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Fermi liquid near Pomeranchuk quantum criticality

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We analyze the behavior of an itinerant two-dimensional Fermi system near a charge nematic ($n=2$) Pomeranchuk instability in terms of the Landau Fermi-liquid (FL) theory. A key object of our study is the fully renormalized vertex function Γ^Ω, related to the Landau interaction function. We derive Γ^Ω for a model case of the long-range interaction in the nematic channel. Already within the random-phase approximation (RPA), the vertex is singular near the instability. The full vertex, obtained by resumming the ladder series composed of the RPA vertices, differs from the RPA result by a multiplicative renormalization factor Z_Γ, related to the single-particle residue Z and effective-mass renormalization m^*/m. We employ the Pitaevski-Landau identities, which express the derivatives of the self-energy in terms of Γ^Ω, to obtain and solve a set of coupled nonlinear equations for Z_Γ, Z, and m^*/m. We show that near the transition the system enters a critical FL regime, where $Z_\Gamma \sim Z \propto (1+g_{c,2})^{1/2}$ and $m^*/m \approx 1/Z$, where $g_{c,2}$ is the $n=2$ charge Landau component which approaches -1 at the instability. We construct the Landau function of the critical FL and show that all but $g_{c,2}$ Landau components diverge at the critical point. We also show that in the critical regime the one-loop result for the self-energy $\Sigma(K) \propto \int dP G(P) D(K-P)$ is asymptotically exact if one identifies the effective interaction D with the RPA form of Γ^Ω.

I. INTRODUCTION

The Fermi-liquid (FL) theory states that low-energy excitations in a system of interacting fermions are represented by fermionic quasiparticles which differ quantitatively but not qualitatively from free fermions.[1–5] Quantitative changes are encoded in the Landau function, $\hat{g}$ (also known as Fermi-liquid interaction function), which is a tensor in the spin space and a function of the momenta of colliding quasiparticles in the orbital space. Angular harmonics of $\hat{g}$ in the charge and spin channels, $g_{c,n}$ and $g_{s,n}$, determine renormalized values of various observables.

The FL theory also allows for instabilities which occur either due to Cooper pairing or to symmetry-breaking deformations of the Fermi surface (FS). The latter are known as Pomeranchuk instabilities.[6] In the FL notations, a Pomeranchuk instability occurs when one of the Landau harmonics approaches -1.

Examples of Pomeranchuk instabilities include phase separation ($g_{c,0}=-1$), a ferromagnetic transition ($g_{s,0}=-1$), at which the FSs of spin-up and spin-down fermions split apart,[7] and nematic-type transitions in the charge[8–18] and spin channels,[19–21] which lower the rotational symmetry of the FS. In modern literature, the point in the parameter space where a Pomeranchuk instability occurs is called a quantum critical point (QCP).

As the system approaches a Pomeranchuk instability, a certain "FS susceptibility," which measures "softness" of the Fermi surface with respect to deformations of particular symmetry, diverges.[6,11] In certain cases, e.g., for a ferromagnetic transition, a diverging FS susceptibility coincides with the thermodynamic susceptibility whose divergence signals a phase transition. In general, however, the FS and thermodynamic susceptibilities are different. For example, any instability resulting in a deformation of the FS in an isotropic FL

is not accompanied by a divergence of a thermodynamic susceptibility, which remains isotropic in the disordered phase. In those cases, a way to detect Pomeranchuk instabilities is to construct the correlation function of the incipient order parameter of certain symmetry.[8] A susceptibility corresponding to this correlation function, χ_{a^*,n^*}, diverges as $1/(1+g_{a^*,n^*})$ at a QCP in the critical channel $\{a^*,n^*\}$, where $a=c,s$. Although it does not follow from any general principle, it is usually assumed that Landau components in all other channels are not affected by a Pomeranchuk instability in the $\{a^*,n^*\}$ channel.

Taken at face value, the last assumption implies that, near a Pomeranchuk QCP, the effective mass $m^*=m(1+g_{c,1})$ remains finite. However, Hertz-Millis-type effective theories of Pomeranchuk QCPs,[10,11,13,17,22–35] in which the original four-fermion interaction is replaced by the interaction of fermions with fluctuations of the incipient order parameter, predict a different behavior. In these theories, the effective mass diverges at a QCP in dimensions $D \leq 3$: as $|\ln(1+g_{a^*,n^*})|$ in $D=3$ and as $(1+g_{a^*,n^*})^{-1/2}$ in $D=2$. As the FL theory is supposed to be valid at $T=0$ down to the very QCP, the divergence of m^* implies that of $g_{c,1}$, in clear disagreement with the assumption that all but g_{a^*,n^*} Landau components are not affected by the Pomeranchuk instability in a particular channel.

As susceptibilities in the FL theory contain the fully renormalized mass, it then becomes an issue whether this singular mass renormalization should be included into the ordinary FL formula for the susceptibility in the critical $\{a^*,n^*\}$ channel $\chi_{a^*,n^*}=(1+g_{c,1})/(1+g_{a^*,n^*})$ and also whether noncritical channels can be "dragged" to criticality simply due to a divergence in the effective mass.

In this paper, we revisit this issue. We consider a Pomeranchuk QCP in the charge channel (a QCP in the spin channel deserves a separate consideration because of subtle issues

related to spin conservation, see Ref. 21). For definiteness, we focus on the charge nematic instability in the $n=2$ channel (Refs. 8–18); our conclusions, however, are also valid for all $n>1$. We show that, for $D \le 3$ the assumption of only one critical Landau component holds only at some "distance" from a QCP in the parameter space. In the immediate vicinity of a QCP, a FL of new type emerges. In what follows, we will refer to this new FL as to a "critical FL." The interactions at high energies, which were the cause of the Pomeranchuk instability in the first place, play the role of "bare" interactions for the critical FL. Accordingly, the enhanced nematic susceptibility $\chi_{c,2}^{\rm FL} \propto 1/(1+g_{c,2})$ is a bare susceptibility of the critical FL. The low-energy interactions, mediated by soft collective fluctuations in the $n=2$ charge channel, lead to further renormalizations of the FL parameters. These renormalizations are encapsulated in the "critical" Landau function, $\bar{g}$, which we show to have both charge and spin components with *any* n, even if the original, "high-energy" FL has only the $n=2$ charge Landau component.

Our key result is that *all* components of $\bar{g}$ diverge in the same way, i.e., as $1/(1+g_{c,2})^{1/2}$, upon approaching a nematic QCP in $D=2$. In particular, a divergence of $\bar{g}_{c,1}$ implies that $m^*/m=1+\bar{g}_{c,1}$ diverges at the QCP as well. At the same time, since divergences are the same for all $\bar{g}_{a,n}$, they cancel out in the expressions for the susceptibilities $\chi_{a,n} \propto (1+\bar{g}_{c,1})/(1+\bar{g}_{a,n})$, which retain the same values $\chi_{a,n}^{\rm FL}$ as in the original FL. In particular, susceptibilities in channels different from the critical one remain finite at a QCP while $\chi_{c,2}$ preserves its $1/(1+g_{c,2})$ form. This means that the divergence of the effective mass near a Pomeranchuk QCP does *not* affect the behavior of any of the susceptibilities.

To obtain these results, we derive diagrammatically an expression for the fully renormalized antisymmetrized interaction vertex $\hat{\Gamma}^{\Omega}(\mathbf{k},\mathbf{p})$ between the particles with momenta $\mathbf{k}$ and $\mathbf{p}$ on the Fermi surface. This vertex is obtained from a more general vertex function in the limit of zero momentum transfer and vanishing frequency transfer and is the "input" parameter for the FL theory: the Landau function is proportional to $\hat{\Gamma}^{\Omega}$

$$\hat{g} = 2\nu Z^2 \frac{m^*}{m} \hat{\Gamma}^{\Omega}, \qquad (1.1)$$

where Z is the quasiparticle residue.

We identify the most relevant part of $\hat{\Gamma}^{\Omega}$ near a QCP and show that it describes an interaction mediated by soft collective bosonic fluctuations in the $n=2$ charge channel. For small $|\mathbf{k}-\mathbf{p}| \approx k_{\rm F}\theta$, we find $\Gamma^{\Omega}(\theta) \propto 1/[1+g_{c,2}+(ak_{\rm F}\theta)^2+\cdots]$ where dots stand for less relevant terms. The length scale a is the effective radius of the interaction $U(q)$ in the d-wave charge channel and the product $ak_{\rm F}$ is a dimensionless parameter of our theory. The calculations are under control if $ak_{\rm F} \gg 1$, which we assume to hold. For $ak_{\rm F} \gg 1$, the system is in the critical FL regime when $(1+g_{c,2}) < 1/(ak_{\rm F})^2 \ll 1$.

Such a singular form of $\hat{\Gamma}^{\Omega}$ was proposed earlier on phenomenological grounds[36] and obtained within the random-phase approximation (RPA) for the Hubbard model near an antiferromagnetic instability.[37] To obtain the full $\hat{\Gamma}^{\Omega}$ in our

case, we first generalize the RPA result to the $n=2$ charge Pomeranchuk instability. We show that the RPA-type formula for the effective interaction $\hat{\Gamma}^{\rm RPA}(\mathbf{k},\mathbf{p}) \propto 1/[1-U(\mathbf{k}-\mathbf{p})\Pi_2(\mathbf{k},\mathbf{p})]$, where $\Pi_d(\mathbf{k},\mathbf{p})$ is the static polarization bubble in the $n=2$ channel, is reproduced by summing up the diagrams which do not renormalize the bare interaction U.

Next, we analyze the diagrams for $\hat{\Gamma}^{\Omega}$ beyond RPA and show that in the critical FL regime the full Γ^{Ω} differs from $\Gamma^{\Omega,\rm RPA}$ by a constant factor $Z_{\Gamma}^{-1} \sim Z^{-1}$. This relation results from resummation of a particular non-RPA series of diagrams which includes renormalization of U into a full dynamic interaction. We show that all other non-RPA diagrams are relatively small in powers of $1/ak_{\rm F}$ and can therefore be neglected. The existence of an extra factor Z_{Γ} between $\hat{\Gamma}^{\Omega}$ and $\hat{\Gamma}^{\Omega,\rm RPA}$ is very important for our analysis—just the RPA form of $\hat{\Gamma}^{\Omega}$ would produce nonsensical results in the FL description.

Another input parameter for a FL theory is the quasiparticle residue Z. To obtain Z, we use the exact Pitaevski-Landau relations[1,38] which express Z in terms of $\hat{\Gamma}^{\Omega}(\mathbf{k},\omega_k;\mathbf{p},\omega_p)$, where now $\mathbf{p}$ and $\mathbf{k}$ are not necessarily at the FS and $\omega_{k,p}$ are finite. We extend the previous calculation of $\hat{\Gamma}^{\Omega}$ to fermions away from the FS and obtain Z as a function of $(1+g_{c,2})^{1/2}$ with $ak_{\rm F}$ as a parameter. This function is rather complex but reduces to a simple form $Z \sim (ak_{\rm F})(1+g_{c,2})^{1/2}$ in the critical FL regime. Using this form of Z and full $\hat{\Gamma}^{\Omega}$, we construct the Landau function of the critical FL and show that all its components diverge in a way discussed above. We also show that the effective mass m^* diverges at but not before a QCP. In this respect, our results do not support the conjecture[35] that the effective mass may diverge before the system reaches a QCP.

We also discuss the relation between the exact formula for the self-energy $\Sigma(k,\omega)$ to linear order in ω and $k-k_{\rm F}$, obtained from the Pitaevski-Landau relations, and a one-loop formula for Σ due to an exchange by soft bosonic collective excitations. We show that the one-loop formula is asymptotically exact in the critical FL regime if the effective interaction is replaced by $Z\hat{\Gamma}^{\Omega}$, i.e., by the RPA form of the effective vertex $\hat{\Gamma}^{\Omega,\rm RPA}$. The corrections to the one-loop formula are small in $1/ak_{\rm F}$ and in $|\omega|/\omega_{\rm FL}$, where $\omega_{\rm FL} \propto (1+g_{c,2})^{3/2}$ is the upper boundary of the FL behavior. We emphasize that the one-loop approximation is valid only for the linear in ω term in the self-energy. The next, $\omega^2 \ln \omega$ term has contributions from all orders even if the theory is extended to a large number of fermionic flavors N.[39] Outside the FL regime, i.e., for $|\omega| > \omega_{\rm FL}$, the self-energy scales as $\omega^{2/3}$. Some vertex corrections in this regime are small in $1/N$ (Refs. 26, 29, 33, and 34) while others remain $O(1)$ even for large N (Ref. 40).

Regarding the full form of the self-energy near a QCP, we show that, for $1+g_{c,2} \ll 1$, the self-energy is "local," in the sense that it depends primarily on ω but not on $\epsilon_k = v_{\rm F}(k-k_{\rm F})$, where $v_{\rm F}$ is the bare Fermi velocity. The prefactor for the ϵ_k term in the self-energy scales as $1/ak_{\rm F}$ and is, therefore, small.

A Pomeranchuk instability in the d-wave charge channel was introduced in the context of the renormalization group

analysis of potential instabilities of a two-dimensional (2D) Hubbard model.[8,9] Shortly thereafter, Oganesyan *et al.* analyzed in detail a d-wave charge instability in isotropic systems.[10] The subject attracted substantial interest[11–18] both from the theoretical perspective and also due to potential relevance to cuprates[41] and ruthenites.[42] There are some subtle differences between a d-wave Pomeranchuk instability in lattice systems[8,9,11,12] and an $n=2$ Pomeranchuk instability in isotropic systems[10,13–17] but our conclusion is that the physics does not change qualitatively between isotropic and lattice cases (see below).

Properties of the Fermi liquid near a charge Pomeranchuk instability were studied by Rosch and Wölfle.[16] They did not consider the Landau function near a nematic QCP but obtained the effective mass in terms of the critical parameter $1+g_{c,2}$ using the one-loop approximation for the self-energy with an RPA form of the effective vertex. Our results agree with Ref. 16 in that the effective mass can be obtained within the one-loop approximation with the RPA vertex but our dependence of the mass on $1+g_{c,2}$ is different from that in Ref. 16. The disagreement originates from the difference in the forms of the static polarization bubble for dressed fermions, which we discuss in Sec. III B.

The structure of the paper is as follows. In Sec. II, we review a general procedure of constructing the FL vertices. Section III is devoted to the critical FL theory. In Sec. III A, we introduce a model for the nematic Pomeranchuk instability in the charge channel. In Sec. III B, we obtain the FL vertex $\hat{\Gamma}^{\Omega}$ near this instability within the RPA approximation. In Sec. III C, we go beyond the RPA level and obtain the full vertex $\hat{\Gamma}^{\Omega}$. In Sec. IV, we obtain the quasiparticle residue Z, the vertex renormalization Z_{Γ}, and effective mass m^* from the exact Pitaevski-Landau relations. In the same section, we also analyze the crossover between ordinary and critical FLs. In Sec. V, we obtain the Landau function in the critical FL using the full vertex found in Sec. III C and show that all components of this Landau function diverge at a QCP. This is the main result of our paper. In Sec. VI, we discuss the relation between the one-loop and exact Pitaveskii-Landau forms of the self-energy near the FS. We present our concluding remarks in Sec. VII. A number of technical issues are discussed in Appendices A–C.

II. DIAGRAMMATIC DESCRIPTION OF AN ORDINARY FERMI LIQUID

We begin with a brief overview of the diagrammatic description of an ordinary FL. The FL theory describes effects of the interactions between fermions confined to a near vicinity of the FS. The interactions which involve fermions away from the Fermi surface are absorbed into the Landau function $g_{\alpha\beta,\gamma\delta}$. This function is related to an exact antisymmetrized vertex $\Gamma^{\Omega}_{\alpha\beta,\gamma\delta}$ via $g_{\alpha\beta,\gamma\delta}=2\nu Z^2(m^*/m)\Gamma^{\Omega}_{\alpha\beta,\gamma\delta}$ ($\nu=m/2\pi$ is the density of states in 2D). The vertex $\Gamma^{\Omega}_{\alpha\beta,\gamma\delta}$ is defined for particles at the FS in the limit of zero momentum transfer q and vanishing energy transfer Ω (Ref. 1)

$$\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}= \lim_{q/\Omega\to 0} [\Gamma(\mathbf{k},0;\mathbf{p},0|\mathbf{k}-\mathbf{q},-\Omega;\mathbf{p}+\mathbf{q},\Omega)\delta_{\alpha\gamma}\delta_{\beta\delta}$$
$$- \Gamma(\mathbf{k},0;\mathbf{p},0|\mathbf{p}+\mathbf{q},\Omega;\mathbf{k}-\mathbf{q},-\Omega)\delta_{\alpha\delta}\delta_{\beta\gamma}]|_{k=p=k_{\mathrm{F}}},$$

(2.1)

where $\Gamma(\mathbf{k},\omega_k;\mathbf{p},\omega_p|\bar{\mathbf{k}},\bar{\omega}_k;\bar{\mathbf{p}},\bar{\omega}_p)$ is an exact nonsymmetrized vertex. The momentum transfer is equal to $\mathbf{q}$ in the first term of Eq. (2.1) and to $\mathbf{k}-\mathbf{p}-\mathbf{q}$ in the second one.

