

**Persistent currents in nanorings and quantum decoherence by Coulomb interaction**Andrew G. Semenov<sup>1</sup> and Andrei D. Zaikin<sup>2,1</sup><sup>1</sup>*I.E. Tamm Department of Theoretical Physics, P.N. Lebedev Physics Institute, 119991 Moscow, Russia*<sup>2</sup>*Forschungszentrum Karlsruhe, Institut für Nanotechnologie, 76021 Karlsruhe, Germany*

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Employing instanton technique we evaluate equilibrium persistent current (PC) produced by a quantum particle moving in a periodic potential on a ring and interacting with a dissipative environment formed by diffusive electron gas. The model allows for detailed nonperturbative analysis of interaction effects and, depending on the system parameters, yields a rich structure of different regimes. We demonstrate that at low temperatures PC is exponentially suppressed at sufficiently large ring perimeters  $2\pi R > L_\varphi$ , where the dephasing length  $L_\varphi$  is set by interactions and does not depend on temperature. This behavior represents a clear example of quantum decoherence by electron-electron interactions at  $T \rightarrow 0$ .

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**I. INTRODUCTION**

Electron decoherence is one of the key ingredients of the many-body ground state in the presence of disorder and electron-electron interactions. The existing nonperturbative theory of this phenomenon at low temperatures in realistic disordered conductors<sup>1-3</sup> is rather complicated to a large extent because of the necessity to properly account for Fermi statistics for interacting electrons. At the same time the main physical reason for electron dephasing appears obvious already without unnecessary technical details: it is the electron interaction with the fluctuating quantum electromagnetic field produced by other electrons moving in a disordered potential.

In order to be able to quantitatively describe and understand the latter effect Guinea<sup>4</sup> suggested a model which mimics all essential features of the “real” problem of interacting electrons in a disordered conductor except for the Pauli exclusion principle. This model describes a quantum particle moving on a ring with radius  $R$  and interacting with quantum dissipative environment. For a system in thermodynamic equilibrium quantum decoherence manifests itself as effective suppression of off-diagonal density-matrix elements beyond a certain length  $L_\varphi$ . Provided there exists nonzero electron dephasing due to its interaction with quantum environment at  $T \rightarrow 0$ , this dephasing length  $L_\varphi$  should stay finite down to zero temperature. Hence, all effects sensitive to quantum coherence, such as, e.g., persistent currents (PC) and Aharonov-Bohm (AB) oscillations in mesoscopic rings, should be suppressed by interactions as soon as the ring perimeter  $2\pi R$  exceeds  $L_\varphi$ .

A great deal of information can be obtained by modeling the environment by a bath of Caldeira-Leggett (CL) oscillators. In this case it was demonstrated<sup>5</sup> that PC is reduced by interactions *in the ground state* implying suppression of quantum coherence exactly at  $T=0$ . For the same CL environment Guinea<sup>4</sup> found that AB oscillations for a quantum particle on a ring are suppressed by the factor  $\sim \exp[-(R/L_\varphi)^2]$ , where the length  $L_\varphi$  is set by interactions and remains finite down to  $T=0$ . A similar result was also obtained earlier from the real-time analysis.<sup>6</sup> Furthermore, the problem<sup>4,6</sup> is exactly equivalent to that of Coulomb

blockade in the so-called single-electron box where exponential reduction in the effective charging energy at large conductances<sup>7,8</sup> is presently considered as a well-established result. Thus, it is now widely accepted that PC for a quantum particle on a ring is exponentially reduced at large ring perimeters down to  $T=0$  due to strong dephasing produced by interaction

between the particle and the CL bath.

It appears that presently no such consensus exists for another important model of the environment<sup>4</sup> formed by a diffusive electron gas. Renormalization-group arguments developed for this model<sup>4</sup> suggest very weak power law  $\sim R^{-\chi}$  suppression of AB oscillations at  $T \rightarrow 0$ , where the factor  $\chi \ll 1$  is set by interactions. On the contrary, the combination of semiclassics, instanton technique, and quantum Monte Carlo (MC) analysis<sup>9</sup> yields much stronger suppression of quantum coherence, namely, exponential suppression  $\sim \exp(-R/L_\varphi)$  (with temperature independent  $L_\varphi$ ) at not too low  $T$  and power-law suppression  $\sim R^{-\chi}$  with  $\chi \approx 1.8$  at  $T \rightarrow 0$  and for  $2\pi R \gtrsim L_\varphi$ .

More recently this problem was reconsidered by means of variational approach,<sup>10</sup> perturbation theory,<sup>11</sup> and MC simulations.<sup>12</sup> In contrast to Ref. 9, either no<sup>10,11</sup> or very weak<sup>12</sup>  $R$ -dependent suppression of PC was found. Note, however, that the variational calculation<sup>10</sup> reduces the PC problem to that of mass renormalization in the  $m=0$  topological sector while, as we show here, PC is determined by other topological sectors ( $m \neq 0$ ) that are distinct from the  $m=0$  sector, unlike the variational result in Ref. 10. Perturbative in the interaction calculations<sup>11</sup> can also miss the correct behavior of PC at not too small  $R$  as it was already demonstrated in Ref. 9 for the problem under consideration and was also discussed elsewhere in a broader context.<sup>1,2</sup> More arguments along the same lines will be presented below in this paper.

As far as numerical MC results are concerned, the authors<sup>12</sup> ascribed the difference between their conclusions and those of the previous work<sup>9</sup> to insufficient Trotter number values employed in the MC analysis.<sup>9</sup> While this particular issue definitely requires further analysis, it is worth pointing out that the MC data<sup>12</sup> cover only the perturbative regime  $R < L_\varphi$  where rather weak suppression of PC was found in Ref. 9 as well, cf. Fig. 3 in that paper. Hence, MC results<sup>12</sup>

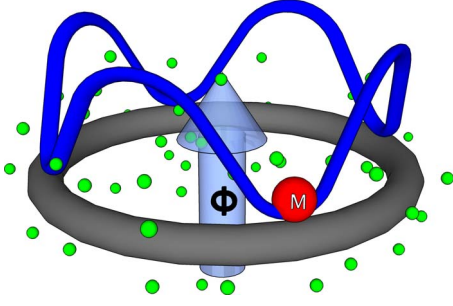


FIG. 1. (Color online) The system under consideration: a particle on a ring in the presence of a periodic potential. The ring is pierced by the magnetic flux and the particle interacts with an effective environment formed by a dirty electron gas.

for small  $R$  do not appear conclusive for the most interesting nonperturbative regime  $R \gg L_\varphi$  where strong (though weaker than exponential) interaction-induced suppression of PC was predicted<sup>9</sup> down to  $T \rightarrow 0$ .

Leaving detailed discussion of MC procedure and results to the future, here we note that—despite significant theoretical activity in the field—currently no well-controlled nonperturbative approach is available which would allow to analytically study PC for the “particle+dirty electron-gas” model<sup>4</sup> in the low-temperature limit and at sufficiently large values of  $R$ . Given both fundamental importance of the problem and remaining controversies in the literature, it is highly desirable to formulate such an approach which would unambiguously resolve the whole issue. A step in this direction is undertaken in the present work.

The structure of our paper is as follows. In Sec. II we define our model which is essentially identical to that pioneered by Guinea<sup>4</sup> except we additionally introduce a periodic potential for the particle on a ring. In Sec. III we formulate our main formalism and evaluate persistent current in the ring without interactions by means of the standard instanton technique. This approach is conveniently generalized to the interacting regime in Sec. IV where we analyze suppression of both quantum coherence and PC by Coulomb interaction at sufficiently low temperatures down to  $T \rightarrow 0$  and arbitrarily large ring perimeters. Discussion of our main observations and conclusions is presented in Sec. V. Some technical details of our calculation are specified in Appendices A and B.

