

Sample-to-sample fluctuations and bond chaos in the m -component spin glass

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We calculate the finite-size scaling of the sample-to-sample fluctuations of the free energy ΔF of the m component vector spin glass in the large- m limit. This is accomplished using a variant of the interpolating Hamiltonian technique which is used to establish a connection between the free energy fluctuations and bond chaos. The calculation of bond chaos then shows that the scaling of the free-energy fluctuations with system size N is $\Delta F \sim N^\mu$ with $\frac{1}{5} \leq \mu \leq \frac{3}{10}$, and very likely $\mu = \frac{1}{5}$ exactly.

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I. INTRODUCTION

Spin-glass physics¹ continue to be a very exciting and difficult topic. One of the important ongoing issues are the finite-size corrections to thermodynamic quantities.^{2–11} Such finite-size corrections are usually impossible to calculate within replica theory (which is otherwise extremely successful for the spin glass and other problems), due to the massless modes which are often encountered and which prevent going beyond zero-loop order in a perturbation expansion. Nevertheless, many results are by now established for the Sherrington-Kirkpatrick spin glass,¹² either numerically or analytically (see references cited above).

An observable which is particularly interesting is the finite-size scaling of the sample-to-sample fluctuations of the free energy. This quantity provides a link between two apparently distant fields, namely, spin-glass physics and extreme value statistics.¹³ In extreme value statistics, the question is the probability distribution of extremal events such as the maximum (or minimum) of a set of random numbers. The classic results of extreme value theory state that such extremal events follow one of three possible limiting distributions (the Weibull, Gumbel, or Fréchet distribution). Recently, a fourth universality class was found, the Tracy-Widom distribution for the smallest (or largest) eigenvalue of a Gaussian random matrix.¹⁴ Similarly, one could ask what the distribution of the ground-state energy of a (Sherrington-Kirkpatrick or other) spin glass is, which is a question of extreme value statistics. In a statistical mechanics setting, however, it can be generalized to the question of what the distribution of the *free* energy is at finite temperature. This is a very difficult question indeed (both for finite and zero temperature). To keep it simpler, we merely ask what the *width* of the distribution is (i.e. the sample-to-sample fluctuations), and this width ΔF will scale in some way with the system size N , i.e., $\Delta F \sim N^\mu$ with an exponent μ . These free-energy fluctuations have been considered for the Sherrington-Kirkpatrick model by numerical investigations, see, e.g., Refs. 6, 7, and 10. There also exist heuristic arguments for $\mu = \frac{1}{4}$ (Ref. 6) and $\mu = \frac{1}{6}$,^{9,11,15–17} and the limit $\mu \leq \frac{1}{4}$ has been

shown.^{18,19} All these results show that the Sherrington-Kirkpatrick model does not fall in any of the four established universality classes of extreme value statistics. For a different famous replica symmetric spin-glass model, the spherical spin glass,²⁰ the situation is different. Since its ground-state energy is the smallest eigenvalue of a Gaussian random matrix, this model falls into the Tracy-Widom universality class.²¹ This implies that the fluctuations of the free-energy scale as $\Delta F \sim N^{1/3}$, which has been confirmed recently by a replica calculation.¹¹

In this work we consider the fully connected m component vector spin glass in the limit of large m . This model is known to be replica symmetric,²² which suggests that its free-energy fluctuations might fall into the same universality class as the spherical model. Furthermore it might be expected that due to replica symmetry the fluctuations in this model are simpler to calculate than in the Ising spin glass. Unfortunately, this hope is not entirely justified, as we will see. As mentioned above, it is usually impossible to calculate subextensive quantities within replica theory. We circumvent this problem by using a connection between the sample-to-sample fluctuations of the free energy and bond chaos, which was derived for the Ising spin glass,¹⁹ and which we generalize here to vector spin glasses. This connection allows us to calculate the sample-to-sample fluctuations by calculating bond chaos instead, and this is possible within a large deviation approximation, combining techniques from Refs. 23 and 24. As it will turn out, the large deviation approximation is not good enough to obtain the final answer, and we must resort to additional resources, such as our previous knowledge of the scaling properties of the large- m model^{25–28} and some additional scaling assumptions. To forestall our main result, we obtain $\mu \leq \frac{1}{5} \leq \frac{3}{10}$ and most likely $\mu = \frac{1}{5}$ exactly. This shows that the large- m spin glass does not belong to any of the four universality classes and probably to a different class than the Ising spin glass (unless it so happens that, e.g., $\mu = \frac{1}{4}$ for both the Ising and the large- m spin glass, which is not entirely ruled out by our results but which we deem unlikely). The result $\mu = \frac{1}{5}$ was first suggested in Ref. 25. However, the derivation required the existence of a gap in

the eigenvalue spectrum of the inverse susceptibility matrix which was later shown not to exist.²⁶ Here, we provide a different explanation for this value of μ .

This paper is organized as follows. In Sec. II we will show how to derive the exact connection between the sample-to-sample fluctuations of the free energy, ΔF_N , and bond chaos for the m -component spin glass using interpolating Hamiltonians. We will see that the fluctuations only depend on the finite-size scaling of averages of powers of the link overlap q_L between two copies of a spin glass with different (but correlated) disorder. The averages of the link overlaps will be calculated in Sec. III by calculating the probability distribution of the link overlap, $P_\epsilon(q_L)$, where the parameter ϵ measures the degree of correlation. We will use these results in Sec. IV to derive the exponent μ of the free-energy fluctuations.

II. CONNECTION BETWEEN THE SAMPLE-TO-SAMPLE FLUCTUATIONS AND BOND CHAOS

We use the technique of interpolating Hamiltonians²⁹ in order to derive an exact connection between the sample-to-sample fluctuations of the free energy, ΔF_N , and bond chaos in the m -component spin glass. To do so, we adapt the method from Ref. 19, which was developed for the Ising spin glass, to the m -component vector spin glass. The Hamiltonian of the m -component spin glass is

$$\mathcal{H} = \sqrt{\frac{1}{N}} \sum_{i < j} J_{ij} \vec{s}_i \vec{s}_j, \quad (1)$$

where \vec{s}_i are m -component spins and J_{ij} are independent Gaussian random variables with unit variance. The spins are assumed to be normalized in such a way that $\vec{s}_i^2 = m$.

