

Robustness of the spin-Chern number

Emil Prodan

Department of Physics, Yeshiva University, New York, New York 10016, USA

(Received 17 April 2009; revised manuscript received 8 July 2009; published 23 September 2009)

The spin Chern (C_s) was originally introduced on finite samples by imposing spin boundary conditions at the edges. This definition leads to confusing and contradictory statements. On one hand, the original paper by Sheng *et al.* revealed robust properties of C_s against disorder and certain deformations of the model and, on the other hand, several people pointed out that C_s can change sign under special deformations that keep the bulk Hamiltonian gap open. Because of the later findings, the spin-Chern number was dismissed as a true bulk topological invariant and now is viewed as something that describes the edge where the spin boundary conditions are imposed. In this paper, we define the spin-Chern number directly in the thermodynamic limit, without using any boundary conditions. We demonstrate its quantization in the presence of strong disorder and we argue that C_s is a true bulk topological invariant whose robustness against disorder and smooth deformations of the Hamiltonian have important physical consequences. The properties of the spin-Chern number remain valid even when the time reversal invariance is broken.

DOI: [10.1103/PhysRevB.80.125327](https://doi.org/10.1103/PhysRevB.80.125327)

PACS number(s): 73.43.-f, 72.25.Hg, 73.61.Wp, 85.75.-d

I. INTRODUCTION

Quantum spin-Hall insulators represent a new state of matter. They were predicted theoretically in Ref. 1 and soon after that they were found experimentally.^{2,3} Samples made of such materials display dissipationless spin currents at the edges, which are robust against continuous deformations and disorder.⁴

The existence of the edge channels is due to the nontrivial topology of the bulk energy bands and two nontrivial topological invariants were proposed to describe this topology virtually at the same time: the Z_2 invariant proposed by Kane and Mele⁵ and the spin-Chern number proposed by Sheng *et al.*⁶ (first mentioned in Ref. 4). In this paper we focus on the later invariant, which came under sustained scrutiny because it promised a finer classification of the spin-Hall insulators. This was later argued not to be the case.^{7,8}

The spin-Chern number was computed by integrating the Berry curvature generated by imposing twisted boundary conditions on a finite sample. The numerical evidence given in Ref. 6 implied that C_s is a robust topological invariant. It was later observed, however, that one can continuously deform the model using spin rotations that keep the bulk Hamiltonian's gap unchanged but switch the sign of the spin-Chern number.^{7,8} This argument shows that sometime the structure proposed in Ref. 6 fails to be a smooth fiber bundle and that C_s may not be well defined over the entire spin-Hall zone of the phase diagram. The current understanding is that whenever one crosses certain zones of the parameter space, C_s jumps, but these jumps are always by an even number. Therefore, one can still use C_s to formulate a Z_2 classification of the spin-Hall insulators and to efficiently compute the Z_2 invariant. For this reason, the interest in the spin-Chern number continues to be strong. For example, an efficient algorithm for numerical evaluations of C_s was proposed by Fukui and Hatsugai⁹ and later the algorithm was used to map C_s for aperiodic systems.¹⁰ But other works totally dismiss the spin-Chern number.^{7,11} For example Fu and Kane¹¹ stated that C_s loses its meaning when the spin is not conserved.

In this paper we redefine the spin-Chern number, this time directly for an infinite sample and without involving any boundary conditions. This shows, univocally, that C_s is a topological invariant describing the bulk and not some edge as previous works implied. In our approach, the spin-Chern number is defined as the half difference between the Chern numbers for the spin-up and spin-down sectors of the occupied space. We demonstrate that the spin-up and down sectors can be rigorously defined even when \hat{s}_z is not conserved.

Furthermore, we state some very general results from the noncommutative theory of the Chern number¹² and then we show how to apply these results to demonstrate the quantization of the spin-Chern number in the presence of strong disorder. Several physical consequences emerge from this analysis such as the existence of critical energies below the Fermi level where the localization length diverges. Unlike the Z_2 invariant, the robust properties of the spin-Chern number remain unchanged when the time reversal symmetry is broken.

