Supercritical Coulomb center and excitonic instability in graphene

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It is well known that there are resonant states with complex energy for the supercritical Coulomb impurity in graphene. We show that opening of a quasiparticle gap decreases the imaginary part of energy, $|\mathrm{Im}\, E|$, of these states and stabilizes the system. For gapless quasiparticles with strong Coulomb interaction in graphene, we solve the Bethe-Salpeter equation for the electron-hole bound state and show that it has a tachyonic solution for strong enough coupling $\alpha = e^2/\kappa\hbar v_F$ leading to instability of the system. In the random-phase approximation, the critical coupling is estimated to be $\alpha_c = 1.62$ and is an analog of the critical charge in the Coulomb center problem. We argue that the excitonic instability should be resolved through the formation of an excitonic condensate and gap generation in the quasiparticle spectrum.

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I. INTRODUCTION

Graphene is a one-atom-thick layer of graphite packed in the honeycomb lattice. Although theoretically considered a long time ago,[1] graphene became an active area of research only recently after the experimental fabrication[2] of this material and because of a variety of its unusual electronic properties.

At low energy the band structure of graphene is formed by the π-electron orbits of carbon and consists of the valence and conduction bands and corresponds to cones touching each other at the so-called Dirac points. Quasiparticle excitations close to these points are described by the massless Dirac equation and have a relativisticlike dispersion $E = \pm\hbar v_F|\mathbf{k}|$, where $v_F \approx 10^6$ m/s is the Fermi velocity and $\mathbf{k}$ is the quasiparticle wave vector. This fact brings an exciting connection between graphene and 3+1-dimensional quantum electrodynamics (QED).

The vanishing density of states at the Dirac points ensures that the Coulomb interaction between the electrons in graphene retains its long-range character in view of vanishing of the static polarization function for $q \to 0$.[3] The large value of the coupling constant $\alpha = e^2/\hbar v_F \sim 1$ means that a strong attraction takes place between electrons and holes in graphene and this resembles strongly coupled QED, thus providing an opportunity for studying the strong-coupling phase experimentally within a condensed-matter laboratory. Given the strong attraction, one may expect an instability in the excitonic channel in graphene with subsequent quantum phase transition to a phase with gapped quasiparticles that may turn graphene into an insulator. This semimetal-insulator transition in graphene is widely discussed now in the literature[4,5] since the first study of the problem in Refs. 6 and 7. The gap opening is similar to the chiral symmetry-breaking phenomenon that occurs in strongly coupled QED and was studied in the 1970s and 1980s.[8–12] In fact, the predicted strong-coupling phase of QED, like other QED effects not yet observed in nature (Klein tunneling, Schwinger effect, etc.), has a chance to be tested in graphene.

We begin our study with the problem of the supercritical Coulomb center in Sec. II in graphene. As is well known,[13,14] for the Coulomb potential, $V_C(r) = -Ze^2/\kappa r$, the spectrum of quasiparticles with a gap Δ contains a continuum spectrum for $|E| > \Delta$ and a discrete one for $0 < E < \Delta$. The lowest bound-state energy equals $E_0 = \Delta\sqrt{1-(2Z\alpha)^2}$ and becomes purely imaginary for $Z\alpha > 1/2$, the "fall into the center" phenomenon. The unphysical complex energies indicate that the Hamiltonian of the system is not a self-adjoint operator for supercritical values $Z\alpha > 1/2$ and should be extended to become a self-adjoint operator. The way out of this situation is well known from the study of the Dirac equation in QED: one should replace the singular $1/r$ potential by a regularized potential which takes into account the finite size of the nucleus, R.[15–17] When the charge Z increases, the energies of discrete states approach the negative-energy continuum, $E = -\Delta$, and then dive into it. Then discrete states turn into resonances with a finite lifetime, which can be described as quasistationary states with complex energies, $\mathrm{Im}\, E \neq 0$. Such states correspond to a rearrangement process when an electron-hole pair is created from the vacuum, the positively charged hole goes to infinity and the electron is coupled to the Coulomb center, thus shielding the charge of the latter. The critical charge Z_c is determined by the condition of appearance of nonzero imaginary part of the energy and increases with the increase of Δ.

Turning to the case of gapless quasiparticles in the regularized Coulomb potential, there are no discrete levels for $Z\alpha < 1/2$ due to scale invariance of the massless Dirac equation, and for $Z\alpha > 1/2$ quasistationary states emerge.[18] The energy of quasistationary levels for the regularized potential has a characteristic exponential-type dependence, $\mathrm{Re}\, E$, $\mathrm{Im}\, E \sim -R^{-1}\exp[-\pi/\sqrt{(Z\alpha)^2-1/4}]$, in the nearcritical regime (according to the analysis in Appendix A, the critical coupling $Z_c\alpha \to 1/2$ for $R\Delta \to 0$). We find that switching on a fermion gap, $\Delta \ll |E|$, decreases $|\mathrm{Im}\, E|$, i.e., increases the stability of the system. The situation here is analogous to the problem of a massless electron in the supercritical Coulomb center in QED first studied in Ref. 19 (for a review, see Ref. 9).

In Sec. III we show that the instability in the supercritical Coulomb center problem is closely related to the excitonic instability in graphene in the supercritical coupling-constant regime $\alpha > \alpha_c \sim 1$. Solving the Bethe-Salpeter (BS) equation for an electron-hole bound state in graphene, we demonstrate that for strong enough coupling constant there are tachyon states with imaginary energy ($E^2<0$) in the spectrum which play here the role of the quasistationary states in the problem of the supercritical Coulomb center. The presence of tachyons signals that the normal state of freely standing graphene is unstable. In fact, the tachyon instability can be viewed as the field theory analog of the fall into the center phenomenon and the critical coupling α_c is an analog of the critical coupling constant $Z_c e^2/\hbar v_F$ in the problem of the Coulomb center. However, in view of the many-body character of the problem, the way of curing the instability in graphene [like in QED (Ref. 9)] is quite different from that in the case of the supercritical Coulomb center. Since the coupling constant in freely standing graphene $\alpha \approx 2.19$ is larger than 1/2, the quasielectron in graphene has the supercritical Coulomb charge. This leads to the production of an electron-hole pair, the hole is coupled to the initial quasielectron forming a bound state but the emitted quasielectron has again a supercritical charge. Thus the process of creating pairs continues leading to the formation of exciton (chiral) condensate in the stable phase, and, as a result, the quasiparticles acquire a gap. The exciton condensate formation resolves the problem of instability, hence a gap generation should take place in a freely standing graphene making it an insulator.

Section IV contains our conclusions. In Appendix A we consider the behavior of bound states for gapped graphene quasiparticles in the regularized Coulomb center and find the critical coupling $Z_c\alpha$ as a function of the parameter $R\Delta$ for the lowest-energy level. In Appendix B we give the exact solution for the tachyon wave function which satisfies the fourth-order differential equation.

II. SUPERCRITICAL COULOMB CENTER: RESONANT STATES

Although the electron-hole problem is a many-body problem in graphene, it is instructive to consider a rather simple one-particle problem of the electron in the field of the supercritical Coulomb center in view of the connection of the latter problem with the excitonic instability in graphene (for a similar problem of instability in the case of a massless fermion in the external field of the supercritical Coulomb center in QED see Refs. 9 and 19).

The Coulomb center problem was studied quite in detail in the literature.[13,14,18,20–24] Since we are mainly interested here in resonant states, we will consider a regularized Coulomb potential

$$V(r) = -\frac{Ze^2}{\kappa r}, \quad (r > R), \quad V(r) = -\frac{Ze^2}{\kappa R}, \quad (r < R) \tag{2.1}$$

(κ is a dielectric constant) because resonant states are connected with diving into the lower continuum that takes place only in the case of a regularized potential.[17,25]

The electron quasiparticle states in vicinity of the K point of graphene in the field of Coulomb impurity are described by the Dirac Hamiltonian in 2+1 dimensions

$$\mathcal{H} = [\sigma^3\Delta + V(r) - i\hbar v_F\sigma^1\partial_x - i\hbar v_F\sigma^2\partial_y], \tag{2.2}$$

where σ^i are Pauli matrices and v_F is the Fermi velocity. [The Hamiltonian of quasiparticle excitations near the K' point is given by Eq. (2.2) with matrices σ^i multiplied by -1.] Note that we introduced the Dirac gap Δ. Although it is absent in the quasiparticle Hamiltonian in graphene in view of the $U(4)$ symmetry, it may appear due to spontaneous symmetry breaking. Since Hamiltonian (2.2) commutes with the total angular momentum operator $J_z = L_z + S_z = -i\hbar\frac{\partial}{\partial\phi} + \frac{\hbar}{2}\sigma_3$, we seek eigenfunctions in the following form,

$$\Psi = \frac{1}{r}\begin{pmatrix} e^{i\phi(j-1/2)}a(r) \\ ie^{i\phi(j+1/2)}b(r) \end{pmatrix}. \tag{2.3}$$

Then we obtain a system of two coupled ordinary differential equations of the first order

$$a' - (j+1/2)\frac{a}{r} + \frac{E+\Delta - V(r)}{\hbar v_F}b = 0,$$

$$b' + (j-1/2)\frac{b}{r} - \frac{E-\Delta - V(r)}{\hbar v_F}a = 0. \tag{2.4}$$

It is convenient to define the variables $\epsilon = E/\hbar v_F$, $m = \Delta/\hbar v_F$, $u = \sqrt{m^2-\epsilon^2}$, $\rho = 2ur$, and $\alpha = e^2/\hbar v_F \kappa$. Equations (2.4) are solved in Appendix A where the discrete spectrum is found in the weak-coupling regime $Z<Z_c$. According to the analysis there, the critical coupling $Z_c\alpha \to 1/2$ for $mR \to 0$.

