primakoff[19] bosons a_i and $a_i^\dagger$, which in turn can be related to the usual spin-ladder operators S_i^+ and S_i^-; to leading order for large spin S,

$$S_i^+ \approx \sqrt{2S} a_i, \quad S_i^- \approx \sqrt{2S} a_i^\dagger. \quad (3)$$

Note, however, that the spin Hilbert space has only $2S+1$ states per site, whereas the bosonic Fock space associated with the canonical boson operators a_i and $a_i^\dagger$ is infinite dimensional; the identification of magnons with canonical bosons is therefore only approximate. For a description of coherence phenomena involving large occupancies of magnon states one should therefore keep in mind that there is a constraint on the magnon Hilbert space. Assuming for simplicity that the parameter $\gamma_k = \gamma$ in Eq. (2) is real and independent of k, the second term on the right-hand side of Eq. (2) can be written as

$$\frac{\gamma}{2}\sum_k \left[e^{-i\omega_0 t} a_k^\dagger a_{-k}^\dagger + e^{i\omega_0 t} a_{-k} a_k\right],$$

$$\approx \frac{\gamma}{4S}\sum_i \left[e^{-i\omega_0 t} S_i^- S_i^- + e^{i\omega_0 t} S_i^+ S_i^+\right],$$

$$= \frac{\gamma}{2S}\sum_i \{\cos(\omega_0 t)[S_i^x S_i^x - S_i^y S_i^y]$$
$$- \sin(\omega_0 t)[S_i^x S_i^y + S_i^y S_i^x]\}. \quad (4)$$

In spin language, the pumping term in Eq. (2) therefore corresponds to a time-dependent single-ion anisotropy whose easy axis rotates with frequency ω_0 around the z axis. Of course, the magnon operators for YIG are not directly related to Holstein-Primakoff bosons because an additional Bogoliubov transformation is necessary to diagonalize the quadratic part of the boson Hamiltonian. Nevertheless, we show in the appendix that also in this case the pumping term in the effective boson Hamiltonian (2) can be related to a rotating easy-axis anisotropy of the above type.

III. KINETIC EQUATIONS

To discuss the time evolution of our toy model defined in Eq. (1) it is convenient to remove the explicit time dependence from the Hamiltonian $\hat{H}(t)$ by performing a canonical transformation to the "rotating reference frame,"

$$\tilde{a} = e^{i/2\omega_0 t} a = \hat{U}_0(t) a \hat{U}_0^\dagger(t), \quad (5a)$$

$$\tilde{a}^\dagger = e^{-i/2\omega_0 t} a^\dagger = \hat{U}_0(t) a^\dagger \hat{U}_0^\dagger(t), \quad (5b)$$

where $\hat{U}_0(t) = e^{-i/2\omega_0 t a^\dagger a}$. The new operators satisfy the Heisenberg equations of motion

$$i\partial_t \tilde{a} = [\tilde{a}, \tilde{H}], \quad i\partial_t \tilde{a}^\dagger = [\tilde{a}^\dagger, \tilde{H}], \quad (6)$$

where the rotated Hamiltonian $\tilde{H}$ of our toy model does not depend explicitly on time,

$$\widetilde{H} = \widetilde{\epsilon}_0 \widetilde{a}^\dagger \widetilde{a} + \frac{\gamma_0}{2} \widetilde{a}^\dagger \widetilde{a}^\dagger + \frac{\gamma_0^*}{2} \widetilde{a}\widetilde{a} + \frac{u}{2} \widetilde{a}^\dagger \widetilde{a}^\dagger \widetilde{a}\widetilde{a}. \tag{7}$$

Here we have introduced the shifted oscillator energy

$$\widetilde{\epsilon}_0 = \epsilon_0 - \frac{\omega_0}{2}. \tag{8}$$

To relate correlation functions in the original model to those in the rotating frame, we simply have to insert the appropriate phase factors. For example, in S-theory one usually considers the normal distribution function,

$$n(t) = \langle a^\dagger(t) a(t) \rangle = \langle \widetilde{a}^\dagger(t) \widetilde{a}(t) \rangle, \tag{9}$$

and its anomalous counter part,

$$p(t) = \langle a(t) a(t) \rangle = e^{-i\omega_0 t} \langle \widetilde{a}(t) \widetilde{a}(t) \rangle \equiv e^{-i\omega_0 t} \widetilde{p}(t), \tag{10}$$

where expectation values are with respect to some density matrix $\hat{\rho}(t_0)$ specified at time t_0,

$$\langle \dots \rangle = \mathrm{Tr}[\hat{\rho}(t_0)\dots]. \tag{11}$$

Throughout this work we shall mark all quantities defined in the rotating reference frame by a tilde.

