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# Gap generation and semimetal-insulator phase transition in graphene

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The gap generation is studied in suspended clean graphene in the continuum model for quasiparticles with the Coulomb interaction. We solve the gap equation with the dynamical polarization function and show that, comparing to the case of the static polarization function, the critical coupling constant lowers to the value $\alpha_c=0.92$, which is close to that obtained in lattice Monte Carlo simulations. It is argued that additional short-range four-fermion interactions should be included in the continuum model to account for the lattice simulation results. We obtain the critical line in the plane of electromagnetic and four-fermion coupling constants and find a second-order phase transition separating zero gap and gapped phases with critical exponents close to those found in lattice calculations.

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## I. INTRODUCTION

Graphene, a one-atom-thick layer of graphite, is a remarkable system with many unusual properties that was fabricated for the first time 5 yrs ago.[1] Theoretically, it was shown long time ago[2] that quasiparticle excitations in graphene have a linear dispersion at low energies and are described by the massless Dirac equation in 2+1 dimensions. The observation of anomalous integer quantum Hall effect in graphene[3] is in perfect agreement with the theoretical predictions[4] and became a direct experimental proof of the existence of gapless Dirac quasiparticles in graphene.

The unusual band structure of graphene has an important consequence for the electron-electron interaction in this material. In the continuum limit, graphene model on a honeycomb lattice, with both on-site and nearest-neighbor repulsions, maps onto a (2+1)-dimensional field theory of Dirac fermions interacting through the Coulomb potential plus, in general, some residual short-range interactions represented by local four-fermion terms. The vanishing density of states at the Dirac points ensures that the Coulomb interaction between the electrons in graphene retains its long-range character due to vanishing of the static polarization function for $q\to 0$.[5] The large value of the "fine-structure" coupling constant $\alpha=e^2/\hbar v_F\sim 1$ means that a strong attraction takes place between electrons and holes in graphene at the Dirac points. As is known, for graphene on a substrate with the effective coupling $\alpha/\kappa\ll 1$, $\kappa$ being a dielectric constant, the system is in a weak-coupling regime and exhibits semimetallic properties due to the absence of a gap in the electronic spectrum. Much less is known about suspended graphene where the coupling constant is large. In fact, suspended graphene provides a condensed-matter analog of strongly coupled quantum electrodynamics (QED) intensively studied in the 1970s and 1980s.[6–10] The dynamics of the vacuum in QED leads to many peculiar effects not yet observed in nature. Some QED-like effects such as zitterbewegung (trembling motion),[11] Klein tunneling,[12] Schwinger pair production,[13] and supercritical atomic collapse[14,15] have a chance to be tested in graphene (for experimental observation of the Klein tunneling in graphene, see Ref. 16). To observe these effects in graphene, it is important to use suspended and clean samples where charges from a substrate do not interfere with the dynamics of electrons.

Recently, it was shown in Ref. 17 that, for strong-enough coupling $\alpha>\alpha_c$, there is a tachyonic solution in the spectrum of the Bethe-Salpeter (BS) equation for the electron-hole bound state signaling the presence of excitonic instability of the zero-gap ground state of monolayer graphene in the supercritical regime. The critical coupling equals $\alpha_c=1/2$ if the vacuum polarization is neglected and $\alpha_c\approx 1.62$ in the random-phase approximation with the static polarization.[18] It was also shown there that physically, the excitonic instability is connected with the well-known supercritical Coulomb center problem[19] where $\alpha_c=1/2$ in two spatial dimensions.[20] The situation is similar to that in the theory of superconductivity,[21] where the four-fermion vertex instability has its origin in the Cooper pair problem. It was argued in Ref. 17 that the formation of an excitonic condensate of electron-hole pairs should cure the excitonic instability and lead to opening of a quasiparticle gap in a free-standing clean graphene resulting in dramatic changes in the transport properties. A similar situation occurs in QED in 3+1 dimensions where the gap generation takes place in the strong-coupling regime[6,7,10] (see also Refs. 8 and 9).

The problem of gap generation in graphene was considered before the actual fabrication of this material in Refs. 22–24, where the random-phase approximation with the static polarization function was used. Recently, lattice Monte Carlo simulations found the value of the critical coupling $\alpha_c=1.08$ for a semimetal-insulator transition[25] and this motivated us to reconsider the problem of gap generation in graphene. It was already indicated in Ref. 23 that taking into account the frequency-dependent polarization function should lower the critical coupling. We investigate this question in the present paper and confirm that the dynamical polarization is indeed quantitatively important: solving the gap equation with frequency-dependent polarization function, we find the critical coupling $\alpha_c=0.92$ instead of $\alpha_c=1.62$ in the case of static polarization. We would like to note also that the presence of a gap would be valuable for electronics applications, in particular, for working graphene transistors.

Another problem studied in the present paper is the order of phase transition connected with the gap generation in

 

graphene. Due to the scale invariance of the model with the Coulomb interaction, an infinite-order phase transition was found in Refs. 23 and 24. Such a phase transition belongs to the class of the so-called conformal phase transitions.[26] According to the recent Monte Carlo (MC) simulations[25] (for related MC simulations, see Ref. 27), the semimetal-insulator phase transition in graphene is of the second order. One of the reasons for such a difference might be lattice finite-size effects which can change the order of phase transition.[28–30] On the other hand, according to Refs. 31–33, the effective continuum theory for quasiparticles in graphene should contain besides the Coulomb interaction some additional contact four-fermion interaction terms that arise from the microscopic graphene lattice interactions. These terms contain a dimensionful parameter, therefore, they explicitly break the scale invariance of the continuum model. In such a case, one may expect a conventional second-order phase transition. In order to take into account these four-fermion interaction terms, we consider in the present paper the simplest Gross-Neveu interaction term and show that the presence of this interaction term plays an important role. First, instead of a critical point, we now have a critical line in the plane of electromagnetic and four-fermion coupling constants separating symmetric and symmetry-broken phases. Second, the inclusion of this term indeed changes the order of phase transition from infinite to the second order along a part of the critical line $0<\alpha<\alpha_c$. Third, it lowers the value of the critical electromagnetic coupling comparing to the case of purely Coulomb interaction. At last, the critical indices stay closer to those obtained in lattice simulations.[25]

The structure of the paper is the following. We begin with presentation in Sec. II of the continuum model describing graphene quasiparticles interacting through the Coulomb potential. In Sec. III, we solve the gap equation with the frequency-dependent one-loop polarization function and determine a critical coupling for the onset of a gap. To get insight into analytical solutions of the gap equation, we then turn back to the case of the static polarization and find asymptotical behavior of the gap function, calculate the excitonic condensate $\langle\bar{\Psi}\Psi\rangle$ of particle-hole pairs, the correlation length, and critical exponents near the phase-transition point $\alpha_c$. In Sec. V, we include the Gross-Neveu four-fermion interaction, find explicitly the critical line in the plane of Coulomb and four-fermion interaction coupling constants, and determine the critical exponents for the phase transition along this line. In Sec. VI, we summarize the main results.

## II. MODEL

For the description of the dynamics in graphene, we will use the same model as in Refs. 22 and 23 in which while quasiparticles are confined to a two-dimensional plane, the electromagnetic (Coulomb) interaction between them is three dimensional in nature. The low-energy quasiparticle excitations in graphene are conveniently described in terms of a four-component Dirac spinor $\Psi_a^T=(\psi_{KAa},\psi_{KBa},\psi_{K'Ba},\psi_{K'Aa})$ which combines the Bloch states with spin indices $a=1,2$ on the two different sublattices $(A,B)$ of the hexagonal graphene lattice and with momenta near the two-nonequivalent valley points $(K,K')$ of the two-dimensional Brillouin zone. In what follows, we treat the spin index as a "flavor" index with $N_f$ components, $a=1,2,\ldots,N_f$, then $N_f=2$ corresponds to graphene monolayer while $N_f=4$ is related to the case of two decoupled graphene layers, interacting solely via the Coulomb interaction.