Let us analyze Eq. (2.4) in the supercritical case $Z\alpha > 1/2$ and show that there are resonant states for $|\epsilon| > m$ (we define the gap $\Delta > 0$). These states describe the instability of the supercritical charge problem with respect to the creation of electron-hole pairs from the vacuum. The created electron is coupled to the Coulomb center, thus shielding the charge of the latter while positively charged hole goes to infinity;[17,25] the process is repeated until the charge of the Coulomb center is reduced to a subcritical value.

The Whittaker function $W_{\mu,\nu}(\rho)$ with $\mu = 1/2 + Z\alpha\epsilon/u$ and $\nu = \sqrt{j^2 - Z^2\alpha^2}$ describes bound states for $|\epsilon| < |m|$ which are situated on the first physical sheet of the variable u and for which Re $u>0$ [see Eq. (A7)]. The quasistationary states are described by the same function $W_{\mu,\nu}(\rho)$ and are on the second unphysical sheet with Re $u<0$. We shall look for the solutions corresponding to the quasistationary states which define outgoing hole waves at $r\to\infty$ with

$$\text{Re}\,\epsilon < 0, \quad \text{Im}\,\epsilon < 0, \quad \text{Re}\,u < 0, \quad \text{Im}\,u < 0. \tag{2.5}$$

For solutions with $Z^2\alpha^2 > j^2$ resonance states are determined by Eq. (A11) for bound states where ν is replaced by $\nu = i\beta$. We will consider the states with $j=1/2$ which correspond to the $nS_{1/2}$ states, in particular, the lowest-energy state belongs to them. The corresponding equation then takes the form

$$\left.\frac{W_{1/2+Z\alpha\epsilon/u,i\beta}(\rho)}{\left(\frac{1}{2} - \frac{Z\alpha m}{u}\right)W_{-1/2+Z\alpha\epsilon/u,i\beta}(\rho)}\right|_{r=R} = \frac{k+1}{k-1},$$

$$k = \frac{m+\epsilon}{u}\sqrt{\frac{\epsilon + Z\alpha/R - m}{\epsilon + Z\alpha/R + m}}\frac{J_1(\tilde{\rho})}{J_0(\tilde{\rho})},$$

$$\tilde{\rho} = \sqrt{(Z\alpha + \epsilon R)^2 - m^2R^2}, \tag{2.6}$$

where $W_{\mu,\nu}(x)$ and $J_a(x)$ are the Whittaker and Bessel functions, respectively.

We are interested in the case of $|\epsilon| \gg m$, and more important, in the case of the massless electron, $m=0$. The analytical results can be obtained for the near-critical values of Z when $Z\alpha - 1/2 \ll 1$. We assume that $|2uR| \ll 1$, then using the asymptotic of the Whittaker function, we find

$$(2uR)^{2i\beta}\frac{\Gamma(1-2i\beta)}{\Gamma(1+2i\beta)}\frac{\Gamma\left(1+i\beta-\frac{Z\alpha\epsilon}{u}\right)}{\Gamma\left(1-i\beta-\frac{Z\alpha\epsilon}{u}\right)} = \frac{\frac{1}{2}-i\beta-\frac{Z\alpha(m-\epsilon)}{u}}{\frac{1}{2}+i\beta-\frac{Z\alpha(m-\epsilon)}{u}}\,\frac{\frac{1}{2}+i\beta-Z\alpha\frac{J_1(Z\alpha)}{J_0(Z\alpha)}}{\frac{1}{2}-i\beta-Z\alpha\frac{J_1(Z\alpha)}{J_0(Z\alpha)}}. \tag{2.7}$$

Expanding Eq. (2.7) in the near-critical region in powers of $\beta = \sqrt{Z^2\alpha^2 - 1/4}$, we find the following equation,

$$(-2i\sqrt{\epsilon^2 - m^2}R)^{2i\beta} = 1 + 4i\beta\left[\frac{J_0(1/2)}{J_0(1/2)-J_1(1/2)} + \Psi(1) - \frac{1}{2}\Psi\left(1 - \frac{i}{2}\frac{\epsilon}{\sqrt{\epsilon^2-m^2}}\right) - \frac{1}{1+i\sqrt{\frac{\epsilon-m}{\epsilon+m}}}\right]. \tag{2.8}$$

Here $\Psi(x)$ is the psi function and we put $u=-i\sqrt{\epsilon^2-m^2}$, where $\mathrm{Im}\sqrt{\epsilon^2-m^2}<0$ on the second sheet. At first we consider the case $m=0$. Writing $\epsilon = |\epsilon|e^{i\gamma}$ Eq. (2.8) takes the form

$$\ln(2|\epsilon|R) + i\left(\gamma - \frac{\pi}{2}\right) \simeq 2\left[\frac{J_0(1/2)}{J_0(1/2)-J_1(1/2)} + \Psi(1) - \frac{1}{2}\Psi\left(1-\frac{i}{2}\right) - \frac{1}{1+i}\right] - \frac{\pi n}{\beta}, \quad n = 1,2,\ldots. \tag{2.9}$$

We find

$$\epsilon_n^{(0)} = aR^{-1}e^{i\gamma}\exp\left[-\frac{\pi n}{\sqrt{Z^2\alpha^2 - 1/4}}\right] = -(1.18 + 0.17i)R^{-1}\exp\left[-\frac{\pi n}{\sqrt{Z^2\alpha^2-1/4}}\right], \quad n = 1,2,\ldots, \tag{2.10}$$

where

$$\gamma = \frac{\pi}{2}\left(1 + \coth\frac{\pi}{2}\right) \approx 3.28, \tag{2.11}$$

$$a = \frac{1}{2}\exp\left[\frac{2J_0(1/2)}{J_0(1/2)-J_1(1/2)} + 2\Psi(1) - 1 - \mathrm{Re}\,\Psi\left(1-\frac{i}{2}\right)\right] \approx 1.19. \tag{2.12}$$

The energy of quasistationary states, Eq. (2.10), has a characteristic essential-singularity-type dependence on the coupling constant reflecting the scale invariance of the Coulomb potential. The infinite number of quasistationary levels is related to the long-range character of the Coulomb potential. Note that a similar dependence takes place in the supercritical Coulomb center problem in QED.[19] Our results are also in agreement with Ref. 18.

Since the "fine-structure constant" $e^2/\hbar v_F \approx 2.19$ in graphene, an instability appears already for the charge $Z=1$. However, in the analysis above we did not take into account the vacuum polarization effects. Considering these effects and treating the electron-electron interaction in the Hartree approximation, it was shown in Ref. 22 that the effective charge of impurity Z_{eff} is such that the impurity with bare charge $Z=1$ remains subcritical, $Z_{eff}e^2/(\kappa\hbar v_F) < 1/2$, for any coupling $e^2/(\kappa\hbar v_F)$, while impurities with higher Z may become supercritical.

For finite m and in the case $|\epsilon| \gg m$, $\mathrm{Re}\,\epsilon < 0$, expanding Eq. (2.8) in m/ϵ we get up to the terms of order m^2/ϵ^2,

$$\epsilon - \frac{m^2}{2\epsilon} = \epsilon_n^{(0)}\left[1 - \frac{m}{\epsilon} + \frac{m^2}{\epsilon^2}(0.29 - 0.23i)\right], \quad n = 1,2,\ldots. \tag{2.13}$$

The resonant states with $\epsilon_n^{(0)}$ describe the spontaneous emission of positively charged holes when electron bound states dive into the lower continuum in the case $m=0$. In order to find corrections to these energy levels due to nonzero m, we seek solution of Eq. (2.13) as a series $\epsilon = \Sigma_{k=0}^{\infty}\epsilon^{(k)}$ with $\epsilon^{(k)}$ of order m^k and easily find the first two terms

$$\epsilon_n = \epsilon_n^{(0)} - m + \frac{m^2}{|\epsilon_n^{(0)}|}(0.24 + 0.20i). \tag{2.14}$$

Since $\mathrm{Im}\,\epsilon_n^{(0)} < 0$, the appearance of a gap results in decreasing the width of resonance and, therefore, increases stability of the system.

We considered above the case $|\epsilon| \gg m$ and analyzed how a nonzero mass affects resonant states. It is instructive to consider resonant states also in the vicinity of the level $\epsilon = -m$ when bound states dive into the lower continuum and determine their real and imaginary parts of energy. First of all, nonzero m increases the value of the critical charge. Let us find it. Using Eq. (A20) in Appendix A, we obtain that the critical value $Z_c\alpha$ for $j=1/2$ scales with m like (see Fig. 1)

$$Z_c\alpha \simeq \frac{1}{2} + \frac{\pi^2}{\log^2(cmR)},$$

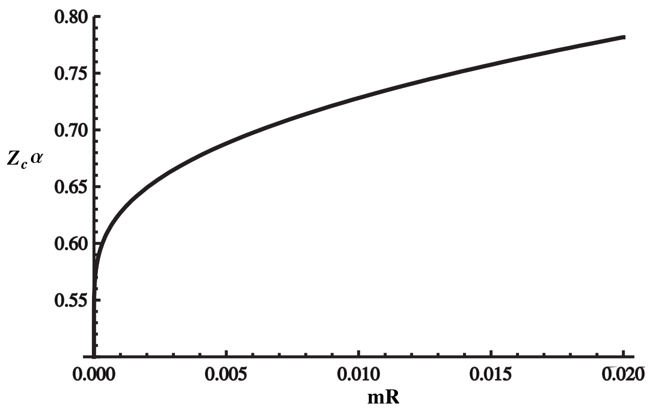

FIG. 1. The critical coupling as a function of mR for the $1S_{1/2}$ level.

$$c = \exp\left[-2\Psi(1) - \frac{2J_0(1/2)}{J_0(1/2) - J_1(1/2)}\right] \approx 0.21. \tag{2.15}$$