A. Instability of the noninteracting system

In the noninteracting limit ($u=0$) the equations of motion for the distribution functions $n(t)$ and $\widetilde{p}(t)$ can be obtained trivially from the equations of motion [Eq. (6)] of the operators $\widetilde{a}(t)$ and $\widetilde{a}^\dagger(t)$ in the rotating reference frame,

$$i\partial_t n(t) = \gamma_0 \widetilde{p}^*(t) - \gamma_0^* \widetilde{p}(t), \tag{12a}$$

$$i\partial_t \widetilde{p}(t) = 2\widetilde{\epsilon}_0 \widetilde{p}(t) + \gamma_0[2n(t) + 1]. \tag{12b}$$

These equations can be solved exactly. For $|\widetilde{\epsilon}_0| > |\gamma_0|$ the solution is oscillatory, while in the strong pumping regime $|\gamma_0| > |\widetilde{\epsilon}_0|$ the solutions grow exponentially. Let us explicitly give the solution of Eqs. (12a) and (12b) with initial conditions $n(0)=n_0$ and $\widetilde{p}(0)=0$. For simplicity, we assume in the rest of this work that γ_0 is real and positive; the case of complex $\gamma_0 = |\gamma_0|e^{i\varphi}$ can be reduced to real $\gamma_0 > 0$ by absorbing the phase factor $e^{i\varphi}$ into a redefinition of the anomalous correlator, $e^{-i\varphi}\widetilde{p}(t) \to \widetilde{p}(t)$. Defining

$$\alpha \equiv \sqrt{\widetilde{\epsilon}_0^2 - \gamma_0^2}, \tag{13}$$

the solution in the weak-pumping regime $\gamma_0 < |\widetilde{\epsilon}_0|$ can be written as

$$\frac{\mathrm{Re}\,\widetilde{p}(t)}{n_0 + \frac{1}{2}} = -\gamma_0 \widetilde{\epsilon}_0 \frac{1 - \cos(2\alpha t)}{\alpha^2}, \tag{14a}$$

$$\frac{\mathrm{Im}\,\widetilde{p}(t)}{n_0 + \frac{1}{2}} = -\gamma_0 \frac{\sin(2\alpha t)}{\alpha}, \tag{14b}$$

$$\frac{n(t) + \frac{1}{2}}{n_0 + \frac{1}{2}} = 1 + \gamma_0^2 \frac{1 - \cos(2\alpha t)}{\alpha^2}. \tag{14c}$$

In the opposite strong-pumping regime $\gamma_0 > |\widetilde{\epsilon}_0|$ the solution can be obtained by replacing $\alpha \to i\beta$ in the above expressions, where

$$\beta = \sqrt{\gamma_0^2 - \widetilde{\epsilon}_0^2}. \tag{15}$$

Then we obtain

$$\frac{\mathrm{Re}\,\widetilde{p}(t)}{n_0 + \frac{1}{2}} = -\gamma_0 \widetilde{\epsilon}_0 \frac{\cosh(2\beta t) - 1}{\beta^2}, \tag{16a}$$

$$\frac{\mathrm{Im}\,\widetilde{p}(t)}{n_0 + \frac{1}{2}} = -\gamma_0 \frac{\sinh(2\beta t)}{\beta}, \tag{16b}$$

$$\frac{n(t) + \frac{1}{2}}{n_0 + \frac{1}{2}} = 1 + \gamma_0^2 \frac{\cosh(2\beta t) - 1}{\beta^2}. \tag{16c}$$

The behavior at the threshold value $\gamma_0 = |\widetilde{\epsilon}_0|$ can be obtained either from Eqs. (14a)–(14c) for $\alpha \to 0$ or from Eqs. (16a)–(16c) for $\beta \to 0$,

$$\frac{\mathrm{Re}\,\widetilde{p}(t)}{n_0 + \frac{1}{2}} = -2\gamma_0 \widetilde{\epsilon}_0 t^2, \tag{17a}$$

$$\frac{\mathrm{Im}\,\widetilde{p}(t)}{n_0 + \frac{1}{2}} = -2\gamma_0 t, \tag{17b}$$

$$\frac{n(t) + \frac{1}{2}}{n_0 + \frac{1}{2}} = 1 + 2\gamma_0^2 t^2. \tag{17c}$$

Physically, the exponential increase of correlations for $\gamma_0 > |\widetilde{\epsilon}_0|$ is a consequence of the fact that in this regime the noninteracting part of the Hamiltonian $\widetilde{H}$ in Eq. (7) is not bounded from below. This is easily seen by setting

$$\widetilde{a} = \frac{\hat{X} + i\hat{P}}{\sqrt{2}}, \quad \widetilde{a}^\dagger = \frac{\hat{X} - i\hat{P}}{\sqrt{2}}, \tag{18}$$

so that

$$\widetilde{\epsilon}_0 \widetilde{a}^\dagger \widetilde{a} + \frac{\gamma_0}{2}[\widetilde{a}^\dagger \widetilde{a}^\dagger + \widetilde{a}\widetilde{a}] = \frac{\widetilde{\epsilon}_0 - \gamma_0}{2} \hat{P}^2 + \frac{\widetilde{\epsilon}_0 + \gamma_0}{2} \hat{X}^2. \tag{19}$$

