Thermodynamic behavior of braiding statistics for certain fractional quantum Hall quasiparticles

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It is widely believed that the braiding statistics of the quasiparticles of the fractional quantum Hall effect is a robust, topological property, independent of the details of the Hamiltonian or the wave function. However, for the quasiparticles of the 1/3 state, an explicit evaluation of the braiding phases using Laughlin's wave function has not produced a well-defined braiding statistics. We revisit this issue and demonstrate that the expected braiding statistics is recovered in the thermodynamic limit for exchange paths that are of finite extent but not for macroscopically large exchange loops that encircle a finite fraction of electrons. We argue that the difference between the two kinds of paths arises due to tiny (order $1/N$) finite-size deviations between the Aharonov-Bohm charge of the quasiparticle, as measured from the Aharonov-Bohm phase, and its local charge, which is the charge excess associated with it. An implication of our work is that models for quasiparticles that produce identical local charge can lead to different braiding statistics, which therefore can, in principle, be used to distinguish between such models.

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I. INTRODUCTION

Among the properties of the fractional quantum Hall effect (FQHE) state believed to be of topological origin are fractional charge,[1] fractional braiding statistics,[2] and composite fermions and their effective magnetic field.[3] This paper concerns an as yet unresolved paradox relating to the braiding statistics[4,5] of the excitations of the fractional quantum Hall states at Landau-level filling factor $\nu=1/m$, where m is an odd integer. Two different physical descriptions have been proposed for the $1/m$ quasiparticle: one as an "antivortex" (Laughlin[1]) and the other as a "composite-fermion quasiparticle," or an excited composite fermion.[3,6] These suggest different wave functions. Even though the composite-fermion (CF) quasiparticle wave function is known to be more accurate,[7] the two have the same fractional charge, and presumably the same topological structure, and hence should produce the same braiding statistics. That, however, was not supported by explicit calculations, which produce very different behavior for the two wave functions.[8–10] The CF quasiparticles possess well-defined braiding statistics when they are nonoverlapping[9–11] but the braiding statistics evaluated using Laughlin's wave function fails to converge to a definite value;[9] see Fig. 15 of Ref. 9 for a comparison of the braiding statistics evaluated from the two wave functions as a function of the quasiparticle separation. Is that behavior real or a finite-size effect? A resolution to this question appears worthwhile, in view of experimental efforts to measure the effects of braiding statistics, as well as the more complicated nonabelian braiding statistics.

We revisit in this paper the issue of the braiding statistics of the Laughlin quasiparticles and conclude that part of the discrepancy found in Ref. 8 is very likely eliminated in the thermodynamic limit but some of it is real. More specifically, when the size of the exchange path is small compared to the system size, the expected value for the braiding statistics is *obtained*, but for macroscopically large exchange paths, which enclose a finite fraction of electrons, the braiding statistics for Laughlin's wave function is in general not well defined. We attribute this to an $\mathcal{O}(1/N)$ correction to the "Aharonov-Bohm (AB) charge" of the Laughlin quasiparticle, noted in Ref. 8. Such macroscopic paths are of possible relevance to experimental geometries that seek to detect the effect of braiding statistics through quasiparticle transport along the edge of the sample, where they can form exchange loops that are comparable to the system size.

It is worth noting that there are other corrections to braiding statistics which are not considered in this paper but may be relevant for an experimental measurement of the braiding statistics. The electric field produced by the charge of the quasiparticle causes Landau-level mixing, which results in a correction to the braiding statistics that decays only as a power law with the separation between the quasiparticles.[12] It has also been argued that the circulating Hall currents induced by the electric field of the charge of a quasiparticle generate a small amount of additional flux, which causes a tiny but nonvanishing change in both the local charge and the braiding statistics of the quasiparticles.[13,14] In addition, proximity to the sample edge can also be a source of correction to the braiding properties.[15] These corrections apply to both quasiparticles and quasiholes but are neglected in our study below, which considers wave functions that are strictly confined to the lowest Landau level and quasiparticle that are far from the edge of the system.