II. FIBER BUNDLES AND THEIR TOPOLOGICAL CLASSIFICATION

The classification of the topological insulators is intrinsically related to the classification of the fiber bundles. Intuitively, a fiber bundle is a collection of spaces, called fibers, which are indexed by a set of parameters that live on a manifold. The parameter space is called the base manifold and the reunion of all fibers is called the total space. An alternative view is to imagine the fibers as being attached to the points of the base manifold and then glued together by a prescribed topology.

The fiber bundles encountered in condensed matter theory are generated by families of projectors P_{θ} , where θ 's live on a manifold M such as the sphere, torus, etc. These projectors act on the same Hilbert space \mathcal{H} and they generate the fibers $\mathcal{H}_{\theta} = P_{\theta}\mathcal{H}$. Since these fibers are embedded in the big Hilbert space \mathcal{H} , one can tell when two fibers \mathcal{H}_{θ} and $\mathcal{H}_{\theta'}$ are close

to each other and when they are not; therefore the bundle of fibers is already equipped with a topology.

Most of the fiber bundles studied in condensed matter are generated for finite samples by imposing parametrically dependent boundary conditions. To be more specific, consider a real space two-dimensional lattice with N quantum states per site and assume that a tight-binding Hamiltonian H has been defined. The wave functions are N -dimensional complex vector fields defined over the finite lattice: $\Psi(n_1, n_2)$, where $n_1=1, \dots, N_1$ and $n_2=1, \dots, N_2$. Assume that we diagonalize H by imposing boundary conditions such as

$$\begin{aligned}\Psi(N_1, n_2) &= \hat{M}_1(\boldsymbol{\theta})\Psi(1, n_2), \\ \Psi(n_1, N_2) &= \hat{M}_2(\boldsymbol{\theta})\Psi(n_1, 1),\end{aligned}\quad (1)$$

where $\hat{M}_1(\boldsymbol{\theta})$ and $\hat{M}_2(\boldsymbol{\theta})$ are two $N \times N$ matrices depending parametrically on $\boldsymbol{\theta}=(\theta_1, \theta_2)$. Assume that, after the diagonalization, we find that the energy spectrum has a gap for all the allowed $\boldsymbol{\theta}$'s. In this case, we can construct a fiber bundle by simply considering the projectors $P_{\boldsymbol{\theta}}$ onto the states with energy below this gap.

While this is a straightforward construction which pertains to easy numerical implementations, it has a few drawbacks which are very often overlooked. The possibilities of choosing the two matrices seem endless and same can be said about the parameter space. How can one then discern between a meaningful construction and the rest? The ultimate goal of these fiber bundle constructions is to characterize the bulk. Therefore, only a rigorous study of the thermodynamic limit can filter out the irrelevant matrices and the wrong parameter spaces. Most of the constructions will not survive in the thermodynamic limit either because the energy gap closes at some point or because the fibers cease to depend smoothly on the parameters. Unfortunately, studying the thermodynamic limit requires tedious numerical simulations, which most of the time can be carried only up to relatively small sample sizes. In fact, it is very rare when such studies are performed at all. We will discuss in Sec. III how to define fiber bundles directly in the thermodynamic limit.

We now turn to the problem of classifying the fiber bundles. If the dimension of the fibers $\mathcal{H}_{\boldsymbol{\theta}}$ is N , the fiber bundle generated by a family of projections is a $U(N)$ fiber bundle because two fibers $\mathcal{H}_{\boldsymbol{\theta}}$ and $\mathcal{H}_{\boldsymbol{\theta}'}$ can be linked by a unitary transformation $U_{\boldsymbol{\theta}\boldsymbol{\theta}'}$. The question of how many topologically nonequivalent $U(N)$ fiber bundles can be build over a manifold \mathbf{M} has been answered long time ago.^{13–17} It turns out that there is a universal manifold \mathbf{X} , called classifying space, such that all $U(N)$ fiber bundles over \mathbf{M} can be generated from continuous functions $f:\mathbf{M} \rightarrow \mathbf{X}$ via the so-called pull-back construction. Describing this pull-back construction is rather technical, but here it is a very simple result: two fiber bundles over \mathbf{M} generated by f and f' can be continuously deformed into each other if and only if the two functions f and f' can be continuously deformed into each other. Therefore, classifying the $U(N)$ fiber bundles over a manifold \mathbf{M} is equivalent to classifying the continuous functions from \mathbf{M} to \mathbf{X} . While a fundamental results in the fiber

bundle theory, this classification algorithm is somewhat formal because the space \mathbf{X} is quite complicated [it is the infinite Grassmannian manifold $\text{Gr}_N(\mathbb{C}^\infty)$].