Obviously, for $\gamma_0 > |\widetilde{\epsilon}_0|$ the noninteracting part of our toy model describes a harmonic oscillator with negative mass. The spectrum of such a quantum-mechanical system is not bounded from below, which gives rise to the exponential growth of correlations discussed above. Fortunately, this pathology of the noninteracting limit is cured for any positive value of the interaction. The physical consequences of this are most transparent if we consider the equations of motion for the expectation values of the creation and annihilation operators, which will be discussed in the following section.

B. Gross-Pitaevskii equation

The toy model Hamiltonian (7) in the rotating reference frame gives rise to the following Heisenberg equation of motion for the annihilation operator:

$$i\partial_t \tilde{a} = \tilde{\epsilon}_0 \tilde{a} + \gamma_0 \tilde{a}^\dagger + u\tilde{a}^\dagger \tilde{a}^2. \quad (20)$$

Taking the expectation value of both sides and factorizing the expectation value of the interaction term as follows:

$$\langle \tilde{a}^\dagger \tilde{a}^2 \rangle \to \langle \tilde{a}^\dagger \rangle \langle \tilde{a} \rangle^2, \quad (21)$$

we obtain the Gross-Pitaevskii equation for the time-dependent order parameter $\phi(t) \equiv \langle \tilde{a}(t) \rangle$ in the rotating reference frame,

$$i\partial_t \phi = \tilde{\epsilon}_0 \phi + \gamma_0 \phi^* + u|\phi|^2 \phi = \frac{\partial H_{cl}(\phi^*, \phi)}{\partial \phi^*}, \quad (22)$$

where the effective classical Hamiltonian H_{cl} is given by

$$H_{cl}(\phi^*, \phi) = \tilde{\epsilon}_0 |\phi|^2 + \frac{\gamma_0}{2}[\phi^{*2} + \phi^2] + \frac{u}{2}|\phi|^4. \quad (23)$$

Writing $\phi = (X + iP)/\sqrt{2}$ we may alternatively write

$$H_{cl}(X, P) = \frac{\tilde{\epsilon}_0 - \gamma_0}{2} P^2 + \frac{\tilde{\epsilon}_0 + \gamma_0}{2} X^2 + \frac{u}{8}(X^2 + P^2)^2. \quad (24)$$

Because the classical Hamiltonian $H_{cl}(X(t), P(t))$ is conserved along the flow defined by the Gross-Pitaevskii equation, the solutions of Eq. (22) are simply given by the curves of constant $H_{cl}(X(t), P(t))$ in phase space. The shape of H_{cl} and typical trajectories are shown in Fig. 1.

Note that in the strong-pumping regime $\gamma_0 > |\tilde{\epsilon}_0|$ the function $H_{cl}(X, P)$ has two degenerate minima at

$$X = 0, \quad P = \pm P_* = \pm \sqrt{\frac{2(\gamma_0 - \tilde{\epsilon}_0)}{u}}, \quad (25)$$

corresponding to stationary points (in the rotating reference frame) of the system. Note that at these special points the expectation value of the annihilation operator is purely imaginary,

$$\langle \tilde{a} \rangle = \pm \frac{i}{\sqrt{2}} P_* = \pm i\sqrt{\frac{\gamma_0 - \tilde{\epsilon}_0}{u}}. \quad (26)$$

The associated stationary points of the dynamical system [Eq. (22)] describe a coherent magnon state where the macroscopic magnetization has a rotating component perpendicular to the static magnetic field. In bosonic language, such a state corresponds to a coherent state, which is an eigenstate of the annihilation operator.[10,21] Whether or not this state should be called a Bose-Einstein condensate of magnons seems to be a semantic question. In our opinion this terminology is somewhat misleading, because this coherent magnon state does not exhibit spontaneous symmetry breaking which is one of the most important properties of a Bose-Einstein condensate in interacting Bose gases. Instead, the coherent magnon state observed by Demokritov and co-workers[11,12] is generated by an external pumping field, which explicitly breaks the $U(1)$ symmetry of the magnon
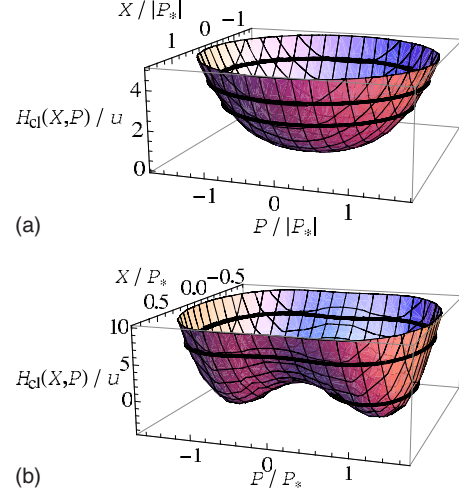