The action describing graphene quasiparticles interacting through the Coulomb potential has the form

$$S=\int dtd^2r\bar{\Psi}_a(t,\mathbf{r})(i\gamma^0\partial_t-iv_F\boldsymbol{\gamma}\boldsymbol{\nabla})\Psi_a(t,\mathbf{r})$$
$$-\frac{1}{2}\int dtdt'd^2rd^2r'\bar{\Psi}_a(t,\mathbf{r})\gamma^0\Psi_a(t,\mathbf{r})U_0(t-t',|\mathbf{r}-\mathbf{r}'|)$$
$$\times\bar{\Psi}_b(t',\mathbf{r}')\gamma^0\Psi_b(t',\mathbf{r}'), \tag{2.1}$$

where $v_F$ is the Fermi velocity, $\bar{\Psi}=\Psi^\dagger\gamma^0$, and the $4\times4$ Dirac $\gamma$ matrices $\gamma^\mu=\tau^3\otimes(\sigma^3,i\sigma^2,-i\sigma^1)$ furnish a reducible representation of the Dirac algebra in $2+1$ dimensions. The bare Coulomb potential $U_0(t,|\mathbf{r}|)$ is given by

$$U_0(t,|\mathbf{r}|)=\frac{e^2\delta(t)}{\kappa}\int\frac{d^2k}{2\pi}\frac{e^{i\mathbf{kr}}}{|\mathbf{k}|}=\frac{e^2\delta(t)}{\kappa|\mathbf{r}|}. \tag{2.2}$$

However, the polarization effects considerably modify this bare Coulomb potential and the interaction will be

$$U(t,|\mathbf{r}|)=\frac{e^2}{\kappa}\int\frac{d\omega}{2\pi}\int\frac{d^2k}{2\pi}\frac{\exp(-i\omega t+i\mathbf{kr})}{|\mathbf{k}|+\Pi(\omega,\mathbf{k})}, \tag{2.3}$$

where $\kappa$ is the dielectric constant due to a substrate and the polarization function $\Pi(\omega,\mathbf{k})$ is proportional (within the factor $2\pi/\kappa$) to the time component of the photon polarization function. Correspondingly, the Coulomb propagator has the form

$$D(\omega,\mathbf{q})=\frac{1}{|\mathbf{q}|+\Pi(\omega,\mathbf{q})}, \tag{2.4}$$

where the one-loop polarization function is[5]

$$\Pi(\omega,\mathbf{k})=\frac{\pi e^2N_f}{4\kappa}\frac{\mathbf{k}^2}{\sqrt{\hbar^2v_F^2\mathbf{k}^2-\omega^2}} \tag{2.5}$$

and in the instantaneous approximation it becomes

$$\Pi(\omega=0,\mathbf{k})=\frac{\pi e^2N_f}{4\kappa\hbar v_F}|\mathbf{k}|. \tag{2.6}$$

The continuum effective theory described by the action (2.1) possesses $U(2N_f)$ symmetry. In the case of graphene, $N_f=2$, the corresponding 16 generators are (see, for example, Ref. 23)

$$\frac{\sigma^\alpha}{2}\otimes I_4,\quad \frac{\sigma^\alpha}{2i}\otimes\gamma^3,\quad \frac{\sigma^\alpha}{2}\otimes\gamma^5,\quad \frac{\sigma^\alpha}{2}\otimes\gamma^3\gamma^5, \tag{2.7}$$

where $I_4$ is the $4\times4$ Dirac unit matrix, and $\sigma^\alpha$, with $\alpha=0$, 1, 2, and 3, are four Pauli matrices connected with the spin degrees of freedom ($\sigma_0$ is the $2\times2$ unit matrix). However, as was pointed out in Ref. 31 (see also Refs. 32 and 33), this symmetry is not exact in the graphene tight-binding model

on lattice. In fact, there are small on-site interaction terms which break the $U(2N_f)$ symmetry; their role will be considered in Sec. V.

## III. GAP GENERATION AND THE CRITICAL COUPLING CONSTANT

In this section, we study spontaneous generation of a gap in the quasiparticle spectrum of graphene. The Schwinger-Dyson equation for the quasiparticle propagator has the form

$$S^{-1}(p_0,\mathbf{p}) = p_0\gamma^0 - \mathbf{p}\boldsymbol{\gamma} - \Delta_0 - ie^2\int \frac{d^3k}{(2\pi)^2} \times D(p_0-k_0,\mathbf{p}-\mathbf{k})\gamma^0 S(k_0,\mathbf{k})\gamma^0, \quad (3.1)$$

where the Coulomb propagator $D(q_0,\mathbf{q})$ is given by Eq. (2.4) and in the random-phase approximation, the polarization is taken as in Eq. (2.5). The vertex corrections are rather small[5] and we neglect them in what follows.

The general form of the propagator of quasiparticles is

$$S^{-1}(p_0,\mathbf{p}) = Z^{-1}p_0\gamma^0 - A\mathbf{p}\boldsymbol{\gamma} - \Delta, \quad (3.2)$$

where $Z$, $A$, and $\Delta$ are functions of $p_0$ and $\mathbf{p}$ and we included also a bare gap in Eq. (3.1) $\Delta_0$. We assume that a dependence of these functions on the energy $p_0$ is rather weak so that we can approximate these functions by their values at $p_0=0$. In this approximation, it is easy to see that $Z=1$, then after the Wick rotation, $k_0=i\omega$, we get a coupled system of equations for $A(\mathbf{p})$, $\Delta(\mathbf{p})$,

$$A(\mathbf{p}) = 1 + \frac{e^2}{\kappa\mathbf{p}^2}\int_{-\infty}^{\infty} d\omega \int \frac{d^2k}{(2\pi)^2} \times D(\omega,\mathbf{p}-\mathbf{k})\frac{\mathbf{pk}A(\mathbf{k})}{\omega^2+\mathbf{k}^2A^2(\mathbf{k})+\Delta^2(\mathbf{k})}, \quad (3.3)$$

$$\Delta(\mathbf{p}) = \Delta_0 + \frac{e^2}{\kappa}\int_{-\infty}^{\infty} d\omega \int \frac{d^2k}{(2\pi)^2} \times D(\omega,\mathbf{p}-\mathbf{k})\frac{\Delta(\mathbf{k})}{\omega^2+\mathbf{k}^2A^2(\mathbf{k})+\Delta^2(\mathbf{k})}. \quad (3.4)$$

We write the integral over $\omega$ as

$$I = \int_{-\infty}^{\infty} d\omega D(\omega,\mathbf{q})\frac{1}{\omega^2+\mathbf{k}^2A^2+\Delta^2} = \int_{-\infty}^{\infty}\frac{dxf(x)}{x^2\mathbf{q}^2+\mathbf{k}^2A^2+\Delta^2}, \quad f(x)=\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}+g}, \quad (3.5)$$

where $g=\pi N_f\alpha/4$. The function $f(x)$ changes slowly from $1/(1+g)$ at $x=0$ [the instantaneous approximation for $D(\omega,\mathbf{q})$] up to 1 at $x=\infty$. The integral in Eq. (3.5) can be evaluated exactly

$$I = \frac{1}{|\mathbf{q}|\sqrt{\mathbf{k}^2A^2+\Delta^2}}J(d,g), \quad d=\frac{\sqrt{\mathbf{k}^2A^2+\Delta^2}}{|\mathbf{q}|},$$

$$J(d,g) = \frac{(d^2-1)(\pi-gc(d))+dg^2c(g)}{d^2+g^2-1}, \quad (3.6)$$

where

$$c(x) = \frac{2\cosh^{-1}(x)}{\sqrt{x^2-1}}, \quad x>1, \quad c(x)=\frac{2\cos^{-1}(x)}{\sqrt{1-x^2}}, \quad x<1, \quad c(1)=2. \quad (3.7)$$