II. DETERMINATION OF BRAIDING STATISTICS FOR TWO GEOMETRIES

The braiding statistics[4,5] is defined equivalently either through an adiabatic exchange of two quasiparticles or through the change in the Berry phase associated with a closed loop of a quasiparticle when another quasiparticle is inserted in the area enclosed by the loop. Because the FQHE quasiparticles are charged, an appropriate Aharonov-Bohm phase must be subtracted from the Berry phase of the exchange or the winding path to extract the statistical contribu-

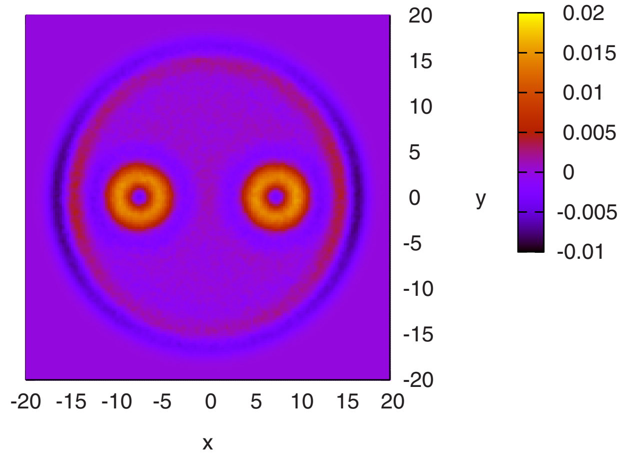

FIG. 1. (Color online) Excess charge density of $\Psi^{\eta\eta'}$ relative to the ground state for $(\eta,\eta')=(7.5,-7.5)$. The distances are measured in units of the magnetic length ℓ. The density is evaluated by Monte Carlo for $N=50$ particles.

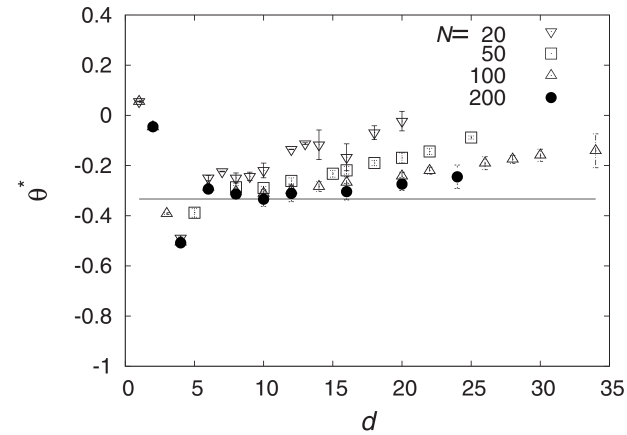

FIG. 2. The statistics parameter θ^* for the path shown in Fig. 1, where two quasiparticles located at η and $-\eta$ are adiabatically exchanged around a circle. The statistics parameter θ^* is given as a function of quasiparticle separation $d\equiv|\eta-\eta'|$, measured in units of the magnetic length ℓ, for $N=20$, 50, 100, and 200. The statistics parameter does not approach a well-defined value with increasing d.

tion. The Berry phases associated with various paths, and hence the braiding statistics, can be evaluated from the knowledge of the wave functions of the quasiparticles, for which we will take below the form proposed by Laughlin. The wave function for a single quasiparticle at η is

$$\Psi_L^{\eta} \equiv e^{-(1/4)\Sigma_j z_j z_j^*} \prod_l (2\partial_l - \eta^*) \prod_{j<k} (z_j - z_k)^3 \quad (1)$$

and that for two quasiparticles at η and η' is

$$\Psi_L^{\eta,\eta'} \equiv e^{-(1/4)\Sigma_j z_j z_j^*} \prod_l (2\partial_l - \eta^*)(2\partial_l - \eta'^*) \prod_{j<k} (z_j - z_k)^3, \quad (2)$$

where $z_j \equiv (x_j - iy_j)/\ell$ denotes the position of the jth electron with $\ell \equiv \sqrt{\hbar c/eB}$, and $\partial_l \equiv \partial/\partial z_l$. For the quasiholes at $\nu=1/m$ the braiding statistics can be evaluated analytically for Laughlin's wave function[16] but for quasiparticles, of interest here, it is necessary to resort to numerical Monte Carlo evaluations of multidimensional integrals.

