

Rising time of entanglement between scattering spins

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We investigate the time evolution of entanglement in a process where a mobile particle is scattered by static spins. We show that entanglement increases monotonically during a transient and then saturates to a steady-state value. For a quasimonochromatic mobile particle, the transient time depends only on the group velocity and width of the incoming wave packet and is insensitive to the interaction strength and spin number of the scattering particles. These features do not depend on the interaction model and can be seen in various physical settings.

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Scattering, an almost ubiquitous mechanism in physics and a rather well-studied topic, has very recently gained renewed interest within the community working on quantum mechanics in virtue of the potential that it has in many respects.¹⁻⁴ Scattering processes between two subsystems are effective in order to probe correlation properties of quantum many-body systems.¹ Moreover, under proper conditions, scattering can be exploited to prepare nonclassical states of inaccessible systems.²⁻⁴ Interesting proposals have been put forward for the generation of quantum correlated light-matter states via off-resonant coherent scattering.⁵ Frequently, a stationary approach to scattering processes is chosen, especially for the sake of entanglement generation: the system under scrutiny is observed at long-time scales, when it should have reached steady conditions. This cuts the (often complicated) time evolution from the effective description of scattering dynamics. While this approach is computationally convenient and frequently useful, it is not fully satisfactory since it leaves some interesting questions, related to the details of the dynamical evolution, unanswered.⁶ In particular, it does not give information on the time needed by entanglement to reach its stationary value, which is a pivotal point for the aims of coherent quantum-information processing (QIP). In fact, an estimate of such transient time will help us to anticipate, quantify, and eventually counteract the effects that decoherence might have in a given process.

In this paper, we focus on a prototypical setting involving both mobile and static spin particles²⁻⁴ to shed some light onto these issues. We clearly identify the physical parameters that determine the duration of the scattering process. *Independently of the Hamiltonian chosen in order to model the spin-spin coupling*, we show that when a quasimonochromatic incoming wave packet of the mobile spin is prepared, with a given average momentum k_0 , stationary conditions are reached in a time dictated only by the wave packet width in momentum space Δk . This parameter can thus be used to tune the duration of the scattering event and the rate of entanglement generation in the system. Counter intuitively, the interaction strength of a given spin-spin Hamiltonian model coupling mobile and static particles does not affect the scattering transient time $\Delta\tau$ but only determines the maximum

entanglement attainable in the process. Although, as anticipated, our conclusions do not depend on the details of the model being considered, most of our quantitative results are presented for spin-spin interactions induced by the Heisenberg exchange coupling. This model arises naturally in the spin-spin interaction of magnetic impurities embedded in a one-dimensional (1D) electron nanowire as well as in other situations, including the interaction between a free and a bound electron in a carbon nanotube.² However, in the second part of our work we quantitatively address the results for an anisotropic XYZ model. Tuning the ratio of its parameters, we are able to span a wide range of significant spin-spin Hamiltonians. Remarkably, our results can be extended to an arbitrary number of static particles and intrinsic spin numbers, which makes them valid in a heterogeneous set of physical settings (ranging from spintronics to cavity-quantum electrodynamics).^{2-5,7} In passing, we also reveal an unexpected and general monotonic time behavior of entanglement, observable under easily matched conditions. Although wiggling of the reflected wave function of the mobile particle is observed due to interference at the scattering site(s), entanglement can only grow in time. First, we empirically observe this behavior for relevant coupling models and then give a clear physical explanation for the case of the Heisenberg exchange coupling and one scattering center. The features described above make it clear that a scattering-based mechanism holds the promises for a genuinely control-limited distribution of entanglement in a partially addressable quantum register. In contrast to procedures based on the temporal gating of spin-spin interactions the process addressed in this paper allows for a dramatic relaxation of the time control, which is a major advantage in light of a potential experimental implementation.

The remainder of the paper is organized as follows. In Sec. I we introduce the single-scatterer version of the system addressed in our work and study the entanglement rising time. An important benchmark is set by comparing the results of this study with those corresponding to the scatteringless case of two exchange-coupled static particles. Sec. II extends the analysis to the two-scatterer case. In Sec. III we quantitatively prove the insensitivity of the entanglement ris-