For a generic ratio of q and Ω, diagrams for the vertex $\Gamma(\mathbf{k},\omega_k;\mathbf{p},\omega_p|\bar{\mathbf{k}},\bar{\omega}_k;\bar{\mathbf{p}},\bar{\omega}_p)$ [and its exchange counterpart $\Gamma(\mathbf{k},\omega_k;\mathbf{p},\omega_p|\bar{\mathbf{p}},\bar{\omega}_p;\bar{\mathbf{k}},\bar{\omega}_k)$] can be separated into two groups depending on whether they contain propagators of "soft" particle-hole pairs

$$\mathcal{P}(\mathbf{q},\Omega,\hat{\mathbf{k}})=-\nu\int d\epsilon_k\int \frac{d\omega}{2\pi}G(\mathbf{k}+\mathbf{q},\omega+\Omega)G(\mathbf{k},\omega)$$

(2.2)

with $q\to 0$ and $\Omega\to 0$, where $\epsilon_k=v_{\mathrm{F}}(k-k_{\mathrm{F}})$. The meaning of $\mathcal{P}$ is the propagator of a particle-hole excitation with momentum $\mathbf{q}$ and energy Ω formed by fermions moving in the direction of $\hat{\mathbf{k}}=\mathbf{k}/k$.

For vanishingly small q and Ω, the dynamic part of the bubble $\mathcal{P}$ is determined by fermions on the FS; therefore, diagrams with soft bubbles are attributes of the FL theory. For these diagrams, the order of limits $q\to 0$ and $\Omega\to 0$ matters because $\mathcal{P}=0$ in the limit of $q/\Omega\to 0$ and is finite (equal to ν) in the limit $\Omega/q\to 0$. The diagrams without soft bubbles generally involve fermions with high energies, of order E_{F}. For these diagrams, the order of limits $q\to 0$ and $\Omega\to 0$ is irrelevant.

Because $\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}$ is the vertex in the limit $q/\Omega\to 0$, it does not include soft particle-hole bubbles ($\mathcal{P}=0$ in this limit) and, in that sense, it is a high-energy property playing a role of the bare vertex in the FL theory. The second-order diagrams for $\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}$ are shown in Fig. 1(a). The wavy line in these diagrams is the bare, static interaction potential $V(\mathbf{k},\mathbf{p};\bar{\mathbf{k}},\bar{\mathbf{p}})$ which may depend not only on the momentum transfer $\mathbf{k}-\bar{\mathbf{k}}$ but also on the incoming momenta themselves.

On the other hand, physical observables, e.g., the specific heat or the nematic charge susceptibility, contain contributions from soft bubbles because the observables are affected by elastic collisions between particles right on the FS, i.e., by processes with $\Omega=0$ and finite q. The corresponding antisymmetrized vertex is called $\Gamma^{q}_{\alpha\beta;\gamma\delta}$. In Fig. 1(b), we show the second-order diagrams for $\Gamma^{q}_{\alpha\beta;\gamma\delta}$. The full vertex $\Gamma^{q}_{\alpha\beta;\gamma\delta}$ is obtained from $\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}$ by summing up an infinite series of ladder diagrams shown in Fig. 1(c), which leads to a familiar integral equation[1]

$$\Gamma^{q}_{\alpha\beta;\gamma\delta}(K_{\mathrm{F}},P_{\mathrm{F}})=\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}(K_{\mathrm{F}},P_{\mathrm{F}})$$
$$-Z^2\nu\int \frac{d\theta}{2\pi}\Gamma^{\Omega}_{\alpha\xi;\gamma\eta}(K_{\mathrm{F}},K'_{\mathrm{F}})\Gamma^{q}_{\eta\beta;\xi\delta}(K'_{\mathrm{F}},P_{\mathrm{F}}),$$

(2.3)

where in shorthand notation $K_{\mathrm{F}},P_{\mathrm{F}}$ denotes the "four momentum" on the FS, i.e., $K_{\mathrm{F}}\equiv\{\mathbf{k}_{\mathrm{F}},\omega_k=0\}$, $\mathbf{k}_{\mathrm{F}}\equiv k_{\mathrm{F}}\hat{k}$, and θ is the angle between $\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}'$.
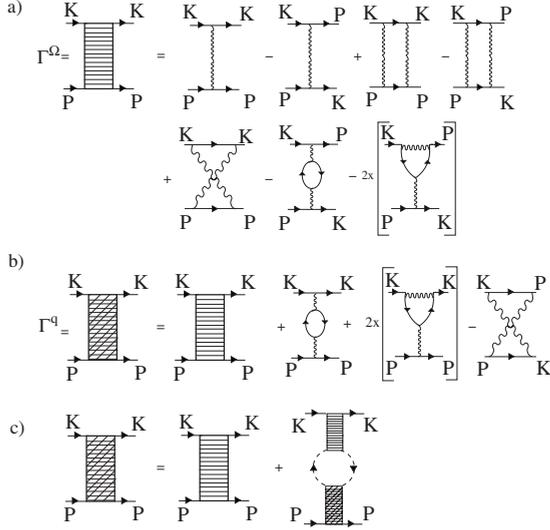
FIG. 1. Fermi-liquid vertices to second order in the bare interaction (wavy line). (a) Diagrams for $\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}$. (b) Diagrams for $\Gamma^{q}_{\alpha\beta;\gamma\delta}$. (c) Diagrammatic representation of the relation between $\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}$ and $\Gamma^{q}_{\alpha\beta;\gamma\delta}$, Eq. (2.3). A bubble composed of dotted lines represents the particle-hole propagator $\mathcal{P}$ [see Eq. (2.2)] in the limit $\Omega/q \rightarrow 0$, where $\mathcal{P}$ reduces to a constant.

The full vertex $\Gamma^{q}_{\alpha\beta;\gamma\delta}$ is related to the scattering amplitude $f_{\alpha\beta;\gamma\delta}$ in the same way as $\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}$ is related to $g_{\alpha\beta;\gamma\delta}$, i.e., via $f_{\alpha\beta;\gamma\delta} = 2\nu Z^2(m^*/m)\Gamma^{q}_{\alpha\beta;\gamma\delta}$. The relation between $\hat{f}$ and $\hat{g}$ takes a particularly simple form when expressed via the partial components in the charge and spin channels, $f_{a,l}$ and $g_{a,l}$,

$$f_{a,l} = \frac{g_{a,l}}{1 + g_{a,l}}. \tag{2.4}$$

Using these relations, one obtains FL formulas for observables in terms of $g_{a,l}$.

III. CRITICAL FERMI-LIQUID THEORY

A. Nematic charge fluctuations

We follow early work[10,11,13,15,16,18] and consider a nematic charge instability described by the model Hamiltonian with the interaction in the d-wave charge channel

$$H = \sum_{\mathbf{k},\mathbf{p},\mathbf{q}} U(\mathbf{q}) d_{\mathbf{k}} d_{\mathbf{p}} c^{\dagger}_{\mathbf{k}+\mathbf{q}/2,\alpha} c^{\dagger}_{\mathbf{p}-\mathbf{q}/2,\beta} c_{\mathbf{p}+\mathbf{q}/2,\beta} c_{\mathbf{k}-\mathbf{q}/2,\alpha}. \tag{3.1}$$

Here

$$d_{\mathbf{k}} = \sqrt{2}\cos(2\phi_{\mathbf{k}}) \tag{3.2}$$

is the "d-wave" form factor and $\phi_{\mathbf{k}}$ is the angle between $\mathbf{k}$ and an arbitrarily chosen x axis. Hamiltonian (3.1) describes the interaction between nematic fluctuations of the electron density $\rho_d(\mathbf{q}) = \sum_{\mathbf{k}} d_{\mathbf{k}} c^{\dagger}_{\mathbf{k}+\mathbf{q}/2,\alpha} c_{\mathbf{k}-\mathbf{q}/2,\alpha}$.

To keep the treatment under control, we assume that the interaction $U(\mathbf{q})$ is sufficiently long ranged in real space, i.e.,

$$U(\mathbf{q}) = U_0 P(qa), \tag{3.3}$$

where the function $P(x)$, subject to $P(0) = 1$, is a decreasing function of its argument, and the effective interaction radius

a is much larger than the inverse average distance between particles, i.e., $k_{\mathrm{F}}a \gg 1$. We also assume that $U(\mathbf{q})$ is analytic for small q,

$$U(\mathbf{q}) = U_0[1 - (qa)^2 + \cdots]. \tag{3.4}$$

Equation (3.1) is a reduced version of a more general interaction between quadrupolar fluctuations of the electron density $c^{\dagger}_{i,\alpha} Q_{ij} c_{j,\alpha}$, where $Q_{ij} = \delta_{ij}\nabla^2 - 2\partial_i\partial_j$, and $i,j = x,y$ (Refs. 10 and 18). Such an interaction can be decoupled into a longitudinal part, which is the same as in Eq. (3.1), and a transverse part with the momentum-dependent factor $\bar{d}_{\mathbf{k}}\bar{d}_{\mathbf{p}}$, where $\bar{d}_{\mathbf{k}} = \sqrt{2}\sin(2\phi_{\mathbf{k}})$. The two terms contribute separately to the fermionic self-energy and Γ^{Ω} and can be treated independently of each other. To simplify the presentation, we consider only the longitudinal part of the quadrupole interaction, given by Eq. (3.1). The effect of the transverse part on thermodynamic properties of a FL has recently been considered by Zacharias $et\ al.$[18] To include transverse fluctuations into our theory, we need to know how the interaction with transverse fluctuations affects the fermionic propagator. This problem has not been solved yet.

The nematic charge susceptibility of free fermions is defined as

$$\chi^{(0)}_{c,2}(\mathbf{q},\Omega) = 2\Pi_d(\mathbf{q},\Omega) = -2\sum_{\mathbf{k},\omega_k} d^2_{\mathbf{k}} G(\mathbf{k}+\mathbf{q}/2,\omega_k+\Omega/2)$$
$$\times G(\mathbf{k}-\mathbf{q}/2,\omega_k-\Omega/2). \tag{3.5}$$

We will be interested in the long-wavelength and low-frequency d-density fluctuations with $q \ll k_{\mathrm{F}}$ and $|\Omega|/v_{\mathrm{F}}q \ll 1$. In this regime, $\chi^{(0)}_{c,2}$ for free 2D fermions with quadratic spectrum $\varepsilon_{\mathbf{k}} = k^2/2m$ is

$$\chi^{(0)}_{c,2}(\mathbf{q},\Omega) = 2\nu\left[1 - \frac{q^2}{2k^2_{\mathrm{F}}} - 2\cos^2(2\phi_{\mathbf{q}})\frac{|\Omega|}{v_{\mathrm{F}}q} + \cdots\right]. \tag{3.6}$$

Notice that Landau damping of nematic fluctuations is anisotropic whereas the dispersion of static $\chi^{(0)}_{c,2}$ with q (absent for $n=0$ density fluctuations for $q \leq 2k_{\mathrm{F}}$) is isotropic.

To first order in the interaction, $\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}$ is given by the first two diagrams in Fig. 1(a),

$$\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}(\mathbf{k},\mathbf{p}) = U_0 d_{\mathbf{k}} d_{\mathbf{p}}\delta_{\alpha\gamma}\delta_{\beta\delta} - U(\mathbf{k}-\mathbf{p})d^2_{\mathbf{k}+\mathbf{p}/2}\delta_{\alpha\delta}\delta_{\beta\gamma}. \tag{3.7}$$

According to our definition of $d_{\mathbf{k}}$, when both $\mathbf{k}$ and $\mathbf{p}$ are on the FS,

$$d^2_{\mathbf{k}+\mathbf{p}/2} = \cos(2\phi_{\mathbf{k}})\cos(2\phi_{\mathbf{p}}) + 1 - \sin(2\phi_{\mathbf{k}})\sin(2\phi_{\mathbf{p}})$$
$$= \frac{1}{2}d_{\mathbf{k}} d_{\mathbf{p}} + \cdots, \tag{3.8}$$

where dots stand for non-$\cos(2\phi_{\mathbf{k}})$ terms which we neglect. Using Eq. (3.8) and the SU(2) identity

$$\delta_{\alpha\delta}\delta_{\beta\gamma} = (\delta_{\alpha\gamma}\delta_{\beta\delta} + \vec{\sigma}_{\alpha\gamma} \cdot \vec{\sigma}_{\beta\delta})/2, \tag{3.9}$$

we separate Eq. (3.7) into the charge and spin channels as

$$\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}(\mathbf{k},\mathbf{p}) = \Gamma^{\Omega}_c \delta_{\alpha\gamma}\delta_{\beta\delta} + \Gamma^{\Omega}_s \vec{\sigma}_{\alpha\gamma} \cdot \vec{\sigma}_{\beta\delta},$$
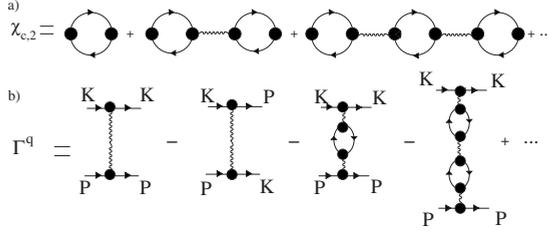
FIG. 2. RPA series for the nematic susceptibility [(a)] and $\Gamma^q_{\alpha\beta;\gamma\delta}$ [(b)]. Solid circles represent d-wave vertices $d_{\mathbf{k}}$ and wavy lines represent the forward-scattering part of the interaction, U_0. The bubbles are with zero vanishing external momentum and zero frequency, the same bubbles as in Figs. 1(b) and 1(c). Since, by assumption $ak_F \gg 1$, only the diagrams that contain U_0 are included.

$$\Gamma^\Omega_c = d_{\mathbf{k}} d_{\mathbf{p}}\left[U_0 - \frac{1}{4}U(\mathbf{k}-\mathbf{p})\right], \quad \Gamma^\Omega_s = -\frac{1}{4}d_{\mathbf{k}}d_{\mathbf{p}}U(\mathbf{k}-\mathbf{p}). \tag{3.10}$$

At this stage, the interaction is static and mostly in the d-wave channel. Hence the effective mass equals to the bare one and the quasiparticle residue $Z=1$. The Landau function is then obtained by simply multiplying Eq. (3.10) by 2ν. Since, by assumption, $U(\mathbf{k}-\mathbf{p})$ is peaked at $\mathbf{k}=\mathbf{p}$, the charge d-wave Landau harmonic, $g_{c,2}$, is much larger than the spin d-wave harmonic $g_{s,2}$. Indeed, for $ak_F \gg 1$,

$$g_{c,2} = 2\nu\int\int\frac{d\phi_{\mathbf{k}}}{2\pi}\frac{d\phi_{\mathbf{p}}}{2\pi}d_{\mathbf{k}}d_{\mathbf{p}}\Gamma^\Omega_c \approx 2\nu U_0\left(1-\frac{c}{ak_F}\right),$$

$$g_{s,2} = 2\nu\int\int\frac{d\phi_{\mathbf{k}}}{2\pi}\frac{d\phi_{\mathbf{p}}}{2\pi}d_{\mathbf{k}}d_{\mathbf{p}}\Gamma^\Omega_s \approx -2\nu U_0\frac{c}{ak_F} \ll g_{c,2}, \tag{3.11}$$

where $c=(3/4\pi)\int_0^\infty dxP(x)$. To leading order in $1/k_Fa$, we then have $g_{c,2}=2\nu U_0$ and $g_{s,2}=0$. We see that $g_{c,2}$ approaches -1 when U_0 approaches $-1/(2\nu)$.