For $\Delta=0$ and setting $A=1$ on the right-hand side of Eq. (3.3), we get the leading one-loop correction,[34] which comes from the range of momenta $k\gg p$ in the integral

$$A(p) = 1 + \frac{2}{\pi^2 gN_f}[\pi-2g+(g^2-1)c(g)]\ln\frac{\Lambda}{p} + \text{finite terms}, \quad (3.8)$$

where $\Lambda$ is a momentum cutoff of order the inverse lattice spacing in graphene. The function $A(p)$ renormalizes the Fermi velocity $v_F^*(p)=v_FA(p)$. The growth of $v_F^*(p)$ in the infrared stops when a nonzero quasiparticle gap is taken into account [see Eq. (3.3)]. In what follows, we assume that the velocity renormalization is already performed[35] and put $A=1$ in Eq. (3.4) which then takes the form

$$\Delta(\mathbf{p}) = \Delta_0 + \frac{e^2}{\kappa}\int \frac{d^2k}{(2\pi)^2}\frac{\Delta(\mathbf{k})}{|\mathbf{p}-\mathbf{k}|\sqrt{\mathbf{k}^2+\Delta^2(\mathbf{k})}}J\left(d=\frac{|\mathbf{k}|}{|\mathbf{p}-\mathbf{k}|},g\right), \quad (3.9)$$

where we set also $\Delta=0$ in the variable $d$. Since the function $J$ depends weakly on the angle between the vectors $\mathbf{p}$ and $\mathbf{k}$, we can approximate $|\mathbf{p}-\mathbf{k}|\to\max(|\mathbf{p}|,|\mathbf{k}|)$. Thus we write

$$J\left(\frac{k}{\max(k,p)},g\right) = J(1,g)\theta(k-p) + J\left(\frac{k}{p},g\right)\theta(p-k). \quad (3.10)$$

Assuming $\Delta(\mathbf{p})=\Delta(|\mathbf{p}|)$ and integrating over the angle in Eq. (3.9), we get

$$\Delta(p) = \Delta_0 + \frac{\alpha}{\pi^2}\int_0^{\Lambda}\frac{dkk\Delta(k)}{\sqrt{k^2+\Delta^2(k)}}\mathcal{K}(p,k), \quad (3.11)$$

where the kernel

$$\mathcal{K}(p,k) = \frac{\theta(p-k)}{p}K\left(\frac{k}{p}\right)J\left(\frac{k}{p},g\right) + \frac{\theta(k-p)}{k}K\left(\frac{p}{k}\right)J(1,g) \quad (3.12)$$

where $K(x)$ is the complete elliptic integral of the first kind and $\theta(x)$ is the Heaviside step function. For zero bare gap, $\Delta_0=0$, Eq. (3.11) admits a nontrivial solution which bifurcates from the trivial one at some critical coupling $\alpha=\alpha_c$. To find this critical point, we neglect the terms quadratic or higher order in $\Delta$ in Eq. (3.11). It must be emphasized that this is *not* an approximation: it is a precise manner to locate

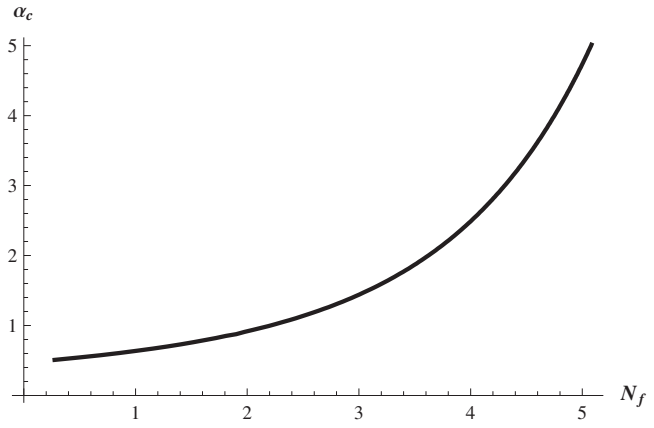


FIG. 1. The critical coupling as a function of $N_f$.

the critical point by applying bifurcation theory.[37] Hence, the bifurcation equation amounts to a linearization of Eq. (3.11) with respect to the gap function. The result reads

$$\Delta(p) = \frac{\alpha}{\pi^2}\int_0^\infty dk\Delta(k)\mathcal{K}(p,k). \tag{3.13}$$

Note that the ultraviolet cutoff, $\Lambda$, has been taken to infinity, which is appropriate at the bifurcation point.[37] This equation is scale invariant and is solved by $\Delta(p)=p^{-\gamma}$ on the condition that the exponent $\gamma$ satisfies the transcendental equation

$$\begin{aligned} 1 &= \frac{4g}{\pi^3 N_f}\int_0^\infty dx x^{-\gamma} \\ &\times\left[\theta(1-x)K(x)J(x,g) + J(1,g)\frac{\theta(x-1)}{x}K\left(\frac{1}{x}\right)\right] \\ &= \frac{4g}{\pi^3 N_f}\int_0^1 dx[x^{-\gamma}J(x,g) + J(1,g)x^{\gamma-1}]K(x), \quad 0<\gamma<1, \end{aligned} \tag{3.14}$$

where $J(x,g)$ is given by Eq. (3.6) and

$$J(1,g) = \begin{cases} \dfrac{2\arccos g}{\sqrt{1-g^2}}, & g \le 1 \\ \dfrac{\ln(g+\sqrt{g^2-1})}{\sqrt{g^2-1}}, & g \ge 1. \end{cases} \tag{3.15}$$

Equation (3.14) defines roots $\gamma$ for any value of the coupling $g$. A bifurcation occurs when two roots in (0,1) become equal. Numerically, we find that this happens for $\gamma=1/2$ and the critical coupling ($N_f=2$),

$$g_c = 1.445, \tag{3.16}$$

which corresponds to $\alpha_c=0.92$. For values $g>g_c$, the roots become complex, indicating that oscillatory behavior of the gap function takes over from nonoscillatory one. Equation (3.14) determines the critical line in the plane $(\alpha,N_f)$ which is presented in Fig. 1. This line should be compared to the critical line

$$\alpha_c = \frac{4\lambda_c}{2-\pi N_f\lambda_c} \tag{3.17}$$

obtained in Ref. 23 using the static polarization function [$\lambda_c=1/4$ in Ref. 23 for the kernel approximation (4,3) used below and $\lambda_c=0.23$ for more refined bifurcation analysis in Ref. 17]. The most crucial difference between two critical lines is that there is a critical number of flavors, $N_{crit}=2/\pi\lambda_c$, for the critical line (3.17) for which $\alpha=\infty$, while $\alpha$ never tends to infinity at finite $N_f$ for the critical line (3.14) presented in Fig. 1.

Recently, in Ref. 36, an approximation for the frequency-dependent one-loop polarization (2.5) was used which reduces it to Eq. (2.6) with additional $\sqrt{2}$ in the denominator, in this case the critical value $\alpha_c=1.13$. The more refined analysis using bifurcation theory gives $\alpha_c=0.93$ very close to the value we found above. We remind also that renormalization-group calculations in two loops yield $\alpha_c=0.833$.[38]

A dynamical gap is generated only if $\alpha>\alpha_c$. Since for suspended clean graphene the fine-structure constant $\alpha\approx 2.19$ is supercritical, the dynamical gap will be generated making graphene an insulator. Note that for graphene on a $SiO_2$ substrate, the dielectric constant $\kappa\approx 2.8$ and $\alpha\approx 0.78$, i.e., the system is in the subcritical regime. The value of $\alpha_c$ is rather large that implies that a weak-coupling approach might be quantitatively inadequate for the problem of the gap generation in suspended clean graphene. Therefore, it is instructive to compare our analytical results to lattice Monte Carlo studies,[25] where $\alpha_c=1.08\pm0.05$ for $N_f=2$ that is rather close to our analytical findings.