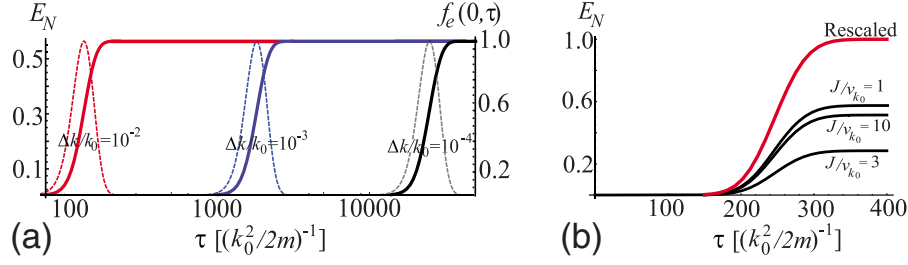


FIG. 1. (Color online) (a) $E_N(\tau)$ (full curves) and *free-propagating* particle $f_e(0, \tau)$ (dashed lines, rescaled to the respective maximum) in a log-in scale for $x_0=5\Delta x$, $J/v_{k_0}=1$ and $\Delta k/k_0=10^{-4}$, 10^{-3} and 10^{-2} . (b) $E_N(\tau)$ for $x_0=5\Delta x$, $\Delta k/k_0=10^{-2}$, and $J/v_{k_0}=1, 3, 10$. The top line shows the same curves rescaled to their maximum. The choice of the parameters used in these plots optimize their visibility (other values are equally valid).

ing time to the spin-spin coupling strength. Finally, in Sec. IV we summarize our findings and remark their most important implications.

I. SCATTERING BY A SINGLE STATIC PARTICLE

In order to provide a significant milestone for our main results, we first review the well-known case of two *static* spin-1/2 particles, e and 1 , interacting via the Heisenberg coupling $\hat{H}=\mathcal{J}\hat{\sigma}\cdot\hat{S}_1$. Here, $\hat{\sigma}$ (\hat{S}_1) is the spin operator of particle e (1) and \mathcal{J} is the coupling rate (we use units such that $\hbar=1$ throughout the paper). We assume the initial state

$$|\chi(0)\rangle = |\uparrow\downarrow\rangle_{e1} = \frac{1}{\sqrt{2}}(|\Psi^+\rangle + |\Psi^-\rangle)_{e1}, \quad (1)$$

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the spin states of each particle and $|\Psi^\pm\rangle_{e1} = (|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle)_{e1}/\sqrt{2}$ are steady states of \hat{H} with energies $\mathcal{J}/4$ and $-3\mathcal{J}/4$, respectively. Upon evolution, the e -1 state reads

$$|\chi(\tau)\rangle = \frac{1}{\sqrt{2}}(e^{-i\mathcal{J}\tau/4}|\Psi^+\rangle + e^{i3\mathcal{J}\tau/4}|\Psi^-\rangle)_{e1}. \quad (2)$$

As a measure of the entanglement between the particles in $\rho_{e1}(\tau) = |\chi(\tau)\rangle\langle\chi(\tau)|$, we use the logarithmic negativity $E_N(\tau)$.⁹ A straightforward calculation gives $E_N(\tau) = \log_2(1 + |\sin \mathcal{J}\tau|)$, which oscillates with characteristic time \mathcal{J}^{-1} . $E_N(\tau)$ is maximized at $\tau = (2q+1)\pi/2\mathcal{J}^{-1}$ ($q \in \mathbb{N}$), which shows that for a set coupling strength \mathcal{J} a fine control on τ is required. This is not the case when the spin-spin interaction takes place during scattering.

To show it, let us address the case of a mobile particle e , interacting with the static spin 1 during scattering. To fix the ideas, we consider a one-dimensional wire along the x axis where spin 1 is embedded at position $x=0$. Spin-spin interaction occurs, via a Heisenberg model, when the mobile spin reaches $x=0$ and is scattered by 1 . Assuming a quadratic dispersion law for e , the Hamiltonian reads

$$\hat{H} = \frac{\hat{p}^2}{2m^*} + J\hat{\sigma}\cdot\hat{S}_1\delta(x), \quad (3)$$

where $\hat{p}(m^*)$ is the momentum operator (effective mass) of e and J is the coupling strength (notice that, unlike \mathcal{J} , J has the dimensions of a frequency times length). The incoming mo-