The main idea for calculating the sample-to-sample fluctuations of the free energy is to use the following two Hamiltonians,

$$\begin{aligned} \mathcal{H}_t &= -\sqrt{\frac{1-t}{N}} \sum_{i < j} J_{ij} \vec{s}_i \vec{s}_j - \sqrt{\frac{t}{N}} \sum_{i < j} J'_{ij} \vec{s}_i \vec{s}_j \quad \text{and} \\ \mathcal{H}'_\tau &= -\sqrt{\frac{1-\tau}{N}} \sum_{i < j} J_{ij} \vec{s}_i \vec{s}_j - \sqrt{\frac{\tau}{N}} \sum_{i < j} J''_{ij} \vec{s}_i \vec{s}_j, \end{aligned} \quad (2)$$

with $0 \leq t, \tau \leq 1$ and J'_{ij} and J''_{ij} additional independent Gaussian random variables with unit variance, and express the free-energy fluctuations in terms of them. These Hamiltonians interpolate between a spin glass with a given set of coupling constants $\{J_{ij}\}$ for $t=0$ (or $\tau=0$) and an identical spin glass with a different, independent set of coupling constants $\{J'_{ij}\}$ or $\{J''_{ij}\}$ for $t=1$ (respectively, $\tau=1$). For all $0 < t, \tau < 1$ we have the same type of spin glass but with coupling constants $\sqrt{1-t}J_{ij} + \sqrt{t}J'_{ij}$ (or $\sqrt{1-\tau}J_{ij} + \sqrt{\tau}J''_{ij}$), which are still Gaussian random variables with unit variance. We denote the disorder average with respect to all coupling constants $\{J_{ij}\}$, $\{J'_{ij}\}$ and $\{J''_{ij}\}$ as $E \cdots$. Thermal averages will be denoted as $\langle \cdots \rangle$.

The free-energy fluctuations can be written as $\beta^2 \Delta F_N^2 = E(\log Z_1 - \log Z_0)(\log Z'_1 - \log Z'_0)$ with Z_t and Z'_τ being the

partition functions with respect to the Hamiltonians \mathcal{H}_t and \mathcal{H}'_τ . This can be seen by writing $\log Z = -\beta F$ and using the independence of the different sets of bonds for $E F_1 F'_0 = E F_0 F'_1 = E F_1 F'_1 = \bar{F}^2$ and $E F_0 F'_0 = \bar{F}^2$ [the overbar denotes a disorder averaged quantity of the original system of interest, i.e., Eq. (1)]. We use the idea in Ref. 29 to represent this expression by integrals in the form

$$\beta^2 \Delta F_N^2 = \int_0^1 dt \int_0^1 d\tau E \frac{\partial}{\partial t} \log Z_t \frac{\partial}{\partial \tau} \log Z'_\tau. \quad (3)$$

To calculate the right-hand side, we follow Ref. 19 and obtain

$$\begin{aligned} E \frac{\partial}{\partial t} \log Z_t \frac{\partial}{\partial \tau} \log Z'_\tau &= \frac{\beta^2}{16N^2} E \sum_{ijkl} (\langle \vec{s}_i \vec{s}_j \rangle_t \langle \vec{s}_k \vec{s}_l \rangle_\tau) \\ &\quad - \langle \vec{s}_i \vec{s}_j \rangle_t \langle \vec{s}_k \vec{s}_l \rangle_\tau (\langle \vec{s}_i \vec{s}_j \rangle_t \langle \vec{s}_k \vec{s}_l \rangle_\tau) \\ &\quad - \langle \vec{s}_i \vec{s}_j \rangle_\tau \langle \vec{s}_k \vec{s}_l \rangle_t + \frac{\beta^2}{8N\sqrt{1-t}\sqrt{1-\tau}} \left(E \sum_{ij} (\langle \vec{s}_i \vec{s}_j \rangle_t \langle \vec{s}_i \vec{s}_j \rangle_\tau) \right. \\ &\quad \left. - m^2 N \right). \end{aligned} \quad (4)$$

The subscript t or τ on the thermal averages indicates whether the average is to be taken in a system with Hamiltonian \mathcal{H}_t or \mathcal{H}'_τ .

We introduce the link overlap between two replicas with potentially different coupling constants as $q_L^{13} = \frac{2}{N(N-1)} \sum_{i < j} \langle \vec{s}_i \vec{s}_j \rangle_t \langle \vec{s}_i \vec{s}_j \rangle_\tau = \frac{1}{N(N-1)} (\sum_{\mu\nu} \langle (\sum_i s_{i\mu}^{(1)} s_{i\nu}^{(3)})^2 \rangle - m^2 N)$. As we will shortly need not only two but up to four replicas, we label the replicas with upper indices (1) to (4), where replicas one and two have Hamiltonian \mathcal{H}_t and replicas three and four have Hamiltonian \mathcal{H}'_τ . Analogously to q_L^{13} , we define the other possible link overlaps q_L^{14} , q_L^{23} , and q_L^{24} . Then Eq. (4) can be expressed in the following way:

$$\begin{aligned} E \frac{\partial}{\partial t} \log Z_t \frac{\partial}{\partial \tau} \log Z'_\tau &= \frac{(N-1)^2 \beta^4}{16} E \langle (q_L^{13} - q_L^{23})(q_L^{13} - q_L^{14}) \rangle \\ &\quad + \frac{(N-1)\beta^2}{8\sqrt{1-t}\sqrt{1-\tau}} E \langle q_L^{13} \rangle \\ &= \frac{(N-1)^2 \beta^4}{16} ([\langle q_L^{13} \rangle^2] - [q_L^{13}]^2) + \frac{(N-1)\beta^2}{8\sqrt{1-t}\sqrt{1-\tau}} [q_L^{13}]. \end{aligned} \quad (5)$$

The notation $[\cdots]$ in the last line stands for the average taken with the bond averaged probability distribution $P_\epsilon(q_L)$ of finding a given link overlap q_L . The parameter ϵ indicates the statistical “distance” between the sets of bonds in the two replicas involved and will be defined in detail below. In principle, one would have to consider simultaneous multireplica overlaps (such as, for instance, the term $E \langle q_L^{13} q_L^{23} \rangle$). Fortunately, it follows from replica symmetry, which holds for the m -component spin glass in the large- m limit, that the corresponding joint probability distribution $P_\epsilon^{123}(q_L^{13}, q_L^{23})$ factorizes into $P_\epsilon^{123}(q_L^{13}, q_L^{23}) = P_\epsilon(q_L^{13}) P_\epsilon(q_L^{23})$. A similar statement holds for the four-replica probability distribution $P_\epsilon^{1234}(q_L^{13}, q_L^{24})$.

Up to now the overlaps q_L^{ab} depend on the two parameters t and τ . However, the only important quantity is how much the sets of bonds in the two replicas a and b differ. For example, when $t=\tau=0$, the bonds in replicas one and three (say) are identical. On the other hand, when $t=0$ and $\tau=1$ (or vice versa), the bonds in replicas one and three are completely uncorrelated. The degree of correlation between the two sets of bonds can be measured by the one parameter ϵ defined by $\frac{1}{\sqrt{1+\epsilon^2}} = \sqrt{1-t}\sqrt{1-\tau}$, as shown in Ref. 19. When $\epsilon=0$, the bonds are identical; when $\epsilon=\infty$, the bonds are completely uncorrelated.