## IV. NONLINEAR EQUATION AND CRITICAL EXPONENTS

The above analysis is adequate precisely at the critical coupling, i.e., at the bifurcation point of the original nonlinear equation. To study momentum dependence of solutions of Eq. (3.11) beyond the critical point, we now turn back to the case of static polarization when $J=\pi/(1+g)$ and Eq. (3.11) is written in the form

$$\Delta(p) = \Delta_0 + \frac{2\lambda}{\pi}\int_0^\Lambda \frac{dkk\Delta(k)}{\sqrt{k^2+\Delta^2(k)}}\mathcal{K}(p,k),$$

$$\lambda = \frac{\alpha}{2(1+\pi N_f\alpha/4)}, \tag{4.1}$$

with the kernel [compare to Eq. (3.12)]

$$\mathcal{K}(p,k) = \frac{1}{p+k}K\left(\frac{2\sqrt{pk}}{p+k}\right) = \frac{\theta(p-k)}{p}K\left(\frac{k}{p}\right) + \frac{\theta(k-p)}{k}K\left(\frac{p}{k}\right). \tag{4.2}$$

The gap equation is essentially different from a gap equation in BCS theory where the gap is momentum independent. In Fig. 2, we present the results of our numerical solution to Eq. (4.1) for $\Delta_0=0$, $N_f=2$, and several values of $\lambda$. The gap is weakly dependent on a momentum up to values $p\sim\Delta(0)$ after which the behavior becomes steeper. To estimate the

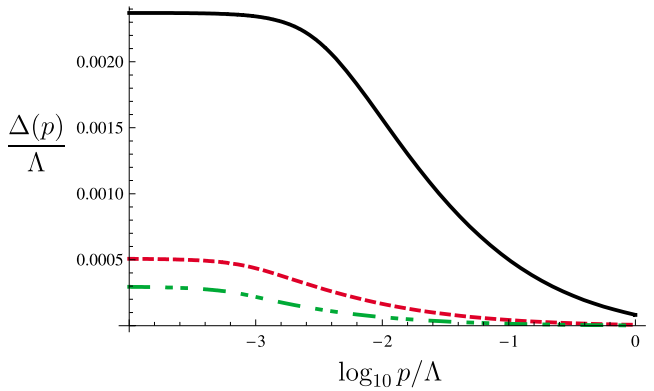


FIG. 2. (Color online) Momentum dependence of the solution to the gap equation (4.1) for $\Delta_0=0$, $N_f=2$: bold (black) line for $\lambda=0.3$, dashed (red) line for $\lambda=0.27$, and dash-dotted (green) line for $\lambda=0.25$.

gap $\Delta(0)$, we need to know the bandwidth parameter $\Lambda$ which can be obtained by equating the wave vector integral over the Brillouin zone to the integral over two Dirac points with a cutoff at $k_c$. We get $k_c=(\pi/\sqrt{3})^{1/2}(2/a)$, where $a$ is the lattice constant, therefore, restoring $\hbar$ and $v_F=\sqrt{3}ta/(2\hbar)$, we find $\Lambda=\hbar v_F k_c=\sqrt{\pi\sqrt{3}}t\approx 2.33t$. For the hopping parameter $t=3$ eV, we obtain $\Lambda\approx 7$ eV. The maximal possible gap $\Delta(0)$ is reached for $\alpha\to\infty$ that corresponds to the value $\lambda=1/\pi\approx 0.32(N_f=2)$. For the values of $\lambda$'s used in Fig. 2, we find the estimates $\Delta(0)=200$, 40, and 25 K for $\lambda=0.3$, 0.27, and 0.25, respectively.

To get insight into analytical solutions of Eq. (4.1), we approximate the elliptic integral functions in Eq. (4.2) by their asymptotical values at $p\ll k$ and $p\gg k$. We obtain the kernel

$$\mathcal{K}(p,k)=\frac{\pi}{2}\left(\frac{\theta(p-k)}{p}+\frac{\theta(k-p)}{k}\right). \tag{4.3}$$

This allows us to reduce the nonlinear integral Eq. (4.1) to the second-order nonlinear differential equation

$$[p^2\Delta'(p)]'+\lambda\frac{p\Delta(p)}{\sqrt{p^2+\Delta^2(p)}}=0, \tag{4.4}$$

with the infrared (IR) and ultraviolet (UV) boundary conditions

$$p^2\Delta'(p)|_{p=0}=0, \tag{4.5}$$

$$[p\Delta(p)]'|_{p=\Lambda}=\Delta_0. \tag{4.6}$$

Equation (4.4) is scale invariant, i.e., if $\Delta(p)$ is a solution, then $l\Delta(p/l)$ is also a solution. The scale invariance is broken by the UV boundary condition only.

The chiral condensate $\langle 0|\bar{\Psi}\Psi|0\rangle$ is the order parameter for the semimetal-insulator transition in graphene. It breaks spontaneously the initial $U(2N_f)$ symmetry down to the $U(N_f)\otimes U(N_f)$ but keeps parity and time-reversal invariances. It is expressed through the full fermion propagator as follows:

$$\begin{aligned}\langle 0|\bar{\Psi}\Psi|0\rangle &= -\lim_{x\to 0}\langle 0|T\Psi(x)\bar{\Psi}(0)|0\rangle\\ &= -i\mathrm{tr}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int^{\Lambda}\frac{d^2p}{(2\pi)^2}G(\omega,\mathbf{p})\\ &= -\frac{N_f}{\pi}\int_0^{\Lambda}\frac{dpp\Delta(p)}{\sqrt{p^2+\Delta^2(p)}}\\ &= \frac{N_f}{\pi\lambda}p^2\Delta'(p)|_{p=\Lambda},\end{aligned} \tag{4.7}$$

where for the last equality, we used Eq. (4.4). Hence, the condensate is nontrivial if a nontrivial solution of the gap equation exists.

One can easily find the solutions of Eq. (4.4) in two asymptotic regions. For $p\ll\Delta(p)$,

$$\Delta(p)=C_1+\frac{C_2}{p}. \tag{4.8}$$

The IR boundary condition (4.5) implies $C_2=0$, therefore, $\Delta(p)\simeq C_1$ for $p\ll\Delta(p)$. For $p\gg\Delta(p)$,

$$\Delta(p)\simeq C_3p^{-\gamma_+}+C_4p^{-\gamma_-},\quad \gamma_\pm=\frac{1}{2}\pm\sqrt{\lambda_c-\lambda}. \tag{4.9}$$

Clearly, in order to find a solution of Eq. (3.11), one needs to show that there exists a solution of the nonlinear differential Eq. (4.4) which connects the asymptotic $\Delta(p)\simeq$const in the infrared region, $p\to 0$, with the asymptotic (4.9) at large momenta. For this, let us define

$$\Delta(p)=e^tu(t+t_0),\quad t=\ln p, \tag{4.10}$$

then the function $u(t)$ satisfies the differential equation

$$u''+3u'+2u+\lambda\frac{u}{\sqrt{1+u^2}}=0. \tag{4.11}$$

The IR boundary condition implies

$$e^{2t}(u'+u)|_{t=-\infty}=0. \tag{4.12}$$

We require that $e^tu(t)\to 1$ as $t\to-\infty$ since all other solutions for $\Delta(p)$ are obtained by varying the constant $t_0$. For this normalization, the infrared scale for the general solution is given by $\Delta(0)=e^{-t_0}$.