A new development in the classification of low dimensional fiber bundles came from the work of Panati.¹⁸ This work provides a computationally more manageable classifying approach and states, among other important things, that the fiber bundles over two-dimensional manifolds without boundaries are completely classified by their Chern numbers. Note that the Chern number is just one invariant among many topologic invariants that can be associated with a fiber bundle. The Chern number can be explicitly computed by integrating the Berry curvature form,

$$dF = \text{Tr}\{P_{\boldsymbol{\theta}}[\partial_{\theta_1}P_{\boldsymbol{\theta}}\partial_{\theta_2}P_{\boldsymbol{\theta}}]\}d\theta_1 \wedge d\theta_2, \quad (2)$$

over the base manifold,

$$C = \frac{1}{2\pi i} \int_{\mathbf{M}} dF. \quad (3)$$

The result of such integration is always an integer and this integer number is called the Chern number.

Given Panati's result,¹⁸ it is clear that in two dimensions there are no magic new topological numbers. Therefore, the spin-Chern number must be related to the classical Chern number. As we shall see, this is indeed the case, but what is new is the structure for which the Chern number is computed.

III. WORKING DIRECTLY IN THE THERMODYNAMIC LIMIT

There is one simple case in which one can work directly with infinite samples. This is the case of translational invariant Hamiltonians where one can use the Bloch fibration. The Bloch fibration is a unitary transformation from the Hilbert space of the infinite sample into a continuum direct sum of N -dimensional complex spaces,

$$U:\mathcal{H} \rightarrow \bigoplus_{\mathbf{k} \in \mathcal{T}} \mathbb{C}^N,$$

$$(U\Psi)(\mathbf{k}) = \frac{1}{2\pi} \sum_n e^{-in\mathbf{k}} \Psi(n), \quad (4)$$

where \mathbf{k} lives on the Brillouin torus $\mathcal{T}=[0, 2\pi][0, 2\pi]$. U transforms the original Hamiltonian in a direct sum of Bloch Hamiltonians,

$$UHU^{-1} = \bigoplus_{\mathbf{k} \in \mathcal{T}} H(\mathbf{k}). \quad (5)$$

Obtaining the explicit expressions of $H(\mathbf{k})$ is, most of the time, a very simple exercise. Since U is an isometry, if H has an energy gap, then all $H(\mathbf{k})$ have an energy gap. We can then compute the projectors $P(\mathbf{k})$ for the states below this gap and define the fiber bundle $\bigoplus_{\mathbf{k} \in \mathcal{T}} P(\mathbf{k})\mathbb{C}^N$, with the Brillouin torus \mathcal{T} as the base manifold. Consequently, we can compute a Chern number directly for the infinite sample.

Consider now the case when the translational symmetry is lost, such as when disorder is present. For simplicity let us

assume that the bulk gap remains open. If this is the case, we can define the total space simply as $P\mathcal{H}$, with P being the projector onto the states below the gap; but how can we define a base manifold without the Brillouin torus? It turns out that the only way out of this dilemma is to define a noncommutative Brillouin torus, which is a torus whose points are operators instead of simple points on a manifold.