With only $g_{c,2}$ being nonzero, the nematic spin susceptibility $\chi_{s,2}=\chi^{(0)}_{s,2}(1+g_{c,1})/(1+g_{s,2})=\chi^{(0)}_{s,2}$ retains its bare value, while the charge susceptibility is

$$\chi_{c,2} = \chi^{(0)}_{c,2}\frac{1+g_{c,1}}{1+g_{c,2}} = \frac{\chi^{(0)}_{c,2}}{1+g_{c,2}} \tag{3.12}$$

diverges when $g_{c,2}$ tends to -1. This corresponds to the "conventional" scenario of a Pomeranchuk QCP, where only one of the Landau components approaches -1 but the effective mass does not diverge. We will see below that such a behavior does not survive in a nonperturbative theory with fully renormalized vertices.

Before moving to the analysis of vertex renormalization, we note that Eq. (3.12) and the FL relation (2.4) can be reproduced diagrammatically by calculating directly the charge susceptibility and $\Gamma^q_{\alpha\beta;\gamma\delta}$. In both cases we need to include only those diagrams that contain $\Pi_d(\mathbf{q},\Omega)$ at vanishing momentum and zero frequency and, thus, do not contribute to renormalization of $\Gamma^\Omega_{\alpha\beta;\gamma\delta}$. Summing up the ladder series for $\chi_{c,2}$ in Fig. 2(a), we obtain
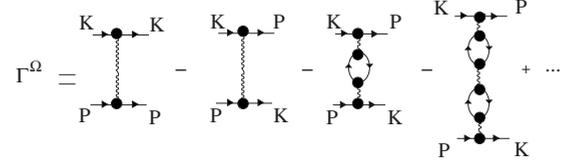

FIG. 3. RPA diagrams for $\Gamma^\Omega_{\alpha\beta;\gamma\delta}$. The difference with the series for $\Gamma^q_{\alpha\beta;\gamma\delta}$ in Fig. 2(b) is in that the polarization bubbles are evaluated at finite external momentum $\mathbf{k}-\mathbf{p}$.

$$\chi_{c,2}(\mathbf{q}\to 0,\Omega=0) = \frac{\chi^{(0)}_{c,2}}{1+2\nu U_0} = \frac{\chi^{(0)}_{c,2}}{1+g_{c,2}}, \tag{3.13}$$

which coincides with Eq. (3.12). The ladder series for $\Gamma^q_{\alpha\beta;\gamma\delta}$ in Fig. 2(b) yields

$$\Gamma^q_{\alpha\beta;\gamma\delta} = d_{\mathbf{k}}d_{\mathbf{p}}\left[\frac{U_0}{1+2\nu U_0}\right]\delta_{\alpha\gamma}\delta_{\beta\delta}. \tag{3.14}$$

Using the relation between $\Gamma^q_{\alpha\beta;\gamma\delta}$ and the scattering amplitude $f_{\alpha\beta,\gamma\delta}=2\nu\Gamma^q_{\alpha\beta,\gamma\delta}$, we find

$$f_{c,2} \approx \frac{2U_0\nu}{1+2U_0\nu}. \tag{3.15}$$

Comparing Eqs. (3.11) and (3.15), we see that the FL relation

$$f_{c,2} = \frac{g_{c,2}}{1+g_{c,2}} \tag{3.16}$$

is indeed reproduced.

B. Random-phase approximation for nematic instability

At the next step, we consider renormalization of $\Gamma^\Omega_{\alpha\beta;\gamma\delta}(\mathbf{k},\mathbf{p})$. An assumption of the long-range interaction in the nematic channel allows one to use the RPA, i.e., to retain only diagrams shown in Fig. 3 with a maximum number of polarization bubbles $\Pi_d(\mathbf{q},0)$ at any given order, now with finite $\mathbf{q}=\mathbf{k}-\mathbf{p}$. While all such diagrams contain the interaction only in the form $U(\mathbf{k}-\mathbf{p})$, non-RPA diagrams involve integrals of U over intermediate momenta. Since each of these integrals contributes a small factor of $1/ak_F$ [see Eq. (3.11)], non-RPA diagrams can be safely neglected. Summing up the RPA series and neglecting the first-order U_0 term, which is irrelevant near a QCP, we obtain

$$\Gamma^{\Omega,\mathrm{RPA}}_{\alpha\beta;\gamma\delta}(\mathbf{k},\mathbf{p}) \approx -\frac{d_{\mathbf{k}}d_{\mathbf{p}}}{2}\frac{U(\mathbf{k}-\mathbf{p})}{1+2U(\mathbf{k}-\mathbf{p})\Pi_d(\mathbf{k}-\mathbf{p},0)}\delta_{\alpha\delta}\delta_{\beta\gamma}. \tag{3.17}$$

Notice a "wrong" combination of spin indices ($\delta_{\alpha\delta}\delta_{\beta\gamma}$ instead of $\delta_{\alpha\gamma}\delta_{\beta\delta}$).

Thermodynamic parameters of a FL (including the effective mass) are determined by $\Gamma^{\Omega,\mathrm{RPA}}_{\alpha\beta;\gamma\delta}(\mathbf{k},\mathbf{p})$ for the particles on the FS and at zero frequencies. However, for the consideration in the next sections, we will also need to know $\Gamma^{\Omega,\mathrm{RPA}}_{\alpha\beta;\gamma\delta}$ away from the FS, i.e., for momenta different from k_F and for nonzero frequencies of incoming and outgoing fermions. A simplification is that, for $1+g_{c,2}\ll 1$, relevant momenta are still close to k_F, while the relevant values of $|\mathbf{k}$

$-\mathbf{p}|$ and ω_k, ω_p are small. In this situation, $\Gamma^{\Omega,\mathrm{RPA}}_{\alpha\beta;\gamma\delta}$ is still given by an expression similar to Eq. (3.17),

$$\Gamma^{\Omega,\mathrm{RPA}}_{\alpha\beta;\gamma\delta}(\mathbf{k},\omega_k;\mathbf{p},\omega_p) = -\frac{1}{2}\frac{d_\mathbf{k}d_\mathbf{p}\delta_{\alpha\delta}\delta_{\beta\gamma}U(\mathbf{k}-\mathbf{p})}{1+2U(\mathbf{k}-\mathbf{p})\Pi_d(\mathbf{k}-\mathbf{p},\omega_k-\omega_p)}$$

$$(3.18)$$

but where now $\Pi_d(\mathbf{k}-\mathbf{p},\omega_k-\omega_p)$ is a dynamic bubble.

$$\Gamma^{\Omega,\mathrm{RPA}}_{\alpha\beta;\gamma\delta}(\mathbf{k},\omega_k;\mathbf{p},\omega_p) = d_\mathbf{k}d_\mathbf{p}\frac{1}{4\nu}\frac{1}{1+g_{c,2}+|\mathbf{k}-\mathbf{p}|^2a^2+2\cos^2(2\phi_\mathbf{q})\dfrac{|\omega_k-\omega_p|}{v_\mathrm{F}|\mathbf{k}-\mathbf{p}|}}\delta_{\alpha\delta}\delta_{\beta\gamma}$$

$$= d_\mathbf{k}d_\mathbf{p}\frac{1}{8\nu}\frac{\delta_{\alpha\gamma}\delta_{\beta\delta}+\vec{\sigma}_{\alpha\gamma}\cdot\vec{\sigma}_{\beta\delta}}{1+g_{c,2}+|\mathbf{k}-\mathbf{p}|^2a^2+2\cos^2(2\phi_\mathbf{q})\dfrac{|\omega_k-\omega_p|}{v_\mathrm{F}|\mathbf{k}-\mathbf{p}|}},\qquad(3.19)$$

where, as before, $\phi_\mathbf{q}$ is the angle between $\mathbf{q}$ and an arbitrary chosen x axis.

This form of $\Gamma^{\Omega,\mathrm{RPA}}_{\alpha\beta;\gamma\delta}(\mathbf{k},\mathbf{p})$ is only valid, however, at some distance away from a QCP, where fermions behave as nearly free quasiparticles. Near a QCP, the quasiparticle mass and residue differ from the free-fermion values and $\Pi_d(\mathbf{q},\Omega)$ is to be treated as a fully renormalized bubble, which we label as $\Pi^*_d(\mathbf{q},\Omega)$.

Renormalization of the polarization bubble is a subtle issue and we pause here to discuss it in some detail. In general, the polarization bubble contains two types of renormalizations: the self-energy insertions, which transform the bare Green's functions into the renormalized ones, and vertex corrections. We will show in Sec. III C 3 that the vertex corrections are small in the quasistatic limit ($\Omega\ll v_\mathrm{F}q$); thus, in this limit, the bubble can be computed as a convolution of two renormalized Green's functions. Still, one has to be careful even with this computation because the quasiparticle residue Z and the effective mass m^* depend on the energy they are measured at: near the FS, they approach the renormalized values of $Z<1$ and $m^*>m$, while at higher energies they approach the free values $Z=1$ and $m^*=m$. It turns out that renormalizations of the three terms in the expansion of the free bubble [Eq. (3.6)] are determined by different energies. First, we consider the constant term $\Pi^*_d(\mathbf{q}\to0,\Omega=0)$. For free fermions, it coincides with the density of states. A product of two Green's functions with close arguments can be represented as a sum of two parts: coherent and incoherent (Ref. 1)

$$G(\mathbf{k}+\mathbf{q},\omega+\Omega)G(\mathbf{k},\omega) = 2\pi Z^2\frac{\mathbf{v}_\mathrm{F}\cdot\mathbf{q}}{\mathbf{v}^*_\mathrm{F}\cdot\mathbf{q}-i\Omega}\delta(\omega)\delta(\epsilon_k)$$

$$+\Phi(\mathbf{k},\omega),\qquad(3.20)$$

where $\mathbf{v}^*_\mathrm{F}=v_\mathrm{F}(m/m^*)\hat{k}$. A contribution of the coherent part to $\Pi^*_d(\mathbf{q}\to0,\Omega=0)$ is equal to the renormalized density of

Polarization bubble

The d-wave bubble of free fermions for $q\ll k_\mathrm{F}$ and $|\Omega|\ll v_\mathrm{F}q$: is given by Eq. (3.6). Substituting this form into Eq. (3.18), replacing $U(\mathbf{k}-\mathbf{p})$ by U_0 in the numerator, and neglecting the $q^2/2k_\mathrm{F}^2$ term compared to the $(aq)^2$ one which comes from the expansion of $U(\mathbf{k}-\mathbf{p})$ in Eq. (3.4), we obtain

states $\nu Z^2(m^*/m)$. However, it would be incorrect to replace $\Pi_d(\mathbf{q}\to0,\Omega=0)$ only by the coherent contribution because there is also a contribution from the incoherent part, $\Phi(\mathbf{k},\omega)$, which cannot be evaluated explicitly. A way to avoid this complication is to integrate over the momentum first. This integral comes from high energies, of order of the ultraviolet cutoff of the theory, Λ. At these energies, one can approximate G by its free-fermion form $G^{-1}(\mathbf{k},\omega)=\omega-\epsilon_k$. Integrating over ϵ_k in between $-\Lambda$ and Λ first and then over the frequency, we find that $\Pi^*_d(\mathbf{q},0)$ is given by the bare density of states

$$\Pi^*_d(q\to0,\Omega=0) = -\nu\int\frac{d\omega}{2\pi}\int_{-\Lambda}^{\Lambda}d\epsilon_k\frac{1}{(i\omega-\epsilon_k)^2}$$

$$= \nu\int\frac{d\omega}{\pi}\frac{\Lambda}{\omega^2+\Lambda^2} = \nu.\qquad(3.21)$$

Both energies (ω and ϵ_k) in this integral are of order Λ, which justifies the replacement of the Green's function by its free-fermion form. This result is known in the theory of electron-phonon interaction:[43] renormalization of the phonon spectrum by particle-hole excitations is the same as if the fermions were free.

Similarly, the q^2 term in Eq. (3.6) is also a high-energy contribution. If the electron spectrum remains quadratic up to Λ, this term is the same as for free fermions; otherwise, k_F in the prefactor of this term is replaced by some momentum on order of either k_F or of the reciprocal-lattice spacing.

In distinction to the static part of Π^*_d, the dynamic part of the bubble, the Landau damping term, comes from low-energy fermions by virtue of energy conservation. Calculating this part using only the coherent term in Eq. (3.20), we see that the Landau damping term is multiplied by an overall factor of $(Zm^*/m)^2$.

The full $\Pi^*_d(\mathbf{q},\Omega)$ is then given by

$$\Pi_d^*(\mathbf{q}, \Omega) = \nu \left[1 - \frac{q^2}{b^2} - 2 \cos^2(2\phi_{\mathbf{q}}) \left(\frac{Zm^*}{m} \right)^2 \frac{|\Omega|}{v_{\mathrm{F}} q} \right],$$
$$(3.22)$$

where $b \sim k_{\mathrm{F}}$.

The q^2 and Landau damping terms in Eq. (3.22) agree with their counterparts in the expression for the polarization

bubble in Ref. 16. The first, constant term differs, however, from that in Ref. 16, where only the contribution from coherent fermions was included and, as a result, the first term had an extra Z factor.

Substituting Eq. (3.22) into Eq. (3.18) and neglecting again the q^2/b^2 term in the bubble compared to the $(aq)^2$ from the interaction, we obtain the dynamic effective interaction as

$$\Gamma_{\alpha\beta;\gamma\delta}^{\Omega,\mathrm{RPA}}(\mathbf{k}, \omega_k; \mathbf{p}, \omega_p) = \frac{1}{4\nu} \frac{d_{\mathbf{k}} d_{\mathbf{p}} \delta_{\alpha\delta} \delta_{\beta\gamma}}{1 + g_{c,2} + |\mathbf{k} - \mathbf{p}|^2 a^2 + 2 \cos^2(2\phi_{\mathbf{q}}) \left(\dfrac{Zm^*}{m} \right)^2 \dfrac{|\omega_k - \omega_p|}{v_{\mathrm{F}} |\mathbf{k} - \mathbf{p}|}} \equiv d_{\mathbf{k}} d_{\mathbf{p}} \delta_{\alpha\delta} \delta_{\beta\gamma} \bar{\Gamma}(\mathbf{k} - \mathbf{p}, \omega_k - \omega_p). \quad (3.23)$$

For $\mathbf{k}$ and $\mathbf{p}$ near the FS, $|\mathbf{k} - \mathbf{p}| = 2k_{\mathrm{F}} \sin(\theta/2)$, where $\theta \equiv \angle(\mathbf{k}, \mathbf{p})$. Because $\mathbf{k}$ and $\mathbf{p}$ are nearly parallel, $|\mathbf{k} - \mathbf{p}| \approx k_{\mathrm{F}} \theta$. Notice that by virtue of Eq. (3.9), the effective interaction has both charge and spin components even though we neglected the $g_{s,2}$ component of the bare interaction.

A cautionary note: Eq. (3.23) is obtained within the RPA, which does not mix channels with different angular momenta. The RPA is valid, strictly speaking, only for a long-range interaction which is not a very realistic assumption. For a generic interaction, there is no proof that the effective interaction does not change quantitatively beyond the RPA, e.g., it is possible that non-RPA renormalizations destroy a simple pole structure of Eq. (3.23). In Appendix A, we discuss the diagrammatic series for Γ^{Ω} beyond RPA and show that the effective interaction does change beyond the RPA.

C. Diagrammatics of the critical FL theory

1. Interaction via dynamic collective fluctuations

We now show that the RPA expression for $\Gamma_{\alpha\beta;\gamma\delta}^{\Omega}$ [Eq. (3.23)] is not the full result, even if the control parameter ak_{F} is large.[44] The reason is that the diagrammatic series for $\Gamma_{\alpha\beta;\gamma\delta}^{\Omega}$ in the bare interaction, considered in the previous section, does not include diagrams with particle-hole bubbles at exactly zero momentum and vanishingly small frequency.
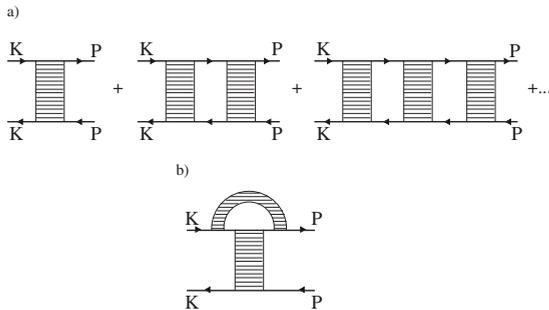

FIG. 4. (a) Ladder series for the full vertex $\Gamma_{\alpha\beta;\gamma\delta}^{\Omega}$ beyond RPA. Each hatched block represents $\Gamma_{\alpha\beta;\gamma\delta}^{\Omega}$ at the RPA level, Eq. (3.23). (b) One-loop vertex correction to the ladder diagram.