The parameter ϵ can be used to eliminate the integration variable τ from Eq. (3) by a variable transformation. The integral over t can then be evaluated exactly, and as a result, the connection between the sample-to-sample fluctuations of the free energy and bond chaos in the m -component spin glass is found to be (see Ref. 19 for details)

$$\begin{aligned} \beta^2 \Delta F_N^2 &= \frac{(N-1)^2 \beta^4}{16} \int_0^\infty d\epsilon f_2(\epsilon) ([q_L^{13}]^2 - [q_L^{13}]^2) \\ &\quad + \frac{(N-1)\beta^2}{4} \int_0^\infty d\epsilon g_2(\epsilon) [q_L^{13}], \\ &=: \frac{(N-1)^2 \beta^4}{16} I_{21} + \frac{(N-1)\beta^2}{4} I_{22} \end{aligned} \quad (6)$$

with the non-negative functions $f_2(\epsilon) = \frac{2\epsilon \log(1+\epsilon^2)}{(1+\epsilon^2)^2}$ and $g_2(\epsilon) = \frac{\epsilon \log(1+\epsilon^2)}{(1+\epsilon^2)^{3/2}}$. Note that bond chaos enters Eq. (6) through the measure of distance ϵ of the bonds of the two replicas between which the link overlap q_L^{13} is calculated. The integrals I_{21} and I_{22} will be calculated below.

The analog of Eq. (6) was called ‘‘second route to chaos’’ in Ref. 19, hence the first index is 2 on I_{21} and I_{22} . In addition to this result, however, there is another exact relation between the fluctuations and bond chaos (the ‘‘first route to chaos’’). It stems from using only the first interpolating Hamiltonian of Eq. (2), \mathcal{H}_t , and the relation $2\beta^2 \Delta F_N^2 = E(\log Z_1 - \log Z_0)^2$, which is easy to derive. Proceeding similarly to above, one can prove the following equality:

$$\begin{aligned} \beta^2 \Delta F_N^2 &= -\frac{(N-1)^2 \beta^4}{16} \int_0^\infty d\epsilon f_1(\epsilon) ([q_L^{13}]^2 - [q_L^{13}]^2) \\ &\quad + \frac{(N-1)\beta^2}{4} \int_0^\infty d\epsilon g_1(\epsilon) [q_L^{13}] \\ &=: -\frac{(N-1)^2 \beta^4}{16} I_{11} + \frac{(N-1)\beta^2}{4} I_{12}. \end{aligned} \quad (7)$$

The only difference between this equation and Eq. (6) is the minus sign in front of the first term and the weight functions $f_1(\epsilon) = \frac{4\epsilon^2}{(1+\epsilon^2)^2} \arcsin \frac{1}{\sqrt{1+\epsilon^2}}$ and $g_1(\epsilon) = \frac{2}{(1+\epsilon^2)^{3/2}} \arcsin \frac{1}{\sqrt{1+\epsilon^2}}$ in the integrals I_{11} and I_{12} instead of $f_2(\epsilon)$ and $g_2(\epsilon)$ as in I_{21} and I_{22} . The minus sign of the first term implies that the second term is an upper bound of $\beta^2 \Delta F_N^2$.

III. CALCULATING $P_\epsilon(q_L)$

Equation (6) involves moments of the link overlap taken with the probability density $P_\epsilon(q_L)$. This function can, in principle, be calculated by taking two replicas with bond realizations drawn with parameter ϵ and constraining the replicas to have link overlap q_L . This constrained system has free energy $F_{\epsilon,J}(q_L)$, and the (nondisorder averaged) probability density $P_{\epsilon,J}(q_L)$ then follows to be

$$P_{\epsilon,J}(q_L) = \frac{\exp[-\beta F_{\epsilon,J}(q_L)]}{\int_0^\infty dq'_L \exp[-\beta F_{\epsilon,J}(q'_L)]}. \quad (8)$$

Finally, $P_{\epsilon,J}(q_L)$ must be averaged over the disorder.

Unfortunately, this task is too difficult in general. Instead, we will calculate only the disorder averaged extensive part of the free energy, denoted by $Nmf_\epsilon(q_L)$, and the probability density defined by

$$P_\epsilon^0(q_L) = \frac{\exp[-\beta Nmf_\epsilon(q_L)]}{\int_0^\infty dq'_L \exp[-\beta Nmf_\epsilon(q'_L)]}. \quad (9)$$

This is the large deviation approximation to $P_\epsilon(q_L)$. Averages taken with respect to $P_\epsilon^0(q_L)$ will be denoted by $[\dots]_0$. The tail of this distribution will be the same as that of $P_\epsilon(q_L)$ but in general there will be deviations. In fact, this is the point where this paper differs most from Ref. 19 since in that publication, the difference between the true and the approximative distribution was only quantitative, whereas here it is substantial. We know that for $\epsilon=0$ the link overlap distribution consists of a δ peak at the Edwards-Anderson value and 0 elsewhere since the large- m spin glass is replica symmetric. We will see below, however, that we do not observe this peak in $P_\epsilon^0(q_L)$ at all. It must therefore be generated from the finite-size corrections to the extensive part of the free energy.

Hence the finite-size corrections are very important in this calculation but we have no direct way of calculating them. In order to overcome this problem, we will show that $P_\epsilon^0(q_L)$ will be valid for ϵ larger than some crossover value, and we will use additional arguments and simulation results to fill the gap for smaller ϵ .

A. Replica calculation

The first task is to calculate the extensive part of the disorder averaged free energy $Nmf_\epsilon(q_L)$ of two replicas constrained to have link overlap q_L . To this end we calculate the partition function $Z_{\epsilon,J}(q_L)$ of this two replica system. This system has vector spins \vec{s}_i^x with $(\vec{s}_i^x)^2 = m$ for every spin $i = 1, \dots, N$ and the two real replicas $x=(10), (2\epsilon)$ (this notation denotes the replica number in its first entry and the value of ϵ in its second entry). These two replicas differ in their coupling constants $K_{ij}(\xi) = \frac{1}{\sqrt{1+\xi^2}} K_{ij}^0 + \frac{\xi}{\sqrt{1+\xi^2}} K'_{ij}$, where K_{ij}^0 and K'_{ij} are independent Gaussian random numbers with unit variance, by choosing $\xi=0$ for the first replica and $\xi=\epsilon$ for the second replica. Furthermore, the two replicas have link overlap q_L , which is enforced by a δ function in the partition function as follows:

$$Z_{\epsilon,J}(q_L) = \text{Tr}_s \left\{ \delta \left[q_L - \frac{1}{N(N-1)} \sum_{\mu\nu} \left(\sum_i s_{i\mu}^{(10)} s_{i\nu}^{(2\epsilon)} \right)^2 - \frac{m^2}{N-1} \right] \right\} \times \exp \left[\frac{\beta}{\sqrt{N}} \sum_{i<j} K_{ij}(0) \bar{s}_i^{(10)} \bar{s}_j^{(10)} + \frac{\beta}{\sqrt{N}} \sum_{i<j} K_{ij}(\epsilon) \bar{s}_i^{(2\epsilon)} \bar{s}_j^{(2\epsilon)} \right]. \quad (10)$$