Describing this noncommutative torus is quite technical, but we can describe, in quite simple terms, how to define the Chern number. Let us show how this definition emerges from the translational invariant case. The Chern number computed as

$$C = \frac{1}{2\pi i} \int_{\mathcal{T}} dF, \quad (6)$$

with the Berry curvature given in Eq. (2) (where \mathbf{k} takes the place of $\boldsymbol{\theta}$), has an expression in the real space,

$$C = 2\pi i \operatorname{tr}\{P[[\hat{n}_1, P], [\hat{n}_2, P]]\}, \quad (7)$$

where tr means trace over the quantum states at the origin, $[\cdot, \cdot]$ denotes the usual commutator, and $\hat{\mathbf{n}}$ is the position operator: $(\hat{n}_i \Psi)(n_1, n_2) = n_i \Psi(n_1, n_2)$ ($i=1, 2$). One can see that we do not really need the Brillouin torus to define the Chern number. Therefore, one could try to define the Chern number for the disordered case using this expression. The main question is will this number be an integer. A fundamental result in noncommutative geometry says that this is indeed the case after averaging over the disorder.

We would like to state this result in more precise terms. For this we need to describe the disorder more precisely. To be specific, let us consider the addition of a random potential to H ,

$$V_\omega = \sum_{n, \alpha} \lambda_n(\omega) |\mathbf{n}, \alpha\rangle \langle \mathbf{n}, \alpha|, \quad (8)$$

where $\lambda_n(\omega)$ is a random variable and α is the index of the quantum states ($\alpha=1, \dots, N$). ω represents a particular disordered configuration and can be viewed as a point in the disorder configuration space Ω , which must be equipped with a probability measure $d\mu(\omega)$ to be used for averaging over the disorder. We assume that the macroscopic state is translational invariant, in which case the lattice translations induce unitary transformations u_n on \mathcal{H} and flows t_n on Ω such that $u_n V_\omega u_n^{-1} = V_{t_n \omega}$. The probability measure $d\mu(\omega)$ is assumed invariant and ergodic with respect to the flows t_n . A concrete example will be the white noise, for which the disorder configuration space is the infinite Cartesian product of $[-\frac{1}{2}, \frac{1}{2}]$ intervals,

$$\Omega = \times_{\mathbf{n} \in \mathbb{Z}^2} \left[-\frac{1}{2}, \frac{1}{2}\right]. \quad (9)$$

A point ω of this space is an infinite sequence,

$$\omega = (\dots, \omega_n, \omega_{n+1}, \dots) = \times_{\mathbf{n} \in \mathbb{Z}^2} \omega_n, \quad (10)$$

and $\lambda_n(\omega)$ is taken as $\lambda_n(\omega) = \omega_n$. The probability measure $d\mu(\omega)$ is given by the infinite product of measures: $d\mu(\omega) = \times_{\mathbf{n} \in \mathbb{Z}^2} d\omega_n$. The ergodic flow is simply $t_{\mathbf{m}}(\times_{\mathbf{n} \in \mathbb{Z}^2} \omega_n) = \times_{\mathbf{n} \in \mathbb{Z}^2} \omega_{\mathbf{n}-\mathbf{m}}$.

With these being said, consider a family of projectors P_ω acting on the whole Hilbert space \mathcal{H} . We can now state the following rigorous result.¹²

Proposition 1. If the matrix element $\langle \mathbf{n} | P_\omega | \mathbf{m} \rangle$ decays sufficiently fast (exponential decay is enough) with the separation $|\mathbf{n}-\mathbf{m}|$ and if P_ω 's satisfy $u_n P_\omega u_n^{-1} = P_{t_n \omega}$, then

$$C = 2\pi i \int_{\Omega} d\mu(\omega) \operatorname{tr}\{P_\omega[[\hat{n}_1, P_\omega], [\hat{n}_2, P_\omega]]\} \quad (11)$$

is an integer that is invariant to smooth deformations of P_ω 's as long as they remain localized.

This is a very general result and it applies to any family of projectors and any type of disorder as long as the conditions inside the proposition's statement are satisfied.

If the projectors P_ω are associated to the spectral projectors onto the occupied states of a Hamiltonian, the Chern number defined above gives the disorder averaged Hall conductance. Proposition 1 tells us that this average remains quantized even in the presence of strong disorder. Since the Hall plateaus cannot be explained without the strong disorder, it was the above statement that completed our understanding of integer quantum hall effect (IQHE).