Note that the dependence of the critical coupling on mR is quite similar to that in the strongly coupled QED.[8,9] For $Z > Z_c$, using Eq. (2.8), we find the following resonant states

$$\epsilon = -m\left(1 + \xi + i\frac{3\pi}{8}e^{-\pi/\sqrt{2\xi}}\right), \quad \xi = \frac{3\pi}{8}\frac{\beta - \beta_c}{\beta\beta_c}, \tag{2.16}$$

where $\beta_c = \sqrt{(Z_c\alpha)^2 - 1/4}$. Like in QED (Ref. 16) the imaginary part of energy of these resonant states vanishes exponentially as $Z \to Z_c$. Such a behavior is connected with tunneling through the Coulomb barrier in the problem under consideration. For the quasielectron in graphene in a central potential $V(r)$, expressing the lower component of the Dirac spinor, Eq. (2.3), through the upper one and following,[16,17] we obtain an effective second-order differential equation in the form of the Schrödinger equation

$$\chi''(r) + k^2(r)\chi(r) = 0,$$

$$a(r) = \exp\left[\frac{1}{2}\int\left(\frac{1}{r} - \frac{\tilde{V}'}{\epsilon + m - \tilde{V}}\right)dr\right]\chi(r). \tag{2.17}$$

Here

$$k^2(r) = 2[\mathcal{E} - U(r)], \quad \mathcal{E} = \frac{\epsilon^2 - m^2}{2}, \quad \tilde{V} = \frac{V}{\hbar v_F} \tag{2.18}$$

and we represent the effective potential as the sum of two terms $U = U_1 + U_2$, where U_1 is the effective potential for the Klein-Gordon equation and U_2 takes into account the spin-dependent effects

$$U_1 = \epsilon\tilde{V} - \frac{\tilde{V}^2}{2} + \frac{j(j-1)}{2r^2}, \tag{2.19}$$

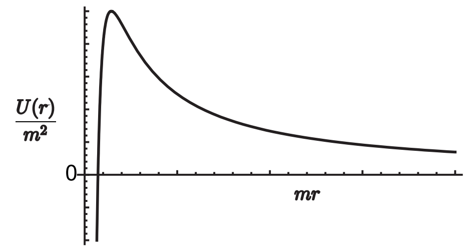

FIG. 2. Effective potential for the Coulomb center in the case $\epsilon = -m$ and $Z = Z_c$.

$$U_2 = \frac{1}{4}\left[\frac{\tilde{V}''}{\epsilon + m - \tilde{V}} + \frac{3}{2}\left(\frac{\tilde{V}'}{\epsilon + m - \tilde{V}}\right)^2 + \frac{2j\tilde{V}'}{r(\epsilon + m - \tilde{V})}\right]. \tag{2.20}$$

Note that Eq. (2.17) and the potentials, Eqs. (2.19) and (2.20), coincide with the corresponding equations in QED.[17] We plot the effective potential $U(r)$ for $Z \to Z_c$, $j = 1/2$, and $\epsilon = -m$ in Fig. 2, where the Coulomb barrier is clearly seen.

Up to now, we considered a one-particle problem in an external field. In the next section, we will consider electrons and holes in graphene which interact by means of the Coulomb interaction and show that an instability develops in the system when the coupling α exceeds some critical value α_c.

III. EXCITONIC TACHYON INSTABILITY

A. Bethe-Salpeter equation

The instability of the supercritical charge problem due to the emission of positively charged holes discussed in the previous section indicates the possibility of the excitonic instability in graphene in the case of a supercritical coupling constant. In this section, we will study the BS equation for an electron-hole state and show that it has a tachyon in the spectrum in the supercritical regime. Before we do this, let us discuss some similarities and differences of the supercritical Coulomb charge problem with the famous Cooper problem in the theory of superconductivity.

Although the Cooper problem is formulated as a quantum-mechanical problem for two particles (electrons), it can be standardly reduced to a one-particle problem in an external potential. Therefore, the Coulomb center problem is similar to the Cooper problem in this respect. However, there are important differences between the two problems. The first one is connected with the fact that the Dirac equation contains the lower continuum with filled negative-energy states. Therefore, if a bound-state energy enters the lower continuum, we are essentially dealing with a many-body problem. This explains why there are resonant states with imaginary energy in the supercritical Coulomb potential unlike the Cooper problem where there are only negative-energy bound states. The second important difference between these two problems is connected with the critical value of coupling constant. It is zero for the Cooper problem because the density of states in this problem is nonzero at the Fermi surface that plays a crucial role in the bound-states formation. On the other hand, $\alpha_c = 1/2$ for the Coulomb center problem in graphene where the density of states is zero at the Dirac point.

The appearance of the Cooper bound state in the theory of superconductivity is directly related to the instability of the normal state of metal. Indeed, according to,[26] the BS equation for an electron-electron bound state in the normal state of metal has a solution with imaginary energy, i.e., a tachyon. This means that normal state is unstable and a phase transition to the superconducting state takes place. As we mentioned above, resonant states in the supercritical Coulomb center problem suggest the excitonic instability in graphene.

For the description of the dynamics in graphene, we will use the same model as in Refs. 6 and 7 in which while quasiparticles are confined to a two-dimensional plane, the electromagnetic (Coulomb) interaction between them is three dimensional in nature. The low-energy quasiparticles excitations in graphene are conveniently described in terms of a four-component Dirac spinor $\Psi_a^T=(\psi_{KAa},\psi_{KBa},\psi_{K'Ba},\psi_{K'Aa})$ which combines the Bloch states with spin indices $a=1,2$ on the two different sublattices (A,B) of the hexagonal graphene lattice and with momenta near the two nonequivalent valley points (K,K') of the two-dimensional Brillouin zone. In what follows we treat the spin index as a "flavor" index with N_f components, $a=1,2,\ldots,N_f$, then $N_f=2$ corresponds to graphene monolayer while $N_f=4$ is related to the case of two decoupled graphene layers, interacting solely via the Coulomb interaction.

The action describing graphene quasiparticles interacting through the Coulomb potential has the form

$$S=\int dtd^2r\bar{\Psi}_a(t,\mathbf{r})(i\gamma^0\partial_t-iv_F\gamma\nabla)\Psi_a(t,\mathbf{r}) -\frac{1}{2}\int dtdt'd^2rd^2r'\bar{\Psi}_a(t,\mathbf{r})\gamma^0\Psi_a(t,\mathbf{r})U_0(t-t',|\mathbf{r}-\mathbf{r}'|) \times\bar{\Psi}_b(t',\mathbf{r}')\gamma^0\Psi_b(t',\mathbf{r}'), \quad (3.1)$$

where $\bar{\Psi}=\Psi^\dagger\gamma^0$ and the 4×4 Dirac γ matrices $\gamma^\mu=\tau^3\otimes(\sigma^3,i\sigma^2,-i\sigma^1)$ furnish a reducible representation of the Dirac algebra in 2+1 dimensions. The Pauli matrices τ,σ act in the subspaces of the valleys (K,K') and sublattices (A,B), respectively. The other two γ matrices which we use are $\gamma^3=i\tau_2\otimes\sigma_0$, $\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3=\tau_1\otimes\sigma_0$ (σ_0 is the 2×2 unit matrix).

The bare Coulomb potential $U_0(t,|\mathbf{r}|)$ takes the simple form

$$U_0(t,|\mathbf{r}|)=\frac{e^2\delta(t)}{\kappa}\int\frac{d^2k}{2\pi}\frac{e^{i\mathbf{kr}}}{|\mathbf{k}|}=\frac{e^2\delta(t)}{\kappa|\mathbf{r}|}. \quad (3.2)$$

However, the polarization effects considerably modify this bare Coulomb potential and the interaction will be

$$U(t,|\mathbf{r}|)=\frac{e^2}{\kappa}\int\frac{d\omega}{2\pi}\int\frac{d^2k}{2\pi}\frac{\exp(-i\omega t+i\mathbf{kr})}{|\mathbf{k}|+\Pi(\omega,\mathbf{k})}, \quad (3.3)$$

where κ is the dielectric constant due to a substrate and the polarization function $\Pi(\omega,\mathbf{k})$ is proportional (within the factor $2\pi/\kappa$) to the time component of the photon polarization function. Correspondingly, the Coulomb propagator has the form

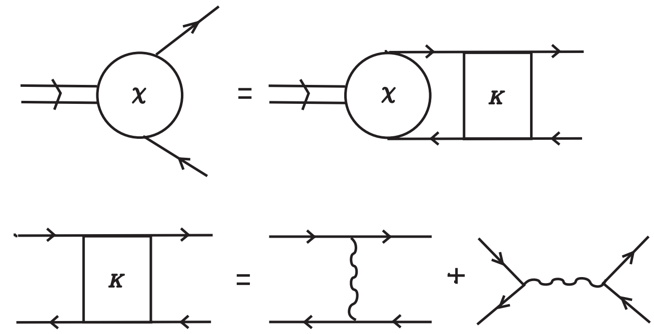

FIG. 3. The BS equation for a bound-electron-hole state χ. The kernel K contains two diagrams: exchange and annihilation ones. The wave line corresponds to the Coulomb propagator.

$$D(\omega,|\mathbf{q}|)=\frac{1}{|\mathbf{q}|+\Pi(\omega,|\mathbf{q}|)}. \quad (3.4)$$

The one-loop polarization function is[3]

$$\Pi(\omega,\mathbf{k})=\frac{\pi e^2N_f}{4\kappa}\frac{\mathbf{k}^2}{\sqrt{\hbar^2v_F^2\mathbf{k}^2-\omega^2}}, \quad (3.5)$$

and in an instantaneous approximation it is

$$\Pi(\omega=0,\mathbf{k})=\frac{\pi e^2N_f}{4\kappa\hbar v_F}|\mathbf{k}|. \quad (3.6)$$

In general, the static polarization operator must have the form $\Pi(0,|\mathbf{q}|)=|\mathbf{q}|F(\alpha,N_f)$ due to dimensional reasons, however its exact form is not known and in the present paper we will use the one-loop approximation.