We follow the replica calculation of Viana²³ for the n times replicated partition function, write the δ function in an integral representation with parameters z_α , and introduce the traceless tensor $T_\alpha^{\mu\nu}$ (Ref. 23) to separate the diagonal part of $Q_{\alpha\beta}^{\mu\nu}$ [with $\sum_{\mu,\nu} \alpha,\beta Q_{\alpha\beta}^{\mu\nu} = 2\sum_{\alpha<\beta} Q_{\alpha\beta}^{\mu\nu} + 2\sum_{\alpha,\mu<\nu} T_\alpha^{\mu\nu} + \sum_{\alpha,\mu} (Q_{\alpha\alpha} + T_\alpha^{\mu\mu})$]. After evaluating the trace and regarding only terms up to third order in the tensors Q , T , and R we get

$$\begin{aligned} EZ_{\epsilon,J}^n(q_L) \propto \int \left(\prod_\alpha dz_\alpha \right) e^{-N\sum_\alpha q_L z_\alpha - m^2 \sum_\alpha z_\alpha} \int d\Lambda_{\alpha\beta}^{\mu\nu} \exp \left(N \left[\tau \sum_{\substack{\alpha<\beta \\ \mu,\nu}} (Q_{\alpha\beta}^{\mu\nu})^2 + \left(\tau + \frac{r}{m+2} \right) \sum_{\alpha,\mu<\nu} (T_\alpha^{\mu\nu})^2 + \left(\tau + \frac{r'}{m+2} \right) \sum_{\alpha,\mu} (T_\alpha^{\mu\mu})^2 \right. \right. \\ \left. \left. + \tau' \sum_{\substack{\alpha\neq\beta \\ \mu,\nu}} (R_{\alpha\beta}^{\mu\nu})^2 + \sum_{\alpha,\mu,\nu} \left(\frac{2z_\alpha}{\beta^2} + \frac{1}{\sqrt{1+\epsilon^2}} \right)^{(-1)} (R_{\alpha\alpha}^{\mu\nu})^2 + \frac{\omega}{3!} \left(\sum_{\substack{\alpha<\beta<\gamma \\ \mu,\nu,\rho}} Q_{\alpha\beta}^{\mu\nu} Q_{\beta\gamma}^{\nu\rho} Q_{\gamma\alpha}^{\rho\mu} + \frac{m^2}{(m+2)(m+4)} \sum_{\substack{\alpha \\ \mu<\nu<\rho}} T_\alpha^{\mu\nu} T_\alpha^{\nu\rho} T_\alpha^{\rho\mu} \right. \right. \right. \\ \left. \left. + \frac{15m^2}{(m+2)(m+4)} \sum_{\alpha,\mu} (T_\alpha^{\mu\mu})^3 + \frac{3m}{m+2} \sum_{\substack{\alpha<\beta \\ \nu,\mu<\rho}} Q_{\alpha\beta}^{\mu\nu} Q_{\beta\alpha}^{\nu\rho} T_\alpha^{\rho\mu} + 3 \sum_{\substack{(\alpha<\beta)\neq\gamma \\ \mu,\nu,\rho}} Q_{\alpha\beta}^{\mu\nu} R_{\beta\gamma}^{\nu\rho} R_{\gamma\alpha}^{\rho\mu} + 3 \sum_{\substack{\alpha<\beta \\ \mu,\nu,\eta}} Q_{\alpha\beta}^{\mu\nu} R_{\beta\alpha}^{\nu\eta} R_{\alpha\alpha}^{\eta\mu} \right. \right. \\ \left. \left. + \frac{3m}{m+2} \sum_{\substack{\alpha\neq\beta \\ (\mu<\nu),\rho}} T_\alpha^{\mu\nu} R_{\alpha\beta}^{\nu\rho} R_{\beta\alpha}^{\rho\mu} + 3 \sum_{\substack{\alpha \\ \mu<\nu,\eta}} T_\alpha^{\mu\nu} R_{\alpha\alpha}^{\nu\eta} R_{\alpha\alpha}^{\eta\mu} \right) \right], \quad (11) \end{aligned}$$

with $\tau = \frac{1}{2}(1 - \frac{1}{\beta^2})$, $\tau' = \frac{1}{2}(1 - \frac{\sqrt{1+\epsilon^2}}{\beta^2})$, $r = -1$, $r' = 2m - 1$, $\omega = 1$, and the matrix $\Lambda_{\alpha\beta}^{\mu\nu}$ is given by

$$\Lambda = \begin{pmatrix} Q^{(10)} & R \\ R & Q^{(2\epsilon)} \end{pmatrix}, \quad (12)$$

with the above separation for $Q_{\alpha\beta}^{\mu\nu}$ and only $T_\alpha^{\mu\mu}$ on the diagonal of the matrices Q . We split the tensor R into its diagonal matrices $p_d^{\mu\nu} = R_{\alpha\alpha}^{\mu\nu}$ and the rest. This equation is solved by a saddle-point integration for the tensors $Q_{\alpha\beta}^{\mu\nu}$, $T_\alpha^{\mu\nu}$, and $R_{\alpha\beta}^{\mu\nu}$ and the parameters z_α .

B. Solving the saddle-point equations

Solving the saddle-point equations is the remaining task to derive the characteristic form of the overlap distribution

$P_\epsilon(q_L)$. As the m -component spin glass was shown to be replica symmetric in the limit $m \rightarrow \infty$,²² we calculate the saddle points in the replica symmetric case. The ansatz for a replica symmetric scenario is as follows:^{23,24}

$$Q_{\alpha\beta}^{\mu\nu} = Q \delta_{\mu\nu}, \quad T_\alpha^{\mu\nu} = 0,$$

$$R_{\alpha\beta}^{\mu\nu} = P \delta_{\mu\nu}, \quad p_d^{\mu\nu} = p_d \delta_{\mu\nu}, \quad \text{and} \quad z_\alpha = z. \quad (13)$$

To briefly justify these equations, one has the usual interpretation $Q_{\alpha\beta}^{\mu\nu(10)} = \langle s_{\alpha\mu}^{(10)} s_{\beta\nu}^{(10)} \rangle$ from the replica calculation. Due to the isotropy of the model, averaged quantities like this reduce to $x \delta_{\mu\nu}$ with some mean value x , depending on the quantity at hand. Thus $EZ_{\epsilon,J}^n(q)$ is given by

$$\begin{aligned} EZ_{\epsilon,J}^n(q_L) \propto \int \left(\frac{N-1}{2\pi} \right)^n dz e^{-2Nnq_L z - m^2 n z} \int d\Lambda_{\alpha\beta}^{\mu\nu} \exp \left(N \left[\tau m n (n-1) Q^2 + \tau' n m (n-1) P^2 + \frac{nm}{2} p_d^2 - m n \frac{1}{2\beta^2} \left(\frac{2z}{\beta^2} + \frac{1}{\sqrt{1+\epsilon^2}} \right)^{(-1)} p_d^2 \right. \right. \\ \left. \left. + 2 \frac{\omega n (n-1) (n-2)}{3!} m Q^3 + \omega n (n-1) (n-2) m Q P^2 + 2 \omega n (n-1) m Q P p_d \right] \right). \quad (14) \end{aligned}$$

