Current dependence of the minimum length of a superconducting wire

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It is shown that superconductivity disappears in small current-carrying samples. The critical size of the superconductor for which a superconducting instanton exists is calculated analytically as a function of the dc bias current within the framework of the Ginzburg-Landau equations.

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Due to recent progress in nanotechnology, experiments can be performed on very small superconducting rings and wires (of the size of the penetration length λ or even of the coherence length ξ), leading to new physical phenomena. These experiments provide a clear demonstration of some interesting properties of superconductors. Phenomena studied by susceptibility or inductive coupling experiments include flux quantization,[1] thermal decay of supercurrents in rings containing weak links,[2] the current-phase relationship for weak links,[2,3] and macroscopic quantum effects.[4] Transport measurements on rings and cylinders have clarified the Little-Parks T_c oscillations[5] and magnetoconductance oscillations due to superconducting fluctuations.[6] Recently, transport experiments involving mesoscopic superconducting rings of 0.1 mm diameter have explored the influence of a superconducting boundary on quantum transport.[7]

It is well known that the proximity effect leads to absence of superconductivity in sufficiently small wires connected to normal metal leads. A quantitative theory of this phenomenon was first developed by De Gennes[8] for a ring of small radius R located in an external magnetic field H and connected to normal electrodes. This theoretical analysis, of De Gennes was based on the linearized Landau-Ginzburg equations, showed that superconductivity in the small rings with $R \sim \xi$ disappeared due to normal electrons from the leads. This prediction has been confirmed by experiments in Al and $Au_{0.7}In_{0.3}$ cylinders.[9]

Using a simple model of a dirty, gapless superconductor, we present the theoretical analysis of the suppression of superconductivity by an electric current in a short superconducting wire.

Consider the four-point experimental setup, where the current is generated by an external battery. In this case, the steady-state equations for the superconducting order parameter $\Delta = |\Delta|\exp(i\chi)$ and the electric potential Φ can be written in the following dimensionless form:[10]

$$\frac{d^2\Psi}{dx^2} - \frac{(J+\nabla\varphi)^2}{\Psi^3} + \Psi - \Psi^3 = 0, \tag{1}$$

$$\varphi = u(\Psi)\frac{d^2\varphi}{dx^2}; \quad u(\Psi) = \frac{112\varsigma(3)}{8\pi^4\Psi^2}\sqrt{1 + \frac{32\pi^2\tau_\varepsilon^2 T_c^2}{7\varsigma(3)}\left(1 - \frac{T}{T_c}\right)\Psi^2}, \tag{2}$$

where

$$x \to x/\xi; \quad \Psi = \frac{|\Delta|}{\Delta_0}; \quad J \to JJ_d; \quad \varphi \to \Phi\frac{c\Phi_0}{8\pi^2\lambda^2\sigma_n};$$
$$\lambda^2 = \frac{T_c c^2}{4\pi^2\sigma_n\Delta_0^2} \tag{3}$$

and

$$\xi^2 = \frac{\pi D}{8(T_c - T)}; \quad J_d = \frac{c\Phi_0}{8\pi^2\lambda^2\xi}; \quad \Delta_0 = \sqrt{\frac{8\pi^2 T_c(T_c - T)}{7\varsigma(3)}}. \tag{4}$$

Here Φ_0 is the unit magnetic flux, D is the electron diffusion constant, σ_n is the normal conductivity, τ_ε is the inelastic relaxation time of the quasiparticles, J_d is the depairing current, and J is the total electric current density with $\nabla J = 0$ due to electroneutrality.

Together with the boundary conditions at the wire edges

$$J = J_n = -\nabla\Phi|_{x=0,L}; \quad \Psi(0) = \Psi(L) = 0. \tag{5}$$

Equations (1) and (2) describe steady-state distribution both of the superconducting order parameter and the current density across the wire.

Assuming the limit $\Delta_0\tau_\varepsilon \ll 1$, (gapless superconductor), we obtain from Eq. (2) that $u(\varphi) = 5.79$ and the normal current $j_n = -\sigma_n\nabla\Phi$ converges into the superconducting current at a distance $l_E \simeq \xi$.[11] Therefore over the distance $x \geq \xi$, far from the wire edge, Eq. (1) can be written in the form

$$\frac{d^2\Psi}{dx^2} - \frac{J^2}{\Psi^3} + \Psi - \Psi^3 = 0 \tag{6}$$

subject to the boundary condition

$$\Psi(0) = \Psi(L) = 0 \tag{7}$$

for different values of the electric current J.

The results used by many researchers[12–16] is based on the analogy between Eq. (6) and a particle moving in a central force field.