bile spin e is supposed to have wave vector ηk ($\eta = \pm 1, k \geq 0$) and be prepared in the spin state $|m'_e = \downarrow, \uparrow\rangle_e$ while the static spin 1 is initially in $|m'_1 = \downarrow, \uparrow\rangle_1$. We call $\mu = \{m'_e, m'_1\}$ ($\nu = \{m_e, m_1\}$) the set of spin quantum numbers of mobile and static particles before (after) the scattering event. Correspondingly, $r_{k,\eta,\nu}^\mu$ ($t_{k,\eta,\nu}^\mu$) is the probability amplitude that e is reflected (transmitted) in state $|m_e = \downarrow, \uparrow\rangle_e$ while the spin state of 1 is changed into $|m_1 = \downarrow, \uparrow\rangle_1$. These probability amplitudes depend on J/v_k ($v_k = k/m^*$) and are computed by solving the time-independent Schrödinger equation (SE) associated with Eq. (3) and imposing proper boundary conditions at $x=0$.¹⁰ The steady state of the system $|\Psi_{k,\eta}^\mu\rangle$ has wave function $\Psi_{k,\eta}^\mu(x, \nu) = \langle x, \nu | \Psi_{k,\eta}^\mu \rangle$ of the form

$$\Psi_{k,\eta}^\mu(x, \nu) = e^{i\eta k x} [(\delta_{\nu,\mu} + r_{k,\eta,\nu}^\mu e^{-2i\eta k x})\theta(-\eta x) + t_{k,\eta,\nu}^\mu \theta(\eta x)], \quad (4)$$

where we have omitted a factor $1/\sqrt{2\pi}$ and $\theta(x)$ is the Heaviside step function. As the kinetic energy of e is the only free-energy term in Eq. (3) the system's spectrum is $\varepsilon_k = k^2/(2m^*)$ and thus coincides with that of a free propagating e . In order to investigate the dynamics of entanglement during scattering, we consider e as prepared in a Gaussian wave packet $|\varphi\rangle$ such that

$$\langle x | \varphi \rangle = \varphi(x) = (2\pi\beta)^{-1/4} e^{ik_0(x+x_0)} e^{-(x+x_0)^2/4\beta} \quad (5)$$

with $x_0, k_0 > 0$ the average position and momentum of e . As for the uncertainties, we have $\Delta x = (\Delta k)^{-1} = \sqrt{2\beta}$. The overall initial state is taken as $|\Psi(0)\rangle = |\varphi\rangle |\uparrow, \downarrow\rangle_{e,1}$, whose time-evolved version $|\Psi(\tau)\rangle$ is found by solving the respective time-dependent SE.

In Fig. 1, we study the evolution of spin-spin entanglement between e and 1 when the former is prepared in a quasimonochromatic wave packet. Clearly, in the processes at hand, the entanglement associated with the full quantum state involves both motional and spin degrees of freedom. The study of this hybrid form of entanglement, which is in general a difficult task requiring *ad hoc* tools for its approach, is beyond the scopes of our work. A possible way to tackle it would be the investigation of the nonlocality content of the state of particles e and 1 and involving both internal and external degrees of freedom along the lines as in Ref. 11. In Fig. 1(a) we set $J/v_{k_0}=1$, $x_0=5\Delta x$, and plot $E_N(\tau)$ for increasing values of $\Delta k/k_0$ up to 10^{-2} . The differences with respect to two static spins are striking. First, there is no os-

cillatory behavior. At small τ , no entanglement is found as far as e has not yet reached 1. At large times, e is far from spin 1 and $E_N(\tau)$ takes a steady value. At intermediate times, the entanglement shows a monotonic increase before eventually saturating to a steady value, which does not depend on Δk . Second, unlike the case of static spins, the characteristic time $\Delta\tau$ of the entanglement evolution is now *independent* of the spin-spin coupling strength J . To illustrate this, in Fig. 1(b) we set Δk and study $E_N(\tau)$ for various J/v_{k_0} 's. Remarkably, although the steady value of $E_N(\tau)$ depends on J/v_{k_0} , the rising time $\Delta\tau$ does not, as it is revealed by the topmost curve in Fig. 1(b). There, for an assigned value of J/v_{k_0} , we have rescaled each $E_N(\tau)$ to the respective stationary value and found that the curves are identical. This marks a profound difference with the static-particle case: the introduction of motional degrees of freedom in our problem of interacting spins does not result in a mere spoiling effect of the system spin coherences, as it might be naively expected, but deeply affects the dynamics of the particles involved in the process. To identify the parameters upon which $\Delta\tau$ depends in the dynamical-scattering process, we first observe that in Fig. 1(a) an increase in Δk of one order of magnitude makes $\Delta\tau$ ten times smaller, suggesting an inverse proportionality between these quantities. Then, in Fig. 1(a) (dashed lines) we plot the probability density $f_e(x=0, \tau)$ of finding particle e at $x=0$ for $J=0$, i.e., the *free-propagation* case. It can be clearly seen that the time needed by $E_N(\tau)$ to reach its steady value coincides with the time during which e is found at $x=0$ with non-negligible probability in the absence of any scattering. Thus, by using the free-particle time-energy uncertainty principle¹³ we obtain $\Delta\tau \sim 1/(v_{k_0} \Delta k)$, which explains¹³ the aforementioned inverse proportionality between $\Delta\tau$ and Δk for a given k_0 . In other words, in the scattering case and for a quasimonochromatic mobile particle the characteristic time over which entanglement changes is decided *only* by the kinematic parameters specifying the incoming wave packet $\varphi(x)$. As for a given J/v_{k_0} the steady value of $E_N(\tau)$ is insensitive to Δk [cf. Figure 1(a)] we conclude that the rising time $\Delta\tau$ can be tuned simply by adjusting Δk with no effect on the stationary value of entanglement (for a set J/v_{k_0}). This result is key to the study of the feasibility of scattering-based QIP protocols²⁻⁴ and the quest for effective ways to relax the control on a system. We have checked that these features hold even when particle 1 has spin number $s > 1/2$. We point out that although Fig. 1 might at first glance suggest that entanglement rises linearly with time in general this is not the case. Rather, our findings show quite clearly that such functional behavior is dictated by the shape of the incoming wave packet of particle e (here assumed to be Gaussian).