The dependence of the integral Eq. (3.11) on the bare gap $\Delta_0$ now becomes an ultraviolet boundary condition for the differential equation; it is

$$u'(t_\Lambda+t_0)+2u(t_\Lambda+t_0)=\Delta_0/\Lambda. \tag{4.13}$$

This condition determines the value of the parameter $t_0=-\ln\Delta(0)$ as a function of the coupling constant, $\lambda$, the bare gap, $\Delta_0$, and the cutoff, $\Lambda$. Equation (4.11) can be rewritten in the form

$$u''+3u'=-\frac{d}{du}V(u),\quad V(u)=u^2+\lambda\sqrt{1+u^2} \tag{4.14}$$

or, equivalently,

$$\left(\frac{1}{2}(u')^2 + V(u)\right)' = -3(u')^2. \tag{4.15}$$

Equation (4.14) is the equation for a particle of unit mass moving in a potential $V$ with friction proportional to velocity. The "energy" $\frac{1}{2}(u')^2+V(u)$ reaches its absolute minimum at $u=0$, hence, the particle moves toward $u=0$ damped by the friction. The asymptotical behavior near $u=0$ is described by the linearized equation

$$u'' + 3u' + (2+\lambda)u = 0 \tag{4.16}$$

and depends on whether the coupling $\lambda > \lambda_c \equiv 1/4$ or $\lambda < \lambda_c$,

$$u(t) \to \frac{B}{\sqrt{\lambda_c - \lambda}} e^{-3t/2} \sinh[\sqrt{\lambda_c - \lambda}(t+\delta)],$$

$$t \to \infty, \quad \text{weak coupling}(\lambda < \lambda_c), \tag{4.17}$$

$$u(t) \to \frac{A}{\sqrt{\lambda - \lambda_c}} e^{-3t/2} \sin[\sqrt{\lambda - \lambda_c}(t+\delta)],$$

$$t \to \infty, \quad \text{strong coupling } \lambda \geq \lambda_c, \tag{4.18}$$

where the constants $A$, $B$, and $\delta$ are functions of the coupling constant $\lambda$. We explicitly singled out the factor $1/\sqrt{\lambda_c - \lambda}$ in front of Eqs. (4.17) and (4.18) since the function $u(t)$ must be nontrivial at $\lambda = \lambda_c$. Obviously, $A(\lambda=\lambda_c) = B(\lambda=\lambda_c)$.

The asymptotics (4.17) and (4.18) imply that at weak coupling, the particle situated initially at $u(-\infty)$ reaches $u=0$ for infinite time, meanwhile, at strong coupling, it will get to $u=0$ in a finite time and then oscillate there with damped amplitude.

It is easy to see that at weak coupling, there are no nontrivial solutions satisfying the UV boundary condition for $\Delta_0 = 0$. For strong coupling, the UV boundary condition (4.13) with $\Delta_0 = 0$ admits an infinite number of solutions for the gap scale $\Delta(0)$, corresponding to different solutions of the equation

$$u'(t_\Lambda + t_0) + 2u(t_\Lambda + t_0) \approx \frac{A\sqrt{\lambda}}{\sqrt{\lambda - \lambda_c}} e^{-3(t_\Lambda + t_0)/2} \sin(\theta + \phi) = 0, \tag{4.19}$$

where

$$\theta = \sqrt{\lambda - \lambda_c}(t_\Lambda + t_0 + \delta) = \sqrt{\lambda - \lambda_c} \ln\left(\frac{e^\delta \Lambda}{\Delta(0)}\right),$$

$$\phi = \arctan(2\sqrt{\lambda - \lambda_c}). \tag{4.20}$$

Hence, the solution is given by $\theta = \pi n - \phi$ or

$$\Delta(0) = \Lambda e^\delta \exp\left(-\frac{\pi n - \phi}{\sqrt{\lambda - \lambda_c}}\right), \quad n = 1, 2, \ldots. \tag{4.21}$$

The solution without nodes, $n=1$, corresponds to the ground state since it generates the largest fermion gap and has the lowest energy. The critical coupling $\lambda_c = 1/4$ is a bifurcation point of the integral Eq. (3.11) with the static vacuum polarization.[39] The expression (4.21) for the gap implies that this bifurcation point corresponds to a continuous phase transition of infinite order. As was shown in Ref. 17, the critical coupling $\lambda_c$ is closely related to the phenomenon "fall into the center" in quantum mechanics problem. A similar situation takes place in the strong coupling QED4 (Ref. 7) where in the ladder approximation (and more generally in quenched approximation when fermions loops are neglected[40]), the phase transition is also of infinite order. The dimensionless correlation length,

$$\xi = \frac{\Lambda}{\Delta(0)} \sim \exp\left(\frac{\pi}{\sqrt{\lambda - \lambda_c}}\right), \tag{4.22}$$

exponentially grows when $\lambda \to \lambda_c$. Such a behavior is inherent for the Berezinskii-Kosterlitz-Thouless phase transition [or the conformal phase transition (CPT) (Ref. 26)] and, obviously, is related to the scale invariance of the problem under consideration. Note, however, that taking into account the finite size of graphene samples should turn this phase transition into a second-order one (as it is shown for QED3 in Ref. 28).

Equation (4.22) means also that in nonperturbative phase, there is a nontrivial running of the coupling $\alpha$ (or the Fermi velocity $v_F$) though we neglected its perturbative running. Defining the $\beta$ function in a standard way, we find

$$\beta(\alpha) \equiv \Lambda \frac{d\alpha}{d\Lambda} = -\frac{\pi}{4}(1 + \pi N_f \alpha/4)^2(\lambda - \lambda_c)^{3/2}, \quad \lambda > \lambda_c, \tag{4.23}$$

where $\lambda$ is defined in Eq. (4.1). The $\beta$ function depends nonanalytically on the coupling $\alpha$ and cannot be obtained in perturbation theory. We expect that if a perturbative running of $\alpha$ is included, the critical point $\lambda_c$ becomes a second-order phase-transition point on which the $\beta$ function (4.23) is continuous when approached from both perturbative and nonperturbative phases.

The order parameter $\langle \bar{\Psi}\Psi \rangle$ in terms of the function $u(t)$ is given by

$$\langle \bar{\Psi}\Psi \rangle = \frac{N_f}{\pi\lambda} e^{2t_\Lambda}[u'(t_\Lambda + t_0) + u(t_\Lambda + t_0)] \tag{4.24}$$

and equals

$$\langle \bar{\Psi}\Psi \rangle = -\frac{N_f A}{\pi\sqrt{\lambda(\lambda - \lambda_c)}} \Lambda^{1/2} \Delta^{3/2}(0) \sin(2\phi)$$

$$= -\frac{N_f A}{\pi\lambda^{3/2}} \Lambda^{1/2} \Delta^{3/2}(0), \tag{4.25}$$

where the relation $\theta = \pi - \phi$ was used.