Our primary objective is to find the inhomogeneous solutions $\Psi(x)$ of Eq. (6) ("instantons"), which will describe nonhomogeneous superconductivity with $\Psi = 0$ at some points. For a sufficiently small sample with periodic Dirichlet boundary conditions, the only solution is $\Psi = 0$ for all x, i.e., there is no superconductivity for such small samples. The idea of this approach came from the series of articles by

Stein and collaborators[17] who studied the disappearance of the instantons in small samples. Integration [Eq. (6)] over x yields

$$x - x_0 = \frac{1}{\sqrt{2}} \int_{\Psi_0^2}^{\Psi^2} \frac{dz}{\sqrt{z^3 - 2z^2 + 4Ez - 2J^2}} \equiv \frac{1}{\sqrt{2}} \int_{\Psi_0^2}^{\Psi^2} \frac{dz}{\sqrt{(z-z_1)(z-z_2)(z-z_3)}}, \quad (8)$$

where E is a constant of integration.

Our aim is to find Ψ as a function of x. Integrating Eq. (8) yields an elliptic integral of the first kind $F(\varphi,\kappa)$, where the arguments φ and κ depend on the following relation between z_1, z_2, z_3, and Ψ^2:

$$\sqrt{2}x = \int_{z_3}^{\Psi^2} \frac{dz}{\sqrt{(z-z_1)(z-z_2)(z-z_3)}} = \frac{2}{\sqrt{z_1 - z_3}} F\left(\arcsin\sqrt{\frac{\Psi^2 - z_3}{z_2 - z_3}}, \sqrt{\frac{z_2 - z_3}{z_1 - z_3}}\right) = \frac{2}{\sqrt{z_1 - z_3}} sn^{-1}\left(\sqrt{\frac{\Psi^2 - z_3}{z_2 - z_3}}, \sqrt{\frac{z_2 - z_3}{z_1 - z_3}}\right), \quad (9)$$

where $sn(k,m)$ is the Jacoby elliptic function of the first type, and we have chosen $x_0=0$. Here $z_1 \geq z_2 \geq z_3$ are the roots of the cubic equation

$$z^3 - 2z^2 + 4Ez - 2J^2 = 0. \quad (10)$$

According to the theory of the Jacoby function, a nonzero solution for $\Psi(x)$ of Eq. (9) with boundary conditions Eq. (7), exists when

$$sn\left(\frac{\sqrt{z_1 - z_3}}{2} L, \sqrt{\frac{z_2 - z_3}{z_1 - z_3}}\right) = 0. \quad (11)$$

This yields[18]

$$\frac{\sqrt{z_1 - z_3}}{\sqrt{2}} L = 2K(m), \quad (12)$$

where

$$K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m^2 \sin^2 \theta}}; \quad m = \sqrt{\frac{z_2 - z_3}{z_1 - z_3}} \quad (13)$$

with $0 < m < 1$ is the complete elliptic integral of the first kind. The function $K(m)$ is a monotonic function of m increasing from $\pi/2$ at $m=0$ to ∞ at $m=1$. We obtain $L_{\min}$, from Eq. (12) with $m=0$ and $K(0)=\pi/2$,

$$L_{\min} = \frac{\sqrt{2}\pi}{\sqrt{z_1 - z_3}} \quad (14)$$

The condition $m=0$, according to Eq. (11), implies that $z_2 = z_3$.

The roots of the cubic Eq. (10) satisfy

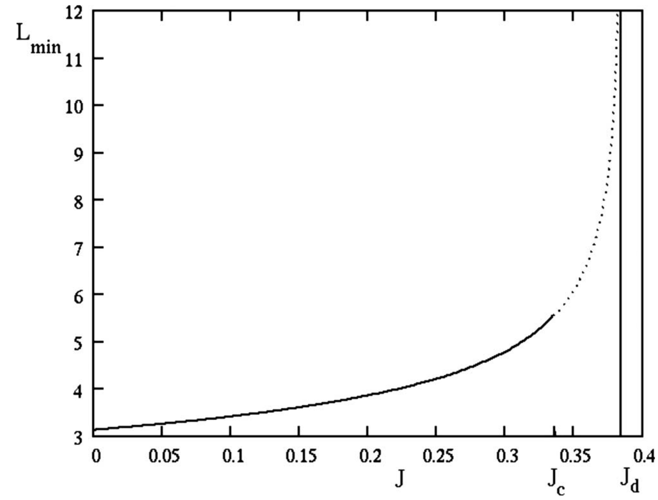

FIG. 1. Minimal size of a superconducting wire $L_{\min}(J)$ as a function of dc bias current in coherence length units. $L_{\min}(J)$ formally diverges as $J \to J_d$, where J_d is the depairing current density. The theory is restricted to the current density $J < J_c = 0.335$. Above this value, the phase slippage occurs (dash line).