## II. MODEL AND EFFECTIVE ACTION

We will consider a quantum particle with mass  $M$  and electric charge  $e$  on a ring with radius  $R$  threaded by external magnetic flux  $\Phi_x$ , see Fig. 1. As before,<sup>4,9</sup> it will be convenient to describe the particle position by a vector  $R = (R \cos \theta, R \sin \theta)$  and consider the angle  $\theta$  as a quantum variable. In contrast to Refs. 4 and 9–12 where the quantum particle on a ring was described only by its kinetic energy (i.e., no potential energy was included into consideration), in this paper we will assume that the particle moves in a periodic potential which—just for the sake of definiteness—is chosen in the form  $U(\theta) = U_0[1 - \cos(\kappa\theta)]$ . Here  $\kappa$  is the total

number of periods of the potential  $U(\theta)$  which the particle should pass before it makes one full circle on the ring. Accordingly, a noninteracting particle on a ring is described by the Hamiltonian

$$\hat{H}_0 = \frac{E_C(\hat{\Phi} + \Phi_x)^2}{\Phi_0^2} + U_0[1 - \cos(\kappa\theta)], \quad (1)$$

where  $\hat{\Phi} = -i\Phi_0 \partial / \partial \theta$  is the magnetic flux operator,  $E_C = 1/(2MR^2)$ , and  $\Phi_0 = 2\pi c/e$  is the flux quantum (here and below we set the Planck’s constant equal to unity  $\hbar = 1$ ).

Now let us include the interaction between the particle on a ring and an effective dissipative environment. Specifically, we will assume that the ring is embedded in the environment formed by the so-called “dirty electron gas.”<sup>4</sup> The total Hamiltonian for our system reads

$$\hat{H} = \hat{H}_0 + \hat{H}_{el} + \hat{H}_{int}, \quad (2)$$

where  $\hat{H}_{el}$  is the standard Hamiltonian for electrons in a disordered conductor and  $\hat{H}_{int}$  describes interaction between the particle and the electronic environment. Fluctuating electrons in this environment produce stochastic electromagnetic field  $V$  described by the equilibrium correlator

$$\langle VV \rangle = T \sum_{\omega_n} \int \frac{d^3k}{(2\pi)^3} \frac{4\pi}{k^2 \epsilon(i|\omega_n|, k)} e^{-i\omega(\tau-\tau') + ikX}, \quad (3)$$

where  $\omega_n = 2\pi nT$  is the Matsubara frequency,  $\epsilon(\omega, k)$  is the dielectric susceptibility of the environment, and  $X = R(\tau) - R(\tau')$ . Similarly to Refs. 4 and 9 we will model the environment by three-dimensional diffusive electron gas with

$$\frac{1}{\epsilon(\omega, k)} \approx \frac{-i\omega + Dk^2}{4\pi\sigma}, \quad (4)$$

where  $\sigma$  is the Drude conductivity of this gas,  $D = v_F l / 3$  is the electron diffusion coefficient, and  $l$  is the electron elastic mean-free path. Interaction between the particle on a ring and fluctuating electrons in the environment is described by the standard Coulomb term

$$\hat{H}_{int} = e\hat{V}. \quad (5)$$

In what follows we will assume that the whole system remains in thermodynamic equilibrium at a temperature  $T$ . Our first and standard step is to integrate out all environmental degrees of freedom effectively described by the collective variable  $V$ . In the limit of weakly disordered environment  $k_F l \gg 1$  to be analyzed below fluctuations of the field  $V$  can be considered Gaussian. In this case integration over this field is carried out exactly.<sup>4,9</sup> After that one arrives at the grand partition function of the system expressed as a single path integral over the angle variable  $\theta(\tau)$

$$\begin{aligned} \mathcal{Z} &\equiv \sum_{m=-\infty}^{\infty} e^{2\pi i m \phi_x} \mathcal{Z}_m \\ &= \sum_{m=-\infty}^{\infty} \int_0^{2\pi} d\theta_0 \int_0^{2\pi m} \mathcal{D}\theta \exp(2\pi i m \phi_x - S[\theta]). \end{aligned} \quad (6)$$

Here we defined  $\beta=1/T$ ,  $\phi_x=\Phi_x/\Phi_0$ , and the winding-number-projected partition functions  $\mathcal{Z}_m$ . The first term in the exponent in Eq. (6) takes care of the magnetic flux while the second term  $S[\theta]$  describes the effective action for our interacting particle on a ring. This action consists of two terms,

$$S[\theta] = S_0[\theta] + S_{\text{int}}[\theta]. \quad (7)$$

The term

$$S_0[\theta] = \int_0^\beta d\tau \left\{ \frac{1}{4E_C} \left( \frac{\partial\theta}{\partial\tau} \right)^2 + U_0 [1 - \cos(\kappa\theta)] \right\} \quad (8)$$

defines the action for a particle in the absence of the environment. This action is identical to one for the Josephson junction<sup>13</sup> where  $E_C$  plays the role of the charging energy. The term  $S_{\text{int}}$  describes the effect of interaction between the particle and the environment. For our model it has the form<sup>4,9</sup>

$$S_{\text{int}}[\theta] = \alpha \int_0^\beta d\tau \int_0^\beta d\tau' \frac{\pi^2 T^2 K[\theta(\tau) - \theta(\tau')]}{\sin^2[\pi T(\tau - \tau')]}, \quad (9)$$

$$K(z) = 1 - \frac{1}{\sqrt{4r^2 \sin^2(z/2) + 1}}, \quad (10)$$

where the constant  $\alpha=3/(8k_F l^2)$  effectively controls the interaction strength in our model and  $r=R/l$ . Note that the integral in Eq. (9) is understood as a principal value. The formal divergence at  $\tau=\tau'$  is regularized by requiring  $K(0)=0$  which explains the origin of the first term in Eq. (10). For the sake of physical consistency of our model below we will set  $1/k_F \ll l \ll 2\pi R/\kappa$ , where the first inequality just means that interaction should remain weak,  $\alpha \ll 1$ , while the second one implies that the distance between the neighboring potential minima should be much larger than  $l$ . Accordingly, the parameter  $r=R/l$  obeys the inequality

$$r \gg \kappa/2\pi. \quad (11)$$

Note that it can also be convenient to rewrite the function  $K(z)$  in terms of the Fourier series

$$K(z) = \sum_n a_n \sin^2 \left[ \frac{nz}{2} \right], \quad (12)$$

where the Fourier coefficients are  $a_n \sim (2/\pi r) \ln(r/n)$  for  $1 \leq n \leq r$  and  $a_n \approx 0$  otherwise.

### III. PERSISTENT CURRENT IN THE ABSENCE OF DISSIPATION

Let us first evaluate persistent current in our system in the absence of interactions, i.e., we set the interaction constant

equal to zero  $\alpha=0$  everywhere in this section. In what follows we will mainly be interested in sufficiently large values of the ring radius  $R$ . Hence, without loss of generality one can consider the limit  $E_C \ll U_0$ . Below we will perform our calculation for a somewhat more stringent condition

$$\kappa^2 E_C \ll U_0 \quad (13)$$

which simplifies our analysis but is by no means important for any of our key conclusions.

In the tight binding limit, Eq. (13), and at sufficiently low temperatures the particle is located at the bottom of one of the potential wells (see Fig. 1), i.e., in the vicinity of the points  $\theta=2\pi p/\kappa$ , where  $0 < p \leq \kappa$  is an integer number. Accordingly, in Eq. (6) one should substitute

$$\int_0^{2\pi} d\theta_0 \rightarrow \sum_{p=1}^{\kappa} \int d\theta_0 \delta(\theta_0 - 2\pi p/\kappa). \quad (14)$$

The particle can move around the ring only by hopping between the neighboring minima  $\theta=2\pi p/\kappa$  and  $\theta=2\pi(p \pm 1)/\kappa$  of the periodic potential  $U(\theta)$ . Each of these tunneling events is described by the well-known instanton (kink) trajectory

$$\tilde{\theta}(\tau) = \frac{4}{\kappa} \arctan[\exp(\omega\tau)] \quad (15)$$

and corresponds to the tunneling rate  $\Delta/2$ , where

$$\Delta = 8 \left( \frac{\kappa U_0 E_C}{\pi} \right)^{1/2} \left( \frac{2U_0}{E_C} \right)^{1/4} \exp \left( -\frac{4}{\kappa} \sqrt{\frac{2U_0}{E_C}} \right) \quad (16)$$

and  $\omega = \kappa \sqrt{2U_0 E_C}$ . In order to evaluate the grand partition function

$$\mathcal{Z} \sim \sum_{p=1}^{\kappa} \langle p | e^{-\beta \hat{H}_0} | p \rangle \quad (17)$$

it is necessary to sum over all possible tunneling events of the particle between all potential minima to all orders in  $\Delta$ . The minimum number of such hops should be equal to  $m\kappa$  for any trajectory corresponding to the winding number  $m$ . Taking into account that effective duration of each tunneling event is  $\sim \omega^{-1}$  and that the total imaginary time span equals to  $\beta \equiv 1/T$  we can distinguish two different limits. In the limit  $m\kappa\omega^{-1} \ll \beta$  the average distance between instantons is large as compared to their typical size, i.e., in this case we are dealing with dilute instanton gas. In the opposite limit  $m\kappa\omega^{-1} \gg \beta$  instantons are very close to each other and essentially merge forming a single trajectory. Below we will also demonstrate that in the above conditions it suffices to set  $m=1$ .