The argument is that such diagrams vanish due to particle-number conservation. This is true, however, only if the interaction is static. Due to Landau damping, the dressed interaction has a singularity in the complex plane, i.e., a branch cut in the Matsubara formalism. In this situation, the frequency integral of the product $G^2(q, \Omega)$ times the dressed interaction has an extra contribution from the Landau damping singularity. To account for this effect, we have to reconsider diagrams with soft particle-hole bubbles and replace each static interaction, $U(\mathbf{k} - \mathbf{p})$, by a dressed one, i.e., by $\Gamma_{\alpha\beta;\gamma\delta}^{\Omega,\mathrm{RPA}}(\mathbf{k}, \omega_k; \mathbf{p}, \omega_p)$ given by Eq. (3.23). Since the RPA interaction already contains all insertions of particle-hole bubbles, the remaining diagrams form the ladder series shown in Fig. 4. In Secs. III C 2 and III C 3 we evaluate this ladder series and show that corrections to ladder diagrams are small.

2. Renormalization of $\Gamma_{\alpha\beta;\gamma\delta}^{\Omega}$ beyond RPA

A building block for ladder diagrams is the product of two Green's functions with the same momentum and frequency and the dynamic interaction, Eq. (3.23). We will analyze explicitly the first few terms in the ladder series and then sum up infinite series of diagrams. Consider first the second-order diagram in Fig. 4(a). Summing over the internal spin indices and neglecting the momentum transfer q compared to k_{F} in the d-wave form factors, we obtain

$$\Gamma_{\alpha\beta;\gamma\delta}^{\Omega,\{2\}}(\mathbf{k}, \mathbf{p}) = d_{\mathbf{k}} d_{\mathbf{p}} \frac{d_{\mathbf{k}}^2 + d_{\mathbf{p}}^2}{2} \delta_{\alpha\delta} \delta_{\beta\gamma} \int \frac{d\Omega}{2\pi} \int \frac{dq q}{2\pi} \int \frac{d\phi}{2\pi}$$
$$\times G^2(\mathbf{k} + \mathbf{q}, \Omega) \bar{\Gamma}(\mathbf{q}, \Omega) \bar{\Gamma}(\mathbf{k} - \mathbf{p} + \mathbf{q}, \Omega),$$
$$(3.24)$$

where $|\mathbf{k}| = |\mathbf{p}| = k_{\mathrm{F}}$, $\phi = \angle(\mathbf{q}, \mathbf{k})$, and $\bar{\Gamma}$ is defined in the second line of Eq. (3.23). Let us integrate over ϕ first. Both the Green's functions and $\bar{\Gamma}$ depend on ϕ. The ϕ dependence of $\bar{\Gamma}$ is in two places: first, in the anisotropic Landau damping term, which depends on ϕ as $\cos^2[2(\phi_{\mathbf{k}} + \phi)]$, where $\phi_{\mathbf{k}} = \angle(\mathbf{k}, \hat{x})$, and, second, in the magnitude of the momentum

$|\mathbf{k}-\mathbf{p}+\mathbf{q}|$. We will see, however, that the integral over ϕ is dominated by narrow regions near $\pm\pi/2$, where $\mathbf{q}$ is almost perpendicular to $\mathbf{k}$. Therefore, the prefactor of the Landau damping term can be replaced by $\cos^2(2\phi_\mathbf{k})$. Expanding next ϕ near $\pm\pi/2$ as $\phi=\pm\pi/2\mp\bar{\phi}$ with $|\bar{\phi}\ll1|$, and using the fact that near a QCP the relevant values of $|\mathbf{k}-\mathbf{p}|\approx k_\mathrm{F}\theta$ (determined by the inverse correlation length) are small, we find that $|\mathbf{k}-\mathbf{p}+\mathbf{q}|^2\approx(k_\mathrm{F}\theta\pm q)^2$ does not depend on $\bar{\phi}$ to leading order. It is thus only $G^2(\mathbf{k}+\mathbf{q},\Omega)$ that depends on $\bar{\phi}$. The integral over $\bar{\phi}$ can then be evaluated; one has to be careful though to resolve an ambiguity at $\Omega=0$, which results in a delta-function term

$$\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}G^2(\mathbf{k}+\mathbf{q},\Omega)=Z^2\left\{2\frac{\delta(\Omega)}{v_\mathrm{F}^* q}-\frac{|\Omega|}{[\Omega^2+(v_\mathrm{F}^* q)^2]^{3/2}}\right\}.$$
(3.25)

The delta-function term will give the leading contribution to the vertex. It is reproduced if the integral over $\bar{\phi}$ is performed as

$$\int_{-\pi}^{\pi}\frac{d\phi}{2\pi}G^2(\mathbf{k}+\mathbf{q},\Omega)\rightarrow2\int_{-\infty}^{\infty}\frac{d\bar{\phi}}{2\pi}\frac{Z^2}{(i\Omega-v_\mathrm{F}^* q\bar{\phi})^2}$$
$$=2i\frac{\partial}{\partial\Omega}\int_{-\infty}^{\infty}\frac{d\bar{\phi}}{2\pi}G(\mathbf{k}+\mathbf{q},\Omega)=2Z^2\frac{\delta(\Omega)}{v_\mathrm{F}^* q}.$$
(3.26)

The second term in Eq. (3.25) is reproduced if one keeps track of the actual limits of the integral over ϕ.

Substituting Eq. (3.25) into Eq. (3.24) and integrating over Ω and q, we find, as advertised, that the leading contribution to $\Gamma_{\alpha\beta;\gamma\delta}^{\Omega,\{2\}}(\mathbf{k},\mathbf{p})$ comes from the δ-functional term, while the contribution from the second term in Eq. (3.25) is proportional to $1+g_{c,2}$, which is small near a QCP. The delta function of Ω eliminates the frequency dependence of the RPA vertices, and, integrating over q, we obtain for the second-order ladder diagram

$$\Gamma_{\alpha\beta;\gamma\delta}^{\Omega,\{2\}}(\mathbf{k},\mathbf{p})=d_\mathbf{k}d_\mathbf{p}\frac{d_\mathbf{k}^2+d_\mathbf{p}^2}{2}\delta_{\alpha\delta}\delta_{\beta\gamma}\left(\frac{Z^2m^*\lambda}{2m}\right)$$
$$\times\frac{1}{4\nu}\left[\frac{2}{4(1+g_{c,2})+(ak_\mathrm{F}\theta)^2}\right],\quad(3.27)$$

where

$$\lambda=\frac{1}{2ak_\mathrm{F}\sqrt{1+g_{c,2}}}.$$
(3.28)

We will see below that $Z^2m^*\lambda/m\approx1$ in the critical FL regime. This implies $\Gamma^{\Omega,\{2\}}$ is of the same order as $\Gamma^{\Omega,\mathrm{RPA}}$ and, thus, ladder renormalizations cannot be neglected. We also note that the parameter λ will play the key role in our further analysis. In particular, it will be shown that the critical FL regime is defined by a condition $\lambda\gg1$ rather than by $1+g_{c,2}\ll1$. Because λ contains a small factor of $1/ak_\mathrm{F}\ll1$, the former is stronger than the latter.

We now return to the ladder series composed from the $G^2(q,\Omega)$ blocks. Repeating the same procedure for the nth term of the series, we obtain

$$\Gamma_{\alpha\beta;\gamma\delta}^{\Omega,\{l\}}(\mathbf{k},\mathbf{p})=d_\mathbf{k}d_\mathbf{p}\frac{d_\mathbf{k}^{2(l-1)}+d_\mathbf{p}^{2(l-1)}}{2}\delta_{\alpha\delta}\delta_{\beta\gamma}$$
$$\times\left(\frac{Z^2m^*\lambda}{2m}\right)^{l-1}\bar{\Gamma}^{\{l\}}(\theta),\quad(3.29)$$

where

$$\bar{\Gamma}^{\{l\}}(\theta)=\frac{1}{4\nu}\left[\frac{l}{l^2(1+g_{c,2})+(ak_\mathrm{F}\theta)^2}\right].$$
(3.30)

Comparing Eq. (3.30) to Eq. (3.23), we see that the structure of the RPA interaction (the $l=1$) term is not reproduced at higher orders: first, the dependence on the angle θ between $\mathbf{k}$ and $\mathbf{p}$ is different and, second, there are additional d factors arising at each order of the perturbation theory. However, the angular dependence can be scaled back to that of the $l=1$ term by replacing $\theta\rightarrow l\theta$. This means that the angular harmonics $\int d\theta\bar{\Gamma}^{\{l\}}(\theta)\cos(n\theta)/2\pi$ of $\bar{\Gamma}^{\{l\}}$ are the same as those of $\bar{\Gamma}$, as long as n and l are not too large: $nl\ll\lambda$. Since $\lambda\gg1$ in the critical FL regime, this condition is always satisfied for not too large l and n. In other words, $\bar{\Gamma}^{\{l\}}$ can be rewritten as the sum of two terms: the first one is the same as for $n=1$ and the second one vanishes on angular integration

$$\bar{\Gamma}^{\{l\}}(\theta)=\bar{\Gamma}(\theta)+R^{\{l\}}(\theta),$$
(3.31)

where

$$R^{\{l\}}(\theta)=(l-1)\frac{(ak_\mathrm{F}\theta)^2-l(1+g_{c,2})}{[l^2(1+g_{c,2})+(ak_\mathrm{F}\theta)^2][1+g_{c,2}+(ak_\mathrm{F}\theta)^2]}$$
(3.32)

and $\int d\theta R^{\{l\}}=0$. Since the observables are determined by harmonics of Γ^Ω, we will neglect the remainder term $R^{\{l\}}$ below, i.e., replace $\bar{\Gamma}^{\{l\}}(\theta)$ by $\bar{\Gamma}(\theta)$.

Still, the subsequent terms in the series contain additional factors $d_\mathbf{k}^{2(l-1)}+d_\mathbf{p}^{2(l-1)}$. These terms account for renormalization of a smooth, regular variation in Γ^Ω along the FS. Such a variation, inherent to a model with anisotropic, d-wave, $\cos(2\phi)$ interaction, implies that both the quasiparticle residue and effective mass are not uniform along the FS. The d-channel vertex can be obtained by extracting contributions proportional to $d_\mathbf{k}d_\mathbf{p}$ at any order, neglecting all other terms, and summing up the series. On the other hand, one can average the vertex over the FS, i.e., restore Galilean invariance broken by neglecting the $\sin(2\phi)$ interaction. The two approaches lead to physically equivalent results which differ only by numerical prefactors. For brevity, we only discuss here the approach based on averaging, which is particularly appealing because it allows one to use the technique developed for isotropic FL systems to obtain m^*/m and Z. Results of alternative approaches are presented in Appendix B. We emphasize that, although no important physics is lost in ei-

ther of the approaches, all of them are still approximate. We will discuss one subtle issue associated with this approximation in Appendix C.

We now apply the averaging procedure. Since relevant $\mathbf{k}$ and $\mathbf{p}$ are almost parallel to each other, we will set $\mathbf{k}=\mathbf{p}$ in the d factors and use an identity

$$\langle d_{\mathbf{k}}^{2l} \rangle = \frac{(2l-1)!!}{l!}. \tag{3.33}$$

Substituting this expression into $\Gamma^{\Omega,\{l\}}$ and summing up the series, we find that the ladder series for $\Gamma_{\alpha\beta;\gamma\delta}^{\Omega}$ result in overall renormalization of $\Gamma_{\alpha\beta;\gamma\delta}^{\Omega,\mathrm{RPA}}$,

$$\Gamma_{\alpha\beta;\gamma\delta}^{\Omega} = \frac{1}{Z_{\Gamma}}\Gamma_{\alpha\beta;\gamma\delta}^{\Omega,\mathrm{RPA}} = \frac{1}{Z_{\Gamma}}\frac{1}{4\nu}\frac{\delta_{\alpha\delta}\delta_{\beta\gamma}}{1+g_{c,2}+(ak_{\mathrm{F}}\theta)^2}, \tag{3.34}$$

where

$$\frac{1}{Z_{\Gamma}} = \sum_{l=1}^{\infty}\left(\frac{Z^2 m^*}{2m}\lambda\right)^{l-1}\frac{(2l-1)!!}{l!}$$

$$= \frac{2}{\sqrt{1-\frac{Z^2 m^*}{m}\lambda}\left(1+\sqrt{1-\frac{Z^2 m^*}{m}\lambda}\right)}. \tag{3.35}$$

Since $Z_{\Gamma} \leq 1$, ladder renormalization enhances the interaction compared to the RPA result.

3. Vertex corrections

We now show that vertex corrections to ladder series are small and thus Eq. (3.34) is a complete result for our model. The lowest-order vertex correction to Γ^{Ω} is shown in Fig. 4(b). When calculating the ladder diagram in the previous section, we saw that the interaction vertices are effectively static ($\Omega=0$) while typical bosonic momenta $q \sim \sqrt{1+g_{c,2}}/a$ are small near a QCP. To estimate the vertex correction, we can then simply calculate a three-leg vertex, shown in Fig. 4(c), at zero external frequency and finite but small external momentum q. With these simplifications, the three-leg vertex reduces to

$$\Gamma_{\triangle}(\mathbf{q}\to 0, \Omega=0) = \int\frac{d\Omega'}{2\pi}\frac{d^2q'}{(2\pi)^2}G(\mathbf{k}+\mathbf{q}',\Omega')$$

$$\times G(\mathbf{k}+\mathbf{q}'+\mathbf{q},\Omega')\bar{\Gamma}(\mathbf{q}',\Omega'). \tag{3.36}$$

We neglected the spin- and d-wave factors as well as the anisotropy of the Landau damping term, all of which give only overall numerical prefactors. Since $\bar{\Gamma}(q',\Omega')$ is now isotropic, the angular integration in Eq. (3.36) can be performed first. Because the momenta in two Green's functions differ by small yet finite q, the angular integral of $G(\mathbf{k}+\mathbf{q}',\Omega')G(\mathbf{k}+\mathbf{q}'+\mathbf{q},\Omega')$ contains only a regular but no $\delta(\Omega')$ term, i.e., the result is given by the second term in Eq. (3.25). Hence

$$\Gamma_{\triangle}(\mathbf{q}\to 0, \Omega=0) = -Z^2\int\frac{d\Omega'}{2\pi}\frac{dq'q'}{2\pi}\frac{|\Omega'|}{[(\Omega')^2+(v_{\mathrm{F}}^*q')^2]^{3/2}}$$

$$\times \bar{\Gamma}(\mathbf{q}',\Omega'). \tag{3.37}$$

We notice immediately that the remaining double integral is finite even right at a QCP, where $1+g_{c,2}=0$. This is to be contrasted with ladder diagrams of the previous section which diverge near a QCP. An explicit calculation can be carried out by rescaling the variables as $q'=xq_0$ and $\Omega'=\Omega_0 y$, where

$$q_0 = Z\sqrt{m^*/m}/a, \quad \Omega_0 = q_0 v_{\mathrm{F}}^*, \tag{3.38}$$

and introducing polar coordinates $x=r\cos\psi$ and $y=r\sin\psi$. We then obtain

$$\Gamma_{\triangle}(\mathbf{q}\to 0, \Omega=0) = -Z\left(\frac{m^*}{m}\right)^{1/2}\frac{1}{ak_{\mathrm{F}}}$$

$$\times \frac{1}{4\pi}\int_0^{\infty}dr\int_0^{\pi/2}d\psi\frac{\sin\psi\cos\psi}{r^2\cos^2\psi+\alpha+\tan\psi}, \tag{3.39}$$

where

$$\alpha = (1+g_{c,2})\frac{m}{m^* Z^2}. \tag{3.40}$$

Integrating over r, we obtain

$$\Gamma_{\triangle}(\mathbf{q}\to 0, \Omega=0) = -\frac{Z}{8}\left(\frac{m^*}{m}\right)^{1/2}\frac{1}{ak_{\mathrm{F}}}\mathcal{F}(\alpha), \tag{3.41}$$

where

$$\mathcal{F}(x) = \int_0^{\pi/2}d\psi\frac{\sin\psi}{\sqrt{x+\tan\psi}}. \tag{3.42}$$

For $x \ll 1$,

$$\mathcal{F}(x) = \frac{\Gamma^2(3/4)}{\sqrt{\pi}} - \frac{\Gamma^2(1/4)}{8\sqrt{\pi}}x + \cdots \approx 0.847 - 0.927x + \cdots \tag{3.43}$$

In the next section, we show that $\alpha \sim 1$ away from a QCP and $\alpha \ll 1$ near a QCP. In both cases, $\mathcal{F}(x) \sim 1$. Therefore, the vertex correction can be estimated as

$$\Gamma_{\triangle}(\mathbf{q}\to 0, \Omega=0) \sim Z\left(\frac{m^*}{m}\right)^{1/2}\frac{1}{ak_{\mathrm{F}}}, \tag{3.44}$$

which is at most of order $1/ak_{\mathrm{F}} \ll 1$.