$$z_1 + z_2 + z_3 = 2; \quad z_1 z_2 + z_1 z_3 + z_2 z_3 = 4E; \quad z_1 z_2 z_3 = 2J^2 \quad (15)$$

or, for $z_2 = z_3$,

$$z_1 + 2z_2 = 2; \quad 2z_1 z_2 + z_2^2 = 4E; \quad z_1 z_2^2 = 2J^2, \quad (16)$$

The first two equations in Eq. (16) yield

$$z_2 = z_3 = \frac{2}{3}(1 - \sqrt{1-3E}); \quad z_1 = \frac{2}{3}(1 + 2\sqrt{1-3E}), \quad (17)$$

which leads, according to Eq. (14), to

$$L_{\min} = \frac{\pi}{(1-3E)^{1/4}}. \quad (18)$$

From the last equation in Eqs. (16) and (17), one obtains the constant of integration E as function of the current J,

$$2(1-3E)^{3/2} - 3(1-3E) + 1 = \frac{27}{4} J^2. \quad (19)$$

If there is no current, $J=0$, Eq. (19) yields $(1-3E)=1$, and Eq. (18) yields $L_{\min} = \pi$. For a given current J, one has to solve Eq. (19) for $(1-3E)$, and insert the result into Eq. (18). This yield, for each current J, the minimal length of wire, $L_{\min}$, which remains superconducting. The $L_{\min}$-J phase diagram is shown in Fig. 1. The value of the critical superconducting length $L_{\min}$ diverges as the superconducting current density reaches the depairing current density $J_d = 2/3\sqrt{3}$ [see Eq. (19)]. There is no superconductivity in the wire when the current density in the wire exceeds the depairing current density [in this case Eq. (6) has the trivial solution $J=0$ and $\Psi = 0$).

The condition $\Delta\tau_\varepsilon \ll 1/40$ [see Eq. (2)] is a very severe constraint for superconductors, which can be satisfied only for so-called gapless superconductors. The latter corresponds to a situation in which the energy gap in the spectrum disappears, but the order parameter and supercurrent still exist. The mechanisms of gapless superconductivity might be either interaction with magnetic impurities which act differently on the electrons with opposite spins in a Cooper pair (in this case the parameter $\tau_\varepsilon\Delta \simeq \epsilon_F\Delta/y\Gamma_{ex}^2$ is usually small, (where ϵ_F is the Fermi energy, y is the concentration of paramagnetic impurities, and Γ_{ex} is the exchange interaction energy) or inelastic interactions with phonons (in this case $\tau_\varepsilon\Delta \simeq \omega_D^2\Delta/T^3$, where ω_D is the Debye energy). In the latter case, this parameter is usually large except in the narrow region at the critical temperature where the energy gap approaches to zero. The conversation length in this case is less than the coherence length only in the vicinity of the critical temperature; otherwise the electric field penetrates deep inside the wire. In this latter case, a substantial fraction of the nanowire is resistive even down to the lowest measured temperature.[19] Therefore, experiments to observe the current dependence of the minimum length of a superconducting wire must be performed under following conditions: (i) the superconducting wire should contain a small concentration of magnetic impurities (gapless superconductor) and (ii) the normal leads should be constructed from a metal having a small Fermi velocity (to keep the boundary condition $\Delta=0$ at the wire edges).

For superconducting Al with parameters: $y=0.01$, $\epsilon_F \simeq 5\times 10^4$ K, $\Gamma_{ex}\simeq 10^5$ K, $\Delta \approx 1$ K $\xi \simeq 10^{-5}$ cm, and $\lambda \simeq 10^{-6}$ cm, one obtains $\tau_\varepsilon\Delta \simeq 5\times 10^{-4}$. In this case the current density is measured in the units of $J_d \simeq 10^9$ A/cm^2.

In conclusion, we have found that if the order parameter is suppressed at the sample edges, then there is no superconductivity in a sample shorter than $L_{\min}$, the critical length of a small, current-carrying superconducting wire. This length, which was calculated analytically for arbitrary current density, depends crucially on the magnitude of the persistent current in the circle (see Fig. 1). In particular, for a gapless superconductor containing magnetic impurities, the critical value $L_{\min}$ is confined to the interval: $\pi \le L_{\min} < 5.54\xi$ $(0 \le J < 0.335)$. Above current density $J_c=0.335$, a phase slippage becomes important and nonstationary effects in the wire dominate.[10]

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