#### A. Dilute instanton gas

We begin with the low-temperature limit  $T \ll \omega/\kappa$ . To proceed let us evaluate a somewhat more general than in Eq. (17) matrix element  $\langle p_1 | e^{-\beta \hat{H}_0} | p_2 \rangle$ . For this purpose we consider multi-instanton trajectories

$$\Theta(\tau) = \frac{2\pi p_1}{\kappa} + \sum_j v_j \tilde{\theta}(\tau - \tau_j), \quad (18)$$

where  $v_j = \pm 1$  and  $\tau_j$  are, respectively, the topological charges and collective coordinates of instantons and  $\tilde{\theta}(\tau)$  is defined in Eq. (15). The trajectory, Eq. (18), describes tunneling of the particle between the states  $|p_1\rangle$  and  $|p_2\rangle$  after  $m$  winds around the ring provided we fix

$$\sum_j v_j = n_2 - n_1 = p_2 - p_1 + m\kappa, \quad (19)$$

i.e., we consider configurations containing totally  $n_1 + n_2$  instantons corresponding to  $n_1$  hops clockwise and  $n_2$  hops counterclockwise. Taking into account all possible tunneling events restricted by the condition (19) and summing over all winding numbers  $m$ , we obtain

$$\begin{aligned} \langle p_1 | e^{-\beta \hat{H}_0} | p_2 \rangle &= \left( \frac{\omega}{\pi} \right)^{1/2} \sum_{m=-\infty}^{\infty} e^{2\pi i m \phi_x - \beta \omega / 2} \\ &\times \sum_{n_1, n_2=0}^{\infty} \frac{(\beta \Delta / 2)^{n_1 + n_2}}{n_1! n_2!} \delta_{n_1 - n_2, p_1 - p_2 - m\kappa}. \end{aligned} \quad (20)$$

Making use of the integral representation for the Kronecker symbol

$$\delta_{j,k} = \int_0^{2\pi} \frac{dy}{2\pi} e^{iy(j-k)}, \quad (21)$$

after performing a summation over  $m$  with the aid of Poisson's resummation formula

$$\sum_{m=-\infty}^{\infty} e^{2\pi i m x} = \sum_{k=-\infty}^{\infty} \delta(x - k) \quad (22)$$

we obtain

$$\begin{aligned} \langle p_1 | e^{-\beta \hat{H}_0} | p_2 \rangle &= \left( \frac{\omega}{\pi \kappa^2} \right)^{1/2} e^{-\beta \omega / 2} \\ &\times \sum_{j=1}^{\kappa} e^{i2\pi(j+\phi_x)(p_1-p_2)/\kappa + \beta \Delta \cos[2\pi(j+\phi_x)/\kappa]}. \end{aligned} \quad (23)$$

This formula allows to easily recover the low-lying energy levels of our problem which contain all necessary information in order to evaluate PC. For instance, the ground-state energy of the particle  $E_0(\phi_x)$  is obtained in a standard way by taking the limit  $T = 1/\beta \rightarrow 0$  in Eq. (23) which yields

$$E_0(\phi_x) = \frac{\omega}{2} - \Delta \cos\left(\frac{2\pi\phi_x}{\kappa}\right) \quad (24)$$

for  $-1/2 < \phi_x < 1/2$ . Equation (24) should be continued periodically outside this interval. This expression determines the periodic dependence (with period equal to the flux quantum  $\Phi_0$ ) of the ground-state energy on the magnetic flux  $\Phi_x$ . In the limit  $\kappa \gg 1$  this dependence reduces to a set of parabolas

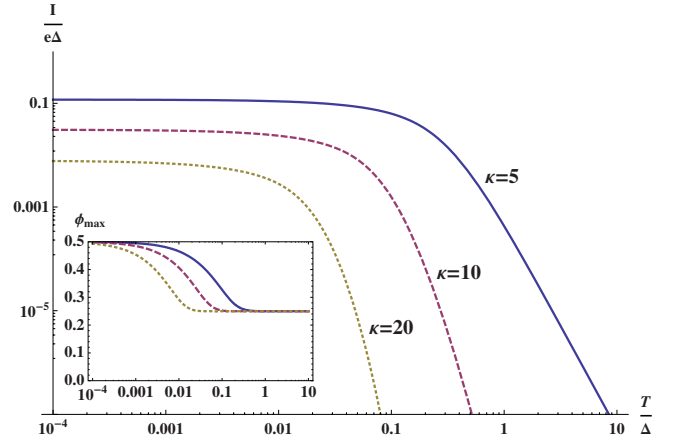


FIG. 2. (Color online) Maximum value of PC as a function of temperature for different values of  $\kappa$ . Inset: The magnetic-flux value  $\phi_{\max}$  at which PC reaches its maximum value as a function of  $T/\Delta$ .

$$E_0(\phi_x) - \frac{\omega}{2} \simeq \frac{2\pi^2 \Delta}{\kappa^2} \min_n (\phi_x - n)^2. \quad (25)$$

Turning back to the grand partition function, Eq. (17), from Eq. (23) we find

$$\mathcal{Z} \sim \sum_{j=1}^{\kappa} e^{\beta \Delta \cos[2\pi(j+\phi_x)/\kappa]}. \quad (26)$$

PC can now be easily obtained from the general formula

$$I = - \frac{eT}{2\pi} \frac{\partial}{\partial \phi_x} \ln \mathcal{Z}, \quad (27)$$

which yields diamagnetic current

$$I = \frac{e\Delta}{\kappa} \frac{\sum_{j=1}^{\kappa} \sin\left[\frac{2\pi(j+\phi_x)}{\kappa}\right] e^{\beta \Delta \cos[2\pi(j+\phi_x)/\kappa]}}{\sum_{j=1}^{\kappa} e^{\beta \Delta \cos[2\pi(j+\phi_x)/\kappa]}}. \quad (28)$$

This expression fully determines PC in the ring at temperatures  $T \ll \omega/\kappa$  and in the absence of interactions with dissipative environment. The dependence of the maximum PC value on temperature is depicted in Fig. 2.

Equation (28) is further simplified at temperatures above and below the interlevel distance  $\sim \Delta/\kappa^2$ . In the limit  $T \gg \Delta/\kappa^2$  the leading contribution to the partition function is defined by configurations with minimal number of instantons. Hence, in the sum over winding numbers in Eq. (6) it is sufficient to keep only the terms with  $m=0, \pm 1$  terms. For the term with  $m=0$  it is necessary to sum over all configurations, whereas for the case  $m=\pm 1$  only configurations with  $\kappa$  instantons contribute. After some algebra we get

$$\mathcal{Z}_0 = \kappa e^{-\beta \omega / 2} I_0(\beta \Delta), \quad (29)$$

where  $I_0(x)$  is the modified Bessel function of imaginary argument. For  $m=\pm 1$  we find

$$\mathcal{Z}_{\pm 1} = \frac{(\beta\Delta)^\kappa}{2^\kappa \kappa!} e^{-\beta\omega/2}. \quad (30)$$

As a result we obtain

$$I = I_{C0}(T) \sin(2\pi\phi_x), \quad (31)$$

where

$$I_{C0}(T) = \frac{e\Delta^\kappa}{(2T)^{\kappa-1} \kappa! I_0(\Delta/T)}. \quad (32)$$

At low temperatures  $T \ll \Delta/\kappa^2$  Eq. (28) reduces to a simple formula

$$I = \frac{e\Delta}{\kappa} \sin\left(\frac{2\pi\phi_x}{\kappa}\right), \quad -1/2 < \phi_x \leq 1/2, \quad (33)$$

which also trivially follows from Eq. (24). This formula demonstrates that at  $T=0$  the magnitude of PC in our system is proportional to  $\Delta$  while its flux dependence deviates from the simple sinusoidal form for all  $\kappa > 1$ . In particular, in the limit  $\kappa \gg 1$  this dependence approaches a sawtooth one

$$I = I_{C0} \min_n(\phi_x - n), \quad I_{C0} = 2\pi e\Delta/\kappa^2. \quad (34)$$