Equation (3.44) can be rewritten via the characteristic momentum q_0 or characteristic frequency Ω_0 from Eq. (3.38) as

$$\Gamma_{\triangle}(\mathbf{q}\to 0, \Omega=0) \sim q_0/k_{\mathrm{F}} \sim \Omega_0/E_{\mathrm{F}}^*, \tag{3.45}$$

where $E_{\mathrm{F}}^* = k_{\mathrm{F}}v_{\mathrm{F}}^*$. When written in this form, it is clear that $\Gamma_{\triangle}(\mathbf{q}\to 0, \Omega=0)$ is an effective Migdal (adiabatic) parameter, i.e., the ratio of characteristic energies of bosons (Ω_0) and fermions (E_{F}^*), of our theory.[29] For $ak_{\mathrm{F}} \gg 1$, this ratio is small both away and near a QCP. The Migdal parameter occurs here in same way as for the case of electrons inter-

acting with *optical* phonons. Namely, if the interaction in Eq. (3.36) is replaced by a propagator of optical phonons $\Omega_0^2/(\Omega^2+\Omega_0^2)$ (where now $\Omega_0 \ll E_F$ is the optical-phonon frequency), the result for the vertex is the same (up to a number). The reason for the smallness of the vertex in both cases is a small phase space defined by q_0 and Ω_0. Although the vertex correction is also small for *acoustic* phonons, the reason is different in this case: a typical momentum of acoustic phonon is on the order of the inverse lattice spacing ($\sim k_F$), while the electron energy at the same momentum is on the order of E_F, i.e., much larger than the phonon energy Ω_0.

Two cautionary notes, both related to the fact that the integral in Eq. (3.41) is determined by $\Omega' \sim v_F^* q'$ $\sim E_F/(ak_F)Z\sqrt{m/m^*}$. First, we used a quasistatic, $\Omega'/v_F^* q'$ form of the Landau damping term, which is not, strictly speaking, justified in this regime. However, because q' is not much smaller than Ω'/v_F^* either, the result for $\Gamma_\triangle$ is still correct up to a numerical prefactor. Second, and more important, we assumed a renormalized FL form of the Green's function $G(\mathbf{k},\omega)=Z/[\omega-v_F^*(k-k_F)]$. Meanwhile, we will see later that (i) a FL behavior extends only up to frequencies of order $\omega_{FL} \sim (v_F/a)(1+g_{c,2})^{3/2}$, and (ii) $\Omega_0 \sim Z\sqrt{m/m^*}(v_F^*/a)$ $\sim (v_F/a)(ak_F)^{3/2}(1+g_{c,2})^{3/4}$. Thus, typical $\Omega' \sim \Omega_0 \ll \omega_{FL}$ are outside the FL regime for small enough $1+g_{c,2}$. However, the non-FL effects are not important here either because the three-leg vertex outside the FL regime is determined by very high energies ($\sim v_F/a$), where fermions behave as almost free quasiparticles with $Z, m^*/m \approx 1$ (Ref. 29). The result is that Eq. (3.45) still holds, but the relevant fermionic energy is E_F while the relevant bosonic energy is $\Omega_0 \sim E_F/ak_F$, and the vertex correction is simply given by

$$\Gamma_\triangle(\mathbf{q} \to 0, \Omega = 0) \sim 1/ak_F. \quad (3.46)$$

Certainly, the same result for the vertex is obtained if one substitutes the free Green's function and bare vertex into Eq. (3.44) from the very beginning. We also emphasize that the vertex correction was found to be small in the limit $\Omega/q \to 0$. In the opposite limit, $q/\Omega \to 0$, vertex corrections are not small and must be included into the theory to ensure that the Ward identities are satisfied.

4. Complete form of the FL vertex

To summarize this section, we conclude that, the full vertex $\Gamma_{\alpha\beta;\gamma\delta}^\Omega$ of a critical FL differs from the RPA result by a constant renormalization factor $1/Z_\Gamma$ given by Eq. (3.35). Hence a complete result for the static d-wave vertex, averaged over the FS, and for $\mathbf{k} \approx \mathbf{p}$ reads

$$\Gamma_{\alpha\beta;\gamma\delta}^\Omega(\mathbf{k},\mathbf{p}) = \frac{1}{4\nu Z_\Gamma}\frac{\delta_{\alpha\delta}\delta_{\beta\gamma}}{1+g_{c,2}+(ak_F\theta)^2}. \quad (3.47)$$

To obtain an explicit form of $\Gamma_{\alpha\beta;\gamma\delta}^\Omega$, we need to know Z and m^*/m. This is what we discuss in the next section. We show there that, near the QCP, the leading terms in Z and m^*/m are fully determined the static vertex, i.e., there is no need to invoke the bosonic dynamics. The subleading terms, which we will also discuss for accuracy, require the knowledge of the dynamic vertex. Following the same steps that led us to

Eq. (3.47), one can show that, for $ak_F\theta \gg \sqrt{1+g_{c,2}}$, the dynamic vertex is given by

$$\Gamma_{\alpha\beta;\gamma\delta}^\Omega(\mathbf{k},\omega_k;\mathbf{p},\omega_p)$$
$$= \frac{1}{4\nu Z_\Gamma}\frac{\delta_{\alpha\delta}\delta_{\beta\gamma}}{1+g_{c,2}+(ak_F\theta)^2+\left(\frac{Zm^*}{m}\right)^2\frac{|\omega_k-\omega_p|}{v_F k_F\theta}}. \quad (3.48)$$

In a general case, the dynamic vertex is a scaling function of the following form

$$\Gamma_{\alpha\beta;\gamma\delta}^\Omega(\mathbf{k},\omega_k;\mathbf{p},\omega_p)$$
$$= \frac{1}{4\nu Z_\Gamma}\frac{\delta_{\alpha\delta}\delta_{\beta\gamma}}{1+g_{c,2}}\mathcal{G}\left(\frac{ak_F}{\sqrt{1+g_{c,2}}},\left(\frac{Zm^*}{m}\right)^2\frac{a|\omega_k-\omega_p|}{v_F\sqrt{1+g_{c,2}}}\right), \quad (3.49)$$

where $\mathcal{G}(x,0)=1/(1+x^2)$. As an explicit form of $\mathcal{G}$ will not be required for estimates of the subleading terms in Z and m^*/m, we will use below the approximate Eq. (3.48).

IV. PITAEVSKI-LANDAU RELATIONS AND SELF-CONSISTENT EQUATIONS FOR Z AND m^*/m

A. Pitaevski-Landau identities for the derivatives of the self-energy

The renormalized mass can be found in two ways. If the Landau function is known, m^*/m can be determined from its $g_{c,1}$ component. Alternatively, one can compute the self-energy in a given microscopic model and extract mass renormalization from the renormalized Green's function. In general, the quasiparticle residue Z can be found only from a microscopic calculation because renormalization of Z is determined by states away from the FS, which are not described by the FL theory. Renormalizations of Z and m^*/m near QCPs have been considered in Refs. 10, 11, 13, 17, and 22–35 via loopwise expansions in the effective interaction. Near a QCP, however, typical energies involved in renormalizations vanish in the inverse proportion to the divergent correlation length. This allows one to determine both m^*/m and Z within the FL framework. To do so, we will use the Pitaevski-Landau identities which relate the derivatives of the self-energy[1,38] to Γ^Ω from Eq. (3.47).

We will need three of the Pitaevski-Landau identities

$$\frac{\partial\Sigma}{\partial\omega} = -i\sum_\beta\int\Gamma_{\alpha\beta,\alpha\beta}^\Omega(K_F,P)[G^2(P)]_\Omega\frac{d^3P}{(2\pi)^3},$$

$$\frac{\partial\Sigma}{\partial\mathbf{k}} = i\sum_\beta\int\frac{\mathbf{p}}{m}\Gamma_{\alpha\beta,\alpha\beta}^q(K_F,P)[G^2(P)]_q\frac{d^3P}{(2\pi)^3},$$

$$\mathbf{k}\frac{\partial\Sigma}{\partial\omega} = -i\sum_\beta\int\mathbf{p}\Gamma_{\alpha\beta,\alpha\beta}^\Omega(K_F,P)[G^2(P)]_\Omega\frac{d^3P}{(2\pi)^3}. \quad (4.1)$$

Here, as before, a shorthand K_F denotes the four momentum on the FS, i.e., $K_F \equiv \{k_F\hat{k},\omega_k=0\}$, while the objects $[G^2(P)]_\Omega$

and $[G^2(P)]_q$ represent the product $G(P)G(P+Q)$ taken at vanishing Q in the limits of $q/\Omega \to 0$ and $\Omega/q \to 0$, respectively. Notice that the internal momentum P is generally not at the FS. Also, we switched here from the Matsubara formalism, employed in the previous sections, to the causal one. To simplify notations, we use the same symbols for Matsubara and real frequencies in all cases when it does not lead to a confusion.

The first two relations are essentially the Ward identities following from particle-number conservation (gauge invariance) while the third one is special for Galilean-invariant system. We remind that Galilean invariance was restored in our problem by averaging over the FS. The vertices Γ^Ω and Γ^q are related by Eq. (2.3).

Equations (2.3) and (4.1) determine a Taylor expansion of the self-energy $\Sigma(k,\omega)$ to first order in ω and $\epsilon_k^* = v_F^*(k-k_F)$,

$$
\Sigma(K) = i\sum_\beta \left[-(\omega - \epsilon_k)\int \Gamma^\Omega_{\alpha\beta,\alpha\beta}(K_F,P)[G^2(P)]_\Omega \frac{d^3P}{(2\pi)^3} \right.
$$
$$
+ \frac{\epsilon_k^*}{Z}\int \Gamma^\Omega_{\alpha\beta,\alpha\beta}(K_F,P)\{[G^2(P)]_q
$$
$$
\left. -[G^2(P)]_\Omega\}\frac{\mathbf{p}\cdot\mathbf{k}_F}{k_F^2}\frac{d^3P}{(2\pi)^3} \right],
\tag{4.2}
$$

where the FS term $\Sigma(K_F)$ is absorbed into a shift of the chemical potential. Using an identity

$$
[G^2(K)]_q - [G^2(K)]_\Omega = -\frac{2\pi i Z^2 m^*}{k_F}\delta(\omega)\delta(k-k_F),
\tag{4.3}
$$

valid for any order of integration over the fermionic momentum and frequency, we can rewrite Eq. (4.2) as

$$
\Sigma(K) = (\omega - \epsilon_k)\left[-i\sum_\beta \int \Gamma^\Omega_{\alpha\beta,\alpha\beta}(K_F,P)[G^2(P)]_\Omega \frac{d^3P}{(2\pi)^3} \right]
$$
$$
+ \epsilon_k \frac{Zm}{4\pi^2}\sum_\beta \int \Gamma^\Omega_{\alpha\beta,\alpha\beta}(\theta)\cos\theta d\theta.
\tag{4.4}
$$

Equivalently,

$$
\Sigma(K) = (\omega - \epsilon_k)\left[\frac{1}{Z} - 1 \right] + \frac{\epsilon_k}{Z}\left[1 - \frac{m}{m^*} \right],
\tag{4.5}
$$

where

$$
\frac{1}{Z} = 1 - i\sum_\beta \int \Gamma^\Omega_{\alpha\beta,\alpha\beta}(K_F,P)[G^2(P)]_\Omega \frac{d^3P}{(2\pi)^3},
\tag{4.6}
$$

$$
\frac{1}{m^*} = \frac{1}{m} - \frac{Z^2}{4\pi^2}\sum_\beta \int \Gamma^\Omega_{\alpha\beta,\alpha\beta}(\theta)\cos\theta d\theta,
\tag{4.7}
$$

and θ is the angle between $\mathbf{k}$ and $\mathbf{p}$ when both momenta are on the FS.

The integral in Eq. (4.7) involves a full *static* vertex of the interaction between the particles on the FS. In our case, this vertex is given by Eq. (3.34) with $\omega_k = \omega_p$. Using this equation, we find

$$
\frac{Z^2}{4\pi^2}\sum_\beta \int \Gamma^\Omega_{\alpha\beta,\alpha\beta}(\theta)\cos\theta d\theta = \frac{Z^2}{4Z_\Gamma}\lambda
\tag{4.8}
$$

and

$$
\frac{m^*}{m} = 1 + \frac{\lambda}{4}\frac{Z^2}{Z_\Gamma}\frac{m^*}{m}.
\tag{4.9}
$$

This is an implicit equation for m^*/m.

Next, we consider the Z factor. In contrast to the effective mass, the Z factor is, in general, determined by the dynamic vertex. However, we will see shortly that the leading term in Z still comes from the static vertex. To see this, we substitute the dynamic interaction from Eq. (3.48) into Eq. (4.6) and relabel the variables as $\mathbf{q} = \mathbf{p} - \mathbf{k}_F$ and $\Omega = \omega_p$, upon which Eq. (4.6) reduces to

$$
\frac{1}{Z} = 1 - \frac{iZ^2}{4Z_\Gamma \nu}\int \frac{d^2q d\Omega}{(2\pi)^3}\frac{1}{(\Omega + \tilde\Omega - \epsilon^*_{\mathbf{k}_F+\mathbf{q}} + i\delta_\epsilon)^2}\Bigg|_{\tilde\Omega \to 0}
$$
$$
\times \frac{1}{1 + g_{c,2} + (aq)^2 - \left(\frac{Zm^*}{m}\right)^2 i\frac{|\Omega|}{v_F q}}.
\tag{4.10}
$$

This integral is similar to that in Eq. (3.24) for the second term in the ladder series for $\Gamma^\Omega_{\alpha\beta;\gamma\delta}$, except for now we have one rather than two interaction vertices. The result of the angular integration is the same as in Eq. (3.25). The leading contribution to $1/Z$ comes from the δ-function term in Eq. (3.25), which renders the vertex static. The contribution from the second, dynamic term Eq. (3.25) is proportional to the three-leg vertex, estimated in Sec. III C 3, except for an extra factor of $1/Z_\Gamma$ because the Z factor contains a full rather than RPA vertex. Since the three-leg vertex is small, so is the dynamic correction to $1/Z$. Collecting all terms, we find

$$
\frac{1}{Z} = 1 + \frac{\lambda}{4}\frac{Z^2}{Z_\Gamma}\frac{m^*}{m} - \frac{m^*}{m}\Gamma_\Delta,
\tag{4.11}
$$

where $\Gamma_\Delta \sim \Gamma_\triangle \sim 1/ak_F \ll 1$ is the dynamic correction due to a three-leg vertex.

B. Solutions for m^*/m, Z, and Z_Γ near a QCP

Equations (3.35), (4.9), and (4.11) form a closed set from which one obtains m^*/m, Z, and Z_Γ as functions of the coupling constant λ, defined by Eq. (3.28). Neglecting Δ in Eq. (4.11) and comparing it to Eq. (4.9) and, we see that

$$
Z\frac{m^*}{m} = 1.
\tag{4.12}
$$

Using this relation, we can rewrite Eqs. (3.35) and (4.11) as

$$
\frac{1}{Z_\Gamma} = \frac{4}{Z}\left(\frac{1-Z}{\lambda Z} \right),
\tag{4.13}
$$

$$
\frac{1}{Z} = 1 + \frac{1}{2}\frac{\lambda Z}{\sqrt{1-\lambda Z}(1 + \sqrt{1-\lambda Z})}.
\tag{4.14}
$$

Introducing a new variable $x = \sqrt{1-\lambda Z}$, we rewrite Eq. (4.14) as a cubic equation for x,

$$x^3 + x^2 + (2\lambda - 1)x = 1. \qquad (4.15)$$

The only real solution of this equation is

$$x = \frac{1}{3}L - \frac{2(\lambda - 2/3)}{L} - 1/3, \qquad (4.16)$$

where $L = (9\lambda + 8 + 3\sqrt{3\lambda}\sqrt{8\lambda^2 - 13\lambda + 16})^{1/3}$. Once x is known, the three parameters of the FL theory m^*/m, Z, and Z_Γ are also known. For $\lambda \ll 1$, $x = 1 - \lambda/2 + \mathcal{O}(\lambda^3)$; for $\lambda \gg 1$, $x = 1/2\lambda + 1/4\lambda^2 + \mathcal{O}(1/\lambda^4)$. Using these asymptotics, we immediately find that at weak coupling

$$\left.\frac{m^*}{m}\right|_{\lambda \ll 1} = 1 + \lambda/4, \quad Z|_{\lambda \ll 1} = 1 - \lambda/4, \quad Z_\Gamma|_{\lambda \ll 1} = 1 - 3\lambda/4,$$
$$(4.17)$$

while at strong coupling

$$\left.\frac{m^*}{m}\right|_{\lambda \gg 1} = \lambda + 1/4\lambda, \quad Z|_{\lambda \gg 1} = 1/\lambda - 1/4\lambda^3, \quad (4.18)$$

$$Z_\Gamma|_{\lambda \gg 1} = 1/4\lambda + 1/4\lambda^2.$$

Notice that $Z_\Gamma \approx Z/4$ in the strong-coupling limit.