Following Kjønsberg and Myrheim,[8] we first calculate the braiding statistics from a direct adiabatic exchange of two quasiparticles around a circle, as shown in Fig. 1. In this case, the two quasiparticles are located at η and $-\eta$, with $\eta \equiv Re^{-i\theta}$, and θ varies from 0 to π. The statistics parameter is given by[8]

$$\theta^* = \frac{1}{\pi}[\beta_2(\pi) - 2\beta_1(\pi)], \quad (3)$$

$$\beta_1(\pi) \equiv \int_0^{\pi} d\theta \frac{\left\langle \Psi_L^{\eta} \middle| i\frac{d}{d\theta} \Psi_L^{\eta} \right\rangle}{\langle \Psi_L^{\eta} | \Psi_L^{\eta} \rangle}, \quad (4)$$

$$\beta_2(\pi) \equiv \int_0^{\pi} d\theta \frac{\left\langle \Psi_L^{\eta,-\eta} \middle| i\frac{d}{d\theta} \Psi_L^{\eta,-\eta} \right\rangle}{\langle \Psi_L^{\eta,-\eta} | \Psi_L^{\eta,-\eta} \rangle}, \quad (5)$$

where Ψ_L^{η} is the wave function of a single quasiparticle at η and $\Psi_L^{\eta,\eta'}$ represents two quasiparticles at η and η'.

We have studied systems with up to $N=200$ particles, evaluating the $2N$-dimensional integrals in Eq. (3) by Monte Carlo method for several values of the interquasiparticle distance $d=2R$, where $R=|\eta|$ is the distance of a quasiparticle from the origin, which is limited by the radius of the 1/3 FQHE droplet, given by $\sqrt{6N}\ell$. Typically 4×10^7 Monte Carlo steps have been performed for the evaluation of integrals. The explicit expressions for various integrals evaluated by the Monte Carlo method are given in Appendix. Figure 1 shows excess charge density for the wave function containing two quasiparticles, confirming that they are located at the intended positions.

Figure 2 shows the braiding statistics parameter for systems with different sizes (different numbers of electrons N) as a function of the quasiparticle separation. No definite value for the braiding statistics is reached as the distance between the quasiparticles is increased for a given N. In fact, the deviation from the expected value *increases* with the separation between the quasiparticles, and θ^* appears to be approaching the value $\theta^*=0$ at large separations (as also found in Ref. 8), suggesting *bosonic* statistics in the asymptotic limit. This demonstrates the basic paradox mentioned in Sec. I.

We note that the braiding statistics parameter is not shown up to the maximum quasiparticle separation in Fig. 2; the calculations extends up to $d\approx35\ell$ for $N=100$ and $d\approx25\ell$ for $N=200$, which are significantly smaller than the maximum available quasiparticle separations ($d_{\max}=2\sqrt{6N}\ell$) of approximately 49ℓ and 69ℓ. The reason is that the Monte Carlo convergence for the relevant integrals becomes ex-

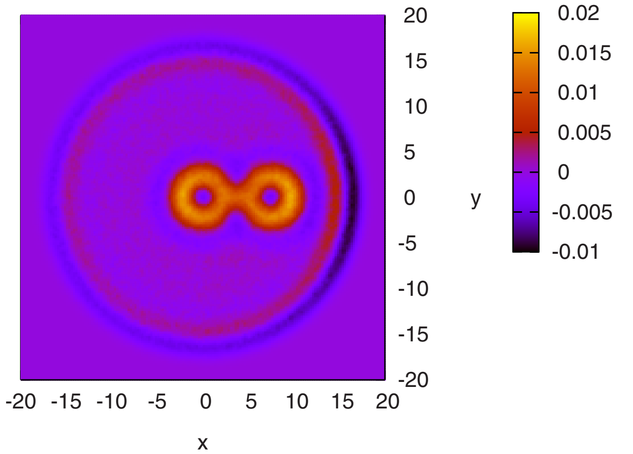

FIG. 3. (Color online) Excess charge density of $\Psi^{\eta\eta'}$ relative to the ground state for $(\eta,\eta')=(7.5,0)$. The distances are measured in units of the magnetic length ℓ. The density is evaluated by Monte Carlo for $N=50$ particles.

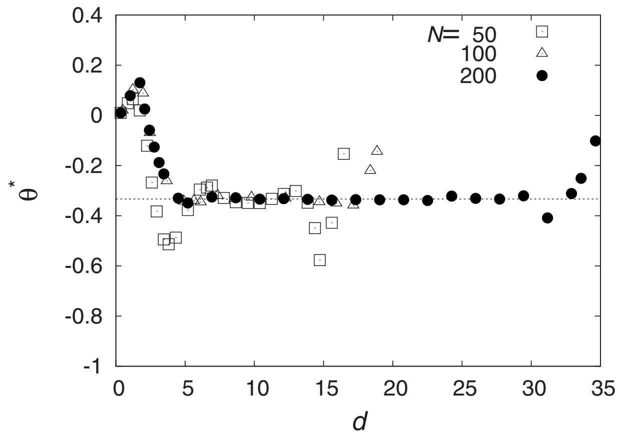

FIG. 4. The statistics parameter θ^* as a function of quasiparticle separation $d\equiv|\eta-\eta'|$, measured in units of the magnetic length, evaluated for the path shown in Fig. 3, where one quasiparticle is adiabatically transported around another. The statistical errors from Monte Carlo sampling are comparable to the symbol size. The deviation at the end of each curve occurs when the quasiparticle gets close to the boundary of the droplet.