Comparing the expressions for PC derived above with those for a free particle on a ring<sup>9-12</sup> we observe that the main physical difference between these two models is the presence of two distinct energy scales,  $\omega$  and  $\Delta$ , in our case whereas only one energy scale  $E_C$  remains in the limit  $U_0=0$ . Otherwise significant features of the effect are essentially the same in both models. Indeed, at high temperatures ( $T > \Delta$  here and  $T > E_C$  for  $U_0=0$ ) the dependence of PC on the magnetic flux  $\Phi_x$  is sinusoidal with the period  $\Phi_0$  and its amplitude decreases with increasing  $T$ , cf. our Eq. (32) with Eq. (11) in Ref. 9. In the low-temperature limit ( $T \ll \Delta/\kappa^2$  here and  $T \ll E_C$  for  $U_0=0$ ) the dependence  $I(\phi_x)$  strongly deviates from sinusoidal (except for a special case  $\kappa=1$ ). In the limit  $\kappa \gg 1$  the sawtooth dependence, Eq. (34), is identical to that for the case  $U_0=0$  where one has  $I_{C0} \sim E_C$ .

Finally, let us compare the dependence of the PC amplitude  $I_{C0}$  on the ring radius  $R$  obtained in these two cases. Provided the parameter  $\kappa$  is fixed and does not change with  $R$ , Eqs. (16) and (34) yield exponential decay of  $I_{C0}$  with increasing  $R$  since  $E_C \propto 1/R^2$ . This exponential decay is due to the fact that for larger  $R$  the potential profile changes in a way that the particle should tunnel at a longer distance  $2\pi R/\kappa$  between the two neighboring states  $|p\rangle$  and  $|p \pm 1\rangle$ . Alternatively, one can keep the distance between the adjacent potential minima  $\theta = 2\pi p/\kappa$  and  $\theta = 2\pi(p \pm 1)/\kappa$  unchanged while increasing  $R$ . This is achieved by varying the parameter  $\kappa$  with  $R$  as  $\kappa \propto R$ . Under this condition the tunneling rate  $\Delta$ , Eq. (16), becomes independent of  $R$  and the magnitude of PC  $I_{C0}$ , Eq. (34), decreases with increasing  $R$  as  $I_{C0} \propto 1/R^2$  exactly as in the case  $U_0=0$ . Note, however, that due to the restriction, Eq. (13), this regime can apply only at not too large values of  $R$ .

### B. Merged instantons

Now let us turn to the case of higher temperatures  $\omega/\kappa \ll T \ll \omega$  which can be realized in the limit  $\kappa \gg 1$ . In this

case the leading contribution to the partition function originates from one multi-instanton trajectory. This trajectory of merged instantons  $\Theta^{(m)}(\tau)$  can easily be evaluated due to the presence of the integral of motion which is the classical energy  $\mathcal{E}_m$  corresponding to the winding number  $m$ . This energy is fixed by the periodic boundary condition

$$\frac{1}{T} = \sqrt{\frac{MR^2}{2}} \int_0^{2\pi|m|} \frac{d\theta}{\sqrt{\mathcal{E}_m + U_0[1 - \cos(\kappa\theta)]}} \quad (35)$$

and the trajectory  $\Theta^{(m)}(\tau)$  is obtained from the equation

$$\tau - \tau_0 = \sqrt{\frac{MR^2}{2}} \int_0^{\Theta^{(m)}} \frac{d\theta}{\sqrt{\mathcal{E}_m + U_0[1 - \cos(\kappa\theta)]}}. \quad (36)$$

Making use of the standard quasiclassical technique one arrives at the following expression for the partition function

$$\mathcal{Z} = \sum_{m=-\infty}^{\infty} \left[ -\frac{\partial^2 S(\mathcal{E}_m)}{\partial \mathcal{E}_m^2} \right]^{-1/2} \frac{e^{2\pi i m \phi_x - |m| S(\mathcal{E}_m) + \mathcal{E}_m/T}}{T \sqrt{2\pi|m|^3}}, \quad (37)$$

where

$$S(\mathcal{E}) = \frac{1}{\sqrt{E_C}} \int_0^{2\pi} \sqrt{\mathcal{E} + U_0[1 - \cos(\kappa\theta)]} d\theta \quad (38)$$

is the classical action. As before, the leading contribution to this partition functions comes from the lowest winding numbers  $m=0, \pm 1$ . For  $m=0$  one recovers the oscillatorlike expression

$$\mathcal{Z}_0 = \frac{\kappa}{2 \sinh \frac{\omega}{2T}} \approx \kappa e^{-\omega/(2T)}, \quad (39)$$

whereas for  $m = \pm 1$  the instanton trajectory approaches the straight line in which case the integrals can be easily evaluated and yield

$$\mathcal{Z}_{\pm 1} = \kappa \sqrt{\frac{2\pi U_0 T}{\omega^2}} e^{-2\pi^2 U_0 T \kappa^2 / \omega^2}. \quad (40)$$

With the aid of these results one again arrives at the expression for PC in the form (31) with

$$I_{C0}(T) = 2eT \sqrt{\frac{2\pi U_0 T}{\omega^2}} \exp\left(\frac{\omega}{2T} - \frac{2\pi^2 U_0 T \kappa^2}{\omega^2}\right). \quad (41)$$

Different regimes considered in this section can also be summarized graphically by means of the diagram depicted in Fig. 3.

## IV. EFFECT OF ELECTRON-ELECTRON INTERACTIONS

### A. Renormalization of the tunneling amplitude

Now let us turn on interactions and analyze the effect of fluctuations in a dissipative environment. To this end we again employ the above instanton technique. Evaluating the path integral in Eq. (6) in the limit, Eq. (13), and for  $T \ll \omega/\kappa$  we follow the same scheme and substitute the trajectory, Eq. (18), describing quantum tunneling of the par-

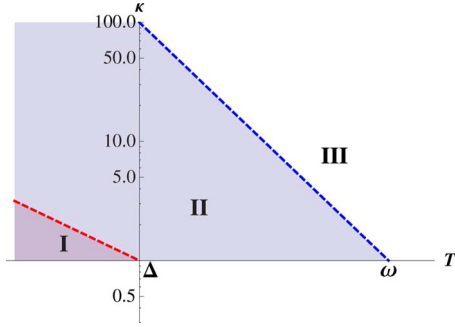


FIG. 3. (Color online) Different regimes for PC in the noninteracting case. The regions I (lowest  $T$ ) and II (higher  $T$ ) are described within the dilute instanton gas approximation [Eqs. (31)–(33), respectively], whereas the high temperature region III corresponds to the case of merged instantons, Eqs. (31) and (41). The dashed lines  $T \approx \Delta/\kappa^2$  and  $T \approx \omega/\kappa$  indicate the crossover between these regimes.

ticle between different potential minima into the full effective action, Eq. (7). As before, let us fix the winding number equal to  $m$ . Then we again arrive at configurations of totally  $k$  instantons restricted by the condition (19) where we now also set  $p_1 = p_2 = p$ . Evaluating the interaction term  $S_{\text{int}}$ , Eqs. (9) and (10), on multi-instanton trajectories, Eq. (18), after some algebra (see Appendix A) we obtain

$$S_{\text{int}}[\Theta(\tau)] = 4\pi\alpha r k / \kappa - 2\alpha \sum_{a,b=1, a < b}^k v_a v_b g(\varphi_{ab}) \ln \left\{ \frac{\sin[\pi T(\tau_b - \tau_a)]}{\pi T \omega^{-1}} \right\}, \quad (42)$$

where

$$g(\varphi) = K(\varphi + 2\pi/\kappa) + K(\varphi - 2\pi/\kappa) - 2K(\varphi) \quad (43)$$

and

$$\varphi_{ab} = \frac{2\pi}{\kappa} \sum_{j=a}^b v_j - \frac{\pi}{\kappa} (v_b + v_a). \quad (44)$$

We observe that the interaction term, Eq. (42), consists of two different contributions. One of them,  $4\pi\alpha r k / \kappa$ , describes interaction-induced suppression of quantum tunneling of the angle variable  $\theta$  between different potential minima. This term yields effective  $r$ -dependent renormalization of the tunneling amplitude

$$\Delta \rightarrow \Delta_r = \Delta e^{-4\pi\alpha r / \kappa}. \quad (45)$$

The remaining contribution in Eq. (42) describes logarithmic interaction between different instantons which occurs for  $\kappa \geq 2$  due to the presence of a dissipative environment. This logarithmic interaction is absent for  $\kappa = 1$  in which case  $g(\varphi) \equiv 0$ .