This leads to four different saddle-point equations (we use Nm as the large parameter)

$$0 = \tau Q - Q^2 - P^2 + P p_d, \quad (15)$$

$$0 = \tau' P - 2QP + Q p_d, \quad (16)$$

$$0 = p_d - \frac{1}{\beta^2} p_d \left(\frac{2z}{\beta^2} + \frac{1}{\sqrt{1+\epsilon^2}} \right)^{(-1)} - 2QP, \quad (17)$$

$$\frac{q_L}{m} = \left[\frac{p_d}{\beta^2 \left(\frac{2z}{\beta^2} + \frac{1}{\sqrt{1+\epsilon^2}} \right)} \right]^2, \quad (18)$$

where line 3 and line 4 can be combined to

$$p_d - 2QP = \sqrt{\frac{q_L}{m}}. \quad (19)$$

In the following, we substitute $\sqrt{\frac{q_L}{m}} \rightarrow q$.

From Eq. (14) the free energy is (after taking the usual replica limit $n \rightarrow 0$)

$$\begin{aligned} \beta f_\epsilon(q) = \text{const} + p_d q - \frac{p_d^2}{2} - \frac{q^2}{2(1-2\tau')} + \tau Q^2 + \tau' P^2 - \frac{2}{3} Q^3 \\ - 2QP^2 + 2QP p_d. \end{aligned} \quad (20)$$

1. Above and at the critical temperature

Above ($\tau < 0$) and at the critical temperature ($\tau = 0$), the Ising spin glass is replica symmetric, just as the m -component spin glass. Therefore the solutions of the saddle-point equations are the same as in Ref. 24. Inserting them into the free energy [Eq. (20)] gives

$$\text{above } T_c: \beta f_\epsilon(q) = \frac{q^2}{2} \left(1 - \frac{\beta^2}{\sqrt{1+\epsilon^2}} \right) + \mathcal{O}(q^4), \quad (21)$$

$$\text{at } T_c: \beta f_\epsilon(q) = \begin{cases} \frac{1}{6} q^3 + \frac{3\epsilon^2}{16} q^2 + \mathcal{O}(\epsilon^4, q^4) & \epsilon^2 \ll q \ll 1 \\ \frac{q^2}{2} \left(1 - \frac{1}{\sqrt{1+\epsilon^2}} \right) + \mathcal{O}(q^4) & q \ll \epsilon^2 \ll 1. \end{cases} \quad (22)$$

At T_c the probability distribution $P_\epsilon(q) \propto e^{-Nm\beta f_\epsilon(q)}$ consists of two parts with different dominating exponents depending on ϵ being smaller or larger than $N^{-1/6}$,

$$P_\epsilon(q) \propto \begin{cases} e^{-Nm(q^3/6)} & \epsilon \ll N^{-1/6} \\ e^{-Nm(q^2\epsilon^2/4)} & N^{-1/6} \ll \epsilon. \end{cases} \quad (23)$$

2. Below the critical temperature

The Eqs. (15), (16), and (19) can not be solved for general q and ϵ . Due to this, we calculate corrections to the two following solvable cases, $q=0$ and $\epsilon=0$, perturbatively in various limits. The exact result for $q=0$ is

$$P = 0,$$

$$Q = \tau,$$

$$p_d = 0. \quad (24)$$

To find the solution for $\epsilon=0$, we rewrite Eqs. (15) and (16) in terms of the new variables $a=Q+P$ and $b=Q-P$,

$$(\tau + p_d)a - \frac{\tau - \tau'}{2}(a - b) - a^2 = 0, \quad (25)$$

$$(\tau - p_d)b - \frac{\tau - \tau'}{2}(a - b) - b^2 = 0 \quad (26)$$

with the solution ($\epsilon=0$, so $\tau=\tau'$)

$$Q = \tau, \quad (27)$$

$$P = p_d = \frac{q}{1-2\tau}, \quad (28)$$

for $a \neq 0$ and $b \neq 0$. With Eq. (20) we get $\beta f_0(q) = \beta f_0 = \frac{1}{3}\tau^3$. We will use this free energy as the reference free energy as it is the energy of the unconstrained system. For $q > \tau - 2\tau^2 = q_{EA}$ the solution with $b=0$ maximizes the free energy (the solution with $a=0$ is an unphysical one). In terms of $\Delta q = q - (\tau - 2\tau^2)$ it is (to lowest order in Δq)

$$Q = P = \frac{\tau + p_d}{2},$$

$$p_d = \tau + \frac{\Delta q}{1-2\tau}, \quad (29)$$

and the difference to βf_0 is

$$\Delta f_0(q) = c_0 \Delta q^3, \quad (30)$$

with $c_0 = \frac{1}{6(1-2\tau)^3}$. Now we will calculate corrections to the first solution, f_0 , in various different limits of ϵ and q .

3. $\epsilon \rightarrow \infty$

In the limit $\epsilon \rightarrow \infty$, i.e., $\tau' \rightarrow -\infty$ Eq. (16) yields the solution (to leading order)

$$P = -\frac{Q p_d}{\tau'} + \mathcal{O}\left[\frac{1}{(\tau')^2}\right]. \quad (31)$$

For Eq. (15) this leads to (again in leading order)

$$0 = \left(\tau + \frac{p_d}{\tau'} \right) Q - Q^2. \quad (32)$$

For $\tau' = -\infty$, p_d is equal to q and the free-energy difference to f_0 is

$$\beta f_\infty(q) - \beta f_0 = q p_d - \frac{p_d^2}{2} + \mathcal{O}(p_d^4) = \frac{q^2}{2} + \mathcal{O}(q^4). \quad (33)$$

4. Perturbative solution for $\epsilon^2 \ll q \ll 1$

We use the saddle-point equations for a Eq. (25) and b Eq. (26), insert $a=a_0+a_1$, $b=b_0+b_1$, and $p_d=p_{d0}+p_{d1}$, where the index 0 denotes the undisturbed solution ($\epsilon=0$) and the index 1 the first-order corrections to it. The results are ($\Delta\tau = \tau - \tau' > 0$),

$$\begin{aligned} a_1 &= -\frac{\Delta\tau p_{d0} - p_{d1} a_0}{\tau + p_{d0}} & Q_1 &= \frac{\epsilon^2 q^2}{4\tau^2(1-2\tau)} \\ b_1 &= \frac{\Delta\tau p_{d0} - p_{d1} b_0}{\tau - p_{d0}} & P_1 &= -\frac{\epsilon^2 q}{4\tau} \end{aligned} \quad (34)$$

and the correction to p_d is $p_{d1} = -\frac{\epsilon^2 q}{2}$. For the free energy these results yield,

$$\beta f_\epsilon(q) - \beta f_0 = \frac{2\tau+1}{16\tau} \epsilon^4 q^2 + \mathcal{O}(\epsilon^4 q^4). \quad (35)$$