II. SCATTERING BY TWO STATIC PARTICLES

We now consider the situation where the mobile spin e is scattered by two static spin-1/2 particles, 1 and 2, placed at $x=0$ and $x=d$, respectively. The Hamiltonian is the same as in Eq. (3) with the inclusion of the term $J\hat{\sigma} \cdot \hat{S}_2 \delta(x-d)$. Each steady state $|\Psi_{k,\eta}^\mu\rangle$ ($\mu = \{m'_e, m'_1, m'_2\}$) has energy $\varepsilon_k = k^2/(2m^*)$ and differs from Eq. (4) for the replacement

$\theta(\eta x) \leftrightarrow \theta[\eta x - (\eta+1)d/2]$ and the addition of $(A_{k,\eta,v}^\mu e^{i\eta kx} + B_{k,\eta,v}^\mu e^{-i\eta kx})[\theta(x) - \theta(x-d)]$. The coefficients $\gamma_{k,\eta,v}^\mu$ ($\gamma = A, B, r, t$) depend implicitly on J/v_k and kd . As shown in Refs. 3 and 4, once boundary conditions at $x=0, d$ are imposed, the steady states can be determined. This configuration has recently been proposed as a potential way to set entanglement between remote spins²⁻⁴ via mediation of e and only mild time control: it is enough to wait for a time $\tau \gg \Delta\tau$ so that a steady state is reached. Clearly, the quantification of $\Delta\tau$ in the case of an incoming wave packet (instead of a plane wave as in Refs. 2-4) is key to estimate the influence of dissipation and decoherence affecting the scattering particles. This would be a pivotal point in view of potential experimental implementations. We take the initial state $|\Psi(0)\rangle = |\varphi, \uparrow\rangle_e |\downarrow, \downarrow\rangle_{12}$, whose time evolution $|\Psi(\tau)\rangle$ is computed through the time-dependent SE. We focus on the dynamics of entanglement $E_N[\rho_{12}(\tau)]$ between static particles 1 and 2. Here

$$\rho_{12}(\tau) = \sum_{m_e} \int dx \langle x, m_e | \Psi(\tau) \rangle \langle \Psi(\tau) | x, m_e \rangle \quad (6)$$

is the 1-2 state obtained by tracing over the spatial and spin degrees of freedom of e . We call

$$p_e(\Omega, \tau) = \int_0^d dx f_e(x, \tau) \quad (7)$$

the probability to find e within the interaction region $\Omega = \{x: 0 \leq x \leq d\}$ at time τ .

In Fig. 2(a) we set $x_0 = 5\Delta x$, $J = v_{k_0}$, $k_0 d = \pi$, and study the behavior of $E_N[\rho_{12}(\tau)]$ and *free-propagating* particle $p_e(\Omega, \tau)$ for increasing values of $\Delta k/k_0$ up to 10^{-2} . On the other hand, Fig. 2(b) considers the effect of different choices of J/v_{k_0} over the entanglement dynamics, for set values of $\Delta k/k_0$ and $k_0 d$. Features similar to those revealed for a single-scattering center are found [see Figs. 1(a) and 1(b)]. Entanglement increases with time to reach a steady value that depends on J/v_{k_0} and $k_0 d$. As in the single-scatterer case, $\Delta\tau$ is determined only by the time needed by the free propagating $\varphi(x)$ to cross Ω . In Fig. 2 our focusing on the regime $k_0 d = \pi$ is due to the fact that such setting allows for efficient entanglement distribution schemes.^{3,4} However, we have checked that other values lead to similar conclusions.