At not too low temperatures  $T \gg \Delta_r / \kappa^2$  interinstanton interactions just provide further renormalization of the tunneling amplitude  $\Delta_r \rightarrow \Delta_r [1 + 2\alpha K(2\pi/\kappa) \ln(2\pi T / \omega)]$ . One can also write down the renormalization group (RG) equation

$$\frac{d\Delta_r}{d \ln \omega} = 2\alpha K(2\pi/\kappa) \Delta_r, \quad (46)$$

which yields both  $r$ - and  $T$ -dependent renormalized tunneling amplitude of the form

$$\Delta_R = \Delta_r \left( \frac{T}{\omega} \right)^{2\alpha K(2\pi/\kappa)}. \quad (47)$$

At even lower temperatures  $T \ll \Delta_r / \kappa^2$  interactions again yield renormalization of the tunneling amplitude which now becomes  $\Delta_r \rightarrow \Delta_r [1 + 2\alpha K(2\pi/\kappa) \ln \frac{2\Delta_r e^\gamma}{\omega}]$ . The corresponding RG Eq. (46) remains the same but relevant energy scale now becomes

$$\Delta_R = \Delta_r \left( \frac{\Delta_r}{\omega} \right)^{\frac{2\alpha K(2\pi/\kappa)}{1 - 2\alpha K(2\pi/\kappa)}}. \quad (48)$$

Equation (48)—together with Eq. (45)—defines the renormalized tunneling amplitude in the limit  $T \rightarrow 0$  and for  $\kappa \geq 2$ . In the particular case  $\kappa = 2$  our results reduce to those established for the so-called spin-boson model with Ohmic dissipation.<sup>14,15</sup>

We also note that Eq. (48) is formally applicable for  $\alpha < 1/[2K(2\pi/\kappa)]$ , while for larger values of the interaction strength  $\alpha$  and at  $T = 0$  the tunneling amplitude is renormalized to zero,  $\Delta_R = 0$ . This is the consequence of the quantum dissipative phase transition which—similarly to other models<sup>13–15</sup>—occurs at the critical interaction strength  $\alpha_c = 1/[2K(2\pi/\kappa)] \approx 1/2$  and implies localization of a quantum particle in one of potential wells at any  $\alpha$  exceeding the critical value  $\alpha_c$ . Accordingly, no PC can flow at  $T = 0$  and  $\alpha \geq 1/[2K(2\pi/\kappa)]$ . This formal conclusion, however, does not appear to be of substantial physical significance, since the applicability range of our model is restricted to small values of  $\alpha \ll 1$ .

Finally, we should point out that, employing a regular perturbation theory in  $\alpha$ , already in the first order one recovers additional terms which cannot be captured within the RG Eq. (46). The corresponding analysis is presented in Appendix B.

## B. Suppression of PC by interactions

It turns out that the behavior of PC may be quite different depending both on temperature and on the parameter  $\kappa$ . Therefore, it is appropriate to distinguish several different cases.

### 1. One potential minimum $\kappa = 1$

As we already discussed, in the case  $\kappa = 1$  logarithmic interaction between instantons is absent and the only effect of interactions is  $r$ -dependent renormalization of the tunneling amplitude, Eq. (45). Accordingly, for PC in this case we obtain

$$I = e\Delta e^{-4\pi\alpha r} \sin(2\pi\phi_x). \quad (49)$$

This result is valid at all temperatures in the range  $T \ll \omega$ . It demonstrates that for  $\kappa = 1$  Coulomb interaction yields expo-

ponential suppression of PC down to  $T=0$  provided the ring perimeter  $2\pi R$  exceeds an effective dephasing length

$$L_\varphi \sim l/\alpha, \quad (50)$$

which is set by the effective interaction strength  $\alpha$  and does not depend on temperature. Note that exactly the same length scale was found in the absence of the periodic potential  $U_0=0$  in Ref. 9.

### 2. Not too low temperatures and $\kappa \geq 2$

In the case  $\kappa \geq 2$  instantons interact logarithmically and—in addition to Eq. (45)—at not too low temperatures the tunneling amplitude is renormalized according to Eq. (47). Combining these two equations and substituting the renormalized tunneling amplitude  $\Delta_R$  into Eq. (32) instead of  $\Delta$  we obtain

$$I = I_C(T) \sin(2\pi\phi_x), \quad (51)$$

where

$$I_C(T) = \begin{cases} I_{C0}(T) \left(\frac{T}{\omega}\right)^{2\alpha\kappa K(2\pi/\kappa)} e^{-4\pi\alpha r}, & T \ll \omega/\kappa, \\ I_{C0}(T) e^{-4\pi\alpha r}, & T \gg \omega/\kappa \end{cases} \quad (52)$$

and  $I_{C0}(T)$  in the corresponding limits is defined, respectively, in Eqs. (32) and (41). In this case we again recover the same exponential suppression of PC at ring perimeters  $2\pi R \geq L_\varphi$  with *temperature-independent* dephasing length defined in Eq. (50). At the same time, the pre-exponent in the expression for  $I_C$  depends on temperature as a power law  $I_C(T) \propto T^{-\mu}$  with  $\mu = \kappa[1 - 2\alpha K(2\pi/\kappa)] - 1$ . For  $\kappa \geq 2$  and small values of the interaction strength  $\alpha \ll 1$  we have  $\mu > 0$ , i.e.,  $I_C(T)$  grows with decreasing temperature. This growth, though somewhat weaker than in the noninteracting case, Eq. (32) (since  $\mu < \kappa - 1$ ), implies that Eqs. (51) and (52) can be trusted only at  $T \geq \Delta_R/\kappa^2$  whereas at even lower temperatures one expects a crossover to a different regime to be discussed below. We also note that qualitatively similar behavior of PC at not too low temperatures follows from the numerical analysis<sup>9</sup> of the model with  $U_0=0$ .

### 3. Zero temperature limit and $\kappa \geq 2$

In order to evaluate PC in the limit  $T \ll \Delta_r/\kappa^2$  we will make use of Eq. (B14) for the free energy. Combining this expression with Eq. (27) and observing that the difference  $\Delta_R - \Delta_r \sim \alpha \ll 1$  in the last term in Eq. (B14) can be safely neglected within the accuracy of our calculation (since it only produces extra terms  $\sim \alpha^2$ ), we obtain

$$I = \frac{e\Delta_R}{\kappa} \left[ \sin\left(\frac{2\pi\phi_x}{\kappa}\right) - \alpha \sum_{n=-r}^r a_n \sin\left(\frac{\pi n}{\kappa}\right) \cos\left(\frac{2\pi\phi_x + \pi n}{\kappa}\right) \right] \times \ln \left| \sin\left(\frac{\pi n}{\kappa}\right) \sin\left(\frac{2\pi\phi_x + \pi n}{\kappa}\right) \right|, \quad (53)$$

where  $\Delta_R$  is defined in Eqs. (45) and (48). This result allows to make the following observations.

First, taking into account the dependence of  $\Delta_R$  on  $r$  we conclude that exactly at  $T=0$  and at sufficiently large ring

perimeters PC is exponentially suppressed as

$$I \propto \exp\left\{-\frac{4\pi\alpha r}{\kappa[1 - 2\alpha K(2\pi/\kappa)]}\right\}, \quad (54)$$

i.e., in this case for any fixed value  $\kappa$  we can define an effective zero-temperature dephasing length

$$\tilde{L}_\varphi = L_\varphi \kappa [1 - 2\alpha K(2\pi/\kappa)]. \quad (55)$$

This result demonstrates that in the limit of weak interactions  $\alpha \ll 1$  the effective length, Eq. (55), turns out to be approximately  $\kappa$  times longer than  $L_\varphi$ , i.e.,  $\tilde{L}_\varphi \approx \kappa l/\alpha$ . Should, however, we assume that  $\kappa \propto r$ , no finite dephasing length could be defined from Eq. (54), although also in this case even for small  $\alpha$  PC can suffer exponentially strong suppression by electron-electron interactions due to the condition (11).