5. Perturbative solution for $q \ll \epsilon^2 \ll 1$

In this case the suitable reference solution is the one with $q=0$, Eq. (24). We again introduce corrections ($Q \rightarrow Q + \Delta Q$) to this solution and combine Eqs. (16) and (19) in lowest order to

$$\tau' \Delta P - 2Q \Delta P + Q(\Delta q + 2Q \Delta P) = 0, \quad (36)$$

where Δq is understood to be the correction to $q=0$. The solution is

$$\Delta P = -\frac{\tau \Delta q}{\tau' - 2\tau + 2\tau^2} \xrightarrow{\epsilon^2 \ll 1} \frac{\Delta q}{1-2\tau} - \frac{\epsilon^2 \Delta q}{4\tau(1-2\tau)}, \quad (37)$$

$\Delta Q=0$ and

$$\Delta p_d = \Delta q \frac{\tau' - 2\tau}{\tau' - 2\tau + 2\tau^2} \xrightarrow{\epsilon^2 \ll 1} \frac{\Delta q}{1-2\tau} - \frac{\epsilon^2 \Delta q}{2(1-2\tau)}. \quad (38)$$

The result for the free energy then is

$$\beta f_\epsilon(q) - \beta f_0 = \frac{1}{16\tau(1-2\tau)} \epsilon^4 q^2 + \mathcal{O}(q^3). \quad (39)$$

The restriction $\epsilon^2 \ll 1$ is basically unnecessary and we could calculate the free energy in the limit $q \ll \min(1, \epsilon^2)$. However, since the regime of large ϵ will not contribute to the sample-to-sample fluctuations we neglect it already at this point and write $\beta f_\epsilon(q) - \beta f_0(q) = f(\epsilon) q^2$ with $f(\epsilon) = \epsilon^4 \frac{1}{16\tau(1-2\tau)}$ for small ϵ .

C. $P_\epsilon^0(q)$

Although we are interested in the probability density of the link overlap q_L , the more practical quantity for the calculation turned out to be $q = \sqrt{q_L/m}$. We therefore formulate our results in terms of q instead of q_L for the time being. This is not a serious restriction since we have the simple relation

$$[(q_L^{13})^n]_0 = m^n \int dq q^{2n} P_\epsilon^0(q) =: m^n [q^{2n}]_0. \quad (40)$$

The probability distribution $P_\epsilon^0(q) \sim e^{-Nm\beta[f_\epsilon(q) - f_0(q)]}$ divides into four parts, depending on the range of ϵ . For $\epsilon \ll N^{-1/4}$ the

contribution to the probability distribution of q of both Eqs. (35) and (39) [both with $f_\epsilon(q) \sim \epsilon^4 q^2$] is negligible, therefore it is approximately a constant in that range for all $q \in [0, q_{EA}]$. Equation (30) implies that $P_\epsilon^0(q)$ has an exponentially decaying tail for $q > q_{EA}$ with $e^{-Nm c_0 (q - q_{EA})^3}$. We define a function $\Theta(q - q_{EA})$ which combines both properties. Instead of this plateau in $P_\epsilon^0(q)$ there should be a δ peak at $q = q_{EA}$ which we do not see in our calculation. This is due to the fact that we have neglected finite-size corrections to the free energy which are dominating in this regime and which we will implement in Sec. III D.

The two solutions we found perturbatively in Eqs. (35) and (39) both produce a probability distribution of the form $e^{-Nm\beta c_x \epsilon^4 q^2}$ with different constants c_x , but hold in different ranges of ϵ , depending on the relation of ϵ^2 and q . The order of ϵ determining the crossover from one regime to the other is where ϵ^2 is of the same order as q and $N\epsilon^4 q^2$ (in the range $N^{-1/4} \ll \epsilon \ll \epsilon_0$) is of order 1 such as to be the dominating part of the free energy. This leads to $\epsilon \sim N^{-1/8}$ as the crossover value. Thus we get the final result

$$P_\epsilon(q) \propto \begin{cases} \Theta(q - q_{EA}) & \epsilon \ll N^{-1/4} \\ e^{-Nm c_1 \epsilon^4 q^2} & N^{-1/4} \ll \epsilon \ll N^{-1/8} \\ e^{-Nm c_2 \epsilon^4 q^2} & N^{-1/8} \ll \epsilon \ll \epsilon_0 \\ e^{-Nm f(\epsilon) q^2} & \epsilon_0 < \epsilon \end{cases} \quad (41)$$

with $c_1 = \frac{2\tau+1}{16\tau}$ and $c_2 = \frac{1}{16\tau(1-2\tau)}$.

D. Overlap distribution at $\epsilon=0$

In order to account for the missing δ peak in $P_\epsilon^0(q)$ for small ϵ , we have to consider the finite-size corrections to the free energy, which are impossible to calculate. However, not all is lost since at least we know that at $\epsilon=0$ they grow as N^{1-y} with $y=2/5$,²⁸ and we expect that this scaling is independent of the value of ϵ . We can therefore trust our result derived above when $Nm\beta[f_\epsilon(q_{EA}) - f_0] \gg N^{3/5}$. Since $q_{EA} = \mathcal{O}(1)$, it follows that this is the case when $N^{3/5} \ll N\epsilon^4$ or $N^{-1/10} \ll \epsilon$. We then have

$$P_\epsilon(q) \propto \begin{cases} \delta_{FS}(q - q_{EA}) & \epsilon \ll N^{-1/10} \\ e^{-N c_2 \epsilon^4 q^2} & N^{-1/10} \ll \epsilon \ll \epsilon_0 \\ e^{-N f(\epsilon) q^2} & \epsilon_0 < \epsilon. \end{cases} \quad (42)$$

The function $\delta_{FS}(q - q_{EA})$ stands for a function which goes to a δ peak in the thermodynamic limit. Note that the new regime completely replaces the first two regimes we calculated in Eq. (41).