The continuum effective theory described by the action, Eq. (3.1), possesses the $U(2N_f)$ symmetry. However, as was pointed out in Ref. 27 (see also Refs. 28 and 29), it is not exact for the Lagrangian on the graphene lattice. In fact, there are small on-site repulsion interaction terms which break the $U(2N_f)$ symmetry.

In order to analyze excitonic instability, we consider the Bethe-Salpeter equation (see, for example, Ref. 12) for an electron-hole bound state which is represented in Fig. 3. The kernel K of the BS equation in the simplest approximation contains two diagrams: the one is due to exchange Coulomb forces and another one is the annihilation diagram. The annihilation diagram does not contribute for the BS wave function considered below. Thus the BS equation takes the following form

$$\left[S^{-1}\left(q+\frac{1}{2}P\right)\chi(q,P)S^{-1}\left(q-\frac{1}{2}P\right)\right]_{\alpha\beta} =\frac{i\alpha}{(2\pi)^2}\int d^3kD(|\mathbf{q}-\mathbf{k}|)[\gamma^0\chi(k,P)\gamma^0]_{\alpha\beta}, \quad (3.7)$$

where $k=(k_0,\mathbf{k})$, α,β are spinor indices, $\chi(q,P)$ is the BS amplitude in momentum space

$$\chi_{\alpha\beta}(q,P) = \int d^3x e^{iqx}\langle 0|T\Psi_\alpha\left(\frac{x}{2}\right)\bar{\Psi}_\beta\left(-\frac{x}{2}\right)|P\rangle, \quad (3.8)$$

$q=(q_0,\mathbf{q})$, $P=(P_0,\mathbf{P})$, $\mathbf{q}$ and $\mathbf{P}$ are relative and total momenta, respectively, and

$$S(p) = \frac{\gamma^0 p_0 - \gamma\mathbf{p} + \Delta}{p_0^2 - \mathbf{p}^2 - \Delta^2 + i0}$$

is the quasiparticle propagator with a gap Δ (the gap Δ is zero in noninteracting graphene, however, may be generated due to the strong Coulomb interaction). In what follows we put $\hbar = v_F = 1$.

Taking into account the static vacuum polarization by massless fermions, i.e., with $\Pi(\omega=0,\mathbf{k})$, corresponds to the replacement of the coupling constant α in Eq. (3.7) by

$$\alpha \to \frac{\alpha}{1+\pi\alpha N_f/4} \equiv 2\lambda.$$

Further, introducing the function

$$\hat{\chi}(q,p) = S^{-1}\left(q+\frac{1}{2}P\right)\chi(q,P)S^{-1}\left(q-\frac{1}{2}P\right)$$

with $p=P/2$, the BS equation can be equivalently rewritten as follows:

$$\hat{\chi}(q,p) = \frac{2i\lambda}{(2\pi)^2}\int \frac{d^3k}{|\mathbf{q}-\mathbf{k}|}\gamma^0 S(k+p)\hat{\chi}(k,p)S(k-p)\gamma^0. \quad (3.9)$$

In general, $\hat{\chi}$ can be expanded in 16 independent matrix structures. In view of the experience in QED,[9] we expect a gap generation in graphene in the supercritical regime. Then the spin-valley $U(4)$ symmetry will be broken (see, e.g., Refs. 6 and 7) that leads to the appearance of massless Nambu-Goldstone bosons in the spectrum. Similar to QED,[9] these Nambu-Goldstone bosons are transformed into tachyons if considered on the wrong vacuum state without a gap generation. In the present paper, we will consider only matrix structures of $\hat{\chi}$ connected with the γ^5 matrix

$$\hat{\chi}(q,p) = \chi_5(q,p)\gamma^5 + \chi_{05}(q,p)q^i\gamma^i\gamma^0\gamma^5, \quad (3.10)$$

where $\chi_5(q,p)$ and $\chi_{05}(q,p)$ are scalar coefficient functions. We will see in the next section that it is enough to consider only χ_5 in order to describe a Nambu-Goldstone excitation in the massive state. However, we retain the function χ_{05} because it is necessary in the study of tachyon. There are also tachyons in other channels which describe different ways of breaking the $U(2N_f)$ symmetry, for example, one can use matrices $I, \gamma^3, \gamma^3\gamma^5$ instead of the matrix γ^5 in Eq. (3.10). To study instability it is enough to find at least one channel with tachyons. The real pattern of a symmetry breaking is defined by solving gap equations for various kinds of order parameters and determining which of them corresponds to the global minimum of the system energy. For simplicity we consider only the channel described by the wave function, Eq. (3.10), which can be treated analytically.

B. Tachyon states

Let us first show that, for $\lambda > \lambda_c$, there is a tachyon in the spectrum of the Bethe-Salpeter equation in the massless theory $\Delta=0$ and determine the critical value λ_c. For the study of tachyon, we can set $\mathbf{p}=0$, however, should keep nonzero p_0. One can check that ansatz, Eq. (3.10), is consistent for Eq. (3.9) and leads to a coupled system of equations for functions $\chi_5(q,p_0), \chi_{05}(q,p_0)$ [in what follows we omit p_0 for brevity in the arguments of the functions $\chi_5(q,p_0), \chi_{05}(q,p_0)$]. Since Eq. (3.9) implies that $\hat{\chi}(q,p)$ does not depend on q_0, we can integrate then over k_0 by using the integrals

$$i\int_{-\infty}^{\infty}\frac{dk_0}{\pi}\frac{c_1+c_2k_0+c_3k_0^2}{[(k_0-p_0)^2-\mathbf{k}^2+i\delta][(k_0+p_0)^2-\mathbf{k}^2+i\delta]} = \frac{c_1+c_3(p_0^2-\mathbf{k}^2)}{2|\mathbf{k}|(p_0^2-\mathbf{k}^2)},$$

where $\delta \to +0$. We obtain the following system of integral equations,

$$\chi_5(\mathbf{q}) = \lambda\int\frac{d^2k}{2\pi}\frac{\mathbf{k}^2[\chi_5(\mathbf{k})+p_0\chi_{05}(\mathbf{k})]}{|\mathbf{q}-\mathbf{k}||\mathbf{k}|(\mathbf{k}^2-p_0^2)},$$

$$\chi_{05}(\mathbf{q}) = \lambda\int\frac{d^2k}{2\pi}\frac{\mathbf{qk}[\mathbf{k}^2\chi_{05}(\mathbf{k})+p_0\chi_5(\mathbf{k})]}{\mathbf{q}^2|\mathbf{q}-\mathbf{k}||\mathbf{k}|(\mathbf{k}^2-p_0^2)}. \quad (3.11)$$

We assume that $\chi_5(\mathbf{q})$ and $\chi_{05}(\mathbf{q})$ depend only on $q=|\mathbf{q}|$, then we can integrate over the angle using

$$\int_0^{2\pi}\frac{d\phi}{\sqrt{q^2+k^2-2qk\cos\phi}} = \frac{4}{q+k}K\left(\frac{2\sqrt{qk}}{q+k}\right) = 4\left[\frac{\theta(q-k)}{q}K\left(\frac{k}{q}\right)+\frac{\theta(k-q)}{k}K\left(\frac{q}{k}\right)\right], \quad (3.12)$$

$$\begin{aligned}\int_0^{2\pi}\frac{d\phi\cos\phi}{\sqrt{q^2+k^2-2qk\cos\phi}} &= \frac{2(q^2+k^2)}{qk(q+k)}\left[K\left(\frac{2\sqrt{qk}}{q+k}\right)-\frac{(q+k)^2}{q^2+k^2}E\left(\frac{2\sqrt{qk}}{q+k}\right)\right]\\ &= \frac{4}{qk}\left\{q\theta(q-k)\left[K\left(\frac{k}{q}\right)-E\left(\frac{k}{q}\right)\right]\right.\\ &\quad\left.+k\theta(k-q)\left[K\left(\frac{q}{k}\right)-E\left(\frac{q}{k}\right)\right]\right\},\end{aligned} \quad (3.13)$$

where $K(x)$ and $E(x)$ are complete elliptic integrals of the first and second kind, respectively, $\theta(x)$ is the Heaviside step function, and for the last equalities in Eqs. (3.12) and (3.13) we used the formulas 8.126.3, 8.126.4 in the book.[30] Further, we approximate the elliptic integrals by their asymptotics

$$K(x) \simeq \frac{\pi}{2}[1 + O(x^2)], \quad E(x) \simeq \frac{\pi}{2}[1 + O(x^2)], \quad x \ll 1. \tag{3.14}$$

This approximation allows one to obtain analytical results for the BS equation. The logarithmic singularity present in elliptic integrals in Eqs. (3.12) and (3.13) at $q=k$ does not influence qualitatively the solution obtained though it is important to take it into account to get correct value of the critical coupling [see the derivation of Eq. (3.30) below]. Thus we find

$$\chi_5(q) = \lambda \int_0^q \frac{k^2 dk}{q(k^2 - p_0^2)}[\chi_5(k) + p_0\chi_{05}(k)] + \lambda \int_q^\Lambda \frac{k dk}{k^2 - p_0^2}[\chi_5(k) + p_0\chi_{05}(k)], \tag{3.15}$$

$$\chi_{05}(q) = \frac{\lambda}{2}\int_0^q \frac{k^2 dk}{q^3(k^2 - p_0^2)}[k^2\chi_{05}(k) + p_0\chi_5(k)] + \frac{\lambda}{2}\int_q^\Lambda \frac{dk}{k(k^2 - p_0^2)}[k^2\chi_{05}(k) + p_0\chi_5(k)]. \tag{3.16}$$

Here we also introduced a finite ultraviolet cutoff Λ which could be taken to be of order π/a, where a is a characteristic lattice size, $a=2.46$ Å for graphene. An alternative, equally good, choice of Λ is related to the energy band, $\Lambda=t/v_F$, where $t=2.4$ eV in graphene.