In reaching our conclusions, we have performed a few technical steps that we address here for pedagogical purposes. In each of the studied configurations, the time evolution of $|\Psi(0)\rangle = |\varphi\rangle |\bar{\mu}\rangle$ (with $|\bar{\mu}\rangle$ the total initial spin state¹⁴) can be expanded over the basis of steady states $\{|\Psi_{k,\eta}^\mu\rangle\}$ with coefficients given by $\langle \Psi_{k,\eta}^\mu | \Psi(0) \rangle$. Provided that $x_0 > 3\Delta x$, i.e., at $\tau=0$ spin e is out of the domain $x \geq 0$, these scalar products are well approximated by taking $\theta(-x) \approx 1$ and neglecting contributions from other θ functions in Eq. (4), so that

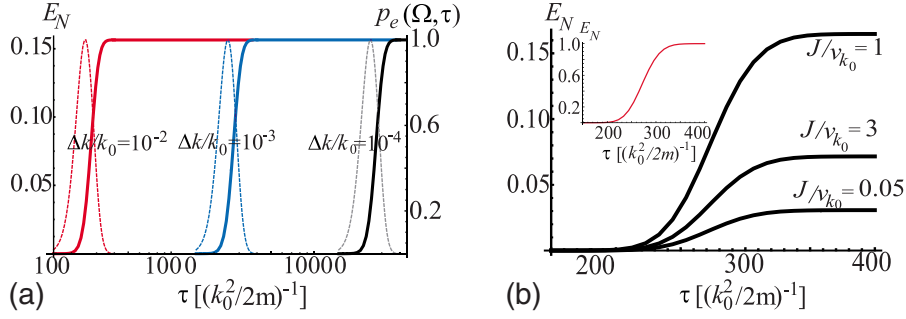


FIG. 2. (Color online) (a) $E_N[\rho_{12}(\tau)]$ for $\Delta k/k_0 = 10^{-2}$, 10^{-3} and 10^{-4} for $J/v_{k_0} = 1$ and $k_0 d = \pi$ (solid lines) and free-propagating particle $p_e(\Omega, \tau)$ (dashed lines, rescaled to their maximum). (b) $E_N[\rho_{12}(t)]$ for $\Delta k/k_0 = 10^{-2}$, $k_0 d = \pi$, $J/v_{k_0} = 0.5, 1$, and 3 . The choice of the coupling strengths used in these plots optimize their visibility (other values are equally valid).

$$\langle \Psi_{k,\eta}^\mu | \Psi(0) \rangle = \delta_{\eta,+} \delta_{\bar{\mu},\mu} \tilde{\varphi}(k) + [\delta_{\eta,+} r_{k,\eta,\bar{\mu}}^{\mu*} + \delta_{\eta,-} t_{k,\eta,\bar{\mu}}^{\mu*}] \tilde{\varphi}(-k) \quad (8)$$

with $\tilde{\varphi}(k')$ the Fourier transform of $\varphi(x)$ (an analogous expression holds in the many-scatterer case). Equation (8) allows us to evaluate

$$\langle x, \mu | \Psi(\tau) \rangle = \sum_{\mu,\eta} \int_0^\infty dk \langle \Psi_{k,\eta}^\mu | \Psi(0) \rangle e^{-i\varepsilon_k \tau} \Psi_{k,\eta}^\mu(x, \mu), \quad (9)$$

which can be solved analytically via a power-series expansion of each spin-dependent amplitude $\gamma_{k,\eta,\nu}^\mu$ ($\gamma = A, B, r, t$) around the carrier wave vector k_0 . The replacement of the n th order expansion of these coefficients in $\langle x, \mu | \Psi(t) \rangle$ results in integrals of the form $\int_0^\infty d\tilde{k} e^{-a\tilde{k}^2 - b\tilde{k}} \tilde{k}^m$ ($m = 0, \dots, n$, $\tilde{k} = k/k_0$, and $\text{Re } a > 0$), which can be computed in terms of exponential and error functions. The behavior of $f_e(x, \tau)$ is then found by tracing the overall state $\rho(\tau) = |\Psi(\tau)\rangle\langle\Psi(\tau)|$ over the spin degrees of freedom. As expected,¹² we find that during scattering $f_e(x, \tau)$ wiggles at $x \leq 0$ due to interference between incoming and reflected waves. The spin state is found by tracing $\rho(\tau)$ over the spatial and, in the many-scatterer case, spin degrees of freedom of e .¹⁵