FIG. 1. (Color online) Graph of the classical Hamiltonian $H_{cl}(X, P)$ defined in Eq. (24). The corresponding classical Hamiltonian equations of motion are equivalent to the Gross-Pitaevskii Eq. (22) for the complex order parameter $\phi(t) = [X(t) + iP(t)]/\sqrt{2}$. The thick black lines are solutions of the equations of motion for different initial conditions. X and P are both measured in units of the momentum scale $|P_*| = \sqrt{2|\gamma_0 - \tilde{\epsilon}_0|/u}$. (a): $\tilde{\epsilon}_0/u = 10$ and $\gamma_0/u = 2$; note that for $|\tilde{\epsilon}_0| > \gamma_0$ our classical Hamiltonian $H_{cl}(X, P)$ has a global minimum for $X = P = 0$. (b): $\tilde{\epsilon}_0/u = 10$ and $\gamma_0/u = 40$; in the regime $\gamma_0 > |\tilde{\epsilon}_0|$ our classical Hamiltonian has two degenerate minima at $(X, P) = (0, \pm P_*)$, so that the graph of $H_{cl}(X, P)$ shown in (b) has some similarity to the shape of Napoleon's hat (Ref. 20).

Hamiltonian. In the static limit, the role of a similar symmetry-breaking term on the Bose-Einstein condensation of magnons has recently been discussed by Dell'Amore *et al.*[20]

C. Time-dependent Hartree-Fock approximation

Let us now take into account the leading fluctuation correction to the replacement Eq. (21) in the derivation of the Gross-Pitaevskii Eq. (22). To first order in u, fluctuations simply renormalize the bare parameters $\tilde{\epsilon}_0$ and γ_0 in Eq. (22) as follows:

$$\tilde{\epsilon}_0 \to \tilde{\epsilon}_c(t) = \tilde{\epsilon}_0 + 2un_c(t), \quad (27a)$$

$$\gamma_0 \to \gamma_c(t) = \gamma_0 + u\tilde{p}_c(t), \quad (27b)$$

where the connected correlation functions $n_c(t)$ and $\tilde{p}_c(t)$ in the rotating reference frame are defined by

$$n_c(t) = \langle \delta \tilde{a}^\dagger(t) \delta \tilde{a}(t) \rangle, \quad (28a)$$

$$\tilde{p}_c(t) = \langle \delta \tilde{a}(t) \delta \tilde{a}(t) \rangle, \quad (28b)$$

with $\delta \tilde{a}(t) = \tilde{a}(t) - \langle \tilde{a}(t) \rangle$. Instead of the Gross-Pitaevskii Eq. (22) we now obtain for the order-parameter dynamics,

$$i\partial_t \phi = \tilde{\epsilon}_c(t)\phi + \gamma_c(t)\phi^* + u|\phi|^2 \phi. \quad (29)$$

Note that this generalized Gross-Pitaevskii equation depends on the connected correlation functions $n_c(t)$ and $\tilde{p}_c(t)$, which

we calculate in self-consistent Hartree-Fock approximation. The resulting equations of motion can be obtained from the corresponding noninteracting kinetic Eqs. (12a) and (12b) by substituting

$$\tilde{\epsilon}_0 \rightarrow \tilde{\epsilon}(t) = \tilde{\epsilon}_0 + 2u[n_c(t) + |\phi(t)|^2], \quad (30a)$$

$$\gamma_0 \rightarrow \gamma(t) = \gamma_0 + u[\tilde{p}_c(t) + \phi^2(t)]. \quad (30b)$$

The kinetic equations for the connected distribution functions are therefore

$$i\partial_t n_c(t) = \gamma(t)\tilde{p}_c^*(t) - \gamma^*(t)\tilde{p}_c(t), \quad (31a)$$

$$i\partial_t \tilde{p}_c(t) = 2\tilde{\epsilon}(t)\tilde{p}_c(t) + \gamma(t)[2n_c(t) + 1]. \quad (31b)$$