We also emphasize that our calculations show that the effective mass m^* diverges right at the critical point but not earlier. In this respect, our results agree with Ref. 30 but disagree with Ref. 35, where it was conjectured that the effective mass may diverge at a topological transition which preempts a QCP. Such a behavior would follow from our formalism only if, for some reason, renormalization of Z and Z_Γ occurs in such a way that $C_Z = Z^2/Z_\Gamma$ were an invariant. According to Eq. (4.9), the effective mass $m^*/m = (1 - \lambda C_Z/4)$ would then diverge at finite $\lambda = 4/C_Z$. However, the solution of the full set of equations shows that the invariant is Z/Z_Γ, in which case the divergence of m^* is only possible at $\lambda = \infty$.

C. Frequency and momentum dependences of the self-energy

We now return to the self-energy given by Eq. (4.5). In the previous section, we found that the relation $Zm^*/m = 1$ holds as long as the vertex correction Δ is small. Substituting this relation into Eq. (4.5), we find that Σ is "local," it depends on ω but not on ϵ_k,

$$\Sigma(K) \approx (m^*/m - 1)\omega \approx \lambda\omega. \qquad (4.19)$$

Introducing the correlation length $\xi = a/\sqrt{1 + g_{c,2}}$, we can rewrite the mass renormalization coefficient as $\lambda = \xi/k_F a^2$. Notice that ξ is the bare correlation length which enters the RPA formula for the vertex. Rosch and Wölfle[16] obtained the same expression for m^*/m in terms of ξ but, for reasons displayed in Sec. III B, their ξ contains an additional factor of $1/\sqrt{Z}$ compared to ours.

The ϵ_k dependence of Σ is determined by the vertex correction Δ. Substituting a full form of $1/Z$ from Eq. (4.11) into Eq. (4.5), we find

$$\Sigma(0, \epsilon_k) = \epsilon_k\left(1 - \frac{m}{m^* Z}\right) = \Gamma_\triangle \epsilon_k \sim \frac{\epsilon_k}{ak_F}. \qquad (4.20)$$

We see that the k-dependent part of the self-energy remains small near a QCP.

V. CRITICAL FL THEORY: LANDAU PARAMETERS

Explicit solutions for m^*, Z, and Z_Γ, obtained in the previous section, complete our task of obtaining the FL vertex: one just needs to substitute these solutions into Eq. (3.47). Having found the full vertex, we can now construct the FL theory of the critical region, i.e., relate harmonics of $\Gamma^\Omega_{\alpha\beta;\gamma\delta}$ to observables. We emphasize again that $\Gamma^\Omega_{\alpha\beta;\gamma\delta}$ plays the role of an effective interaction between quasiparticles of an ordinary FL with bare nematic susceptibility $\chi_{c,2} \propto 1/(1 + g_{c,2})$ already enhanced by the interactions. We label bare susceptibilities as $\chi_{a,n}$ and the ones renormalized by $\Gamma^\Omega_{\alpha\beta;\gamma\delta}$ as $\bar{\chi}_{a,n}$. Applying the standard FL phenomenology, we obtain

$$\frac{m^*}{m} = 1 + \bar{g}_{c1},$$

$$\bar{\chi}_{c,n} = \chi_{c,n}\frac{1 + \bar{g}_{c1}}{1 + \bar{g}_{c,n}}, \quad \bar{\chi}_{s,n} = \chi_{s,n}\frac{1 + \bar{g}_{c1}}{1 + \bar{g}_{s,n}}, \qquad (5.1)$$

where $\bar{g}_{c,n}$ and $\bar{g}_{s,n}$ are charge and spin components of the Landau function for the critical FL

$$\bar{g}_{\alpha\beta;\gamma\delta}(\theta) = 2\nu Z^2\frac{m^*}{m}\Gamma^\Omega_{\alpha\beta;\gamma\delta}(\theta)$$

$$= \frac{Z^2 m^*}{4Z_\Gamma m}\frac{1}{1 + g_{c,2} + (ak_F\theta)^2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \vec{\sigma}_{\alpha\gamma}\cdot\vec{\sigma}_{\beta\delta})$$

$$\approx \frac{1}{1 + g_{c,2} + (ak_F\theta)^2}(\delta_{\alpha\gamma}\delta_{\beta\delta} + \vec{\sigma}_{\alpha\gamma}\cdot\vec{\sigma}_{\beta\delta}). \quad (5.2)$$

In 2D, the harmonics of $\bar{g}(\theta)$ are given by $\bar{g}_n = \oint \bar{g}(\theta)\cos(n\theta)d\theta/(2\pi)$. Integrating over θ, we find that the first

$$n \lesssim n_c \equiv 2\lambda(ak_F)^2 \qquad (5.3)$$

harmonics of $\bar{g}$ both in the charge and spin channels diverge concurrently with the effective mass upon approaching QCP as

$$\bar{g}_{a,n} = \lambda e^{-n/n_c}, \qquad (5.4)$$

where λ is given by Eq. (3.28). At a QCP, where $\lambda \to \infty$, *all* harmonics of $\bar{g}$ diverge. We notice in passing that all harmonics diverge also for $D \leq 3$ albeit the divergence in only logarithmic for $D = 3$. We also see, however, the susceptibilities retain their bare values despite the divergence of the Landau components. Indeed, because all Landau components diverge in the same way, renormalization of the effective mass in the numerator cancels with that of the effective "g factors,"

$$\bar{\chi}_{a,n} = \chi_{a,n}\frac{1 + \bar{g}_{c,1}}{1 + \bar{g}_{a,n}} \approx \chi_{a,n}\frac{\bar{g}_{c,1}}{\bar{g}_{a,n}} \approx \chi_{a,n}. \qquad (5.5)$$

In particular, the nematic susceptibility remains equal to

$\chi_{c,2} \propto 1/(1+g_{c,2})$, i.e., *it is not affected by mass renormalization.*

One can also introduce an "actual" Landau function, g^*, which describes a combined effect of renormalizations in the ordinary and critical FLs and includes contributions from both the regular and collective mode parts of $\Gamma^\Omega_{\alpha\beta;\gamma\delta}$. The nematic susceptibility can be written equivalently either in terms of either $\bar{g}$ or g^*,

$$\bar{\chi}_{c,2} = \chi_{c,2} \frac{1+\bar{g}_{c,1}}{1+\bar{g}_{c,2}} = \frac{\chi^{\{0\}}_{c,2}}{1+g_{c,2}} \frac{1+\bar{g}_{c,1}}{1+\bar{g}_{c,2}} = \chi^{\{0\}}_{c,2} \frac{1+g^*_{c,1}}{1+g^*_{c,2}}. \tag{5.6}$$

All other susceptibilities $\bar{\chi}_{a,n}$ with $\{a,n\}$ different from $\{c,2\}$ can be written as

$$\bar{\chi}_{a,n} = \chi^{\{0\}}_{a,n} \frac{1+\bar{g}_{c,1}}{1+\bar{g}_{a,n}} = \chi^{\{0\}}_{a,n} \frac{1+g^*_{c,1}}{1+g^*_{a,n}}. \tag{5.7}$$

Comparing the expressions for χ in terms of $\bar{g}$ and g^*, we read off the components of g^* as

$$g^*_{a,n} = g_{a,n} + \bar{g}_{a,n}(1+g_{a,n}). \tag{5.8}$$

For all partial components different from the $n=2$ charge one, the regular contribution $g_{a,n}$ is absent, and $g^*_{a,n} = \bar{g}_{a,n}$. The divergence of $\bar{g}_{a,n}$ implies that *all components of the actual Landau function g^* different from the nematic one diverge at a Pomeranchuk instability.* The $n=2$ component behaves as

$$g^*_{c,2} = (1+\bar{g}_{c,2})(1+g_{c,2}) - 1 \approx \lambda(1+g_{c,2}) - 1$$
$$= \frac{(1+g_{c,2})^{1/2}}{2ak_F} - 1. \tag{5.9}$$

The nematic component of the actual Landau function approaches -1 at a QCP, where $g_{c,2} = -1$. We show the behavior of $\bar{g}_{a,n}$ and $g^*_{a,n}$ in Fig. 5.

Note in passing that, although $\chi_{c,2}$ diverges as $1/(1+g_{c,2})$ in both the ordinary and critical FL regimes, this divergence has different origins in the two regimes. In an ordinary FL (where $m^*/m \sim 1$), the divergence of $\chi_{c,2}$ is entirely due to that of the d-wave charge g factor $1/(1+g_{c,2})$, while in the critical FL the divergence is equally "shared" between the effective mass and the g factor, each contributing a factor of $1/(1+g_{c,2})^{1/2}$. The analogous "sharing" holds for the spin susceptibility near a ferromagnetic instability.

There is an interesting relation between our results and the Mermin sum rule[1,45] for the forward-scattering amplitude for parallel spins: $f_{\alpha\alpha;\gamma\delta}(\theta=0)=0$, which is a manifestation of the Pauli principle. In terms of the components of the actual Landau function, the sum rule implies that

$$\sum_{a,n} \frac{g^*_{a,n}}{1+g^*_{a,n}} = 0. \tag{5.10}$$

Using Eq. (5.8), we find that the sum rule for forward scattering requires that
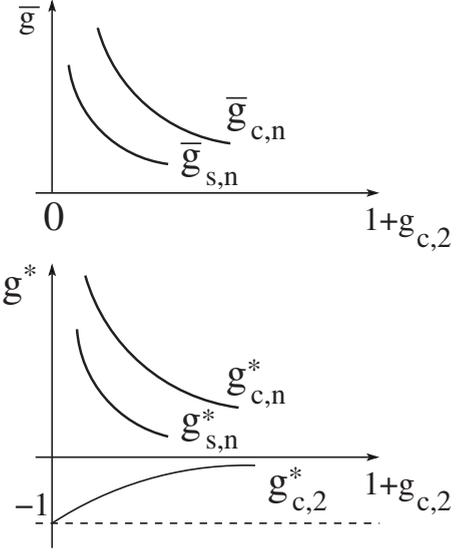

FIG. 5. Top: schematic behavior of the charge (c) and spin (s) components of the critical Landau function $\bar{g}$ as a function of $g_{c,2}$. Bottom: same for the actual Landau function g^*.

$$\sum_{a,n\neq 2} \frac{\bar{g}_{a,n}}{1+\bar{g}_{a,n}} + \frac{\bar{g}_{s,2}}{1+\bar{g}_{s,2}} = -\frac{g^*_{c,2}}{1+g^*_{c,2}}. \tag{5.11}$$

The right-hand side (rhs) of this equation is evaluated using Eq. (5.9) and, for large λ, equals to

$$4\lambda(ak_F)^2. \tag{5.12}$$

For the left-hand side (lhs) we obtain, using Eq. (5.4), that each term in the sum becomes $\lambda/(1+\lambda) \approx 1$, so that the sum diverges. The divergence is artificial, however, because harmonics with $n \geq n_c$ fall off with n exponentially, as specified by Eq. (5.4). Furthermore, the very expression for the critical Landau function, Eq. (5.2), is only valid up to $n = n_{max} \geq n_c$. At larger n, the terms $R^{\{l\}}(\theta)$ neglected in the derivation of Z_Γ [cf. Eq. (3.32)] become relevant as $\int \cos(n\theta) R^{\{l\}}(\theta)$ is no longer a small correction.

As $\bar{g}_{a,n}$ are close to unity for $n \ll n_c$ and cut at $n_{max} \geq n_c$, the sum in Eq. (5.11) is on order of n_c. To see this in more detail, we substitute $\bar{g}_{a,n}$ from Eq. (5.4) into Eq. (5.11), and replace the sum over n by an integral because only large n matter. This yields for the lhs of Eq. (5.11),

$$\sum_{a,n\neq 2} \frac{\bar{g}_{a,n}}{1+\bar{g}_{a,n}} \approx 2 \int_{\sim 1}^{n_{max}} dn \frac{\lambda}{\lambda + e^{n/n_c}}$$
$$= 4(ak_F)^2 \lambda \int_{\sim 1}^{e^{n_{max}/n_c}} \frac{dy}{y(1+y/\lambda)} = 4 C_n \lambda(ak_F)^2, \tag{5.13}$$

where $C_n \sim n_{max}/n_c \sim 1$. Comparing the expressions for the rhs and lhs of the sum-rule formula, Eqs. (5.12) and (5.13), we see that both of them are of the same order $\lambda(ak_F)^2$. The forward scattering sum rule requires C_n to be equal to one, but a precise value of this number is beyond the accuracy of our calculation of Z_Γ, which accounts only for harmonics with $n \leq n_c$.

VI. EQUIVALENCE OF THE ONE-LOOP AND EXACT RESULTS FOR THE SELF-ENERGY

In effective low-energy theories of QCPs, e.g., in the spin-fermion model, the effective interaction is $D(Q) = g_{FB}^2 \chi(Q)$, where $\chi(Q)$ is the susceptibility of the divergent order parameter and g_{FB} is the fermion-boson coupling. The self-energy is obtained via a loopwise expansion in D, which is usually truncated at the one-loop order [cf. Fig. 7(a)]

$$\Sigma_{1L}(K) = \int D(K-P)G(P)\frac{d^3P}{(2\pi)^3}. \quad (6.1)$$

Such a formula is used in the Eliashberg and fluctuation exchange (FLEX) theories [with the bare or full $G(P)$, respectively[46,47]].

On the other hand, we showed in the previous section that the linear in ω and ϵ_k parts of the Σ can be found from the Pitaevski-Landau identities using a fully renormalized vertex. Now we can ask the following question: how does the one-loop result for Σ correspond to that obtained from the Pitaevski-Landau identities? In this section, we show that Eq. (6.1) is asymptotically exact in the critical FL regime with corrections small in $1/ak_F$ and ω/ω_{FL}, if the effective interaction $D(Q)$ is identified with $Z\Gamma^\Omega$ (more accurately, with $\Sigma_\beta Z\Gamma_{\alpha\beta,\alpha\beta}^\Omega$).