These equations are equivalent to the system of differential equations

$$\chi_5'' + \frac{2}{q}\chi_5' + \lambda\frac{\chi_5 + p_0\chi_{05}}{q^2 - p_0^2} = 0,$$

$$\chi_{05}'' + \frac{4}{q}\chi_{05}' + \frac{3\lambda}{2}\frac{q^2\chi_{05} + p_0\chi_5}{q^2(q^2 - p_0^2)} = 0 \tag{3.17}$$

with the following boundary conditions

$$q^2\chi_5'|_{q=0} = 0, \quad [q\chi_5(q)]'|_{q=\Lambda} = 0,$$

$$q^4\chi_{05}'|_{q=0} = 0, \quad [q^3\chi_{05}(q)]'|_{q=\Lambda} = 0. \tag{3.18}$$

The system of differential equations, Eqs. (3.17), can be reduced to one equation of the fourth order whose solutions are given in terms of generalized hypergeometric functions ${}_4F_3(q^2/p_0^2)$ and the Meijer functions with the corresponding boundary conditions (for this analysis see Appendix B). However, since we seek for the solution with $p_0 \to 0$, it is simpler to straightforwardly analyze the system, Eqs. (3.17), itself, in this regime the system decouples

$$\chi_5'' + \frac{2}{q}\chi_5' + \lambda\frac{\chi_5}{q^2 - p_0^2} = 0, \quad \chi_{05}'' + \frac{4}{q}\chi_{05}' + \frac{3\lambda}{2}\frac{\chi_{05}}{q^2 - p_0^2} = 0, \tag{3.19}$$

where we keep p_0 in the denominators because it regularizes singularities for $q \to 0$.

Obviously, Eqs. (3.19) are differential equations for the hypergeometric function $F(a,b;c;z)$.[30] The solutions that satisfy the infrared boundary conditions are

$$\chi_5 = C_1 F\left(\frac{1+\gamma}{4}, \frac{1-\gamma}{4}; \frac{3}{2}; \frac{q^2}{p_0^2}\right),$$

$$\chi_{05} = C_2 F\left(\frac{3(1+\tilde{\gamma})}{4}, \frac{3(1-\tilde{\gamma})}{4}; \frac{5}{2}; \frac{q^2}{p_0^2}\right), \tag{3.20}$$

where $\gamma=\sqrt{1-4\lambda}$ and $\tilde{\gamma}=\sqrt{1-2\lambda/3}$. Using the asymptotic of the hypergeometric functions, one may easily check that the ultraviolet boundary conditions for the function χ_5 can be satisfied only for $\lambda > 1/4$, therefore, 1/4 is the critical coupling for the approximation that we use. (Note that if we neglect the vacuum polarization contribution, then $\lambda=\alpha/2$ and the critical value 1/4 coincides with the critical coupling $Z_c\alpha=1/2$ obtained in Sec. II for the Coulomb center problem.) The UV boundary condition for the function χ_{05} can be satisfied for the values of $\lambda > 3/2$ but not for $\lambda < 3/2$. Therefore, for $1/4 < \lambda < 3/2$ we take a trivial solution $\chi_{05}=0$ and we are left only with the equation for the function χ_5. Knowing the function χ_5 we then solve an inhomogeneous equation, Eqs. (3.17), for χ_{05}, in this way we find that the function $\chi_{05} \sim p_0$. The critical value $\lambda_c=1/4$ coincides with the critical coupling constant found in Ref. 7 where the same approximation for the kernel was made. In the supercritical regime $\gamma=i\omega$, $\omega=\sqrt{4\lambda-1}$ and the function $\chi_5(q)$ behaves asymptotically as

$$\chi_5(q) \sim q^{-1/2}\cos(\sqrt{\lambda - 1/4}\ln q + \text{const}). \tag{3.21}$$

Such oscillatory behavior is typical for the phenomenon known in quantum mechanics as the collapse (fall into the center) phenomenon: in this case the energy of a system is unbounded from below and there is no ground state. Nodes of the wave function of the bound state signify the existence of the tachyon states with imaginary energy p_0, $\text{Im}\, p_0^2 < 0$. Indeed, the UV boundary condition for χ_5 leads to the equation

$$\frac{(1+i\omega)\Gamma\left(1+\frac{i\omega}{2}\right)\Gamma\left(\frac{1-i\omega}{4}\right)\Gamma\left(\frac{5-i\omega}{4}\right)}{(1-i\omega)\Gamma\left(1-\frac{i\omega}{2}\right)\Gamma\left(\frac{1+i\omega}{4}\right)\Gamma\left(\frac{5+i\omega}{4}\right)}\left(-\frac{\Lambda^2}{p_0^2}\right)^{i\omega/2} = 1. \tag{3.22}$$

Then we find the following tachyon solution,

$$p_0^2 = -\Lambda^2 \exp\left[-\frac{4\pi n}{\omega} + \delta(\omega)\right],$$

$$\delta(\omega) = \frac{4}{\omega}\left\{\arctan\omega + \arg\left[\Gamma\left(1+\frac{i\omega}{2}\right) \times\Gamma\left(\frac{1-i\omega}{4}\right)\Gamma\left(\frac{5-i\omega}{4}\right)\right]\right\}. \tag{3.23}$$

If λ tends to 1/4 from the above, i.e., $\omega \to 0$,

$$p_0^2 = -\Lambda^2 \exp\left[-\frac{4\pi n}{\omega} + \delta(0)\right],$$

$$\delta(0) = 4 + 2\Psi(1) - \Psi(1/4) - \Psi(5/4) \approx 7.3, \quad n = 1,2,\ldots . \tag{3.24}$$

Thus, we see that the strongest instability, i.e., the smallest negative value of p_0^2 is given by the solution for the function χ_5 with $n=1$. The tachyon states play here the role of the quasistationary states in the problem of supercritical Coulomb center resulting in the vacuum instability. In fact, the tachyon instability can be viewed as the field theory analog of the fall into the center phenomenon and the critical coupling α_c is an analog of the critical coupling $Z_c\alpha$ in the problem of the Coulomb center.

The tachyon energy Eq. (3.24) has a characteristic essential singularity of the kind $1/\sqrt{\lambda-\lambda_c}$ in the exponent. It can be argued that this behavior reflects a scale invariance in the problem under consideration and keeps its form for any approximation which does not introduce new scale parameter except the cutoff.[31]

There are two possibilities for the system with the supercritical charge to become stable: to shield spontaneously the charge or to generate spontaneously the fermion gap. The first possibility is realized in the problem of the supercritical Coulomb center which is due to the formulation of the problem as the one-particle one. The second possibility, dynamical generation of the fermion gap, is realized for quasiparticles in graphene interacting through supercritical Coulomb interaction. The situation here is completely analogous to the strongly coupled QED (Refs. 8, 9, and 12) where it is shown that the vacuum stabilization by generating dynamical fermion gap is a rather universal phenomenon.

The critical value λ_c determines the critical coupling α_c as a function of the fermion number N_f,

$$\alpha_c = \frac{4\lambda_c}{2 - \pi N_f \lambda_c} \tag{3.25}$$

[compare with Eq. (28) in Ref. 7]. The critical value $\lambda_c = 1/4$ in the approximation, Eq. (3.14), used for kernels. The more precise value of λ_c can be found if one notes that λ_c corresponds to the limit $p_0=0$. Taking this limit in the system, Eq. (3.17), we get

$$\chi_5(q) = \frac{2\lambda}{\pi}\int_0^\infty dk\chi_5(k)\left[\frac{\theta(q-k)}{q}K\left(\frac{k}{q}\right) + \frac{\theta(k-q)}{k}K\left(\frac{q}{k}\right)\right], \tag{3.26}$$

$$\chi_{05}(q) = \frac{2\lambda}{\pi q}\int_0^\infty dk\chi_{05}(k)\left\{\theta(q-k)\left[K\left(\frac{k}{q}\right) - E\left(\frac{k}{q}\right)\right] + \frac{k\theta(k-q)}{q}\left[K\left(\frac{q}{k}\right) - E\left(\frac{q}{k}\right)\right]\right\}. \tag{3.27}$$

Note that the ultraviolet cutoff, Λ, has been taken to infinity, which is appropriate at the critical point. These equations are scale invariant and are solved by $\chi_5(q)=q^{-\gamma}$ and $\chi_{05}(q) = q^{-\rho}$ on the condition that the exponents γ,ρ satisfy the transcendental equations

$$1 = \frac{2\lambda}{\pi}\int_0^1 dx[x^{-\gamma} + x^{\gamma-1}]K(x), \quad 0 < \gamma < 1, \tag{3.28}$$

$$1 = \frac{2\lambda}{\pi}\int_0^1 dx[x^{-\rho} + x^{\rho-3}][K(x) - E(x)], \quad 0 < \rho < 3. \tag{3.29}$$

These equations define roots γ,ρ for any value of the coupling λ. An instability is signalized by oscillatory behavior of the functions $\chi_5(q)$ and $\chi_{05}(q)$. For the function $\chi_5(q)$ this occurs when two of the roots of Eq. (3.28) in the interval (0,1) coalesce and then become complex conjugate. We find that this happens when $\gamma=1/2$, for this value the integral in Eq. (3.28) is exactly evaluated (see, the book[32]) and we obtain the critical value

$$\lambda_c = \frac{4\pi^2}{\Gamma^4(1/4)} \approx 0.23. \tag{3.30}$$

The second equation, Eq. (3.29), gives higher critical value $\lambda_c=0.91$ therefore the instability is determined by the value $\lambda_c=0.23$. The critical value $N_{crit}\approx 2.8$ corresponds to $\alpha=\infty$ in Eq. (3.25). Since for graphene the number of flavors $N_f=2$, the critical coupling is estimated to be $\alpha_c\approx 1.62$ in the considered approximation.[33] Because the coupling constant in freely standing graphene $\alpha\approx 2.19(\kappa\approx 1)$ the system is in the unstable phase. On the other hand, for graphene on a SiO_2 substrate the dielectric constant $\kappa\approx 2.8$, therefore, $\alpha\approx 0.78$, i.e., the system is in the stable phase.