For $\phi = 0$ these equations reduce to the kinetic equations obtained within S-theory.[4] The numerical solution of Eqs. (29), (31a), and (31b) for $n_c(0) = n_0$, $\tilde{p}_c(0) = 0$, and infinitesimal Im $\phi(t) > 0$ is shown in Fig. 2. Obviously, for sufficiently strong pumping an infinitesimal initial value of $\phi(0)$ builds up to a finite oscillation. Moreover, the connected correlation functions $n_c(t)$ and $\tilde{p}_c(t)$ remain always bounded, in contrast to the exponentially growing correlations in the noninteracting limit given in Eqs. (16a)–(16c). Note also that the time evolution of the connected correlation functions appears to be rather irregular as soon as the order parameter has built up to a finite value. In the conventional S-theory the quantities n_c and $\tilde{p}_c$ are periodic [Fig. 2(c)], while including the order-parameter dynamics disturbs this strict periodicity [Fig. 2(b)]. This feature is still missing within the usual S-theory in the strong-pumping regime.

IV. SUMMARY AND CONCLUSIONS

Let us briefly summarize the two main results of this work. First of all, we have shown that a complete theoretical description of the coherent magnon state emerging in YIG for sufficiently strong microwave pumping requires an extension of the usual S-theory, which includes the Gross-Pitaevskii type of equation for the expectation values of the magnon operators. Within a simple toy model consisting only of a single magnon mode we have shown how to construct such an extension. The explicit solution of the resulting kinetic equations shows that the order-parameter dynamics strongly influences the distribution functions.

Our second main result is the observation that in spin language the usual bosonic resonance Hamiltonian (2) corresponds to a time-dependent rotating easy-axis anisotropy whose axis is perpendicular to the direction of the external field. If this anisotropy is sufficiently strong, it gives rise to a forced oscillation of the macroscopic magnetization around the direction of the static external field. Although this phenomenon can be described in terms of a coherent magnon state, it should not be called a Bose-Einstein condensate, because the emergence of this state is not associated with any kind of spontaneous symmetry breaking.

In future work, we shall further extend our approach in two directions: on the one hand, a realistic model for YIG involves a quasicontinuum of magnon modes, which can
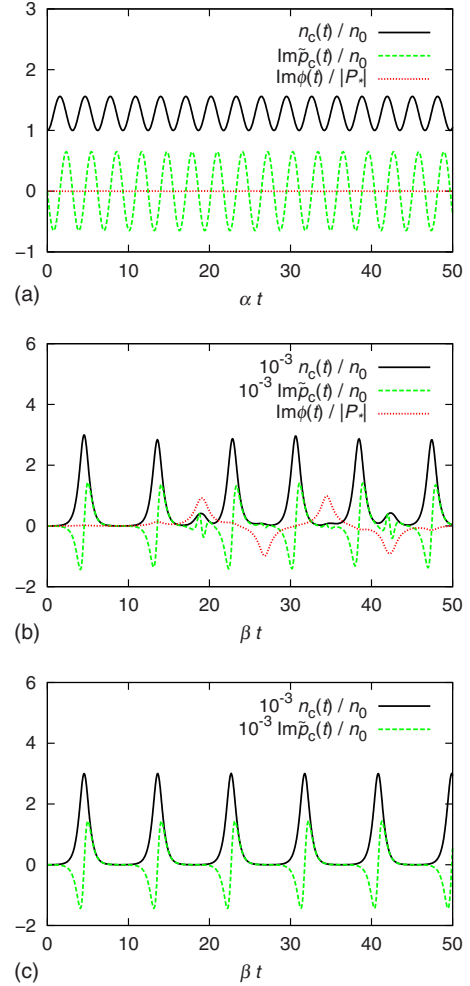

FIG. 2. (Color online) Numerical solution of the coupled kinetic Eqs. (29), (31a), and (31b) with initial conditions $n_c(0) = n_0 = 1$, $\tilde{p}_c(0) = 0$, and $\phi(0) = 0.05i$. The characteristic energy scales α and β are defined in Eqs. (13) and (15). (a): $\tilde{\epsilon}_0/u = 500$ and $\gamma_0/u = 200$. Recall that in the absence of interactions there is no instability as long as $|\tilde{\epsilon}_0| > \gamma_0$. (b): $\tilde{\epsilon}_0/u = 500$ and $\gamma_0/u = 5000$. In this regime there would be an instability in the noninteracting limit, but in the interacting system all correlations remain finite. (c): same parameters as (b) but without finite expectation values, like in the conventional S-theory.

condense at finite wave vectors $\pm \boldsymbol{k}_*$. For a more realistic quantitative description of the experiments, we should therefore generalize our extended S-theory to include all magnon modes relevant to the experiments on YIG. This would also allow us to distinguish between the "primary magnons" created by the external pumping and the "condensing magnons" with wave vectors at the minima of the dispersion. The second direction for improving our approach is to include correlation effects beyond the self-consistent Hartree-Fock approximation into the kinetic equations. For example, to second order in u the kinetic equations will contain relaxation terms, which will damp the oscillatory time dependence found at the Hartree-Fock level. Work in both directions is in progress.