tremely slow for large values of N and d. A similar problem was encountered in Ref. 8, where results were shown for $N=100$ and $N=200$ for up to quasiparticle separations of $d\approx 23\ell$ and $d\approx 17\ell$, respectively. The reason why we can go to somewhat larger values of d than Ref. 8, with a number of Monte Carlo iterations that is typically an order of magnitude smaller, has to do with a technical difference between how the braiding statistics is evaluated. Reference 8 relates the braiding statistics to certain normalization integrals, which requires a Monte Carlo evaluation of $2N$ quantities. This method has the advantage that it produces the statistics parameter as a continuous function of the quasiparticle separation but the statistical errors from Monte Carlo sampling mount rapidly with N. In contrast, we evaluate only two integrals. While we must perform a separate calculation for each value of the quasiparticle separation, our method allows a better convergence for a given separation. Our results are of course fully consistent with those of Ref. 8 wherever the two can be compared.

To ascertain how generic this behavior is, we proceed to deduce the braiding statistics from a different geometry in which a quasiparticle at η is transported along a circular path enclosing another quasiparticle at η', as shown in Fig. 3. The change in the Berry phases caused by the presence of the quasiparticle η' is attributed to the braiding statistics. More explicitly, the braiding statistics parameter θ^* is defined as

$$\theta^* = \oint_{\mathcal{C}} \frac{d\theta}{2\pi} \frac{\left\langle \Psi_L^{\eta,\eta'} \middle| i\frac{d}{d\theta}\Psi_L^{\eta,\eta'} \right\rangle}{\langle \Psi_L^{\eta,\eta'}|\Psi_L^{\eta,\eta'}\rangle} - \oint_{\mathcal{C}} \frac{d\theta}{2\pi} \frac{\left\langle \Psi_L^{\eta} \middle| i\frac{d}{d\theta}\Psi_L^{\eta} \right\rangle}{\langle \Psi_L^{\eta}|\Psi_L^{\eta}\rangle} \equiv \frac{1}{2\pi}(\gamma_2-\gamma_1). \quad (6)$$

The quasiparticle position is defined as $\eta\equiv Re^{-i\theta}$ and $\mathcal{C}$ refers to the circular path with R fixed and θ varying from 0 to 2π. We choose $\eta'=0$ for convenience of calculation.

The braiding statistics parameter obtained from this geometry is shown in Fig. 4. It is seen to approach the expected value of $-1/3$ as soon as the separation between the quasiparticles is large enough that the overlap between them is negligible. We note that we are able to obtain the braiding statistics up to the maximum separation of $d_{\max}=\sqrt{6N}\ell$ for this path, which ought to be contrasted to the first geometry for which the convergence becomes very slow for large d. The deviation near the end of the curves in Fig. 4 indicates proximity of the quasiparticle to the boundary of the droplet.

The qualitative difference between the statistics calculated from two paths is puzzling, and the remainder of the paper is concerned with its possible origin and implications. We have studied above two geometries for which the calculation is simplified by symmetry. It would be natural to consider other paths but the significant increase in the computation time for nonsymmetric paths precludes an estimation of the braiding statistics that is sufficiently accurate, especially for large paths, to capture the subtle corrections of interest here.

III. FINITE-SIZE CORRECTIONS

It was found in Ref. 10 that the actual distance between two CF quasiparticles may be slightly different from the naive expectation. To elaborate, suppose we know how to construct a localize CF quasiparticle at η or the origin. Now, when we construct a wave function with *both* CF quasiparticles present, the distance between the two is not exactly $|\eta|$ but slightly larger; the correlations between the two CF quasiparticles cause them to move slightly farther apart. The use of the actual distance between them is crucial for obtaining the correct braiding statistics. (Interestingly, however, even with the naive distance, the braiding statistics for the CF quasiparticles has a well-defined thermodynamic limit but with a wrong sign for the statistics parameter.)