IV. DEFINING THE SPIN-CHERN NUMBER

The spin-Chern number was originally defined for a finite sample by choosing $M_1(\boldsymbol{\theta}) = e^{i\theta_1}$ and $M_2(\boldsymbol{\theta}) = e^{2i\theta_2 \hat{s}_z}$ in the boundary conditions [Eq. (1)]. Here we define the spin-Chern number directly in the thermodynamic limit.

Let us start our discussion from the concrete model of electrons in graphene,¹

$$\begin{aligned} H_0 = & -t \sum_{\langle mn \rangle, \sigma} |\mathbf{m}, \sigma\rangle \langle \mathbf{n}, \sigma| \\ & + i\lambda_{SO} \sum_{\langle\langle mn \rangle\rangle, \sigma\sigma'} [\mathbf{s} \cdot (\mathbf{d}_{km} \times \mathbf{d}_{nk})]_{\sigma, \sigma'} |\mathbf{m}, \sigma\rangle \langle \mathbf{n}, \sigma'| \\ & + i\lambda_R \sum_{\langle mn \rangle, \sigma\sigma'} [\hat{\mathbf{z}} \cdot (\mathbf{s} \times \mathbf{d}_{mn})]_{\sigma, \sigma'} |\mathbf{n}, \sigma\rangle \langle \mathbf{n}, \sigma'|. \end{aligned} \quad (12)$$

Here, \mathbf{m} and \mathbf{n} denote the sites of the honeycomb lattice and σ and σ' denote the electron spin degrees of freedom, taking the values ± 1 . The Hamiltonian acts on the Hilbert space \mathcal{H} spanned by the orthonormal basis $|\mathbf{n}, \sigma\rangle$. The simple angular brackets denote the nearest neighbors and double angular brackets denote the second nearest neighbors. In the second sum, \mathbf{k} represents the unique common nearest neighbor of \mathbf{m} and \mathbf{n} . The electrons are considered noninteracting. The three terms in Eq. (12) are the usual nearest-neighbor hopping term, the intrinsic spin-orbit coupling preserving the lattice symmetries, and the Rashba potential induced by the substrate supporting the graphene sheet.¹ We assume that the parameters in the Hamiltonian are chosen so that we are in the spin-Hall zone of the phase diagram.⁵

There are two sites per unit cell and two spin states per site, therefore the model has four states per unit cell. Let us neglect the disorder for a moment and let P denote the spectral projector onto the states below the insulating gap of H_0 . Under the Bloch fibration $U: \mathcal{H} \rightarrow \oplus_{\mathbf{k} \in \mathcal{T}} \mathcal{C}^4$, this projector be-

comes $UPU^{-1} = \oplus_{\mathbf{k} \in \mathcal{T}} P(\mathbf{k})$. The Bloch Hamiltonians $H(\mathbf{k})$ display two upper and two lower bands separated by an insulating gap. The Chern number of the projectors $P(\mathbf{k})$ onto the lower (and for that matter also of the upper) bands is zero, as it will generically be for any time reversal invariant band model. According to Panati's result,¹⁸ the fiber bundle of the occupied states is trivial, i.e., it has the simplest topological structure possible, that of $T \times \mathcal{C}^2$. How can there be nontrivial topological structures? The answer is because in the spin-Hall effect the nontrivial topological structures appear only in the spin sectors when taken separately.

We now show how one can define spin-up and spin-down sectors even for the case when \hat{s}_z does not commute with the Hamiltonian. The spin operator is defined as

$$\hat{s}_z |n, \sigma\rangle = \frac{1}{2} \sigma |n, \sigma\rangle. \quad (13)$$