To prove this assertion, we substitute $\Sigma_\beta Z\Gamma_{\alpha\beta,\alpha\beta}^\Omega$ for D into Eq. (6.1), use Eq. (3.47) for Γ^Ω, replace G by its coherent part, and take the limit of small ω and ϵ_k. The effective interaction can be decomposed into static and dynamic parts. The static part gives an "anomalous" ϵ_k term in Σ_{1L}, which comes from the immediate vicinity of the FS. The dynamic part gives a regular $\omega - \epsilon_k$ term, which comes from the entire phase space. Explicitly, $\Sigma_{1L}(K) = \Sigma_{1L}^{an}(K) + \Sigma_{1L}^{reg}(K)$,

$$\Sigma_{1L}^{an}(K) = \epsilon_k \frac{Z^2 m}{4\pi^2} \sum_\beta \int \Gamma_{\alpha\beta,\alpha\beta}^\Omega(\theta)\cos\theta d\theta, \quad (6.2a)$$

$$\Sigma_{1L}^{reg}(K) = \left(\omega - \epsilon_k \frac{m}{m^*}\right)(-i) \sum_\beta \int \Gamma_{\alpha\beta,\alpha\beta}^\Omega(K_F, P)$$

$$\times [G^2(P)]_\Omega \frac{d^3P}{(2\pi)^3}. \quad (6.2b)$$

We see immediately that the ω terms in Eqs. (4.4), (6.2a), and (6.2b) are the same, whereas the ϵ_k terms are the same if

$$\frac{Z(Z-1)m}{4\pi^2} \sum_\beta \int \Gamma_{\alpha\beta,\alpha\beta}^\Omega(\theta)\cos\theta d\theta$$

$$+ \left(\frac{m}{m^*} - 1\right) i \sum_\beta \int \Gamma_{\alpha\beta,\alpha\beta}^\Omega(K_F, P)[G^2(P)]_\Omega \frac{d^3P}{(2\pi)^3} = 0. \quad (6.3)$$

Expressing the integrals of the vertices in Eq. (6.3) via m^* and Z using Eqs. (4.6) and (4.7), we see that Eq. (6.3) reduces to an identity

$$-\left(\frac{m}{m^*} - 1\right)\left[\frac{1}{Z} - 1\right] + (Z-1)\left[\frac{1}{Z}\left(1 - \frac{m}{m^*}\right)\right] = 0. \quad (6.4)$$

We thus see that both ω and ϵ_k terms in the one-loop self-energy coincide with the exact expressions obtained using the Pitaevski-Landau identities, if $Z\Gamma^\Omega$ is identified with the effective interaction. This equivalence indeed holds only as long as m^*/m and Z are energy independent and only a coherent part of $G(Q)$ is relevant, which is the case for the critical FL regime of a QCP. In Appendix C, we show explicitly how the self-energy is reproduced by the diagrammatic expansion for Γ^Ω.

VII. CONCLUSIONS

In this paper, we analyzed properties of the Fermi liquid near a quantum phase transition using a simple model of the nematic $n=2$ charge Pomeranchuk instability as an example. Our main result is that, near a phase transition, the system enters into a new critical FL regime, in which all spin Landau components and all charge components with $n \neq 2$ increase and eventually diverge at the critical point. This behavior is the consequence of a singular momentum dependence of the Landau function in a critical FL. Therefore, a common assumption that all Landau components, other than the one corresponding to the critical channel ($g_{c,2}$ for the nematic charge instability), are featureless near a transition is incorrect in $D \leq 3$. The divergence of the Landau components, including the one controlling renormalization of the effective mass, has no consequences for susceptibilities channels because the divergent effective mass cancels out with the divergent effective g factor.

To prove these statements, we derived the Landau function for a critical Fermi liquid, related to the vertex Γ^Ω via $g = 2\nu Z^2(m^*/m)\Gamma^\Omega$, where ν is the density of states, Z is the quasiparticle residue, m^* is the effective mass, and Γ^Ω is defined in the limit of zero momentum transfer and vanishingly small energy transfer. Our starting point is a model of 2D fermions with a d-wave interaction, which we assumed to be of sufficiently long range, such that $ak_F \gg 1$, where a is the effective radius of the interaction. We computed the Fermi-liquid vertex Γ^Ω in two stages. First, we considered only those diagrams for Γ^Ω that do not contain soft particle-hole bubbles with zero momentum and vanishing frequency. For $ak_F \gg 1$, diagrams of that type can be summed up in the RPA. The RPA vertex $\Gamma^{\Omega,RPA}$ contains a component which diverges at the critical point and can be interpreted as arising from the exchange by soft collective excitations in the $n=2$ charge channel. Next, we included the diagrams with soft particle-hole bubbles. In an ordinary FL, the full interaction is approximated by its static term, and such diagrams vanish due to particle-number conservation. In a critical FL, however, the effective interaction is dynamic, and the diagrams with soft particle-hole bubbles are finite. We showed that the relevant diagrams of that kind form a ladder series. Summing up this series, we found that non-RPA diagrams with soft bubbles renormalize the RPA vertex by a constant factor, i.e., the full vertex Γ^Ω is equal to $\Gamma^{\Omega,RPA}/Z_\Gamma$, where Z_Γ is a function of Z, m^*/m, and of the dimensionless coupling constant

$\lambda = 1/(2ak_{\rm F}\sqrt{1+g_{c,2}})$. Vertex corrections to ladder diagrams were shown to be small as $1/ak_{\rm F}$. Having found Γ^{Ω}, we employed the Pitaevski-Landau identities for the derivatives of the self-energy in terms of Γ^{Ω} to obtain coupled equations for Z and m^*/m. This gave us a closed set of equations for Z_{Γ}, Z, and m^*/m with λ as a parameter. Solutions of these equations allows one to follow the evolution from an ordinary FL behavior at $\lambda < 1$, where m^*/m, Z, and Z_{Γ} are all close to 1, to a critical FL behavior at $\lambda \gg 1$, where $m^*/m = 1/Z \approx \lambda$ and $Z_{\Gamma} \sim Z$. We also showed that the self-energy of a critical FL is "local," i.e., $\Sigma(k,\omega) \approx \Sigma(\omega)$ and that m^* diverges only at $\lambda = \infty$ rather than at finite λ. The latter result implies that no preemptive transitions occur before the nematic one is reached.

Finally, we showed that the exact self-energy (as given by the Pitaevski-Landau identities) is asymptotically close to the one-loop expression $\Sigma(K) \propto \int G(Q)D(K-Q)$ if one identifies the effective interaction D with $Z\Gamma^{\Omega} = \Gamma^{\Omega,{\rm RPA}}$. We have shown explicitly that in the limit of $\omega, \epsilon_k \to 0$, corrections to the one-loop result are small in powers of $1/ak_{\rm F}$ and $\omega/\omega_{\rm FL}$, where $\omega_{\rm FL}$ is an upper boundary for the FL behavior.

ACKNOWLEDGMENTS

APPENDIX A: DIAGRAMMATIC SERIES FOR Γ^{Ω} BEYOND RPA

In the main part of the text, we adopted a model case of the long-range interaction in the d-wave channel to justify the RPA. Such a model, although a necessary prerequisite for an RPA-type treatment in any channel, is hardly realistic and one is naturally led to wonder as to what extent the main features of the effective interaction, Eq. (3.23), survive beyond the RPA level. In this appendix, we address this issue by evaluating Γ^{Ω} in a direct order-by-order perturbation theory for a short-range, Hubbard-type interaction (U = const). Notice that this issue is different from renormalization of the RPA interaction via ladder series in the dynamic interaction, considered in Sec. III C of the main text. Here, we discuss the full Γ^{Ω} without contributions from the soft particle-hole bubbles, which is the input for the calculations in Sec. III C. The issue is whether this full Γ^{Ω} has the same structure as the RPA result, Eq. (3.23), i.e., whether it can be interpreted as the exchange by soft collective excitations.

The issue of validity of the RPA is not specific to a charge nematic QCP. To avoid unnecessary complications associated
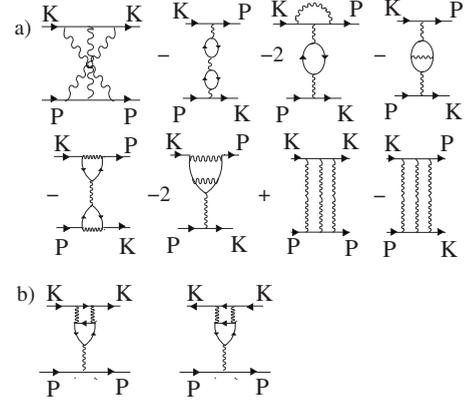

FIG. 6. (a) Third-order diagrams for $\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}$ that contain only the particle-particle and particle-hole bubbles at combinations of external momenta $K \pm P$. (b) Examples of the third-order diagrams that contain convolutions of particle-particle and particle-hole bubbles with Green's functions over the internal momenta.

with the d-wave factors in the vertices, we consider here a simpler case of an instability in the $n=0$ channel (charge instability for $U<0$ or spin instability for $U>0$).

Diagrams for Γ^{Ω} to second order are shown in Fig. 1(a). Explicitly,

$$
\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}(K,P) = -\frac{U}{2}[1 - U\Pi_{ph}(K-P) + U\Pi_{pp}(K+P)]\delta_{\alpha\delta}\delta_{\beta\gamma}
$$
$$
+ \frac{U}{2}[1 + U\Pi_{ph}(K-P)
$$
$$
+ U\Pi_{pp}(K+P)]\sigma_{\alpha\delta}\sigma_{\beta\gamma}, \tag{A1}
$$

where $\Pi_{ph}(K) = -\int[d^3L/(2\pi)^3]G_L G_{L+K}$ and $\Pi_{pp}(K) = -\int[d^3L/(2\pi)^3]G_L G_{-L+K}$ are particle-hole and particle-particle bubbles. Notice that, to this order, all contributions to $\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}$ contain either $\Pi_{ph}(K-P)$ or $\Pi_{pp}(K+P)$ but no bubbles with other momenta.

The new physics emerges at the third order of the interaction. The third-order diagrams can be divided into two classes. Diagrams from the first class, shown in Fig. 6(a) still contain either $\Pi_{ph}(K-P)$ or $\Pi_{pp}(K+P)$. Diagrams from the second class contain convolutions of Π_{ph} and Π_{pp}.

Adding the contributions of the third-order diagrams from the first class to Eq. (A1), we obtain

$$
\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}(K,P) = -\frac{U}{2}\{1 - U[\Pi_{ph}(K-P) - \Pi_{pp}(K+P)]
$$
$$
+ U^2[\Pi^2_{ph}(K-P) + \Pi^2_{pp}(K+P)]\}\delta_{\alpha\delta}\delta_{\beta\gamma}
$$
$$
+ \frac{U}{2}\{1 + U[\Pi_{ph}(K-P) + U\Pi_{pp}(K+P)]
$$
$$
+ U^2[\Pi^2_{ph}(K-P) + \Pi^2_{pp}(K+P)]\}\vec{\sigma}_{\alpha\delta}\cdot\vec{\sigma}_{\beta\gamma}. \tag{A2}
$$

This result for $\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}(K,P)$ can be rewritten as an expansion of the following formula

$$\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}(K,P) = -\frac{U}{2}\left[\frac{1}{1-U\Pi_{pp}(K+P)} + \frac{1}{1+U\Pi_{ph}(K-P)} - 1\right]$$

$$\times \delta_{\alpha\delta}\delta_{\beta\gamma}$$

$$+\frac{U}{2}\left[\frac{1}{1-U\Pi_{pp}(K+P)} + \frac{1}{1-U\Pi_{ph}(K-P)} - 1\right]$$

$$\times \vec{\sigma}_{\alpha\delta}\cdot\vec{\sigma}_{\beta\gamma}. \tag{A3}$$

Although the last formula is valid only to order U^3, it is very likely that higher-order diagrams for Γ^{Ω} of the same type, i.e., containing only $\Pi_{ph}(K-P)$ or $\Pi_{pp}(K+P)$, are described by this expression. We see from Eq. (A3) that the structure of the vertex is virtually the same as in the RPA in a sense that $\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}$ contains three separate geometric series, describing exchange by charge, spin, and pairing fluctuations. The three separate contributions to $\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}$ diverge near a corresponding QCP, e.g., near $U\Pi_{ph}(0)=-1$ for a charge QCP and $U\Pi_{ph}(0)=1$ for a spin QCP (the former can occur if pairing instability is suppressed by, e.g., magnetic field). We emphasize that, at this level, the particle-particle and particle-hole channels entirely decouple, i.e., the interaction in the pairing channel does not affect the structure of the effective interaction mediated by soft collective excitations in the particle-hole channel.

The situation is changed by diagrams from the second class. There are 24 topologically distinct third-order diagrams which contains momentum integrals of the polarization bubbles and we refrain from presenting all of them. As an example, two of the 24 diagrams are shown in Fig. 6(b). We computed analytically all 24 diagrams. There are numerous cancellations between the diagrams, and the final result is rather compact

$$\bar{\Gamma}^{\Omega}_{\alpha\beta;\gamma\delta}(K,P) = \Gamma^{\Omega}_{\alpha\beta;\gamma\delta}(K,P) - U^3\left[\int\frac{d^3L}{(2\pi)^3}[\Pi_{ph}(L)\right.$$

$$\times(2G_{L+K}G_{L+P} - G_{L+K}G_{P-L})$$

$$+ \Pi_{pp}(L)G_{L-P}G_{L-K}]\delta_{\alpha\delta}\delta_{\beta\gamma}\bigg]$$

$$+ U^3\int\frac{d^3L}{(2\pi)^3}[\Pi_{ph}(L)G_{L+K}G_{P-L}$$

$$+ \Pi_{pp}(L)G_{L-P}G_{L-K}]\vec{\sigma}_{\alpha\delta}\cdot\vec{\sigma}_{\beta\gamma}, \tag{A4}$$

where $\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}$ is given by Eq. (A3).

We see from Eq. (A4) that there are cross terms which involve both particle-hole and particle-particle bubbles, i.e., the terms with $\Pi_{ph}(L)G_{L+K}G_{P-L}$ and $\Pi_{pp}(L)G_{L-P}G_{L-K}$. The presence of such terms implies that the particle-particle and particle-hole channels do couple beginning from the third order in U. This coupling should affect the interaction mediated by near-critical charge- and spin-density fluctuations.

Note that $\bar{\Gamma}$ contains a term which involves only particle-hole bubbles $[(\Pi_{ph}(X)G_{X+K}G_{X+P}]$. To understand qualitatively the effect of this term, we neglect momentarily other terms and approximate Π_{ph} by a constant [we recall that a static bubble $\Pi_{ph}(\mathbf{q},0)$ is a constant for $q<2k_F$ in 2D]. After simple manipulations, we then obtain

$$\bar{\Gamma}^{\Omega}_{\alpha\beta;\gamma\delta}(k,p) = -U\left[\frac{1}{1+U^2\Pi_{ph}^2} - \frac{1}{2}\frac{1}{1-U\Pi_{ph}}\right]\delta_{\alpha\delta}\delta_{\beta\gamma}$$

$$+ \frac{U}{2}\frac{1}{1-U\Pi_{ph}(K-P)}\vec{\sigma}_{\alpha\delta}\cdot\vec{\sigma}_{\beta\gamma}. \tag{A5}$$

We see that the term $1/(1+U\Pi_{ph})$ describing the interaction via soft bosons in the charge channel is no longer there. Although disappearance of this term well may be an artifact of the approximation [Eq. (A5) is, strictly speaking valid only to order U^3], this is still clearly a warning sign for the whole approach, as it shows that additional terms with particle-hole bubbles, not included into RPA-type analysis may be relevant. Note also that there is a "leakage" of the $1/(1-U\Pi_{ph})$ term from the spin channel into the charge channel, i.e., interaction mediated by a soft boson in the spin channel induces the same interaction in the charge channel. This effect was considered in detail in Ref. 21.

APPENDIX B: ALTERNATIVE AVERAGING PROCEDURES FOR Γ^{Ω}

For completeness, we present here the results for Z_{Γ}, Z, and m^*/m obtained within alternative approaches for summing up the diagrammatic series, Eq. (3.30), for Γ^{Ω}. We remind that the uncertainty is related to the presence of the factors $d_{\mathbf{k}}d_{\mathbf{p}}(d_{\mathbf{k}}^{2(l-1)}+d_{\mathbf{p}}^{2(l-1)})$ in the diagrammatic series for $\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}(\mathbf{k},\mathbf{p})$. In the main part of text, we used the approximation in which the angular dependence of $\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}$ was averaged over the FS. An alternative is to keep only the $d_{\mathbf{k}}d_{\mathbf{p}}$ term at each order and neglect other (non-d-wave) terms. A simple exercise in trigonometry shows that this amounts to replacing

$$d_{\mathbf{k}}d_{\mathbf{p}}(d_{\mathbf{k}}^{2(l-1)}+d_{\mathbf{p}}^{2(l-1)}) \rightarrow (2l-1)d_{\mathbf{k}}d_{\mathbf{p}} \tag{B1}$$

at each order of the perturbation theory. Summing up the series for $\Gamma^{\Omega\{n\}}_{\alpha\beta;\gamma\delta}(\mathbf{k},\mathbf{p})$, we find that Eq. (3.34) is still valid, i.e., $\Gamma^{\Omega}_{\alpha\beta;\gamma\delta}=\Gamma^{\Omega,\mathrm{RPA}}_{\alpha\beta;\gamma\delta}/Z_{\Gamma}$, where Z_{Γ} is a constant, but now the expression for Z_{Γ} is different

$$\frac{1}{Z_{\Gamma}} = 1 + \sum_{l=1}^{\infty}\left(\frac{Z^2 m^*}{2m}\lambda\right)^l\left(l+\frac{1}{2}\right) = \frac{2 - \frac{Z^2 m^*\lambda}{2m}\left(1 - \frac{Z^2 m^*\lambda}{2m}\right)}{2\left(1 - \frac{Z^2 m^*\lambda}{2m}\right)^2}. \tag{B2}$$

Using this form and isotropic FL formulas for m^*/m and Z, we obtain after simple algebra that in the critical FL regime, where λ is large, $m^*/m=1/Z=\lambda/2$ and $Z_{\Gamma}\approx 1/2\lambda\approx Z/2$.