Second, the result, Eq. (53), demonstrates that at  $T \rightarrow 0$  the effect of electron-electron interactions on PC does not just reduce to renormalization of the tunneling amplitude  $\Delta \rightarrow \Delta_R$ . We observe that Eq. (53) also contains additional terms evaluated here within the first-order perturbation theory in  $\alpha$ . This  $r$ -dependent first-order contribution turns out to be singular at values of  $\phi_x$  close to  $\pm 1/2$  and at all other half-integer numbers which indicates insufficiency of the first-order perturbation theory, at least for such values of  $\phi_x$ . Higher-order terms of the perturbation theory in the interaction may contain similar (or even stronger) singularities and, on top of that, may grow with increasing  $r$ . Hence, at  $T \rightarrow 0$  no perturbation theory in  $\alpha$  can in general be trusted, in particular, at sufficiently large  $r$ . A more detailed analysis of this issue is beyond the scope of the present paper. Here we only conjecture that such analysis might yield an additional dependence of PC on the ring radius  $r$  not accounted for in the expression for  $\Delta_R$ . It is quite likely that such kind of  $r$  dependence of PC at  $T=0$  was also observed within a numerical treatment<sup>9</sup> in the case  $U_0=0$ . Additional support for this conjecture is provided by the exact solution presented below.

### 4. Toulouse limit

As we already pointed out, for  $\kappa=2$  our problem is exactly mapped onto the well known spin-boson model with Ohmic dissipation.<sup>14,15</sup> In this case interaction between instantons is given by Eq. (A9) and the grand partition function reads (see also Appendices A and B)

$$Z[z; \beta] = \sum_{n=0}^{\infty} (\Delta_r \cos z)^{2n} \int_0^\beta d\tau_1 \int_{\tau_1}^\beta d\tau_2 \dots \int_{\tau_{2n-1}}^\beta d\tau_{2n} \times \exp\left(4\alpha \sum_{j < k=1}^{2n} (-1)^{j+k} \ln \left[ \frac{\sin[\pi T(\tau_k - \tau_j)]}{\pi T \omega^{-1}} \right] \right). \quad (56)$$

In order to nonperturbatively evaluate PC at all values of  $r$  and at all temperatures including  $T=0$  we can profit from the exact solution known for the particular value of the interaction strength  $\alpha=1/4$ , the so-called Toulouse limit.

Introducing the parameter  $\frac{1}{2\pi\xi(z)} = \frac{\Delta_r^2 \cos^2 z}{T\omega}$  one can conveniently write down the exact expression for the partition function, Eq. (56), with  $\alpha=1/4$  in the form

$$Z[z; \beta] = \exp \left[ \int_{\pi T/\omega}^{\infty} dx \frac{1 - e^{-x/2\pi\xi(z)}}{x \sinh x} \right]. \quad (57)$$

This result allows to immediately establish PC which reads

$$I = \frac{e\Delta_r^2}{2\omega} \sin(2\pi\phi_x) \int_{\pi T/\omega}^{\infty} dx \frac{e^{-\frac{x\Delta_r^2}{\omega T} \cos^2(\pi\phi_x)}}{\sinh x}. \quad (58)$$

At not too low temperatures  $\omega \gg T \gg \Delta_r$  we obtain

$$I = \frac{e\Delta_r^2}{2\omega} e^{-\pi r} \ln \frac{2\omega}{\pi T} \sin(2\pi\phi_x). \quad (59)$$

In low-temperature limit  $T \rightarrow 0$  one finds

$$I = \frac{e\Delta_r^2}{2\omega} \sin(2\pi\phi_x) \int_{\pi}^{\infty} dy \frac{e^{-\frac{y\Delta_r^2}{\omega^2} \cos^2(\pi\phi_x)}}{y}. \quad (60)$$

Evaluating the integral in Eq. (60) we arrive at the final result

$$I = \frac{e\Delta_r^2}{2\omega} e^{-\pi r} \left\{ \pi r + \ln \left[ \frac{\omega^2 e^{-\gamma}}{\pi \Delta_r^2 \cos^2(\pi\phi_x)} \right] \right\} \sin(2\pi\phi_x). \quad (61)$$

We observe that both at nonzero temperatures and exactly at  $T=0$  the above exact expressions for PC demonstrate its exponential suppression at ring perimeters exceeding  $L_\varphi = \tilde{L}_\varphi$  [these two length scales coincide for  $\kappa=2$  and  $\alpha=1/4$ , see Eq. (55)]. In addition, the result, Eq. (61), demonstrates that at  $T=0$  (a) the dependence of PC on  $r$  deviates from purely exponential (which is in agreement with our above conjecture) and (b) the dependence of PC on  $\phi_x$  deviates from purely sinusoidal and contains the logarithmic singularity at half-integer values of  $\phi_x$ , cf. also Eq. (53).

## V. DISCUSSION

In this paper we proposed a model which allows for detailed nonperturbative treatment of the effect of electron-electron interactions on PC in normal nanorings at low temperatures. Our investigation employs a well-controlled instanton technique and yields a rich structure of different regimes. The main features observed within our analysis can be summarized as follows: (i) Coulomb interaction yields  $R$ -dependent renormalization of the tunneling amplitude, Eq. (45), which, in turn, results in exponential suppression of PC at large enough  $R$ , (ii) logarithmic interaction between instantons yields additional renormalization of the tunneling amplitude described by the RG Eq. (46), and (iii) electron-electron interactions generate yet additional contributions not captured by Eqs. (45) and (46), see Eq. (53) and Appendix B. These contributions may become particularly important at  $T \rightarrow 0$  indicating the failure of the naive perturbation theory in the interaction at sufficiently large  $R$ .

Although the effect (iii) still requires additional nonperturbative analysis, already (i) and (ii) result in exponential suppression of PC at any temperature including  $T \rightarrow 0$  for any given  $\kappa$  and at ring perimeters exceeding the dephasing length set by interactions and defined in Eqs. (50) and (55). Note that the length scale identical to Eq. (50) also follows from the earlier nonperturbative analysis<sup>9</sup> developed for the case  $U_0=0$ . Thus, similarly to Ref. 9 the decoherence effect in our model is controlled by the parameter

$$\alpha r \sim \alpha \sum_{n=1}^r n a_n$$

rather than by  $\alpha$  or  $\alpha \ln r$  as it was sometimes suggested in the literature in the case  $U_0=0$ .

It is worthwhile to stress that exponential dependence of PC on  $R$  of the form  $I \propto \exp(-AR)$  by itself does not yet necessarily imply decoherence. For instance, even in the absence of interactions PC  $I \propto \Delta$ , Eq. (33), can decrease exponentially with increasing  $R$  provided the parameter  $\kappa$  is fixed to be independent of the ring radius. Obviously, quantum coherence is not destroyed in this case. Exponential reduction in PC with increasing  $R$  at  $T \rightarrow 0$  can also occur in superconducting nanorings due to proliferation of quantum phase slips.<sup>16,17</sup> Also in that case the dependence  $I \propto \exp(-AR)$  can be interpreted just as a nontrivial coordinate-dependent renormalization effect.<sup>17</sup>

An important qualitative difference between nanorings with dissipation considered in Sec. IV and the two last examples is that in our problem dissipation explicitly *violates time-reversal symmetry* (thus causing genuine decoherence of a quantum particle), while no such symmetry is violated in superconducting nanorings<sup>16,17</sup> or in the absence of dissipation (Sec. III). Hence, quantum coherence remains fully preserved in the last two cases despite exponential suppression of PC at large  $R$ .

One can discriminate between decoherence and pure renormalization in a number of ways. For instance, one can drive the system out of equilibrium and investigate its relaxation by means of a real-time analysis. This approach was employed in Refs. 9 and 18 where a finite dephasing time  $\tau_\varphi = L_\varphi/v$  was found at  $T=0$  with  $L_\varphi$  defined in Eq. (50) and  $v$  being the particle velocity. This observation allows to unambiguously identify quantum decoherence.