III. MODEL INDEPENDENCE OF ENTANGLEMENT RISING TIME

We are now in a position to explain the insensitivity of $\Delta\tau$ to the values of J . Key to this task is the observation that as long as the wave packet is narrow enough around k_0 the term proportional to $\tilde{\varphi}(-k)$ (with $k \geq 0$) in Eq. (8) can be neglected. Thus $\langle \Psi_{k,\eta}^\mu | \Psi(0) \rangle \approx \delta_{\mu,+} \delta_{\bar{\mu},\mu} \tilde{\varphi}(k)$ meaning that, within the limits of our study, the spectral decomposition of $|\Psi(0)\rangle$ is the one corresponding to $J=0$. On the other hand, the spectrum ε_k does not depend on J [see discussion after Eq. (4)] so that the energy and time uncertainties¹³ $\Delta\varepsilon$ and $\Delta\tau$, are the same as in the free-particle case, *i.e.*, $\Delta\tau = 1/\Delta\varepsilon \sim 1/(v_{k_0} \Delta k)$. These features hold for a quasimonochromatic incoming wave packet. Most importantly, it is clear that our proof does not rely on the specific form of the interaction Hamiltonian, the number of scattering particles as well as their intrinsic spin numbers. While all these parameters affect the shape of $r_{k,\eta,\bar{\mu}}^\mu$'s and $t_{k,\eta,\bar{\mu}}^\mu$'s, we have just shown that, for quasimonochromatic wave packets, they cannot influence the entanglement rising time.

To further illustrate the discussed insensitivity of $\Delta\tau$, in Fig. 3(a) we set $\Delta k/k_0 = 10^{-2}$ and analyze the entanglement between 1 and 2 when these are coupled to e with strengths $J_{e1} \neq J_{e2}$. The case of spin-1 scattering centers and equal couplings is also reported. In Fig. 3(b) we set the same $\Delta k/k_0$ as in Fig. 3(a) and address the XYZ spin-spin model

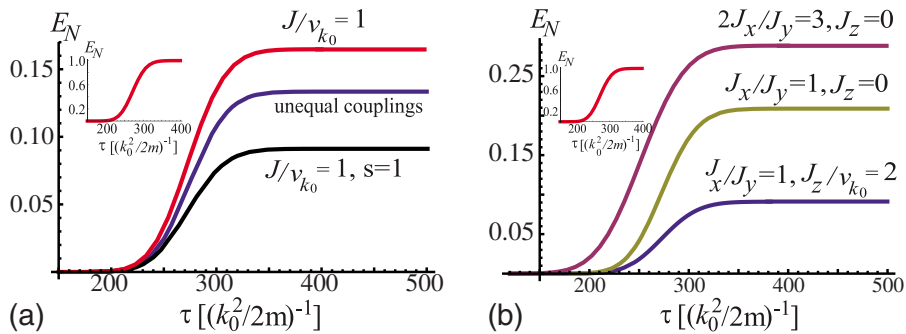


FIG. 3. (Color online) (a) Two-static-spin case, Heisenberg model. From top to bottom: $E_N[\rho_{12}(t)]$ for $J/v_{k_0} = 1$ (equal couplings, spin-1/2), unequal couplings $J_{e1} = 2J_{e2} = 2.6v_{k_0}$ (spin-1/2) and $J/v_{k_0} = 1$ for spin-1 scattering centers (equal couplings). (b) Two-static-spin case, spin-1/2, XYZ model. From top to bottom: $J_x = J_y/2 = 3v_{k_0}$ with $J_z = 0$ (anisotropic XY), $J_x = J_y = v_{k_0}$ with $J_z = 0$ (isotropic XY) and $J_x = J_y = J_z/2 = v_{k_0}$ (XYZ). Insets: curves rescaled to their maximum. We have taken $\Delta k/k_0 = 10^{-2}$ and $k_0 d = \pi$.

$$\hat{H}_{XYZ} = \sum_{l=x,y,z} J_l [\hat{\sigma}_l \hat{S}_{1,l} \delta(x) + \hat{\sigma}_l \hat{S}_{2,l} \delta(x-d)]. \quad (10)$$