Finally, since the $U(2N_f)$ symmetry is spontaneously broken, there must exist Nambu-Goldstone excitations in the stable phase where a quasiparticle gap arises. Let us show that the BS Eq. (3.9) indeed admits such solutions. To see this, according to Ref. 9 we set $p_0=\mathbf{p}=0$. Then, Eq. (3.9) has a solution of the form $\chi(q,0)=\chi_5(q,0)\gamma_5$ for which we obtain the equation

$$\chi_5(q,0) = \frac{\lambda}{2\pi}\int \frac{d^2k}{|\mathbf{q}-\mathbf{k}|}\frac{\chi_5(k,0)}{\sqrt{\mathbf{k}^2+\Delta^2(k)}}, \tag{3.31}$$

or, after integrating over the angle,

$$\chi_5(q,0) = \lambda\int_0^\Lambda \frac{dkk\chi_5(k,0)}{\sqrt{k^2+\Delta^2(k)}}\mathcal{K}(q,k), \tag{3.32}$$

with the kernel

$$\mathcal{K}(q,k) = \frac{\theta(q-k)}{q}K\left(\frac{k}{q}\right) + \frac{\theta(k-q)}{k}K\left(\frac{q}{k}\right). \tag{3.33}$$

On the other hand, the equation for a gap function obtained in Ref. 7 has the form

$$\Delta(q) = \lambda\int_0^\Lambda \frac{dkk\Delta(k)}{\sqrt{k^2+\Delta^2(k)}}\mathcal{K}(q,k). \tag{3.34}$$

One can see that Eq. (3.32) has the solution $\chi_5(q,0) = C\Delta(q)$ where the gap function $\Delta(q)$ satisfies Eq. (3.34) and

C is a constant. Thus the wave function $\chi_5(q,0)$ describes a gapless Nambu-Goldstone excitation. Solving the BS equation at nonzero $p_0, \mathbf{p}$ one can obtain a dispersion law $p_0 \sim |\mathbf{p}|$ for a Nambu-Goldstone excitation.

IV. CONCLUSIONS

In this paper we studied instabilities in graphene which arise at strong Coulomb coupling. For the supercritical Coulomb center problem, it was known before that the fall into the center instability arises if $Z\alpha$ exceeds the critical value 1/2 leading to the appearance of quasistationary levels with complex energies. The energy of quasistationary states in the case of gapless quasiparticles has a characteristic essential-singularity-type dependence on the coupling constant reflecting the scale invariance of the Coulomb potential. We showed that a quasiparticle gap stabilizes the system decreasing the imaginary part $|\mathrm{Im}\, E|$ of quasistationary states, thus increasing their lifetime.

Considering the many-body problem of strongly interacting gapless quasiparticles in graphene, we showed that the Bethe-Salpeter equation for an electron-hole bound state contains a tachyon in its spectrum in the supercritical regime $\alpha > \alpha_c$ and found the critical constant $\alpha_c = 1.62$ in the static random-phase approximation. The tachyon states play the role of quasistationary states in the problem of the supercritical Coulomb center and lead to the rearrangement of the ground state and the formation of exciton condensate. Thus, there is a close relation between the two instabilities, in fact, the tachyon instability can be viewed as the field theory analog of the fall into the center phenomenon and the critical coupling α_c is an analog of the critical coupling $Z_c\alpha$ in the problem of the Coulomb center. The physics of two instabilities is related to strong Coulomb interaction.

The calculated critical value $\alpha_c = 1.62$ should be compared with the value $\alpha_c = 1.08$ found in Monte Carlo simulations[4] for the rearrangement of the ground state of graphene and appearance of a gap. The obtained value of α_c is rather large that indicates that the ladder approximation is not quantitatively good enough for the problem of excitonic instability and gap generation in freely standing graphene. Certainly, both higher-order corrections and improving the instantaneous approximation can vary the value of critical coupling. It is essential however that a ground-state rearrangement at strong coupling is connected with the "fall into the supercritical Coulomb center" phenomenon. Therefore, such an rearrangement in graphene with large Coulomb interaction seems to be very plausible for strong enough coupling even if one goes beyond the ladder approximation. Finally, the physical picture of instabilities in graphene is quite similar to that elaborated earlier in strongly coupled QED (Refs. 8, 9, and 12) (see, also, Refs. 10 and 11). In QED, the ladder approximation is not reliable quantitatively also because the critical coupling constant for chiral symmetry breaking is of order 1. However, the main results of the ladder approximation survive when all diagrams with photons exchanges are included (the so-called quenched approximation without fermion loops).[31] Further, the existence of the critical point is exactly proved in the lattice version of QED.[34] We note also that in the presence of an external magnetic field the value of the critical coupling reduces to zero (magnetic catalysis phenomenon[35]) so that the gap generation takes place already in the weak-coupling regime.

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APPENDIX A: DISCRETE SPECTRUM FOR A REGULARIZED COULOMB POTENTIAL

The discrete spectrum of Eq. (2.4) exists for $|\epsilon| < m$. In this case it is convenient to define

$$\rho = 2ur, \quad u = \sqrt{m^2 - \epsilon^2}, \quad a = \frac{\sqrt{m+\epsilon}}{2}(g - f),$$

$$b = \frac{\sqrt{m-\epsilon}}{2}(g + f) \tag{A1}$$

and rewrite Eq. (2.4) as follows:

$$\rho g' + g\left(\frac{\rho}{2} - \frac{1}{2} - Z\alpha\frac{\epsilon}{u}\right) + f\left(j + Z\alpha\frac{m}{u}\right) = 0,$$

$$\rho f' - f\left(\frac{\rho}{2} + \frac{1}{2} - Z\alpha\frac{\epsilon}{u}\right) + g\left(j - Z\alpha\frac{m}{u}\right) = 0. \tag{A2}$$

Substituting f from the first equation into the second one, we obtain the equation for the g component

$$\frac{d^2 g}{d\rho^2} + \left(-\frac{1}{4} + \frac{\frac{1}{2} + Z\alpha\frac{\epsilon}{u}}{\rho} + \frac{\frac{1}{4} - j^2 + Z^2\alpha^2}{\rho^2}\right) g = 0, \tag{A3}$$

which is the well-known Whittaker equation.[30] Its general solution is

$$g = C_1 W_{\mu,\nu}[\rho] + C_2 M_{\mu,\nu}[\rho], \quad \mu = \frac{1}{2} + \frac{Z\alpha\epsilon}{u}, \tag{A4}$$

where $\nu = \sqrt{j^2 - Z^2\alpha^2}$. Taking into account the asymptotic of the Whittaker functions $W_{k,\nu}(z), M_{k,\nu}(z)$ at infinity,

$$W_{\mu,\nu}(\rho) \simeq e^{-ur}(2ur)^{\mu}, \quad r \to \infty, \tag{A5}$$

$$M_{\mu,\nu}(\rho) \simeq \frac{\Gamma(1+\nu)}{\Gamma\left(\frac{1}{2} - \mu + \nu\right)} e^{ur}(2ur)^{-\mu}, \quad r \to \infty, \tag{A6}$$

we find that the regularity condition at infinity requires $C_2 = 0$. Then the first equation in Eq. (A2) gives the following solution for the f component in region II ($r>R$),

$$f_{\rm II}=C_1\left(j-Z\alpha\frac{m}{u}\right)W_{-1/2+Z\alpha\epsilon/u,\nu}[\rho]. \tag{A7}$$

Solutions in region I ($r<R$) can be easily obtained from Eq. (2.4)

$$b_{\rm I}=A_1 rJ_{|j+1/2|}\left(r\sqrt{\left(\epsilon+\frac{Z\alpha}{R}\right)^2-m^2}\right), \tag{A8}$$

$$a_{\rm I}=A_1\,\mathrm{sgn}(j)\sqrt{\frac{\epsilon+Z\alpha/R+m}{\epsilon+Z\alpha/R-m}}rJ_{|j-1/2|}\left(r\sqrt{\left(\epsilon+\frac{Z\alpha}{R}\right)^2-m^2}\right), \tag{A9}$$

where A_1 is a constant and we took into account the infrared boundary condition which selects only regular solution for $b_{\rm I}$ and $a_{\rm I}$. Energy levels are determined through the continuity condition of the wave function at $r=R$,

$$\left.\frac{b_{\rm I}}{a_{\rm I}}\right|_{r=R}=\left.\frac{b_{\rm II}}{a_{\rm II}}\right|_{r=R}, \tag{A10}$$

that gives the equation

$$\left.\frac{W_{1/2+Z\alpha\epsilon/u,\nu}(\rho)}{\left(j-\frac{Z\alpha m}{u}\right)W_{-1/2+Z\alpha\epsilon/u,\nu}(\rho)}\right|_{r=R}=\frac{k+1}{k-1},$$

$$k=\mathrm{sgn}(j)\frac{m+\epsilon}{u}\sqrt{\frac{\epsilon+Z\alpha/R-m}{\epsilon+Z\alpha/R+m}}\frac{J_{|j+1/2|}(\tilde\rho)}{J_{|j-1/2|}(\tilde\rho)},$$

$$\tilde\rho=\sqrt{(Z\alpha+\epsilon R)^2-m^2R^2}. \tag{A11}$$

We analyze this equation in the limit $R\to 0$ where we can use the asymptotical behavior of the Whittaker function at $\rho\to 0$,

$$W_{\mu,\nu}(\rho)\simeq\frac{\Gamma(2\nu)}{\Gamma\left(\frac{1}{2}-\mu+\nu\right)}\rho^{1/2-\nu}+\frac{\Gamma(-2\nu)}{\Gamma\left(\frac{1}{2}-\mu-\nu\right)}\rho^{1/2+\nu}. \tag{A12}$$