That raises the question if the ill-defined braiding statistics parameter for the Laughlin quasiparticle might also arise from an $\mathcal{O}(1/N)$ difference between the actual and the ap-

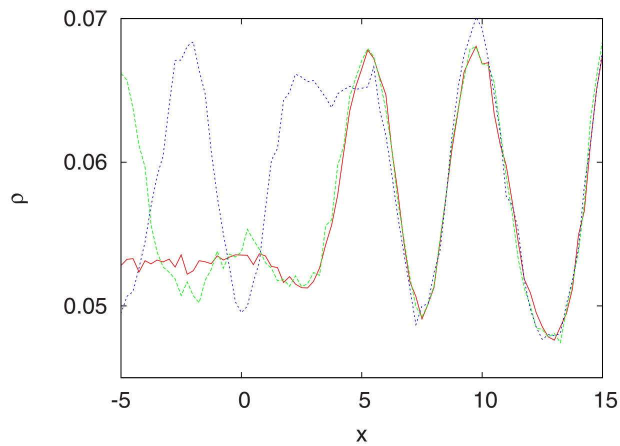

FIG. 5. (Color online) Density profiles along the x axis for (i) Ψ^{η} with a single quasiparticle at η=7.5 (red solid curve); (ii) $\Psi^{\eta\eta'}$ with two quasiparticles at $(\eta,\eta')=(7.5,-7.5)$ (green dashed); and (iii) $\Psi^{\eta\eta'}$ with two quasiparticles at $(\eta,\eta')=(7.5,0)$ (blue dotted). The density is evaluated by Monte Carlo for N=50 particles, and has a minimum at the position of the quasiparticle. The density profile of the quasiparticle at position η=7.5 is seen to be unaltered by the presence of the other quasiparticle at $\eta=-7.5$, and only slightly affected by the other quasiparticle at η=0 in spite of significant overlap between the two quasiparticles.

parent distances between the quasiparticles. As shown in Fig. 5, that is not the case; the density profile of the quasiparticle at $\eta=7.5\ell$ remains unchanged (within the statistical uncertainty of our Monte Carlo) when another quasiparticle is added either at $\eta=-7.5\ell$ or at the origin. The absence of any rigid translation implies that we are subtracting the correct Aharonov-Bohm phase from the Berry phase associated with the exchange path.

Some insight into the origin of the finite-size corrections can be gained from the following observation. We note that Fig. 2 shows the behavior of θ^* as a function of the quasiparticle separation for a given N. Figure 6 displays how the statistics parameter θ^* for a *fixed* quasiparticle separation

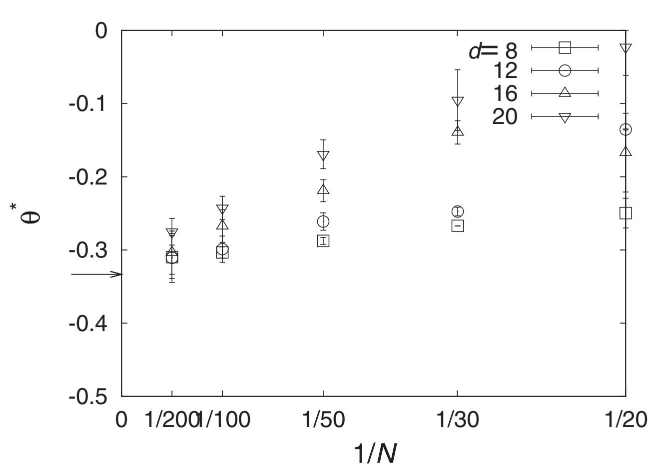

FIG. 6. Same data as in Fig. 2 but with the statistics parameter θ^* now plotted as a function of $1/N$ for fixed quasiparticle separations $d/\ell=8,12,16,20$. The arrow indicates the position $\theta^*=-1/3$ at the vertical axis.

TABLE I. The statistics parameter θ^*_{T} in the thermodynamic limit for quasiparticle separations $d/\ell=8,12,16,20$ with the path shown in Fig. 1. The thermodynamic values are estimated by the best fits of three values for N=50,100,200 to a function $\theta^*(N)=a/N+\theta^*_{\mathrm{T}}$. All the values are consistent with the expected value $-1/3$ within the numerical uncertainty.

d/ℓ	8	12	16	20
θ^*_{T}	−0.32(1)	−0.33(2)	−0.33(2)	−0.31(3)

evolves as a function of $1/N$ (for the first geometry). The thermodynamic limit is now consistent with the expected value of $\theta^*=-1/3$. In order to estimate thermodynamic values θ^*_{T} of the statistic parameter we have fitted the data for the three largest systems (N=50,100,200) in Fig. 6 to a curve $a/N+\theta^*_{\mathrm{T}}$ for each quasiparticle separation. The estimated values which are tabulated in Table I are in agreement with the value $-1/3$ within numerical errors. From a combination of our numerical results and the reasons given below, we believe that θ^* for a fixed path converges to $-1/3$ in the limit $N\to\infty$.