What we need in the actual calculation of the sample-to-sample fluctuations of the free energy are the expressions $[(q_L^{13})^2] - [q_L^{13}]^2$ and $[q_L^{13}]$. In the regimes where we have an explicit expression for $P_\epsilon^0(q)$, we can calculate this directly (see below). In the regime just found, however, we do not have this information available. We must therefore resort to other methods and use a finite-size scaling ansatz of the form

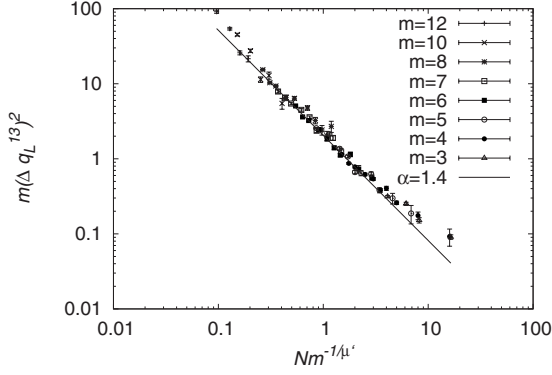


FIG. 1. Finite-size scaling plot of the variance of the link overlap at temperature $T=0.6$. See text for explanation.

$$\frac{1}{m^2}([\langle q_L^{13} \rangle^2] - \langle q_L^{13} \rangle^2) = N^{-\alpha} \mathcal{F}_m(N^\beta \epsilon) \quad (43)$$

for $\epsilon \ll N^{-1/10}$ and with a scaling function $\mathcal{F}_m(x)$ whose properties will be discussed below. Similarly, we assume that in this regime $\langle q_L^{13} \rangle$ can be written as

$$\frac{1}{m} \langle q_L^{13} \rangle = \mathcal{G}_m(N^\rho \epsilon) \quad (44)$$

with an exponent ρ and a scaling function $\mathcal{G}_m(x)$. These scaling functions and exponents will enter the calculation of the fluctuations below.

We assume $m \rightarrow \infty$ such that the system is replica symmetric. Therefore, the left-hand side of Eq. (43), which is the variance of q_L^{13}/m , goes to 0 for $N \rightarrow \infty$ since it is just the squared width of the peak in $P_\epsilon^0(q_L)$. The exponents α and β are unknown but we can obtain α from a simulation at $\epsilon = 0$ by measuring the width of the peak in the distribution of q_L . This is shown in Fig. 1. There is, however, a complication involved. In a simulation, m must be finite, and for finite m , the thermodynamic limit is a replica symmetry broken phase and the variance of q_L^{13} will *not* tend to 0, thus apparently making $\alpha=0$. In order to overcome this problem, we recall that in Refs. 25 and 26 it was shown that at $T=0$ there is a critical number of spin components $n_0 \sim N^{\mu'}$ (with an exponent $\mu' = 2/5$) above which the system does not depend on the number of components any more and is thus identical to the replica symmetric $m=\infty$ limit. We conjecture that something similar happens at finite temperature, i.e., when $m \gg N^{\mu'}$, the system is in a replica symmetric phase, whereas for $m \ll N^{\mu'}$ it is in a different phase. For this reason we have plotted the variance $(\Delta q_L^{13})^2$ of q_L^{13} against $x = Nm^{-1/\mu'}$, such that we can expect replica symmetric behavior for small x and a crossover to a constant variance for large x , and this is precisely what can be observed (although the crossover is so slow that we do not see the expected plateau yet). For the determination of α only the data for small x are relevant.

For the simulation, we have implemented a parallel tempering Monte Carlo algorithm using system sizes from $N=64$ to $N=216$ with up to 29 different temperatures in the range of $[0.6; 1.6]$. In that way we produced for two different replicas (with the same set of coupling constants since we are

considering $\epsilon=0$) at least 64 statistically independent sets of spin configuration for every temperature. From these, we calculated the variance of the link overlap $(\Delta q_L^{13})^2 = [\langle q_L^{13} \rangle^2] - \langle q_L^{13} \rangle^2$. According to the scaling ansatz above, $m^{-2}(\Delta q_L^{13})^2 \sim (Nm^{-1/\mu'})^{-\alpha} m^{-\alpha/\mu'}$ at $\epsilon=0$, such that $m^{\alpha/\mu'-2}(\Delta q_L^{13})^2 \sim (Nm^{-1/\mu'})^{-\alpha}$. The data in Fig. 1 shows on the one hand that $\alpha/\mu'-2 \approx 1$ and on the other hand that $\alpha \approx 1.4$. The observation $\alpha/\mu'-2 \approx 1$ implies $\alpha \approx \frac{6}{5}$. Together with the second, more direct, observation of α , the data shows that $\alpha \geq \frac{6}{5}$, and as we will see below, this is all we need to know.

IV. CALCULATING $[q^n]_0$ AND THE SAMPLE-TO-SAMPLE FLUCTUATIONS OF THE FREE ENERGY

The remaining task is to calculate $[q^n]_0$ and insert it into Eq. (6). To do so, we use steepest descent methods to write for the regime ϵ with $N^{-1/10} \ll \epsilon \ll \epsilon_0$ (with $P_\epsilon^0(q) = \frac{1}{\mathcal{N}_q} e^{-\beta N m f_\epsilon(q)}$ and the normalization constant \mathcal{N}_q)

$$[q^n]_0 = \frac{1}{\mathcal{N}_q} \int_0^\infty dq q^n e^{-N m c_x \epsilon^4 q^2 + \mathcal{O}(q^4)}, \quad (45)$$

neglect the term of order q^4 in the exponent (note that we set the upper bound from 1 to ∞ , which introduces only exponentially small errors), and get [with $\mathcal{N}_q = \frac{\Gamma(\frac{1}{2})}{2\sqrt{N m c_x \epsilon^4}}$]

$$\begin{aligned} [q^n]_0 &= \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty dx x^{n-1/2} (N m c_x \epsilon^4)^{-n/2} e^{-x} \\ &= \frac{(N m c_x \epsilon^4)^{-n/2}}{\Gamma(\frac{1}{2})} \Gamma\left(\frac{n+1}{2}\right). \end{aligned} \quad (46)$$

We then have

$$[q^2]_0 = \frac{1}{2 N m c_x \epsilon^4}, \quad (47)$$

$$[q^4]_0 = \frac{3}{(2 N m c_x \epsilon^4)^2}. \quad (48)$$

With this, we calculate the sample-to-sample fluctuations through Eqs. (6) and (7) by taking the leading order in N of every integral into account. The first integrals, I_{21} , [with $f_2(\epsilon) = 2\epsilon^3 + \mathcal{O}(\epsilon^5)$] separates into three integration intervals corresponding to our three regimes. We neglect the range of $\epsilon > \epsilon_0$ because it gives a contribution of order 1 which we are not interested in. For the part $\epsilon \ll N^{-1/10}$ we have the scaling ansatz for $[\langle q_L^{13} \rangle^2] - \langle q_L^{13} \rangle^2 = m^2 N^{-\alpha} \mathcal{F}_m(N^\beta \epsilon)$. This ansatz must match $m^2([q^4]_0 - [q^2]_0^2)$ from the neighboring regime at the crossover point $\epsilon = N^{-1/10}$, which there goes as $N^{-2} \epsilon^{-8} \sim N^{-6/5}$ [see Eqs. (47) and (48)]. This implies that the scaling function $\mathcal{F}_m(x)$ decays as $x^{-\gamma}/m^2$ for $x \rightarrow \infty$ with an exponent γ obeying $\gamma(\beta - \frac{1}{10}) = \frac{6}{5} - \alpha$.