For nonzero bare gap, $\Delta_0 \neq 0$, we obtain the following equation for the scale $\Delta(0)$:

$$\Delta_0 = \frac{A\sqrt{\lambda}}{\sqrt{\lambda - \lambda_c}} \frac{\Delta^{3/2}(0)}{\sqrt{\Lambda}} \sin(\theta + \phi) \tag{4.26}$$

and the order parameter

$$\langle\bar{\Psi}\Psi\rangle = \frac{N_f}{\pi\lambda}\Lambda\left[\Delta_0 - \frac{A}{\sqrt{\lambda-\lambda_c}}\frac{\Delta^{3/2}(0)}{\sqrt{\Lambda}}\sin\theta\right]. \quad (4.27)$$

Let us write $\theta+\phi=\pi-\epsilon$, where $\epsilon$ tends to zero when $\Delta_0\to 0$ and $\lambda\to\lambda_c$. Then the above equations are rewritten as

$$\Delta_0 = \frac{A\sqrt{\lambda}}{\sqrt{\lambda-\lambda_c}}\frac{\Delta^{3/2}(0)}{\sqrt{\Lambda}}\sin\epsilon, \quad (4.28)$$

$$\langle\bar{\Psi}\Psi\rangle = \frac{N_f\Lambda}{\pi\lambda}\left[\frac{2\lambda-1}{2\lambda}\Delta_0 - A\frac{\Delta^{3/2}(0)}{\sqrt{\Lambda}}\frac{\cos\epsilon}{\sqrt{\lambda}}\right]. \quad (4.29)$$

In such a form, the equations are convenient for finding critical exponents near the phase-transition point $\lambda_c$. Critical exponents describe the approach to criticality of such quantities as the correlation length, the order parameter, the susceptibility, etc.; they are defined in a standard way[10,41]

$$\xi = \frac{\Lambda}{\Delta(0)} \sim (\lambda-\lambda_c)^{-\nu}, \quad \frac{\langle\bar{\Psi}\Psi\rangle}{\Lambda^2} \sim (\lambda-\lambda_c)^{\beta},$$

$$\chi = \left.\frac{\partial\langle\bar{\Psi}\Psi\rangle}{\partial\Delta_0}\right|_{\Delta_0=0} \sim (\lambda-\lambda_c)^{-\gamma}, \quad \lambda\to\lambda_c, \quad (4.30)$$

$$\langle\bar{\Psi}\Psi\rangle|_{\lambda=\lambda_c} \sim \Delta_0^{1/\delta}, \quad \Delta_0\to 0. \quad (4.31)$$

If the theory of second-order phase transition is applicable, then the exponents are assumed to obey the following hyperscaling relations in spaces of arbitrary dimension $D$:

$$2\beta+\gamma = D\nu, \quad 2\beta\delta-\gamma = D\nu, \quad \frac{\delta-1}{\delta+1} = \frac{2-\eta}{D},$$

$$\beta = \nu\frac{D-2+\eta}{2}. \quad (4.32)$$

Here, the exponent $\eta$ describes the behavior of the correlation function

$$\langle\bar{\Psi}\Psi(r)\bar{\Psi}\Psi(0)\rangle|_{\lambda=\lambda_c} \propto r^{-D+2-\eta}, \quad r\to\infty. \quad (4.33)$$

Using Eq. (4.29), we find

$$\langle\bar{\Psi}\Psi\rangle|_{\lambda=\lambda_c} = -\frac{4N_f\Lambda}{\pi}\left[\Delta_0 + \frac{2A\Delta^{3/2}(0)}{\sqrt{\Lambda}}\right] \quad (4.34)$$

and due to Eq. (4.26),

$$\Delta(0) \sim \left(\frac{\Delta_0}{\ln(\Lambda/\Delta_0)}\right)^{2/3}, \quad \lambda=\lambda_c. \quad (4.35)$$

The critical exponent $\delta=1$ and from hyperscaling relations we obtain

$$\eta = 2, \quad \gamma = 0, \quad \beta = \frac{3\nu}{2}. \quad (4.36)$$

The infinite-order phase transition with the correlation length (4.22) formally corresponds to the limit

$$\beta = \frac{3\nu}{2} \to \infty. \quad (4.37)$$

Certainly, the infinite-order phase transition is quite different from that one studied in lattice simulations[25] where a second-order phase transition was found with the critical exponents $\delta\sim 2.3$, $\beta\sim 0.8$, and $\gamma\sim 1 (N_f=2)$. One of the reasons for such a difference might be a finite size of a lattice that changes the kind of phase transition. Effectively, the finite size of a lattice can be taken into account by introducing an infrared cutoff ($k_0\sim\pi/L$, $L$ is a linear size of the system) in the integral Eq. (3.11).[28,29] Another reason could be that one should take into account residual lattice interactions, i.e., the present analysis has to be further refined by incorporating effective four-fermion terms (see the next section).

## V. PHASE DIAGRAM IN THE MODEL WITH ADDITIONAL FOUR-FERMION INTERACTION

As discussed in Sec. I, when comparing the results of lattice simulations[25] to analytical calculations, one should have in mind that the continuum theory described by Lagrangian (2.1) putted on a lattice contains unavoidably additional interaction terms, in particular, local four-fermion interaction terms. This means that it would be appropriate to add to the continuum theory some local four-fermion interaction terms in addition to the long-range Coulomb interaction in Eq. (2.1). The amount of induced couplings depends of course on the particular lattice regularization employed. Furthermore, according to Refs. 31–33, the effective continuum model for quasiparticles in graphene in addition to the Coulomb interaction should contain contact four-fermion interaction terms which arise from the original lattice tight-binding model. Usually, these residual four-fermion terms are irrelevant operators from the point of view of the renormalization group, however, we will show that they can play a significant role in the critical behavior. In order to study how these four-fermion terms influence the gap generation, we will consider in this section a continuum model with the Coulomb interaction and the Gross-Neveu four-fermion interaction of the type

$$\mathcal{L}_4 = \frac{G}{2}(\bar{\Psi}_a\Psi_a)^2, \quad (5.1)$$

where the four-fermion coupling constant $G$ is of the order of the lattice constant and the "flavor" index $a=1,2,\ldots,N_f$, $N_f=2$ for physical spin-$\frac{1}{2}$ electrons. The interaction term (5.1) breaks the initial $U(2N_f)$ symmetry of the action (2.1) down to the $U(N_f)\otimes U(N_f)\otimes Z_2$ symmetry. While the gap term $\Delta\bar{\Psi}\Psi$ is invariant under the $U(N_f)\otimes U(N_f)$, it is not under the discrete chiral $Z_2$ symmetry: $\Psi\to\gamma_5\Psi$, $\bar{\Psi}\to-\bar{\Psi}\gamma_5$. In the absence of the bare gap term, $\Delta_0\bar{\Psi}\Psi$, the $Z_2$ symmetry forbids the fermion gap generation in perturbation theory. The appearance of the energy gap is due to the spontaneous breaking of the above discrete chiral symmetry that leads to a neutral condensate $\langle\bar{\Psi}\Psi\rangle$ of fermion-antifermion pairs (excitonic condensate).

The gap Eq. (3.11) is modified in the presence of the interaction (5.1) in the following way:

$$\Delta(p) = \Delta_0 - G\left(1 - \frac{1}{4N_f}\right)\langle\bar{\Psi}\Psi\rangle + \frac{\alpha}{\pi^2}\int_0^\Lambda \frac{dkk\Delta(k)}{\sqrt{k^2+\Delta^2(k)}}\mathcal{K}(p,k), \tag{5.2}$$

where the condensate $\langle\bar{\Psi}\Psi\rangle$ contributes like a bare fermion gap and can be computed from the fermion self-energy; the factor $1-1/4N_f$ in the second term on the right-hand side takes into account both Hartree and Fock $(-1/4N_f)$ contributions. For the sake of simplicity, we consider only the Hartree term, if necessary, the Fock contribution can be easily restored in final formulas.[42] In the approximation to the kernel used above [Eq. (4.3)], the condensate is given by the expression (4.7). The condensate does not change the differential Eq. (4.4), however, it modifies the ultraviolet boundary condition (4.6):

$$\left[\left(1 + \frac{\tilde{g}N_f}{\lambda}\right)p\Delta'(p) + \Delta(p)\right]\Big|_{p=\Lambda} = \Delta_0, \tag{5.3}$$

where we introduced the notation $\tilde{g}=G\Lambda/\pi$ and $\lambda$ is defined in Eq. (4.4). Using the definition of the gap function in terms of the $u(t)$ function (4.10) and the asymptotic behavior of the last one, Eqs. (4.17) and (4.18), Eq. (5.3) can be written for $\Lambda \gg \Delta(0)$ in the following form:

$$B\frac{\Delta^{3/2}(0)}{\sqrt{\Lambda}}\left[\left(1 + \frac{\tilde{g}N_f}{\lambda}\right)\cosh\left(\omega \ln\frac{\Lambda e^{\delta}}{\Delta(0)}\right) + \frac{1-\tilde{g}N_f/\lambda}{2\omega}\sinh\left(\omega \ln\frac{\Lambda e^{\delta}}{\Delta(0)}\right)\right] = \Delta_0,$$

$$\omega = \sqrt{\lambda_c - \lambda}, \quad \lambda < \lambda_c, \tag{5.4}$$

$$B\frac{\Delta^{3/2}(0)}{\sqrt{\Lambda}}\left[1 + \frac{\tilde{g}N_f}{\lambda} + \frac{1-\tilde{g}N_f/\lambda}{2}\ln\frac{\Lambda e^{\delta}}{\Delta(0)}\right] = \Delta_0, \quad \lambda = \lambda_c, \tag{5.5}$$

$$A\frac{\Delta^{3/2}(0)}{\sqrt{\Lambda}}\left[\left(1 + \frac{\tilde{g}N_f}{\lambda}\right)\cos\left(\tilde{\omega} \ln\frac{\Lambda e^{\delta}}{\Delta(0)}\right) + \frac{1-\tilde{g}N_f/\lambda}{2\tilde{\omega}}\sin\left(\tilde{\omega} \ln\frac{\Lambda e^{\delta}}{\Delta(0)}\right)\right] = \Delta_0,$$

$$\tilde{\omega} = \sqrt{\lambda - \lambda_c}, \quad \lambda > \lambda_c, \tag{5.6}$$

and we remind that in the utilized approximation $\lambda_c=1/4$. These equations imply the following solutions for the dynamical gap in the case $\Delta_0=0$:

$$\Delta(0) = \Lambda e^{\delta}\left[\frac{\tilde{g}N_f(1-2\omega) - \lambda(1+2\omega)}{\tilde{g}N_f(1+2\omega) - \lambda(1-2\omega)}\right]^{1/2\omega}, \quad \lambda < \lambda_c, \tag{5.7}$$

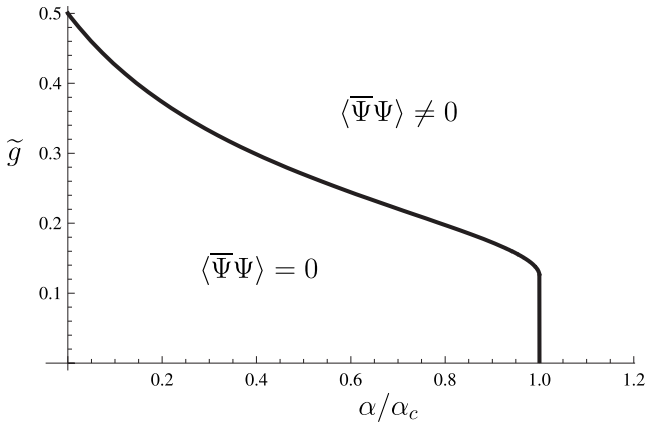


FIG. 3. Phase diagram for $N_f=2$.

$$\Delta(0) = \Lambda e^{\delta}\exp\left[-2\frac{\tilde{g}N_f + 1/4}{\tilde{g}N_f - 1/4}\right], \quad \lambda = \lambda_c = \frac{1}{4}, \quad \tilde{g}N_f > \frac{1}{4}, \tag{5.8}$$

$$\Delta(0) = \Lambda e^{\delta}\exp\left[-\frac{\pi n}{\tilde{\omega}} - \frac{1}{\tilde{\omega}}\arctan\left(2\tilde{\omega}\frac{\tilde{g}N_f + \lambda}{\tilde{g}N_f - \lambda}\right)\right],$$

$$\lambda > \lambda_c, \quad n = 1,2,\ldots. \tag{5.9}$$

Note that solutions (5.8) and (5.9) contain essential singularity; the first solution at $\tilde{g}=1/4N_f$ and the second one at $\lambda=\lambda_c$.

Setting $\Delta(0)=0$, we find the critical line separating the spontaneously broken and unbroken phases of the chiral symmetry

$$\begin{cases} \tilde{g} = \dfrac{1}{4N_f}(1+\sqrt{1-\lambda/\lambda_c})^2, & \text{for} \quad \lambda \le \lambda_c = \dfrac{1}{4} \\ \tilde{g} < \dfrac{1}{4N_f}, & \lambda = \lambda_c = \dfrac{1}{4}. \end{cases} \tag{5.10}$$

The phase diagram in the plane of two coupling constants is displayed in Fig. 3.[42] Above the critical line, the gap equation for the fermion self-energy $\Delta(p)$ has a nontrivial solution. Thus, the chiral symmetry is dynamically broken that implies the existence of a nonzero vacuum condensate $\langle\bar{\Psi}\Psi\rangle$. For $\tilde{g}=0$, the condition for a gap generation becomes $\lambda>\lambda_c$ and the corresponding critical coupling coincides with Eq. (3.17). On the other hand, in the other limiting case $\alpha=0$, $\tilde{g}_c=1/N_f$ coincides with the critical coupling in the Gross-Neveu model. In the part of the phase diagram above the critical line $\lambda<\lambda_c$, the short-range four-fermion interactions play important role for the condensate formation, meanwhile, in the region $\lambda>\lambda_c$, Coulomb forces are mainly responsible for the condensate formation.

We consider now the phase transition along the upper part of the critical line and compute the critical exponents. Since we consider nonrunning coupling $\alpha$ (in absence of running of

the Fermi velocity $v_F$), the renormalization-group flow can be determined from Eq. (5.7) which near a critical point takes the form

$$\frac{\Delta(0)}{\Lambda} \sim \left(\frac{\tilde{g}-\tilde{g}_1}{\tilde{g}-\tilde{g}_2}\right)^{1/2\omega}, \quad \tilde{g}_1 = \frac{(1+2\omega)^2}{4N_f}, \quad \tilde{g}_2 = \frac{(1-2\omega)^2}{4N_f},$$

$$\tilde{g} > \tilde{g}_1 > \tilde{g}_2. \tag{5.11}$$

It implies an explicit form of the $\beta$ function for the coupling $\tilde{g}$,

$$\beta(\tilde{g},\alpha) \equiv \Lambda \frac{\partial \tilde{g}}{\partial \Lambda}\bigg|_{\alpha,\Delta(0)} = -N_f(\tilde{g}-\tilde{g}_1)(\tilde{g}-\tilde{g}_2), \quad \tilde{g} > \tilde{g}_1. \tag{5.12}$$

Equation (5.12) has indeed a nontrivial fixed line at $\tilde{g}=\tilde{g}_1$. We stress that the $\beta$ function (5.12) is obtained in nonperturbative phase where a quasiparticle gap is spontaneously generated. In perturbative phase, the $\beta$ function was calculated in Ref. 43 [see Eq. (7) there], in the leading order in $1/N_f$ and small coupling $\alpha$, both $\beta$ functions behave as $\beta \simeq -(g-g_0)$ near the fixed point $g_0 \sim (1-\alpha)/N_f$. As is seen from Eq. (5.11), the phase transition is of the second order. Denoting the deviation from the critical line as $\tau \equiv \tilde{g}-\tilde{g}_1$ and because $\Delta(0) \sim \tau^{1/2\omega} (\tau \to 0)$, we find the exponent

$$\nu = \frac{1}{2\omega}. \tag{5.13}$$

The condensate is given by

$$\langle \bar{\Psi}\Psi \rangle = \frac{N_f B}{\pi\lambda} \Delta^{3/2}(0) \Lambda^{1/2} \times \left[ \cosh\left(\omega \ln \frac{\Lambda e^{\delta}}{\Delta(0)}\right) - \frac{1}{2\omega} \sinh\left(\omega \ln \frac{\Lambda e^{\delta}}{\Delta(0)}\right)\right]. \tag{5.14}$$