What is the origin of these finite-size corrections and do they always disappear in the thermodynamic limit? The contribution of braiding statistics to the Berry phase associated with an exchange is obtained by removing the AB contribution due to the charge of the quasiparticles. However, for Laughlin's wave function, the AB phase of a closed loop has finite-size corrections, as noted in Ref. 8, and also found in our calculations [see Fig. 7(a)]. The AB phase for a single quasiparticle has been estimated[8] to be

$$2\beta_1(\pi)=\gamma_1\approx-2\pi\frac{BA}{\phi_0}\left(\frac{1}{3}+\frac{0.13}{N}\right), \tag{7}$$

where $\phi_0=hc/e$ and A is the area of the circular loop. The last factor is the "AB charge" of the quasiparticle,[16] and indeed approaches 1/3 in the thermodynamic limit but has finite-size corrections that vanish only as $1/N$. In contrast, the "local charge"[17] of the Laughlin quasiparticle, defined as the charge *excess* relative to the FQHE ground state, does not have a similar correction—it has a well-defined value so long as the system size is large compared to the size of the quasiparticle.[9] The $\mathcal{O}(1/N)$ difference between the AB charge and the local charge of the quasiparticle is surprising but reveals long-range phase structure in the quasiparticle wave function, relative to the ground state, which manifests in the AB charge but not in the local charge.

We now assume that Eq. (7) is correct and ask what its implications are. For a *fixed* area A, the term of order $BA/N\phi_0$ (whose coefficient may depend on the geometry of the path) vanishes in the limit $N\to\infty$ and thus does not make any contribution to the braiding statistics, as found in Fig. 6. However, this term is nonzero for macroscopically large loops enclosing a finite fraction of electrons, which will in general result in a nonzero correction to the braiding statistics; the only exception is for paths for which this term is canceled *exactly* by a similar term in the Berry phase associated with exchange or winding. Our calculations indicate

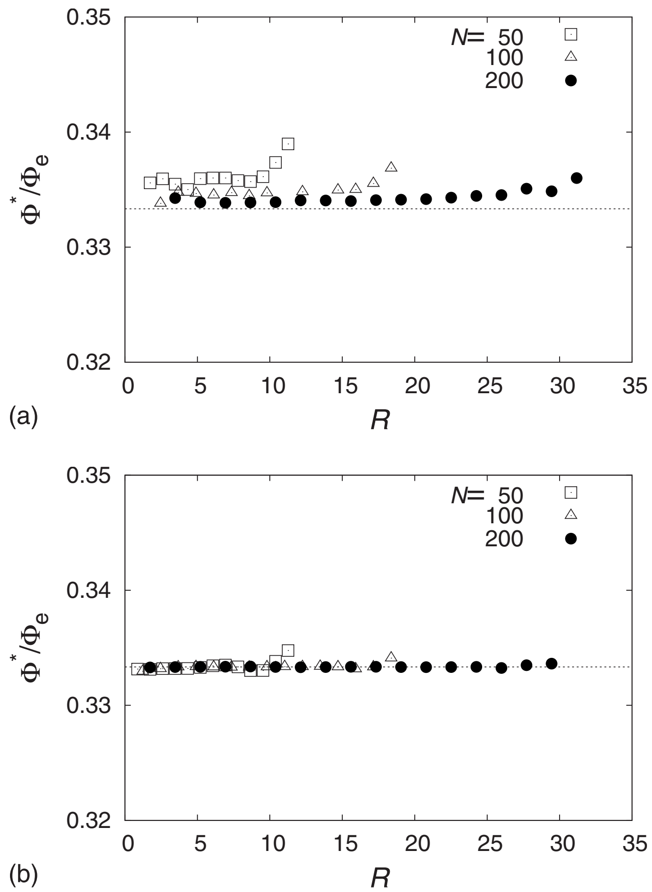

FIG. 7. The Aharonov-Bohm phase Φ^* for (a) the Laughlin quasiparticle and (b) the CF quasiparticle, for a closed loop of radius $|\eta|=R$, quoted in units of the AB phase for an electron, $\Phi_e=-2\pi BA/\phi_0$ ($\phi_0=hc/e$). The panel (b) is taken from Ref. 10, reproduced here for comparison. The large deviations at the end are due to proximity to the boundary. There are finite-size corrections to the AB phase of Laughlin's quasiparticle but not to the CF quasiparticle.