We see that, as in Eq. (4.18), $Z_{\Gamma}|_{\lambda\gg 1}\sim Z$, the only difference between Eq. (4.18) and the present case being the numerical prefactor. This proves our point that both procedures for summing up the ladder series for Γ^{Ω} yield physically equivalent results.

We also obtain very similar results by neglecting the d-wave structure of the intermediate vertices in the ladder series, i.e., replacing $(d_{\mathbf{k}}^{2(l-1)}+d_{\mathbf{p}}^{2(l-1)})/2$ by $(1/2)^{l-1}$. In this situation, we obtain $1/Z_{\Gamma}=1/(1-\frac{Z^2 m^*\lambda}{4m})$. Combining this

with the equations for Z and m^*/m and solving the full set, we obtain $m^*/m = 1/Z = \lambda/4$, and $Z_\Gamma|_{\lambda \gg 1} = Z$ for $\lambda \gg 1$.

APPENDIX C: LOOP EXPANSION FOR Σ

1. Relation between vertex renormalization and single-particle residue

We begin by discussing a subtle point in the relation between the effective fermion-boson interaction $D = g_{FB}^2 \chi$ and $Z\Gamma^\Omega$. We found that in the critical FL regime, the vertex renormalization constant Z_Γ is related to the single-particle residue Z as $Z_\Gamma = CZ$, where $C \sim 1$ depends on which of the approximate averaging procedures, discussed in Sec. III C and Appendix B, is employed. One of such procedures yields $C = 1$, whereas the other two yield $C \neq 1$. The uncertainty is related to the fact that we used anisotropic d-wave form of the interaction between fermions, yet approximated the quasiparticle residue Z and the effective mass m^* by constants independent of the position on the FS. Regardless of a particular value of C, proportionality between Z_Γ and Z implies that $Z\Gamma^\Omega = (Z/Z_\Gamma)\Gamma^{\Omega,RPA}$ is proportional to $\Gamma^{\Omega,RPA}$. It is then instructive to verify whether the exact Pitaevski-Landau formula for Σ is reproduced if the diagrammatic expansion for the vertex Γ^Ω in terms of $\Gamma^{\Omega,RPA}$ is transformed into an expansion for the self-energy by contracting a pair of external legs of a vertex. The first term in the series is $\Gamma^{\Omega,RPA}$, and hence the bare self-energy is the one with $\Gamma^{\Omega,RPA}$ instead of D. This would be the final result if Z and Z_Γ were equal in the critical FL [in that case, $\chi = Z\Gamma^\Omega = (Z/Z_\Gamma)\Gamma^{\Omega,RPA} = \Gamma^{\Omega,RPA}$].

Whether Z_Γ is equal to Z or just proportional to Z does not affect our main result that all Landau parameters diverge upon approaching a nematic QCP. It turns out, however, that only an exact equality $Z_\Gamma = Z$ is consistent with the loopwise expansion for the self-energy. This is not surprising because an equality $Z_\Gamma = Z$ is the result of a particular averaging procedure in which the d-wave vertices arising from intermediate states were replaced by constants when evaluating the ladder series for Γ^Ω. This is consistent with replacing Z and m^* by angle-independent constants. In the other two procedures, we either averaged the angle-dependent interaction over the FS at each order or extracted the d-wave component from the interaction. In both cases, there are extra combinatorial factors associated with the d-wave nature of the interaction.

To see that only an equality $D = \Gamma^{\Omega,RPA}$ is consistent with diagrammatics, we consider explicitly diagrams for Σ. To second-loop order, there are two diagrams, b and c in Fig. 7. Diagram b is a part of the self-consistent renormalization of the Green's function in the one-loop diagram. Diagram c is a vertex correction to the one-loop self-energy. In two subsequent sections, we will compute these two diagrams and show that they are indeed small, at least as along as $ak_F \gg 1$.

2. Green's-function renormalization in the one-loop self-energy

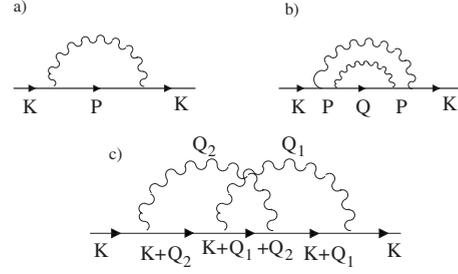We start with diagram b in Fig. 7,

FIG. 7. Diagrams for the self-energy at the one-loop [(a)] and two-loop orders [(b) and (c)]. The wavy line is the RPA interaction, Eq. (3.23).

$$\Sigma_{2L}^{\{b\}} \sim \int d^3P d^3Q [G^2(P)]_\Omega G(Q) D(K-P) D(Q-P).$$

$$(C1)$$

Integrating over Q first, we obtain

$$\Sigma_{2L}^{\{b\}} \sim \int d^3P [G^2(P)]_\omega D(K-P) \Sigma^{\{1\}}(P), \qquad (C2)$$

where $\Sigma_{1L}(P)$ is one-loop self-energy with $D = \Gamma^{\Omega,RPA}$. The leading term in $\Sigma_{1L}(P)$ is $\lambda\omega$. This term, however, vanishes upon integration over P. The most straightforward way to see this is to perform momentum integration first. Replacing $\int d^2p$ in Eq. (C2) by $\int d\epsilon_p$ and integrating over ϵ_p from $-\Lambda$ to Λ, where $\Lambda \sim E_F$, we find after simple algebra that, because of the double pole in $G^2(P)$, the integral comes either from vanishingly small ϵ_p or from $|\epsilon_p| \sim \Lambda$. The high-energy contribution to $\Sigma^{\{2\}}$ is of order

$$\frac{Z^2 m^*}{m} \lambda \int d\omega_p \omega_p \int_{-\Lambda}^\Lambda \frac{d\epsilon_p^*}{(\omega_p - \epsilon_p^*)^2} \sim \int_0^\Omega d\omega_p \omega_p \frac{2\Lambda}{\omega_p^2 - \Lambda^2} \sim \frac{\omega^2}{\Lambda}$$

$$\ll \Sigma_{1L}. \qquad (C3)$$

The contribution from vanishingly small ϵ_p,

$$\bar{I} = \frac{Z^2 m^*}{m} \lambda^2 \int d\omega_p \omega_p \int \frac{d\epsilon_p^*}{(\omega_p - \epsilon_p^* + i\delta \, \mathrm{sgn} \, \epsilon_p)^2} \qquad (C4)$$

comes from the branch cut at $\epsilon_p = 0$, caused by $\mathrm{sgn} \, \epsilon_p$ term in Eq. (C4). Evaluating the integral, we find that it vanishes in the limit of $\delta \to 0$,

$$\bar{I} \sim i \frac{Z^2 m^*}{m} \lambda^2 \int_0^\Omega d\omega_q \omega_q \frac{\delta}{\omega_p^2 + \delta^2} \to 0. \qquad (C5)$$

A nonzero contribution to $\Sigma_{2L}^{\{b\}}$ comes only from the ϵ_k-dependent term in Σ_{1L}, given by Eq. (4.20),

$$\Sigma_{2L}^{\{b\}} \sim \frac{Z^2 \Delta m^*}{m} \int d\omega_p d^2p \frac{\epsilon_p^*}{(\omega - \epsilon_p^* + i\delta_\epsilon)^2} D(K-P)$$

$$\sim \frac{Z\Gamma_\triangle m^*}{m} \int d\omega_p d^2p \, G(P) D(K-P) \sim \Gamma_\triangle \Sigma_{1L}.$$

$$(C6)$$

As we have shown in Sec. IV C, $\Gamma_\triangle \sim 1/ak_F \ll 1$, i.e., $\Sigma_{2L}^{\{b\}}$ is

indeed small compared to Σ_{1L}. The same conclusion holds for higher-order diagrams of the same type.

3. Vertex corrections to the one-loop self-energy

Next, we consider the vertex correction to the one-loop self-energy given by diagram c in Fig. 7. Integrating over P, we obtain

$$\Sigma_{2L}^{\{c\}}(K) = -\int \frac{d^3 Q}{(2\pi)^3} \Gamma_\triangle(Q;K) D(Q) G(K+Q). \quad (C7)$$

In Sec. III C 3, we computed the three-leg vertex $\Gamma_\triangle(Q;K)$ in the limit of $\Omega/q \to 0$. In this limit, $\Gamma_\triangle \sim 1/ak_F \ll 1$ coincides with the effective Migdal parameter of our theory. To calculate the self-energy, however, we need to know $\Gamma_\triangle(Q;K)$ at arbitrary Ω/q. When evaluating the dynamic part of the vertex, it is important to account for finite curvature of the Fermi surface, i.e., to keep a quadratic term in the expansion of the dispersion

$$\epsilon_{\mathbf{k}_F+\mathbf{q}}^* = v_F^* q_\parallel + \frac{q_\perp^2}{2m^*}, \quad (C8)$$

where $q_\parallel$ and $q_\perp$ are the components of $\mathbf{q}$ along and transverse to $\mathbf{k}_F$, correspondingly. In general, $\Gamma_\triangle(Q;K)$ is an involved function of its arguments; however, a useful estimate can be written as

$$\Gamma_\triangle(\mathbf{q},\Omega;\mathbf{k}_F,\omega_k=0) = \Gamma_\triangle + \lambda \frac{Z^2 m^*}{m} \frac{|\Omega|}{\max\{|\Omega|,|\epsilon_{\mathbf{k}_F+\mathbf{q}}^*|\}}, \quad (C9)$$

where the first term is the vertex in the limit $\Omega/q \to 0$. As it is always the case in Migdal-Eliashberg-type theories, the vertex depends crucially on the ratio of energies entering the dynamic term. Substituting the dynamic term into the self-energy, one can readily see that Ω is on order of the external frequency ω_k, $q_\parallel$ is of order ω/v_F^*, and $q_\perp$ is of order of the

inverse correlation length $\xi^{-1} = (1+g_{c,2})^{1/2}/a$, so that a typical value of $|\epsilon_{\mathbf{k}_F+\mathbf{q}}^*|$ is the largest of the two energies, $|\omega_k|$ and ω_{FL}, where

$$\omega_{FL} = \frac{1}{m^*\xi^2} = \frac{E_F}{ak_F}(1+g_{c,2})^{3/2}. \quad (C10)$$

Therefore, the vertex can be estimated as

$$\Gamma_\triangle(\mathbf{q},\Omega;\mathbf{k}_F,\omega_k=0) \sim \Gamma_\triangle + \frac{|\Omega|}{\max\{|\Omega|,\omega_{FL}\}}. \quad (C11)$$

The physical meaning of ω_{FL} is that it is an upper boundary for a FL behavior near a QCP. For energies higher than ω_{FL}, Landau damping becomes the dominant term in $\Gamma^{\Omega,RPA}$ and, consequently, Z and m^* become energy dependent. The FL theory is therefore valid only at energies smaller than ω_{FL}. We see from Eq. (C11) that vertex corrections are irrelevant for $\Gamma_\triangle \ll 1$ and $|\omega| \ll \omega_{FL}$.

Notice that the smallness of dynamic term in $\Gamma_\triangle$ is due to the presence of the quadratic term $q_\perp^2/(2m^*)$ in the fermionic dispersion, Eq. (C8). This term reflects finite curvature of the FS in larger than one dimensions. Without such a term, vertex correction would be of order 1 for any frequency. Importance of the FS curvature for vertex corrections in 2D has been discussed in Refs. 26, 29, 31, 34, and 40.

To verify this reasoning, we computed diagram c in Fig. 7 explicitly. There are two contributions to the self-energy: the first one is linear in ω and gives a correction to mass renormalization; the second one is quadratic in ω and gives a correction to damping. Although the ω^2 contribution is formally smaller than the linear one, the coefficient of the ω^2 term diverges in the non-FL regime. The linear contribution can be extracted by keeping only $\Gamma_\triangle$ in Eq. (C9) and substituting it into Eq. (C7). This gives

$$\Sigma_{2L}^{\{c,1\}}(\mathbf{k}_F,\omega) \sim \Gamma_\triangle \Sigma_{1L} \sim \Gamma_\triangle \lambda \omega. \quad (C12)$$

The ω^2 contribution is obtained from the entire expression for the self-energy

$$\Sigma_{2L}^{\{c,2\}}(\mathbf{k}_F,\omega) \sim Z^3 \int d^2 q_1 d^2 q_2 d\Omega_1 d\Omega_2 \frac{\Gamma^{\Omega,RPA}(q_1,\Omega_1)\Gamma^{\Omega,RPA}(q_2,\Omega_2)}{[i(\omega+\Omega_1) - \epsilon_{\mathbf{k}_F+\mathbf{q}_1}^*][i(\omega+\Omega_1+\Omega_2) - \epsilon_{\mathbf{k}_F+\mathbf{q}_1+\mathbf{q}_2}^*][i(\omega+\Omega_2) - \epsilon_{\mathbf{k}_F+\mathbf{q}_2}^*]}, \quad (C13)$$

where $\epsilon_{\mathbf{k}_F+\mathbf{q}}^*$ is given by Eq. (C8). Integrating over $q_\parallel$ first, we see that the region of integration over the internal frequencies, Ω_1 and Ω_2, is bounded by external ω. This immediately shows that the double integral $\iint d\Omega_1 d\Omega_2$ contributes a factor of ω^2. Rescaling variables as $x=q_{1\perp}\xi$ and $y=q_{2\perp}\xi$, we obtain

$$\Sigma_{2L}^{\{2,c\}}(\mathbf{k}_F,\omega) \sim i\lambda \frac{|\omega|^3}{\omega_{FL}^2} \int_0^\infty \frac{dx}{1+x^2} \int_0^\infty \frac{dy}{1+y^2} \frac{1}{\beta^2+x^2 y^2}, \quad (C14)$$

where $\beta = |\omega|/\omega_{FL}$. For $\beta \ll 1$, the double integral in Eq. (C14) behaves as $\ln|\beta|/|\beta|$. Collecting the two contributions, the final result for diagram c reads

$$\Sigma_{2L}^{\{c\}} \sim \left(\frac{v_F/a}{E_F} + i\frac{\omega}{\omega_{FL}}\right)\Sigma_{1L}. \quad (C15)$$

We see that $\Sigma^{\{2\}}$ is indeed parametrically smaller than Σ_{1L} in the FL regime, i.e., vertex corrections are also irrelevant for the fermionic self-energy.

To conclude this section, we compare our results with the canonical Migdal-Eliashberg theory of the electron-phonon

interaction. In that theory, the vertex correction to the self-energy has a similar form, except for the denominators of both terms in Eq. (C15) contain the same energy scale: the Fermi energy. In this case, the second term is always smaller than the first one as long as ω is smaller than the typical phonon frequency. In our case, however, the effective "Fermi energies" in the static and dynamic terms are different and, what is most important, they behave differently as a QCP is approached: whereas E_F remains finite, the upper boundary of the FL behavior, ω_{FL}, vanishes as $1/\lambda^3$. In the non-FL regime, the one-loop self-energy behaves as $\Sigma_{1L} \sim \omega_0^{1/3}\omega^{2/3}$

for $\omega \ll \omega_0$, where $\omega_0 = E_F/(ak_F)^4$. This behavior can be interpreted as resulting from the energy dependences of the mass renormalization coefficient $\lambda(\omega) \sim (\omega_0/\omega)^{1/3}$ and of the quasiparticle residue $Z(\omega) \sim 1/\lambda(\omega)$. Rewriting ω_{FL} as ω_0/λ^3 with ω-dependent λ, we see that $\omega_{FL} \sim \omega$. Therefore, the second term in Eq. (C15) is of order unity, i.e., vertex corrections are, in general, important in the non-FL region of a QCP and also for the $\omega^2 \ln \omega$ term in the FL regime.[39] Some of the higher-loop diagrams can be regularized by expanding the theory to the large N case;[26,29,34] however, there are still diagrams that are not small in the large N approximation.[39,40]

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