APPENDIX: PARALLEL PUMPING OF MAGNONS IN YIG

It is generally accepted that the magnetic properties of YIG in the parallel pumping geometry can be modeled by the following time-dependent quantum spin model:[23,24]

$$\hat{H}_{\mathrm{YIG}}(t) = -\frac{1}{2}\sum_{ij}\sum_{\alpha\beta}[J_{ij}\delta^{\alpha\beta} + D_{ij}^{\alpha\beta}]S_i^\alpha S_j^\beta$$
$$- [h_0 + h_1\cos(\omega_0 t)]\sum_i S_i^z, \quad (A1)$$

where $\alpha, \beta = x, y, z$ label the three spin components, and the exchange couplings $J_{ij} = J(\mathbf{r}_i - \mathbf{r}_j)$ are only finite if the lattice sites $\mathbf{r}_i$ and $\mathbf{r}_j$ are nearest neighbors on a cubic lattice with lattice spacing $a \approx 12.376$ Å. The value of the nearest-neighbor exchange is $J \approx 1.29$ K. The dipolar tensor $D_{ij}^{\alpha\beta} = D^{\alpha\beta}(\mathbf{r}_i - \mathbf{r}_j)$ is explicitly

$$D_{ij}^{\alpha\beta} = (1 - \delta_{ij})\frac{\mu^2}{|\mathbf{r}_{ij}|^3}[3\hat{r}_{ij}^\alpha\hat{r}_{ij}^\beta - \delta^{\alpha\beta}], \quad (A2)$$

where $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ and $\hat{r}_{ij} = \mathbf{r}_{ij}/|\mathbf{r}_{ij}|$. If we arbitrarily set the magnetic moment $\mu = 2\mu_B = e\hbar/(mc)$, then we should work with an effective spin $S \approx 14.2$, as discussed in Ref. 24. Here h_0 and h_1 are the amplitudes of the static and oscillating magnetic field (multiplied by μ). We assume that $h_0 > |h_1|$ and that both the static and the oscillating magnetic field point into the direction of the macroscopic magnetization which we call the z axis. At this point one might already wonder how in this parallel pumping geometry one can possibly arrive at a bosonic resonance Hamiltonian of the form (2), which according to Eq. (4) can be related to some rotating easy-axis anisotropy. In fact, we shall show shortly that the spin Hamiltonian (A1) with parallel pumping *cannot* be reduced to the bosonic resonance Hamiltonian (2).

To bosonize Hamiltonian (A1) we express the spin operators in terms of boson operators b_i and $b_i^\dagger$ by means Holstein-Primakoff transformation,[19]

$$S_i^+ = \sqrt{2S}\sqrt{1 - \frac{b_i^\dagger b_i}{2S}}b_i = (S_i^-)^\dagger, \quad (A3a)$$

$$S_i^z = S - b_i^\dagger b_i. \quad (A3b)$$

As usual, the square roots are then expanded in powers of $1/S$, resulting in a Hamiltonian of the form

$$\hat{H}_{\mathrm{YIG}}(t) = H_0(t) + \hat{H}_2(t) + \hat{H}_{\mathrm{int}}, \quad (A4)$$

where $H_0(t)$ is a time-dependent constant, $\hat{H}_2(t)$ is quadratic in the boson operators, and the time-independent interaction $\hat{H}_{\mathrm{int}}$ involves three and more boson operators. After Fourier transformation to momentum space the quadratic part of the Hamiltonian can be written as

$$\hat{H}_2(t) = \sum_k \left[A_k b_k^\dagger b_k + \frac{B_k}{2}b_k^\dagger b_{-k}^\dagger + \frac{B_k^*}{2}b_{-k}b_k \right]$$
$$+ h_1\cos(\omega_0 t)\sum_k b_k^\dagger b_k, \quad (A5)$$

where

$$A_k = A_{-k} = \sum_i e^{-i k \cdot r_{ij}}A_{ij}, \quad (A6a)$$

$$B_k = B_{-k} = \sum_i e^{-i k \cdot r_{ij}}B_{ij} \quad (A6b)$$

with

$$A_{ij} = \delta_{ij}h_0 + S\left(\delta_{ij}\sum_n J_{in} - J_{ij}\right) + S\left[\delta_{ij}\sum_n D_{in}^{zz} - \frac{D_{ij}^{xx} + D_{ij}^{yy}}{2}\right], \quad (A7a)$$

$$B_{ij} = -\frac{S}{2}[D_{ij}^{xx} + 2iD_{ij}^{xy} - D_{ij}^{yy}]. \quad (A7b)$$