This leads to

$$\begin{aligned}
I_{21} &= \int_0^{N^{-1/10}} d\epsilon 2\epsilon^3 N^{-\alpha} \mathcal{F}_m(N^\beta \epsilon) + \int_{N^{-1/10}}^{\epsilon_0} d\epsilon 2\epsilon^3 \frac{1}{2(Nc_2\epsilon^4)^2} \\
&\sim 2N^{-\alpha-4\beta} \int_0^{N^{\beta-1/10}} dx \mathcal{F}_m(x) x^3 + N^{-2} \int_{N^{-1/10}}^{\epsilon_0} d\epsilon \frac{\epsilon^{-5}}{c_2^2} + \dots \\
&\sim \text{const}_1 N^{-\alpha-4/\gamma(6/5-\alpha)-2/5} + \text{const}_2 N^{-8/5}, \quad (49)
\end{aligned}$$

as long as $\gamma > 4$. If $\gamma < 4$, the first part of the integral is of order $\mathcal{O}(N^{-8/5})$.

The second integral of Eq. (6), I_{22} , can be estimated in a similar way with the scaling ansatz $\frac{1}{m}[q_L^{13}] = \mathcal{G}_m(N^\rho \epsilon)$. The scaling function $\mathcal{G}_m(x)$ should have the properties $\mathcal{G}_m(x) \xrightarrow{x \rightarrow \infty} x^{-\eta}$ and $\mathcal{G}_m(x) \xrightarrow{x \rightarrow 0} \text{const}$ and should scale as $N^{-1}\epsilon^{-4}$ for $\epsilon = N^{-1/10}$ in order to match the neighboring regime, which yields the scaling relation $\rho = \frac{3}{5\eta} + \frac{1}{10}$. This gives rise to a contribution

$$\begin{aligned}
I_{22} &= \int_0^{N^{-1/10}} d\epsilon \epsilon^3 \mathcal{G}_m(N^\rho \epsilon) + \int_{N^{-1/10}}^{\epsilon_0} d\epsilon \frac{\epsilon^3}{2Nc_2\epsilon^4} + \dots \\
&= \text{const}_3 N^{-12/5\eta-2/5} + \text{const}_4 \frac{\log N}{N}, \quad (50)
\end{aligned}$$

provided that $\eta > 4$. If $\eta < 4$ the first part of the integral is $\mathcal{O}(N^{-1})$ instead. We therefore find the scaling exponent of the sample-to-sample fluctuations with the system size, $\Delta F_N \sim N^\mu$, to be

$$\mu = \max \left[\frac{1}{5}, \frac{4}{5} - \frac{\alpha}{2} - \frac{2}{\max(\gamma, 4)} \left(\frac{6}{5} - \alpha \right), \frac{3}{10} - \frac{6}{5 \max(\eta, 4)} \right]. \quad (51)$$

As described in Sec. III, we have another, slightly different route to chaos and will use it now to check for consistency with the above result. From Eq. (7) we take the first integral $I_{11} = m^2 \int_0^{\epsilon_0} d\epsilon f_1(\epsilon) ([q^4] - [q^2]^2)$ and use the same scaling ansatz as above for Eq. (49) and obtain contributions of the form

$$\begin{aligned}
I_{11} &= m^2 \int_0^{\epsilon_0} d\epsilon f_1(\epsilon) ([q^4] - [q^2]^2) \sim \text{const}_5 N^{-\alpha-3/\gamma(6/5-\alpha)-3/10} \\
&+ \text{const}_6 N^{-3/2} + \dots \quad (52)
\end{aligned}$$

This integral is positive and has a negative prefactor in Eq. (7). The fluctuations can not be negative, however, therefore the leading order of this term *must* be compensated by the second integral I_{12} . We use the scaling function $\mathcal{G}_m(N^\rho \epsilon)$ again, which yields

$$I_{12} = m \int_0^{\epsilon_0} d\epsilon g_1(\epsilon) [q^2] \sim \text{const}_7 N^{-3/5\eta-1/10} + \text{const}_8 N^{-7/10}. \quad (53)$$

This second integral, together with its leading prefactor N from Eq. (7), must at least cancel the term of order $\mathcal{O}(N^{1/2})$ which is contained in $N^2 I_{11}$. This is only possible if $\eta \geq \frac{3}{2}$. Hence we obtain a limit on η and no contradiction to the result derived above.

V. CONCLUSION

The purpose of this work was to calculate the finite-size scaling of the sample-to-sample fluctuations of the free energy in the m -component vector spin glass in the limit of large m . The result is Eq. (51). Although this equation looks unpromising at first sight, it is in fact very informative. First, we have the solid result $\mu \geq \frac{1}{5}$. Second, from our numerical work we know that $\alpha \geq \frac{6}{5}$. But under this condition, the second term in the max function in Eq. (51) is $\leq \frac{1}{5}$ and can thus simply be omitted. Third, the last term in the max function is $\leq \frac{3}{10}$, such that we obtain $\frac{1}{5} \leq \mu \leq \frac{3}{10}$. Even better, the third term is greater than $\frac{1}{5}$ only for $\eta > 12$, which seems an unlikely large value. We therefore conjecture that $\mu = \frac{1}{5}$ is in fact the exact answer. But be that as it may, the exponent η could easily be measured in simulations, and work along these lines is in progress.

Since our result is not an exact mathematical proof, we will summarize here the main assumptions on which it rests because they may have been obscured by the technicalities. The first ingredient is the connection between the fluctuations and bond chaos of the link overlap, Eqs. (6) and (7). They are mathematically exact equalities and pose no problem. The second ingredient is the calculation of bond chaos. This is done using large deviation statistics and replica theory. We believe that replica theory, in principle, gives the correct results. It became apparent, however, that for the m component spin glass the ‘‘small’’ deviations play a crucial role. The small deviations statistics are caused by finite-size corrections of the free energy, which can not be calculated within replica theory. However, by reference to earlier results²⁸ we know at least the finite-size scaling of the free-energy corrections and can thus estimate the point where our large deviation calculation becomes valid. The region of the small deviations is then covered by a scaling ansatz, which is assumed to cross over smoothly to the region of the large deviations. The introduction of the scaling ansatz also introduced a number of unknown exponents. However, only three of these exponents are actually relevant for our results, and one of them, α , has been measured experimentally. Moreover, the precise values of the exponents are largely irrelevant: if $\alpha \geq \frac{6}{5}$ (as we have checked numerically) and $\eta \leq 12$, then $\mu = \frac{1}{5}$ is exact.

The result $\mu = \frac{1}{5}$ (or even the range $\frac{1}{5} \leq \mu \leq \frac{3}{10}$) is very interesting because it demonstrates that the large- m model is fundamentally different from the spherical model,²⁰ even though their free energies are identical.²² While the width of the distribution of ground-state energies of the spherical spin glass scales as $N^{1/3}$, this behavior is definitely ruled out by our results.

It would be interesting to see whether a result similar to ours could be obtained using the methods of Ref. 11. We believe a corresponding replica calculation for the large- m model ought to be feasible.

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