We study the cases of $J_x=J_y=J_z/2=v_{k_0}$ (embodying an XXZ model), $J_x=J_y=v_{k_0}$ with $J_z=0$ (isotropic XY) and $J_x=J_y/2=3v_{k_0}$ with $J_z=0$ (anisotropic XY). The ratios of the parameters are chosen so as to provide the best visibility of each plot. The insets in Figs. 3(a) and 3(b) report all the curves rescaled to their maximum value, showing that the entanglement rising time is not affected by the specific quantum spin number or the interaction model, which only affect the stationary value of entanglement. In particular, the applicability of our results to the XY model is remarkable since an effective XY model is indeed found⁴ considering the dispersive interaction of a single photon traveling across a 1D waveguide (a GaAs/GaN nanowire, for instance) and static atom-like systems (such as InAs/GaN quantum dots or nitrogen-vacancy centers in diamond⁷). The (typical) case of linear dispersion law for a photon crossing a waveguide matches the requirements of our study, i.e., a narrow-bandwidth mobile particle. These considerations demonstrate the broad applicability of our investigation and results, which cover a wide range of experimental situations, from spintronics to quantum optics.^{2-5,7} In the second scenario, in particular, the proven ability to experimentally engineer the temporal shape of photonic wave packets⁸ makes scattering-based techniques analogous to the one discussed here and in Refs. 2-4 quite advantageous against strategies using time-dependent “modulation” or “pulsing” of spin-spin interactions. In the former case, once the kinematic properties of the mobile-particle wave packets are set, there is no necessity for time control of the entanglement evolution. We remark that, in order for the steady-state entanglement not to depend on Δk [as in Figs. 1 and 2], this has to be smaller than the inverse of the characteristic length associated with γ amplitudes. In particular, for two static spins, it must be $\Delta k \leq 1/d$.

For the Heisenberg model, the monotonic rise of entanglement¹⁶ can be interpreted as due to the progressive construction of a phase difference between spin components of the overall wave function. Here, we discuss the steps required in order to prove this feature. For the sake of argument, we focus on the case of Eq. (3) and we assume that the incoming wave packet $\varphi(x)$ is quasimonochromatic. The Hamiltonian \hat{H} in Eq. (3) commutes with \hat{S}^2 and \hat{S}_z , where $\hat{S} = \hat{\sigma} + \hat{S}_1$ is the total spin of e and 1. It follows that for the initial spin state $|\Psi(0)\rangle = |\varphi\rangle |\uparrow, \downarrow\rangle_{e,1}$ [see also Eq. (1)]

$$\langle x | \Psi(\tau) \rangle = \frac{1}{\sqrt{2}} \sum_{\sigma=\pm} \varphi_{\sigma}(x, \tau) |\Psi^{\sigma}\rangle_{e,1}, \quad (11)$$

where $\varphi_{\pm}(x, \tau)$ are evolved wave packets fulfilling the condition $\varphi_{\sigma}(x, 0) = \varphi(x)$ and $|\Psi^{\pm}\rangle_{e,1} = (|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle)_{e,1} / \sqrt{2}$. Upon trace over the spatial variable of e we get $E_N(\tau) = \log_2(1 + |\text{Im } \Sigma(\tau)|)$ with $\Sigma(\tau)$ the spatial overlap between $\varphi_{-}(x, \tau)$ and $\varphi_{+}(x, \tau)$. Now, for $|\Psi^{\pm}\rangle_{e,1}$ the spin-spin interaction in Eq. (3) reduces to an effective static potential $\Gamma_{\pm} \delta(x)$ with $\Gamma_{+} = J/4$ and $\Gamma_{-} = -3J/4$ (Ref. 10) with associated reflection and transmission probability amplitudes

$$r_k^{\sigma} = t_k^{\sigma} - 1 = - \frac{1}{\sqrt{1 + (v_k/\Gamma_{\sigma})^2}} e^{i \arccot(\Gamma_{\sigma}/v_k)}. \quad (12)$$

The steady states $|\Psi_{k,\eta}^{\pm}\rangle$ are similar to those in Eq. (4) with associated energies $\varepsilon_k = k^2/(2m^*)$. Assuming $x_0 \geq 3\Delta x$, the projections $\langle \Psi_{k,\eta}^{\pm} | \varphi^{\pm} \rangle$ of $\varphi^{\pm}(x)$ onto the stationary states are well approximated by a form analogous to Eq. (4) (with due replacements). Provided that the width of the incoming wave packet around k_0 is narrow enough, we find $\langle \Psi_{k,\eta}^{\pm} | \varphi^{\pm} \rangle \approx \delta_{\eta,\pm} \tilde{\varphi}(k)$. Moreover, $r_k^{\pm} \approx r_{k_0}^{\pm} = r^{\pm} [t_k^{\pm} \approx t_{k_0}^{\pm} = t^{\pm}]$. Thus, by calling Δ_r (Δ_t) the phase difference between r^+ and r^- (t^+ and t^-), we get