Finally, we use a Bogoliubov transformation to diagonalize the time-independent part of $\hat{H}_2(t)$,

$$\begin{pmatrix} b_k \\ b_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} u_k & -v_k \\ -v_k^* & u_k \end{pmatrix}\begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix}, \quad (A8)$$

where

$$u_k = \sqrt{\frac{A_k + \epsilon_k}{2\epsilon_k}}, \quad v_k = \frac{B_k}{|B_k|}\sqrt{\frac{A_k - \epsilon_k}{2\epsilon_k}}, \quad (A9)$$

and

$$\epsilon_k = \sqrt{A_k^2 - |B_k|^2}. \quad (A10)$$

After this transformation the Hamiltonian reads[9]

$$\hat{H}_2(t) = \sum_k \left[\epsilon_k a_k^\dagger a_k + \frac{\epsilon_k - A_k}{2} \right]$$
$$+ h_1\cos(\omega_0 t)\sum_k \left[\frac{A_k}{\epsilon_k}a_k^\dagger a_k + \frac{A_k - \epsilon_k}{2\epsilon_k} \right]$$
$$+ \sum_k [\gamma_k \cos(\omega_0 t)a_k^\dagger a_{-k}^\dagger + \gamma_k^* \cos(\omega_0 t)a_{-k}a_k], \quad (A11)$$

where

$$\gamma_k = -\frac{h_1 B_k}{2\epsilon_k}. \tag{A12}$$

To obtain the quadratic part of the resonance Hamiltonian (2) from Eq. (A11) two additional approximations are necessary: the second line in Eq. (A11) involving the combination $\cos(\omega_0 t) A_k a_k^\dagger a_k$ has to be dropped while in the last line one should substitute

$$\gamma_k \cos(\omega_0 t) \rightarrow \frac{\gamma_k}{2} e^{-i\omega_0 t}, \quad \gamma_k^* \cos(\omega_0 t) \rightarrow \frac{\gamma_k^*}{2} e^{i\omega_0 t}. \tag{A13}$$

Apparently this approximation has been accepted for many decades in the literature.[4–10] However, a thorough study of the nonresonant terms neglected in this approximation has been performed by Zvyagin *et al.*,[22] who showed that the neglected terms can qualitatively change the results obtained in resonance approximation. Here we would like to point out that the approximations leading to Eq. (A13) amount to an essential modification of the original spin Hamiltonian. To see this, let us for the moment accept the validity of these approximations, thus replacing Eq. (A11) by the noninteracting part of the resonant Hamiltonian (2),

$$\hat{H}_2(t) \approx \sum_k \epsilon_k a_k^\dagger a_k + \frac{1}{2} \sum_k \left[\gamma_k e^{-i\omega_0 t} a_k^\dagger a_{-k}^\dagger + \gamma_k^* e^{i\omega_0 t} a_{-k} a_k \right], \tag{A14}$$

where we have dropped the constant terms. Using now the inverse of the Bogoliubov transformation Eq. (A8) to re-express the magnon operators in Eq. (A14) in terms of Holstein-Primakoff bosons and assuming for simplicity that γ_k is real, the second term in Eq. (A14) can be written as

$$\frac{1}{2} \sum_k \left[\gamma_k e^{-i\omega_0 t} a_k^\dagger a_{-k}^\dagger + \gamma_k e^{i\omega_0 t} a_{-k} a_k \right],$$

$$= \frac{1}{2} \sum_k \left\{ \frac{\gamma_k A_k}{\epsilon_k} \cos(\omega_0 t) [b_k^\dagger b_{-k}^\dagger + b_{-k} b_k] \right.$$

$$\left. + i\gamma_k \sin(\omega_0 t) [b_k^\dagger b_{-k}^\dagger - b_{-k} b_k] \right\}$$

$$+ \sum_k \frac{\gamma_k B_k}{\epsilon_k} \cos(\omega_0 t) \left[b_k^\dagger b_k + \frac{1}{2} \right]. \tag{A15}$$

Only the last term on the right-hand side has the form of the boson representation of the Zeemann term associated with an external pumping field parallel to the magnetization, while the first two terms can be identified with the boson representation of spin anisotropies associated with a rotating easy axis perpendicular to the z axis, see Eq. (4). We thus conclude that the time-dependent part of the resonant Hamiltonian (2) does not represent the time-dependent Zeemann energy associated with a harmonically oscillating magnetic field in the direction of the magnetization. Instead, the time-dependent off-diagonal pumping terms arise from a rotating easy-axis anisotropy perpendicular to the magnetization. The microscopic origin of such a term is not clear to us; possibly the time-dependent electric field associated with the harmonically varying magnetic field parallel to the magnetization can indirectly induce such a term in the spin Hamiltonian, similar to the second-order interaction Hamiltonian in the theory of two-magnon Raman scattering in antiferromagnets.[25,26] Moreover, in real materials crystallographic or shape anisotropies can give rise to further contributions to the effective spin Hamiltonian, which after Holstein-Primakoff transformation might have the same form as the terms in Eq. (A15).