Another way<sup>5</sup> amounts to analyzing fluctuations of PC in the ground state of an interacting system. For instance, one can study the correlator  $\langle (\hat{I} - \langle \hat{I} \rangle)^2 \rangle$ , where  $\hat{I}$  is the current operator which expectation value  $\langle \hat{I} \rangle$  defines PC in the ground state. Within the model<sup>5</sup> it was demonstrated that, while the average PC in the ground state decreases with increasing interaction strength, its fluctuations *increase*, thus implying genuine decoherence rather than pure renormalization. A similar situation occurs within the model studied here.

Following<sup>4</sup> here we intentionally disregarded Fermi statistics by suppressing electron exchanges between the ring and the environment. The question arises if inclusion of the Pauli principle into the model could alter our main conclusion about nonvanishing electron decoherence at zero tempera-



ture. Golubev and one of the present authors addressed this issue by developing two entirely different nonperturbative in the interaction techniques.<sup>1,2</sup> Both these methods yield the same conclusion: Although Fermi statistics is crucially important for other properties of a dirty interacting electron gas, it practically *does not affect* interaction-induced quantum decoherence at  $T \rightarrow 0$ . Quantitative agreement was demonstrated<sup>9</sup> between the corresponding results derived for the models with<sup>1</sup> and without<sup>9</sup> the Pauli principle.

Despite all these developments (including recently obtained *exact* solution of the problem<sup>2</sup>) it is sometimes argued in the literature that the Pauli principle can preclude from electron decoherence at  $T \rightarrow 0$ . E.g., this conclusion was reached on the basis of the first-order perturbation theory in the interaction.<sup>19</sup> Insufficiency of such kind of perturbation theory for the problem in question was repeatedly demonstrated elsewhere<sup>1,2,9,20</sup> and was also observed here within the model considered. More recently, von Delft and coauthors<sup>21</sup> reiterated the same incorrect conclusion.<sup>19</sup> Unfortunately the analysis<sup>21</sup> is fundamentally flawed since it violates causality effectively implying that dynamics of interacting electrons would be affected by photons coming both from the past and from the future. Hence, the results<sup>21</sup> cannot be considered seriously. For more details on this issue we refer the reader to the paper.<sup>22</sup>

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#### APPENDIX A: INSTANTON GAS IN THE PRESENCE OF INTERACTIONS

Let us analyze the effect of electron-electron interactions on the dilute instanton gas employed in our work. Combining Eqs. (9) and (12) we can rewrite the dissipative part of the action describing such interactions within our model in the following form

$$S_{\text{int}}[\theta] = -\frac{\alpha}{2} \sum_n a_n \int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau' \times \frac{\pi^2 T^2 [1 - e^{in\theta(\tau)}][1 - e^{-in\theta(\tau')}]}{\sin^2[\pi T(\tau - \tau')]} \quad (\text{A1})$$

Substituting multi-instanton trajectories  $\Theta(\tau)$ , Eq. (18), into Eq. (A1) and employing the approximation

$$1 - e^{in\theta(\tau)} \approx \sum_j e^{2\pi i n / \kappa \sum_{l=1}^{j-1} \nu_l} (1 - e^{in\nu_j \tilde{\theta}(\tau - \tau_j)}) \equiv \sum_j C_j f_{nj}(\tau) \quad (\text{A2})$$

appropriate for well-separated instantons with the aid of the identity

$$\frac{\pi^2 T^2}{\sin^2[\pi T(\tau - \tau')]} = \frac{\partial^2}{\partial \tau \partial \tau'} \ln |\sin[\pi T(\tau - \tau')]| \quad (\text{A3})$$

we obtain

$$S_{\text{int}}[\Theta] = -\frac{\alpha}{2} \sum_n a_n \sum_{j_1 < j_2} C_{j_1} \bar{C}_{j_2} \int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau' f'_{nj_1}(\tau) \bar{f}'_{nj_2}(\tau') \times \ln |\sin[\pi T(\tau - \tau')]|. \quad (\text{A4})$$

Let us first consider the terms in the above sum which correspond to  $j_1 = j_2$ . They are

$$-\frac{\alpha}{2} \sum_n a_n \int_{-\beta/2}^{\beta/2} d\tau \int_{-\beta/2}^{\beta/2} d\tau' f'_{nj}(\tau) \bar{f}'_{nj}(\tau') \ln |\sin[\pi T(\tau - \tau')]|. \quad (\text{A5})$$

Since an effective instanton width  $1/\omega$  is much smaller than the inverse temperature  $\beta$  it is possible to split the contribution, Eq. (A5), into regular and singular (in the limit  $T \rightarrow 0$ ) parts, respectively,

$$S_{\text{reg}} = -\frac{\alpha}{2} \sum_n a_n \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \frac{\partial^2}{\partial \tau \partial \tau'} \cos\{n[\tilde{\theta}(\tau) - \tilde{\theta}(\tau')]\} \times \ln |\omega(\tau - \tau')| \approx \frac{4\pi\alpha r}{\kappa} \quad (\text{A6})$$

and

$$S_{\text{sing}} = -2\alpha \sum_n a_n \sin^2\left(\frac{\pi n}{\kappa}\right) \ln \frac{\pi T}{\omega} = -2\alpha K \left(\frac{2\pi}{\kappa}\right) \ln \frac{\pi T}{\omega}. \quad (\text{A7})$$

Consider now the remaining terms with  $j_1 \neq j_2$ . Provided the distance between instantons remains much larger than their width  $\sim 1/\omega$  we find

$$-\alpha \sum_n a_n \sum_{j_1 < j_2} \Re[C_{j_2} \bar{C}_{j_1} (1 - e^{\frac{2\pi i n}{\kappa} \nu_{j_2}})(1 - e^{-\frac{2\pi i n}{\kappa} \nu_{j_1}})] \times \ln |\sin[\pi T(\tau_{j_2} - \tau_{j_1})]|. \quad (\text{A8})$$

Performing summation over  $n$  in Eq. (A8) and combining the result with Eqs. (A6) and (A7) we arrive at Eq. (42).

Let us note that in the particular case  $\kappa=1$  the logarithmic interaction between instantons vanish while in the case of  $\kappa=2$  and for large enough  $r$  we get  $S_{\text{int}}[\Theta] = 2\pi\alpha r(n_1 + n_2) + \tilde{S}(\alpha)$  where the logarithmic interinstanton interaction  $\tilde{S}(\alpha)$  takes the form

$$\tilde{S}(\alpha) = -4\alpha \sum_{j_1 < j_2} (-1)^{j_1 - j_2} \ln \left\{ \frac{\sin[\pi T(\tau_{j_2} - \tau_{j_1})]}{\pi T \omega^{-1}} \right\}. \quad (\text{A9})$$

Hence, the partition function for our model with  $\kappa=2$  is formally equivalent to that for the well-known spin-boson model with Ohmic dissipation.<sup>14,15</sup>

#### APPENDIX B: PERTURBATION THEORY

Let us rewrite the partition function of our problem as

$$\begin{aligned} \mathcal{Z} &= \kappa \sum_{n=0}^{\infty} \sum_{\nu_1=\pm 1} \dots \sum_{\nu_n=\pm 1} \left(\frac{\Delta}{2}\right)^n \int_0^{\beta} d\tau_1 \int_{\tau_1}^{\beta} d\tau_2 \dots \int_{\tau_{n-1}}^{\beta} d\tau_n \\ &\times \sum_{m=-\infty}^{\infty} e^{2\pi i m \phi_x - S_{\text{int}}[\Theta(\tau)]} \delta_{\sum_i \nu_i, m\kappa}. \end{aligned} \quad (\text{B1})$$

With the aid of the Poisson's resummation formulas Eq. (B1) can be transformed to

$$\begin{aligned} \mathcal{Z} &= \kappa \int_0^{2\pi} \frac{dz}{2\pi} \sum_{m=-\infty}^{\infty} e^{2\pi i m \phi_x - im\kappa z} Z[z; \beta] \\ &= \sum_{k=1}^{\kappa} Z[2\pi(\phi_x - k)/\kappa; \beta], \end{aligned} \quad (\text{B2})$$

where

$$\begin{aligned} Z[z; \beta] &= \sum_{k=0}^{\infty} \sum_{\nu_1=\pm 1} \dots \sum_{\nu_k=\pm 1} \left(\frac{\Delta_r}{2}\right)^k \int_0^{\beta} d\tau_1 \int_{\tau_1}^{\beta} d\tau_2 \dots \int_{\tau_{k-1}}^{\beta} d\tau_k \\ &\times e^{iz \sum_{j=1}^k \nu_j + 2\alpha \sum_{a<b=1}^k \nu_a \nu_b g(\varphi_{ab}) W(\tau_a - \tau_b)} \end{aligned} \quad (\text{B3})$$

and

$$W(\tau_a - \tau_b) = \ln \left\{ \frac{\sin[\pi T(\tau_b - \tau_a)]}{\pi T \omega^{-1}} \right\} = - \sum_{n=-\infty}^{\infty} W_n e^{2\pi i n T(\tau_a - \tau_b)}. \quad (\text{B4})$$