In the limit $R\to 0$ Eq. (A11) reduces to the following one,

$$\frac{\Gamma(-2\nu)}{\Gamma(2\nu)}\frac{\Gamma\left(1+\nu-Z\alpha\frac{\epsilon}{u}\right)}{\Gamma\left(1-\nu-Z\alpha\frac{\epsilon}{u}\right)}(2uR)^{2\nu}=-\frac{j+\nu-\frac{Z\alpha(m+\epsilon)}{u}+k_0\left[j-\nu-\frac{Z\alpha(m-\epsilon)}{u}\right]}{j-\nu-\frac{Z\alpha(m+\epsilon)}{u}+k_0\left[j+\nu-\frac{Z\alpha(m-\epsilon)}{u}\right]}+O(R), \tag{A13}$$

where

$$k_0=\mathrm{sgn}(j)\frac{m+\epsilon}{u}\frac{J_{|j+1/2|}(Z\alpha)}{J_{|j-1/2|}(Z\alpha)}\equiv\frac{m+\epsilon}{u}\sigma(Z\alpha,j). \tag{A14}$$

Using the relationships

$$\frac{j+\nu-\frac{Z\alpha(m-\epsilon)}{u}}{j-\nu-\frac{Z\alpha(m+\epsilon)}{u}}=-\frac{Z\alpha}{j-\nu}\frac{u}{m+\epsilon},$$

$$\frac{j-\nu-\frac{Z\alpha(m-\epsilon)}{u}}{j+\nu-\frac{Z\alpha(m+\epsilon)}{u}}=-\frac{Z\alpha}{j+\nu}\frac{u}{m+\epsilon}, \tag{A15}$$

Eq. (A13) can be rewritten in more convenient form

$$\frac{\Gamma(-2\nu)}{\Gamma(2\nu)}\frac{\Gamma\left(1+\nu-Z\alpha\frac{\epsilon}{u}\right)}{\Gamma\left(1-\nu-Z\alpha\frac{\epsilon}{u}\right)}(2uR)^{2\nu}=-\frac{j-\nu-\frac{Z\alpha(m-\epsilon)}{u}}{j+\nu-\frac{Z\alpha(m-\epsilon)}{u}}\frac{j+\nu-Z\alpha\sigma(Z\alpha,j)}{j-\nu-Z\alpha\sigma(Z\alpha,j)}. \tag{A16}$$

In the limit $R\to 0$ the energy levels are determined by the poles of the gamma function $\Gamma(1+\nu-Z\alpha\frac{\epsilon}{u})$ and by a zero of the right-hand side of Eq. (A16), this leads to the familiar result (analog of the Balmer formula in QED) (Ref. 13) (rederived also in Ref. 14),

$$\epsilon_{n,j}=m\left[1+\frac{Z^2\alpha^2}{(\nu+n)^2}\right]^{-1/2},\quad\begin{cases}n=0,1,2,3,\ldots,\ j>0,\\ n=1,2,3,\ldots,\ j<0.\end{cases} \tag{A17}$$

The bound states for $n\geq 1$ are doubly degenerate, $\epsilon_{n,j}=\epsilon_{n,-j}$. The lowest-energy level is given by

$$\epsilon_{0,j=1/2}=m\sqrt{1-(2Z\alpha)^2}. \tag{A18}$$

If $Z\alpha$ exceeds 1/2, then the energy, Eq. (A18), becomes imaginary, i.e., the fall into the center phenomenon[18,23,24] occurs. According to Refs. 15 and 17, nonzero R resolves this problem. For $Z\alpha>1/2$, ν is imaginary for certain j and for such j we denote $\nu=i\beta$, $\beta=\sqrt{Z^2\alpha^2-j^2}$. For finite R discrete levels exist for $Z\alpha>1/2$. Their energy decreases with increasing of $Z\alpha$ until they reach the lower continuum. The behavior of lowest-energy levels with $j=1/2$ as functions of the coupling $Z\alpha$ is shown in Fig. 4.

The critical charge Z_c that corresponds to diving into the continuum is obtained from Eq. (A16) setting $\epsilon=-m$ there and using the corollary of the Stirling formula: $\frac{\Gamma(x+iy)}{\Gamma(x-iy)}\to e^{2iy\log x}$, $x\to+\infty$. We come to the equation

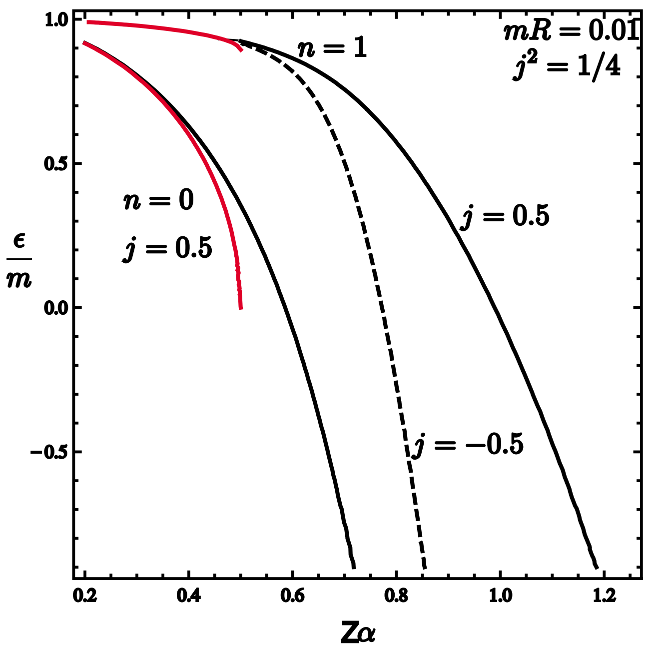

FIG. 4. (Color online) The lowest-energy levels as functions of $Z\alpha$. Red lines correspond to the pure Coulomb potential (they exist for $Z\alpha < 1/2$); black solid lines are numerical solutions for $j = 1/2$, $mR = 0.01$; black dashed line is numerical solutions for $j = -1/2$, $mR = 0.01$.

$$e^{-2i\beta \log(2Z\alpha mR)} = \frac{i\beta - j + Z\alpha\sigma(Z\alpha,j)}{-i\beta - j + Z\alpha\sigma(Z\alpha,j)} \frac{\Gamma(1-2i\beta)}{\Gamma(1+2i\beta)}, \tag{A19}$$

or,

$$-\beta \log(2Z\alpha mR) = \arg[Z\alpha\sigma(Z\alpha,j) - j + i\beta] + \arg \Gamma(1-2i\beta) + \pi n, \tag{A20}$$

where n is integer. It is not difficult to check that for $j = 1/2$ and $n = 1$ the critical coupling $Z_c\alpha$ approaches the value 1/2 for $mR \to 0$. The dependence of the critical coupling $Z_c\alpha$ on mR for $j = 1/2$ is shown in Fig. 1.

The bound and quasistationary states in gapped graphene in the case of the supercritical Coulomb impurity were also numerically calculated in the tight-binding lattice model which has a natural lattice scale cutoff that provides an important control of the validity of the Dirac equation approach.[21,36]

APPENDIX B: FOURTH-ORDER DIFFERENTIAL EQUATION

The system of equations, Eq. (3.17), with boundary conditions, Eq. (3.18), is reduced to the following fourth-order differential equation for the function $\chi_5(q)$,

$$\chi_5^{IV} + \frac{2(5q^2 - 3p_0^2)}{q(q^2 - p_0^2)}\chi_5''' + \frac{(44 + 5\lambda)q^2 - 8p_0^2}{2q^2(q^2 - p_0^2)}\chi_5'' + \frac{4p_0^2 + (8 + 7\lambda)q^2}{q^3(q^2 - p_0^2)}\chi_5' + \frac{3\lambda^2\chi_5}{2q^2(q^2 - p_0^2)} = 0, \tag{B1}$$

with the corresponding boundary conditions

$$q^2\chi_5'|_{q=0} = 0,$$

$$q^4 \frac{d}{dq}\left[(q^2 - p_0^2)\left(\chi_5'' + \frac{2}{q}\chi_5' + \lambda\frac{\chi_5}{q^2 - p_0^2}\right)\right]\Bigg|_{q=0} = 0, \tag{B2}$$

$$[q\chi_5(q)]'|_{q=\Lambda} = 0,$$

$$\frac{d}{dq}\left[q^3(q^2 - p_0^2)\left(\chi_5'' + \frac{2}{q}\chi_5' + \lambda\frac{\chi_5}{q^2 - p_0^2}\right)\right]\Bigg|_{q=\Lambda} = 0. \tag{B3}$$

In terms of the variable $z = q^2/p_0^2$ these equations are rewritten as

$$\left\{z^3(z-1)\frac{d^4}{dz^4} + 2z^2(4z-3)\frac{d^3}{dz^3} + \frac{5}{8}z[(\lambda + 22)z - 10]\frac{d^2}{dz^2} + \frac{19\lambda + 60}{16}z\frac{d}{dz} + \frac{3\lambda^2}{32}\right\}\chi_5 = 0, \tag{B4}$$

and boundary conditions

$$z^{3/2}\left.\frac{d\chi_5}{dz}\right|_{z=0} = 0,$$

$$z^{5/2}\frac{d}{dz}\left[(z-1)\left(4z\frac{d^2\chi_5}{dz^2} + 6\frac{d\chi_5}{dz} + \frac{\lambda\chi_5}{z-1}\right)\right]\Bigg|_{z=0} = 0, \tag{B5}$$

$$\left(2z\frac{d\chi_5}{dz} + \chi_5\right)\Bigg|_{z=\Lambda^2} = 0,$$

$$z^{1/2}\frac{d}{dz}\left[z^{3/2}(z-1)\left(4z\frac{d^2\chi_5}{dz^2} + 6\frac{d\chi_5}{dz} + \frac{\lambda\chi_5}{z-1}\right)\right]\Bigg|_{z=\Lambda^2} = 0. \tag{B6}$$