$$\begin{aligned} \Sigma(\tau) \approx & \int_{-\infty}^0 dx [|\varphi_{f,R}(x, \tau)|^2 + |r^+ r^-| e^{i\Delta_t} |\varphi_{f,L}(x, \tau)|^2] \\ & + \int_0^{\infty} dx |t^+ t^-| e^{i\Delta_t} |\varphi_{f,R}(x, \tau)|^2, \end{aligned} \quad (13)$$

where $\varphi_{f,p}(x, \tau)$ ($p=R,L$) are evolved free wave packets with average positions $x_p = s_p x_0$ at $\tau=0$ and Fourier transforms $\tilde{\varphi}(-s_p k')$ with $s_L = -s_R = 1$. In our derivation we have neglected terms proportional to $\int_0^{\infty} dx \varphi_{f,R}^*(x, \tau) \varphi_{f,L}(x, \tau)$ due to the assumption of dealing with an incoming wave packet narrow enough around k_0 . As $\varphi_{f,L}(x, \tau)$ and $\varphi_{f,R}(x, \tau)$ are left- and right-propagating wave packets, respectively, both $\int_{-\infty}^0 dx |\varphi_{f,L}(x, \tau)|^2$ and $\int_0^{\infty} dx |\varphi_{f,R}(x, \tau)|^2$ are increasing functions of time whose shapes are similar to those in Fig. 2(b) for a Gaussian wave packet $\varphi(x)$. Hence, the behavior of $|\text{Im } \Sigma(\tau)|$ depends on $\sin \Delta_{r(t)}$. As $\Delta_{r(t)} \in [0, \pi/2]$ (regardless of J and k_0) the sine functions are always positive. Therefore, $|\text{Im } \Sigma(\tau)|$ is an increasing function of time, proving the monotonic character of $E_N(\tau)$. Physically, it is now clear that the buildup of entanglement relies on the phase difference acquired by the singlet and triplet components of the incoming wave packet once e is scattered off. Dynamically, as soon as the scattering process starts and the reflected and transmitted waves are progressively generated, the singlet and triplet components start to build up the mentioned phase difference and entanglement grows. For J/v_{k_0} up to ≈ 10 and $\Delta k/k_0$ up to 10^{-2} , the largest relative increments in r_k^{\pm} and t_k^{\pm} in the range $[k_0 - 3\Delta k, k_0 + 3\Delta k]$ are on the order of $\approx 10\%$, confirming the validity of our arguments.

The study of wave packets having $\Delta x \rightarrow 0$, namely the regime opposed to quasi-monochromaticity, is under ongoing investigations.

IV. CONCLUSIONS

Our work reveals that in a scattering process involving one mobile and many static spins, a situation that can be engineered in various physical settings^{2-5,7} the time required for entanglement to reach its steady value can be *tuned* by preparing the mobile spin's wave packet, regardless of the specific spin-spin interaction model. If a wave packet narrow

enough in frequency is prepared, the strength of the interaction between mobile and static spins determines just the stationary value of the entanglement which while scattering takes place can only grow in time. Our findings show that the degree of control on the class of systems we have addressed can be significantly reduced down to the off-line engineering of a proper mobile spin's wave packet. The quantification of the scattering transient time enables the analysis of noise effects, such as phase-damping affecting the static spins, on protocols for entanglement distribution via scattering.²⁻⁴

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- ¹³We point out that in this work we make use of the *heuristic* time-energy uncertainty principle (Ref. 12) rather than the rigorous inequality by Mandelshtam and Tamm [L. I. Mandelshtam and I. E. Tamm, *J. Phys. (USSR)* **9**, 249 (1945)]. Our goal here is indeed to merely estimate the characteristic *time scale* $\Delta\tau$ over which entanglement builds up. A more accurate quantification of such a time is under ongoing investigation.
- ¹⁴We assume that $|\bar{\mu}\rangle$ belongs to the uncoupled-spin basis $\{|\mu\rangle\}$. The generalization to arbitrary (pure or mixed) initial spin states is straightforward.
- ¹⁵We numerically trace over the spatial variable using a mesh of the region where $f_c(x, \tau)$ is non negligible and checking the stability of the outcomes vs the number of mesh points.
- ¹⁶The monotonic rise of entanglement observed in Figs. 1–3 clearly depends on the initial spin state (if the static spins are already entangled, a decrease may take place).