[1] V. Cherepanov, I. Kolokolov, and V. S. L'vov, Phys. Rep. **229**, 81 (1993).

[2] H. Suhl, J. Phys. Chem. Solids **1**, 209 (1957).

[3] E. Schlömann, J. J. Green, and U. Milano, J. Appl. Phys. **31**, 386S (1960); E. Schlömann and R. I. Joseph, *ibid.* **32**, 1006 (1961); E. Schlömann and J. J. Green, *ibid.* **34**, 1291 (1963).

[4] V. E. Zakharov, V. S. L'vov, and S. S. Starobinets, Zh. Eksp. Teor. Fiz. **59**, 1200 (1970) [Sov. Phys. JETP **32**, 656 (1971)]; Usp. Fiz. Nauk **114**, 609 (1974) [Sov. Phys. Usp. **17**, 896 (1975)].

[5] V. M. Tsukernik and R. P. Yankelevich, Zh. Eksp. Teor. Fiz. **68**, 2116 (1975) [Sov. Phys. JETP **41**, 1059 (1976)].

[6] I. A. Vinikovetskii, A. M. Frishman, and V. M. Tsukernik, Zh. Eksp. Teor. Fiz. **76**, 2110 (1979) [Sov. Phys. JETP **49**, 1067 (1979)].

[7] S. P. Lim and D. L. Huber, Phys. Rev. B **37**, 5426 (1988); **41**, 9283 (1990).

[8] Yu. D. Kalafati and V. L. Safanov, Zh. Eksp. Teor. Fiz. **95**, 2009 (1989) [Sov. Phys. JETP **68**, 1162 (1989)].

[9] V. S. L'vov, *Wave Turbulence under Parametric Excitations* (Springer, New York, 1994).

[10] S. M. Rezende, Phys. Rev. B **79**, 060410(R) (2009); **79**, 174411 (2009).

[11] S. O. Demokritov, V. E. Demidov, O. Dzyapko, G. A. Melkov, A. A. Serga, B. Hillebrands, and A. N. Slavin, Nature (London) **443**, 430 (2006).

[12] V. E. Demidov, O. Dzyapko, S. O. Demokritov, G. A. Melkov, and A. N. Slavin, Phys. Rev. Lett. **99**, 037205 (2007); **100**, 047205 (2008).

[13] A. S. Borovik-Romanov, Y. M. Bunkov, V. V. Dmitriev, and Y. M. Mukharskii, Pis'ma Zh. Eksp. Teor. Fiz. **40**, 256 (1984) [JETP Lett. **40**, 1033 (1984)].

[14] G. E. Volovik, J. Low Temp. Phys. **153**, 266 (2008).

[15] Y. M. Bunkov and G. E. Volovik, arXiv:0904.3889 (unpublished).

[16] A. A. Zvyagin, Fiz. Nizk. Temp. **33**, 1248 (2007) [Sov. J. Low Temp. Phys. **33**, 948 (2007)].

[17] A. I. Bugrij and V. M. Loktev, Fiz. Nizk. Temp. **33**, 51 (2007) [Low Temp. Phys. **33**, 37 (2007)]; Fiz. Nizk. Temp. **34**, 992 (2008) [Low Temp. Phys. **34**, 1259 (2008)].

[18] See, for example, L. Pitaevskii and S. Stringari, *Bose-Einstein Condensation* (Clarendon, Oxford, 2003).

[19] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).

[20] R. Dell'Amore, A. Schilling, and K. Krämer, Phys. Rev. B **79**, 014438 (2009).

[21] S. M. Rezende and N. Zagury, Phys. Lett. A **29**, 47 (1969); C. B. de Araujo, Phys. Rev. B **10**, 3961 (1974).

[22] A. A. Zvyagin, V. Ya. Serebryannyi, A. M. Frishman, and V. M. Tsukernik, Fiz. Nizk. Temp. **8**, 1205 (1982) [Sov. J. Low Temp. Phys. **8**, 612 (1982)].

[23] I. S. Tupitsyn, P. C. E. Stamp, and A. L. Burin, Phys. Rev. Lett. **100**, 257202 (2008).

[24] A. Kreisel, F. Sauli, L. Bartosch, and P. Kopietz, Eur. Phys. J. B **71**, 59 (2009).

[25] P. A. Fleury, Phys. Rev. Lett. **21**, 151 (1968); P. A. Fleury and R. Loudon, Phys. Rev. **166**, 514 (1968).

[26] R. J. Elliott and M. F. Thorpe, J. Phys. C **2**, 1630 (1969).