Equation (B4) is the Pochhammer-type equation,[37] its canonical form is

$$\left(\prod_{k=0}^{3}(\theta + b_k - 1) - z\prod_{k=1}^{4}(\theta + a_k)\right)\chi_5 = 0, \quad \theta \equiv z\frac{d}{dz}, \tag{B7}$$

where the parameters

$$b_0 = 1, \quad b_1 = 3/2, \quad b_2 = 3/2, \quad b_3 = 0, \tag{B8}$$

$$a_{1,2}=\frac{1}{4}(1\pm\gamma),\quad \gamma=\sqrt{1-4\lambda},$$

$$a_{3,4}=\frac{3}{4}(1\pm\tilde{\gamma}),\quad \tilde{\gamma}=\sqrt{1-\frac{2\lambda}{3}} \tag{B9}$$

describe the behavior of $\chi_5(z)$ at the points $z=0$ and $z=\infty$, respectively. The general solution of Eq. (B4) at the point $z=0$ can be written in terms of four linearly independent solutions,

$$\begin{aligned}\chi_5=&\frac{C_1}{\sqrt{z}}{}_4F_3\left(a_1-\frac{1}{2},a_2-\frac{1}{2},a_3-\frac{1}{2},a_4-\frac{1}{2};-\frac{1}{2},\frac{1}{2},1;z\right)\\&+C_2z\,{}_4F_3\left(a_1+1,a_2+1,a_3+1,a_4+1;\frac{5}{2},\frac{5}{2},2;z\right)\\&+C_3G^{24}_{44}\left(z\left|\begin{matrix}1-a_1, & 1-a_2, & 1-a_3, & 1-a_4\\ -1/2, & -1/2, & 0, & 1\end{matrix}\right.\right)\\&+C_4G^{34}_{44}\left(-z\left|\begin{matrix}1-a_1, & 1-a_2, & 1-a_3, & 1-a_4\\ -1/2 & 0, & 1, & -1/2\end{matrix}\right.\right),\end{aligned} \tag{B10}$$

where ${}_{q+1}F_q[(a)_{q+1};(b)_q;z]$ is higher hypergeometric function and

$$G^{mn}_{pq}\left(z\left|\begin{matrix}a_1, & \ldots a_n, & a_{n+1}, & \ldots a_p\\ b_1, & \ldots b_m, & b_{m+1}, & \ldots b_q\end{matrix}\right.\right) \tag{B11}$$

is the Meijer G function.[32] Leading asymptotic of the each term at $z\to 0$ is

$$\begin{aligned}\chi_5=&\frac{C_1}{\sqrt{z}}[1+O(z)]+zC_2[1+O(z)]\\&+C_3\frac{1}{\sqrt{z}}\frac{1}{2\pi}\prod_{i=1}^{4}\Gamma(a_i-1/2)[\log z+D+O(z)]\\&+C_4\left[-\frac{i\pi}{2\sqrt{z}}\prod_{i=1}^{4}\Gamma(a_i-1/2)+O(1)\right],\end{aligned}$$

$$D=4\gamma-2+4\log 2+\sum_{i=1}^{4}\psi(a_i-1/2), \tag{B12}$$

and γ is the Euler constant. Hence from the boundary conditions, Eq. (B6), we find that $C_3=0$ and

$$C_1=C_4\frac{i\pi}{2}\prod_{i=1}^{4}\Gamma(a_i-1/2). \tag{B13}$$

Asymptotical behavior of the function ${}_4F_3$ can be found from Eq. 7.2.3.77 in the book,[32] thus we obtain

$$\begin{aligned}&{}_4F_3(a_1,a_2,a_3,a_4;b_1,b_2,b_3;z)\\&\simeq\frac{\prod_{i=1}^{3}\Gamma(b_i)}{\prod_{i=1}^{4}\Gamma(a_i)}\sum_{k=1}^{4}(-z)^{-a_k}\frac{\Gamma(a_k)\prod_{i=1}^{4}\Gamma(a_i'-a_k)}{\prod_{i=1}^{3}\Gamma(b_i-a_k)},\end{aligned} \tag{B14}$$

if no two a_k, $k=1,\ldots,4$, differ by an integer, the prime in the product $\prod_{i=1}^{4}\Gamma(a_i'-a_k)$ means that the term with $i=k$ is absent. Thus we obtain

$$\begin{aligned}&z\,{}_4F_3\left(a_1+1,a_2+1,a_3+1,a_4+1;\frac{5}{2},\frac{5}{2},2;z\right)\\&\simeq-\frac{\Gamma^2(5/2)}{\prod_{i=1}^{4}\Gamma(a_i+1)}\\&\times\left[(-z)^{-a_4}\frac{\Gamma(a_4+1)\Gamma(a_1-a_4)\Gamma(a_2-a_4)\Gamma(a_3-a_4)}{\Gamma^2\left(\frac{3}{2}-a_4\right)\Gamma(1-a_4)}\right.\\&\left.+(3\ \text{cyclic permutations}\ 1\to 2\to 3\to 4\to 1)\vphantom{\frac{1}{1}}\right].\end{aligned} \tag{B15}$$

Similarly, for the Meijer G function we use Eq. 8.2.1.4 in the book[32] to find the asymptotic at large z,

$$\begin{aligned}&G^{mn}_{pq}\left(z\left|\begin{matrix}a_1, & \ldots a_n, & a_{n+1}, & \ldots a_p\\ b_1, & \ldots b_m, & b_{m+1}, & \ldots b_q\end{matrix}\right.\right)\\&\simeq\sum_{k=1}^{n}z^{a_k-1}\frac{\prod_{i=1}^{n}\Gamma(a_k-a_i')\prod_{i=1}^{m}\Gamma(1+b_i-a_k)}{\prod_{j=m+1}^{q}\Gamma(a_k-b_j)\prod_{j=n+1}^{p}\Gamma(1+a_j-a_k)},\end{aligned} \tag{B16}$$

if no two a_k, $k=1,\ldots n$, differ by an integer.

In our case we obtain

$$G_{44}^{34}\left(-z\left|\begin{matrix}1-a_1, & 1-a_2, & 1-a_3, & 1-a_4\\ -1/2 & 0, & 1, & -1/2\end{matrix}\right.\right) \simeq (-z)^{-a_4}\frac{\Gamma(a_1-a_4)\Gamma(a_2-a_4)\Gamma(a_3-a_4)\Gamma(a_4)\Gamma(1+a_4)\Gamma(a_4-1/2)}{\Gamma(3/2-a_4)} + (3\ \text{cyclic permutations}\ 1\to 2\to 3\to 4\to 1). \qquad (B17)$$

Hence the function $\chi_5(z)$ behaves at $z\to\infty$ as

$$\chi_5(z) = \sum_{i=1}^{4} A_i z^{-a_i}[1+O(1/z)],$$

$$A_i = (-1)^{-a_i}\left[-C_4\pi^2\cot(\pi a_i) - C_2\frac{\Gamma^2(5/2)}{\prod_{i=1}^{4}\Gamma(a_i+1)}\right]F_i, \qquad (B18)$$

where

$$F_i = \frac{\Gamma(a_j-a_i)\Gamma(a_k-a_i)\Gamma(a_l-a_i)\Gamma(1+a_i)}{\Gamma^2(3/2-a_i)\Gamma(1-a_i)}, \quad k\neq l\neq j\neq i. \qquad (B19)$$

The UV boundary conditions lead to the following equations:

$$A_1(1-\gamma)z^{-\gamma/4} + A_2(1+\gamma)z^{\gamma/4} - A_3(1+3\tilde{\gamma})z^{-1/2-3\tilde{\gamma}/4} - A_4(1-3\tilde{\gamma})z^{-1/2+3\tilde{\gamma}/4} = 0, \qquad (B20)$$

$$A_1(1-\gamma)z^{-\gamma/4} + A_2(1+\gamma)z^{\gamma/4} + A_3\frac{3}{2}(3+\tilde{\gamma})z^{1/2-3\tilde{\gamma}/4} + A_4\frac{3}{2}(3-\tilde{\gamma})z^{1/2+3\tilde{\gamma}/4} = 0, \qquad (B21)$$

where $z=\Lambda^2/p_0^2$. This system of equations does not have solutions for $\lambda<1/4$. Near the critical value, $\lambda \gtrsim 1/4$, we can neglect the terms with A_3, then we get $A_4=0$ and

$$A_1(1-i\omega)z^{-i\omega/4} + A_2(1+i\omega)z^{i\omega/4} = 0, \quad \omega = -i\gamma. \qquad (B22)$$

This gives

$$C_2\frac{\Gamma^2(5/2)}{\prod_{i=1}^{4}\Gamma(a_i+1)} = -C_4\pi^2\cot(a_4), \qquad (B23)$$

and the equation

$$\left(-\frac{\Lambda^2}{p_0^2}\right)^{i\omega/2} = -\frac{1-i\omega}{1+i\omega}\frac{F_1}{F_2}\frac{\cot(\pi a_1)-\cot(\pi a_4)}{\cot(\pi a_2)-\cot(\pi a_4)}. \qquad (B24)$$

From this equation we get when $\omega \ll 1$,

$$p_0^2 = -\Lambda^2\exp\left[-\frac{4\pi n}{\omega}+a\right], \quad a\approx 7.14, \quad n=1,2,\ldots. \qquad (B25)$$

Comparing this result with Eq. (3.24) we see that the approximation that decouples the system, Eq. (3.17), works nicely near the critical coupling λ_c.

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