that γ_1 and γ_2 have the same finite-size correction but β_2 and $2\beta_1$ do not. One may ask if the deviation in θ^* from its expected value in Fig. 2 can be attributed to a term of this type. We find that a slightly different definition of the braiding statistics

$$\bar{\theta}^* \equiv \frac{1}{\pi}\beta_2(\pi) - 2\left(-\frac{1}{3}\frac{BA}{\phi_0}\right) - \left(-\frac{0.13}{N}\frac{BA}{\phi_0}\right) \qquad (8)$$

(which assumes that the finite-size correction in β_2 is only half as large as that in $2\beta_1$) produces the "correct" value of the braiding statistics in a somewhat wider region, as shown in Fig. 8 but the braiding statistics is still not as well defined as it is for the second geometry. Because the physical origin of the finite-size correction to the AB phase of the Laughlin quasiparticles is not understood, we are unable to explain at this stage why the two geometries behave differently with regard to such corrections but we suspect that generically the braiding statistics will have nonzero corrections for macroscopically large exchange paths.

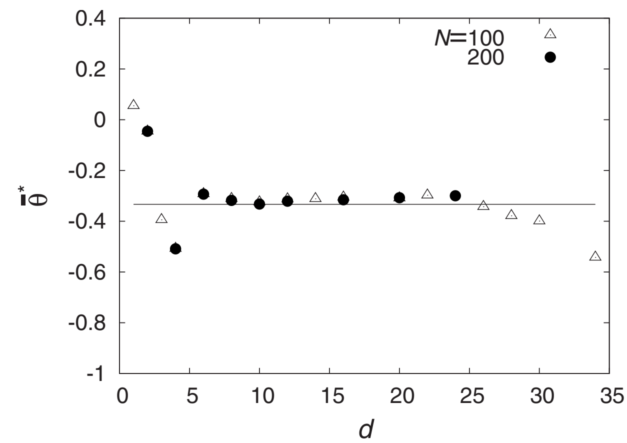

FIG. 8. The statistics parameter $\bar{\theta}^*$ defined in Eq. (8) as a function of quasiparticle separation $d\equiv|\eta-\eta'|$, measured in units of the magnetic length, evaluated for the path shown in Fig. 3, where one quasiparticle is adiabatically transported around another. For the $N=100$ particle curve, the deviation from the expected value at large d is *not* a boundary effect; we expect that, if we could compute it, $\bar{\theta}^*$ would exhibit a similar deviation at large d for $N=200$ particles.

In contrast, the AB phase for the CF quasiparticle does not contain any finite-size correction,[10] as seen in Fig. 7(b). Consistent with the above interpretation, its braiding is not sensitive to the geometry even for finite N; it has been evaluated for both of the paths considered above[9,10] and does not show significant dependence on the path in finite-size calculations (so long as the quasiparticles do not overlap).

We conclude with a comment on the value of the statistics. With our convention, the statistics parameter for the quasiholes is also $\theta^*=-1/3$. One indeed expects the same value for the braiding statistics of quasiparticles and quasiholes because they are related by a combination of charge and the flux conjugation. In contrast, the braiding statistics parameter for the CF quasiparticles[7] is $\theta^*_{\mathrm{CF}}=2/3$. The difference arises because the CF quasiparticle wave function is, by construction, antisymmetric with respect to an interchange in η and η', and, as a result, the Berry phases is given by $\theta^*_{\mathrm{CF}}=1+\theta^*$.

IV. SUMMARY

In summary, our study shows that the braiding statistics is an extremely sensitive test of the phase correlations built in a quasiparticle wave function, and that two quasiparticle wave functions that have identical local charge can have different behavior under braiding. In particular, tiny finite-size differences between the Aharonov-Bohm charge and the local charge can spoil the notion of braiding statistics for large exchange loops. A consideration of large exchange loops thus allows, *in principle*, a way of distinguishing between models for the quasiparticles of the 1/3 state. While such macroscopic loops appear naturally in certain experimental geometries, an extraction of an $\mathcal{O}(1)$ contribution from the $\mathcal{O}(N)$ Berry phase is likely to be nontrivial.