On the critical line, Eq. (5.4) implies $\Delta(0) \sim \Delta_0^{1/(3/2+\omega)}$. Substituting it into the expression for the fermion condensate gives the critical scaling relation

$$\langle \bar{\Psi}\Psi \rangle \sim \Delta^{3/2-\omega} \sim \Delta_0^{(3/2-\omega)/(3/2+\omega)}, \tag{5.15}$$

thus the critical exponent

$$\delta = \frac{3/2+\omega}{3/2-\omega}. \tag{5.16}$$

It is equal to $\delta=2$ in the case of the Gross-Neveu model when $\alpha=0$ and $\delta \to 1$ for $\alpha \to \alpha_c$. It is easy to find also the $\beta$ exponent

$$\beta = \frac{1}{2\omega}\left(\frac{3}{2}-\omega\right). \tag{5.17}$$

Finally, it follows from Eqs. (5.4) and (5.14) that

$$\partial_{\Delta_0} \langle \bar{\Psi}\Psi \rangle|_{\Delta_0=0} \sim \tau^{-1}, \quad \tau \to 0, \tag{5.18}$$

hence the exponent $\gamma=1$. The found critical exponents satisfy the hyperscaling relations (4.32). The additional critical exponent $\eta$ may be calculated independently or using hyperscaling relations, $\eta=2-2\omega$. By definition, the anomalous dimension $\gamma_m$ of the composite operator is given by $\dim(\bar{\Psi}\Psi)=D-1-\gamma_m$, then the correlator (4.33) implies the relation $\eta=D-2\gamma_m$. In our case, $D=3$, we obtain $\gamma_m=1/2+\omega$. The dynamical dimension of the four-fermion interaction term equals $\dim(\bar{\Psi}\Psi)^2=2\dim(\bar{\Psi}\Psi)=4-2\gamma_m$. Because $1/2 \le \gamma_m \le 1$ along the critical line, $\dim(\bar{\Psi}\Psi)^2 \le 3$ and the four-fermion operator $(\bar{\Psi}\Psi)^2$ acts as a renormalizable one. In the renormalization-group terminology, the $(\bar{\Psi}\Psi)^2$ becomes a relevant operator in the scaling region while it is irrelevant away from the critical line, in accordance with standard renormalization-group approach,[32] as its effects are suppressed by powers of cutoff. On the other hand, the anomalous dimension $\gamma_m$ governs the behavior of the amputated Bethe-Salpeter wave function (form factor) of bound states, $\chi^{(amp)}(q) \sim [q/\Delta(0)]^{\gamma_m-1}$, in the range of momenta $\Delta(0) \ll q \ll \Lambda$. The "critical" value $\gamma_m=1/2$ separates loose ($\gamma_m<1/2$) and tight ($\gamma_m \ge 1/2$) bound states. The wave functions with large $\gamma_m (\gamma_m>1/2)$ slowly decrease with momentum. They describe tight bound states which are relevant for critical scaling laws of a theory.[44] Since such bound states resemble pointlike particles, the scaling properties of a theory can be described by an effective Lagrangian with elementary scalar fields (for recent such an approach, see, Ref. 45). The computer simulations of lattice graphene model may reveal in principle the existence of such tight bound states.

We see that the additional Gross-Neveu four-fermion interaction plays an important role. First, it changes the order of a phase transition from the infinite to the second-order one. Second, the critical coupling becomes lower than in the model with the pure Coulomb interaction. Third, the critical exponents stay closer to those obtained in lattice simulations.[25] Further, the critical indices depend on the coupling $\alpha$ along the critical line $0<\alpha<\alpha_c$ and satisfy the hyperscaling relations. The phase diagram (5.10) resembles closely those obtained in the strong coupling QED4 (Ref. 46) and QED3.[47] Since the phase transition is of second order along the $0<\alpha<\alpha_c$ part of the critical curve, Eq. (5.10), resonances should exist on the symmetric side of the curve, whose masses tend to zero as the critical curve is approached.[48,49] The part of the critical curve with $\tilde{g}<1/4N_f$ is rather special and is related to the conformal phase transition.[26] It is characterized by a gap function having an essential singularity at the transition point and by abrupt change of the spectrum of light excitations as the critical point is crossed: light bound states near the critical line are absent in the symmetric phase, however, they are present in the phase with broken symmetry (for a discussion in detail of the CPT in QED3, see Refs. 48 and 50). The corresponding effective potential for the order parameter $\langle \bar{\Psi}\Psi \rangle$, unlike the familiar Ginzburg-Landau potential, was shown to have a branched fractal structure in the region $\alpha>\alpha_c$, where the Coulomb interaction is mainly responsible for the bound states formation.[24] We hope that at least some features of the picture outlined above will be confirmed in experiments with suspended clean graphene.

## VI. CONCLUSIONS

In this paper, we studied the gap generation in suspended clean graphene at neutral point. Solving the Schwinger-Dyson equation with the frequency-dependent polarization function, we found analytically that the critical coupling constant for onset of a gap equals $\alpha_c=0.92$ which is close to the value obtained in Monte Carlo simulations. We showed that the critical coupling $\alpha_c$ corresponds to the infinite-order phase transition in the case of purely Coulomb interaction with peculiar critical exponents while Monte Carlo simulations point to the second-order phase transition with different critical exponents.

Adding the Gross-Neveu four-fermion interaction that is present in the continuum limit of the lattice model, we found the critical line Eq. (5.10) in the plane of Coulomb and four-fermion coupling constants separating zero-gap and gapped phases. We showed that the order of a phase transition changes from the infinite to the second-order one along the part $0<\alpha<\alpha_c$ of the critical line and the critical coupling becomes lower than in the model with pure Coulomb interaction. The critical exponents $\nu$, $\delta$, and $\beta$ along the line of second-order phase transition are given by the expressions (5.13), (5.16), and (5.17), respectively, and the exponent $\gamma=1$. These exponents satisfy hyperscaling relations and characterize the transition between phases with distinct symmetry properties and become, in general, functions of the Coulomb coupling $\alpha$ or the four-fermion coupling $\tilde{g}$. They are close to the critical exponents obtained in lattice simulations.[25]

The other part of the critical curve with $\tilde{g}<1/4N_f$ is rather special and is related to the conformal phase transition characterized by an essential singularity at the transition point and by abrupt change of the spectrum of light excitations as the critical point is crossed. However, the shape of the last part of phase-transition curve might be strongly influenced by the finite-size effects which appear to be nontrivial.[28] Also, the running of the coupling $\alpha$, due to the running of the Fermi velocity $v_F$, may change the shape of the vertical part of the critical line. These effects most likely change the kind of phase transition to the second-order one. This would indicate that the semimetal-insulator transition in graphene is likely to be of second order. We expect that the form of the critical curve in graphene can be checked in further lattice simulations.

Also, our results maybe important for the proper interpretation of lattice simulations of low-energy field-theory model for quasiparticles in graphene interacting through the long-range Coulomb potential[25] because local four-fermion terms are expected to be generated by the lattice regularization procedure. We showed that in spite of being small ($G\sim$ lattice constant), the induced local interactions can play a significant role in the critical behavior observed in lattice simulations. A related aspect of the near-critical behavior is the appearance of composite electron-hole degrees of freedom whose form factors slowly decrease with momentum (tight bound states) and the momentum behavior is governed by large anomalous dimension. Their dynamics can be studied similarly to that in strongly coupled QED (Ref. 51) but this remains a problem for future investigations.

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