Expanding the exponent in Eq. (B3) one recovers the perturbation series for  $Z[z; \beta]$ . Extending the definition of the Fourier coefficients  $a_n$  to negative  $n$  in such a way that  $a_{-n} = a_n$  and  $\alpha_0 = 0$  we may write

$$\begin{aligned} g(\varphi_{ab}) &= 2 \sum_{n=1}^r a_n \sin^2\left(\frac{\pi n}{\kappa}\right) \cos(n\varphi_{ab}) \\ &= \sum_{n=-r}^r a_n \sin^2\left(\frac{\pi n}{\kappa}\right) e^{in\varphi_{ab}}. \end{aligned} \quad (\text{B5})$$

With the aid of this equation it is easy to define all orders of the perturbative expansion  $Z = Z^{(0)} + Z^{(1)} + \dots$ , where



FIG. 4. (Color online) First-order diagram of the perturbation theory in  $\alpha$ .

$$Z^{(0)}[z; \beta] = e^{-2\beta\Delta_r \sin^2(z/2)} \quad (\text{B6})$$

and

$$\begin{aligned} Z^{(1)}[z; \beta] &= -2\alpha\Delta_r^2 \sum_{n=-r}^r a_n \sin^2\left(z + \frac{\pi n}{\kappa}\right) \sin^2\left(\frac{\pi n}{\kappa}\right) \\ &\times \int_0^{\beta} d\tau_1 \int_{\tau_1}^{\beta} d\tau_2 Z_0[z; \tau_1] Z_0\left[z + \frac{2\pi n}{\kappa}; \tau_2 - \tau_1\right] \\ &\times Z_0[z; \beta - \tau_2] W(\tau_2 - \tau_1). \end{aligned} \quad (\text{B7})$$

The last term can also be represented graphically by the diagram in Fig. 4.

Performing the Fourier transformation of the partition function

$$Z[z, \beta] = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{-i\lambda\beta} \tilde{Z}[z, \lambda] \quad (\text{B8})$$

and introducing the self-energy  $\tilde{\Sigma}$  as a sum of all irreducible diagrams, we obtain

$$\tilde{Z}[z, \lambda] = \frac{1}{-i\lambda + 2\Delta_r \sin^2(z/2) - \tilde{\Sigma}[z; \lambda]}. \quad (\text{B9})$$

To the first order in the interaction one finds

$$\begin{aligned} \tilde{\Sigma}_1[z; \lambda] &= 2\alpha\Delta_r^2 \sum_{n=-r}^r a_n \sin^2\left(z + \frac{\pi n}{\kappa}\right) \sin^2\left(\frac{\pi n}{\kappa}\right) \\ &\times \sum_{k=-\infty}^{\infty} \frac{W_k}{-i\lambda - 2\pi i k T + 2\Delta_r \sin^2\left(\frac{z}{2} + \frac{\pi n}{\kappa}\right)}, \end{aligned} \quad (\text{B10})$$

where  $W_0 = \ln[2\pi T/\omega]$  and  $W_k = 1/2|k|$  for  $k \neq 0$ . Performing summation over  $k$ , we get

$$\begin{aligned} \tilde{\Sigma}_1[z; \lambda] &= 2\alpha\Delta_r^2 \sum_{n=-r}^r a_n \sin^2\left(z + \frac{\pi n}{\kappa}\right) \sin^2\left(\frac{\pi n}{\kappa}\right) \\ &\times \frac{\ln \frac{2\omega}{\pi T} + \gamma + \frac{1}{2}\psi \left[ 1 + \frac{\lambda + 2i\Delta_r \sin^2\left(\frac{z}{2} + \frac{\pi n}{\kappa}\right)}{2\pi T} \right] + \frac{1}{2}\psi \left[ 1 - \frac{\lambda + 2i\Delta_r \sin^2\left(\frac{z}{2} + \frac{\pi n}{\kappa}\right)}{2\pi T} \right]}{-i\lambda + 2\Delta_r \sin^2\left(\frac{z}{2} + \frac{\pi n}{\kappa}\right)}, \end{aligned} \quad (\text{B11})$$

where  $\psi$  is the digamma function and  $\gamma \approx 0.577$  is the Euler constant. Substituting this expression into Eq. (B9), we observe that  $\tilde{Z}[z, \lambda]$  has a pole at  $\lambda_p = -2i\Delta_r \sin^2(z/2) + \delta\lambda_p$ , where

$$\begin{aligned} \delta\lambda_p = & i\alpha\Delta_r \sum_{n=-r}^r a_n \sin\left(z + \frac{\pi n}{\kappa}\right) \sin\left(\frac{\pi n}{\kappa}\right) \\ & \times \left\{ \ln \frac{2\pi T}{\omega} + \gamma + \frac{1}{2} \psi \left[ 1 + \frac{i\Delta_r}{\pi T} \sin\left(z + \frac{\pi n}{\kappa}\right) \sin\left(\frac{\pi n}{\kappa}\right) \right] \right. \\ & \left. + \frac{1}{2} \psi \left[ 1 - \frac{i\Delta_r}{\pi T} \sin\left(z + \frac{\pi n}{\kappa}\right) \sin\left(\frac{\pi n}{\kappa}\right) \right] \right\}. \quad (\text{B12}) \end{aligned}$$

Combining the above results with Eq. (B8) one arrives at the final perturbative in the interaction expression for the partition function. Of particular interest is the low-temperature limit  $T \ll \Delta_r / \kappa^2$ . In this case one has  $\psi(1+z) \approx \ln(z) + O(1/z)$  and the free energy reduces to

$$\begin{aligned} -T \ln \mathcal{Z} = & 2\Delta_r \sin^2\left(\frac{\pi\phi_x}{\kappa}\right) \\ & - \alpha\Delta_r \sum_{n=-r}^r a_n \sin\left(\frac{\pi n}{\kappa}\right) \sin\left(\frac{2\pi\phi_x + \pi n}{\kappa}\right) \\ & \times \left[ \ln \frac{2\Delta e^\gamma}{\omega} + \ln \left| \sin\left(\frac{\pi n}{\kappa}\right) \sin\left(\frac{2\pi\phi_x + \pi n}{\kappa}\right) \right| \right]. \quad (\text{B13}) \end{aligned}$$

It is straightforward to observe that the first logarithmic term in the square brackets simply yields the renormalization  $\Delta_r \rightarrow \Delta_r [1 + 2\alpha\kappa(2\pi/\kappa) \ln \frac{2\Delta_r e^\gamma}{\omega}]$  and does not alter the dependence of PC on the flux  $\phi_x$ . Hence, this perturbative contribution can be absorbed in the first term in Eq. (B13) simply by substituting  $\Delta_R$  instead of  $\Delta_r$ , where  $\Delta_R$  is defined in Eq. (48).

The second logarithmic term in the square brackets in Eq. (B13) contains an additional flux dependence which turns singular at  $\phi_x$  close to the half-integer numbers. Clearly, this perturbative contribution cannot be just reduced to the renormalization, Eq. (48). As a result, we obtain

$$\begin{aligned} -T \ln \mathcal{Z} = & 2\Delta_R \sin^2\left(\frac{\pi\phi_x}{\kappa}\right) \\ & - \alpha\Delta_r \sum_{n=-r}^r a_n \sin\left(\frac{\pi n}{\kappa}\right) \sin\left(\frac{2\pi\phi_x + \pi n}{\kappa}\right) \\ & \times \ln \left| \sin\left(\frac{\pi n}{\kappa}\right) \sin\left(\frac{2\pi\phi_x + \pi n}{\kappa}\right) \right|. \quad (\text{B14}) \end{aligned}$$

Employing this expression we arrive at Eq. (53).

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