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APPENDIX: EXPRESSIONS USED FOR MONTE CARLO

Useful expressions for Monte Carlo calculation for the first path, which considers an exchange of two quasiparticles around a circle, are

$$\langle\Psi_L^{\eta}|\Psi_L^{\eta}\rangle = \int \mathcal{D}z \exp\left(-\frac{1}{2}\sum_j z_j z_j^*\right)\prod_l \left(2\frac{\partial}{\partial z_l} - \eta^*\right)\prod_l \left(2\frac{\partial}{\partial z_l^*} - \eta\right)\left|\prod_{j<k}(z_j - z_k)^3\right|^2$$
$$= \int \mathcal{D}z \exp\left(-\frac{1}{2}\sum_j z_j z_j^*\right)\left|\prod_{j<k}(z_j - z_k)^3\right|^2 \prod_l (|z_l - \eta|^2 - 2), \tag{A1}$$

$$\left\langle \Psi_L^{\eta}\middle|i\frac{d}{d\theta}\Psi_L^{\eta}\right\rangle = \int \mathcal{D}z \exp\left(-\frac{1}{2}\sum_j z_j z_j^*\right)\prod_l \left(2\frac{\partial}{\partial z_l^*} - \eta\right)\left(-\eta^*\frac{\partial}{\partial \eta^*}\right)\prod_l \left(2\frac{\partial}{\partial z_l} - \eta^*\right)\left|\prod_{j<k}(z_j - z_k)^3\right|^2$$
$$= \eta^* \int \mathcal{D}z \exp\left(-\frac{1}{2}\sum_j z_j z_j^*\right)\left|\prod_{j<k}(z_j - z_k)^3\right|^2 \prod_l (|z_l - \eta|^2 - 2)\sum_m \frac{z_m - \eta}{|z_m - \eta|^2 - 2},$$

$$\langle\Psi_L^{\eta,-\eta}|\Psi_L^{\eta,-\eta}\rangle = \int \mathcal{D}z \exp\left(-\frac{1}{2}\sum_j z_j z_j^*\right)\left|\prod_{j<k}(z_j - z_k)^3\right|^2 \prod_l (|z_l - \eta|^2|z_l + \eta|^2 - 8|z_l|^2 + 8),$$

$$\left\langle \Psi_L^{\eta,-\eta}\middle|i\frac{d}{d\theta}\Psi_L^{\eta,-\eta}\right\rangle = 2(\eta^*)^2 \int \mathcal{D}z \exp\left(-\frac{1}{2}\sum_j z_j z_j^*\right)\left|\prod_{j<k}(z_j - z_k)^3\right|^2$$
$$\times \prod_l (|z_l - \eta|^2|z_l + \eta|^2 - 8|z_l|^2 + 8)\sum_m \frac{z_m^2 - \eta^2}{|z_m - \eta|^2|z_m + \eta|^2 - 8|z_m|^2 + 8}. \tag{A2}$$

For the second path, in which one quasiparticle winds around another, we have used the following expressions for our Monte Carlo calculations:

$$\langle\Psi_L^{\eta,\eta'=0}|\Psi_L^{\eta,\eta'=0}\rangle = \int \mathcal{D}z \exp\left(-\frac{1}{2}\sum_j z_j z_j^*\right)\left|\prod_{j<k}(z_j - z_k)^3\right|^2 \prod_l (|z_l|^2|z_l - \eta|^2 - 4|z_l - \eta|^2 - 4|z_l|^2 + 2|\eta|^2 + 8), \tag{A3}$$

$$\left\langle \Psi_L^{\eta,\eta'=0}\middle|i\frac{d}{d\theta}\Psi_L^{\eta,\eta'=0}\right\rangle = \eta^* \int \mathcal{D}z \exp\left(-\frac{1}{2}\sum_j z_j z_j^*\right)\left|\prod_{j<k}(z_j - z_k)^3\right|^2$$
$$\times \prod_l (|z_l|^2|z_l - \eta|^2 - 4|z_l - \eta|^2 - 4|z_l|^2 + 2|\eta|^2 + 8)$$
$$\times \sum_m \frac{(|z_m|^2 - 2)(z_m - \eta) - 2z_m}{|z_m|^2|z_m - \eta|^2 - 4|z_m - \eta|^2 - 4|z_m|^2 + 2|\eta|^2 + 8}. \tag{A4}$$

Because all the integrals above are independent of θ for our choice of $\mathcal{C}$ and η', the integrals over θ in Eqs. (3) and (6) are trivial.

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