The goal is to slash the fiber bundle $\oplus_{\mathbf{k} \in \mathcal{T}} P(\mathbf{k}) \mathcal{C}^4$ into two nontrivial fiber bundles, which make the smooth decomposition $P(\mathbf{k}) = P_-(\mathbf{k}) \oplus P_+(\mathbf{k})$ for all the \mathbf{k} 's of the Brillouin torus. At the first sight, this seems a very complicated job with no clear chances of success. The key idea is to use the operator $P\hat{s}_zP$ to do just that. Indeed, after the Bloch fibration, $UP\hat{s}_zPU^{-1}$ becomes $\oplus_{\mathbf{k} \in \mathcal{T}} P(\mathbf{k})\hat{s}_zP(\mathbf{k})$ and we can diagonalize each of the operators $P(\mathbf{k})\hat{s}_zP(\mathbf{k})$. As we shall see, if the Rashba term does not exceed a threshold value, the spectrum of $P(\mathbf{k})\hat{s}_zP(\mathbf{k})$ consists of two isolated eigenvalues, positioned symmetrically and away from the origin, for all the \mathbf{k} 's of the Brillouin torus. The spectral projectors $P_{\pm}(\mathbf{k})$ onto the negative and positive eigenvalues are smooth of \mathbf{k} and can be used to achieve the decomposition $P(\mathbf{k}) = P_-(\mathbf{k}) \oplus P_+(\mathbf{k})$. At this point we can define the Chern numbers C_{\pm} for the fiber bundles $\oplus_{\mathbf{k} \in \mathcal{T}} P_{\pm}(\mathbf{k}) \mathcal{C}^4$ via Eq. (2) and define the spin-Chern number as $C_s = \frac{1}{2}(C_+ - C_-)$. This is our construction in a nutshell.

We now step back and give a more general construction, without involving the Brillouin torus. Assume that the disorder is turned on and that we work now with the Hamiltonian $H(\omega) = H_0 + \lambda V_{\omega}$. For simplicity, we assume that the amplitude of the disorder is not very large so that $H(\omega)$ still has an energy gap. Let $P(\omega)$ be the projector onto the states with energy below this gap and consider the operator $P(\omega)\hat{s}_zP(\omega)$. When the Rashba term is zero, \hat{s}_z commutes with $H(\omega)$ and for that reason it also commutes with $P(\omega)$. Consequently, its spectrum consists of just two points, $\pm \frac{1}{2}$. These eigenvalues are highly degenerate. When the Rashba interaction is turned on, \hat{s}_z and $P(\omega)$ no longer commute; therefore the degeneracy is lifted and the eigenvalues of $P(\omega)\hat{s}_zP(\omega)$ spread toward the origin. As we shall see, if the amplitude of the Rashba term does not exceed a threshold value, the spectrum of $P(\omega)\hat{s}_zP(\omega)$ remains confined into two isolated islands, with the origin separating them. Therefore, we can define the spectral projectors onto the states with eigenvalues below the origin, $P_-(\omega)$, and above the origin, $P_+(\omega)$. To be able to define the noncommutative Chern numbers, we must demonstrate that the matrix elements $\langle \mathbf{m} | P_{\pm}(\omega) | \mathbf{n} \rangle$ decay exponentially fast with the separation $|\mathbf{m} - \mathbf{n}|$.

The exponential decay is required by the conditions stated in the proposition 1. However, we would like to pause here

and discuss what is the implication of this exponential decay for the translational invariant case. In this case, we have $P_{\pm} = \int P_{\pm}(\mathbf{k}) d\mathbf{k}$ and it is known since the work of Kohn¹⁹ that only smooth integrals of \mathbf{k} lead to exponentially localized functions in real space. The reciprocal statement is also known to be true since the work of des Cloizeaux,^{20,21} namely, if a total projector is exponentially localized then its Bloch decomposition is smooth with \mathbf{k} . The smoothness of $P_{\pm}(\mathbf{k})$ on \mathbf{k} is essential because the Berry curvature involves derivatives over \mathbf{k} . Therefore, the exponential localization of $\langle \mathbf{m} | P_{\pm}(\omega) | \mathbf{n} \rangle$ is essential for both translational invariant and disordered cases. The good news is that exponential localization is always simpler to demonstrate than the smoothness with \mathbf{k} .

Once we demonstrate the exponential localization of the kernels, we can define the Chern numbers for the spin-up and spin-down sectors via

$$C_{\pm} = 2\pi i \int_{\Omega} d\mu(\omega) \text{tr} \{ P_{\pm}(\omega) [[\hat{n}_1, P_{\pm}(\omega)], [\hat{n}_2, P_{\pm}(\omega)]] \}. \quad (14)$$

Since \hat{s}_z commutes with the translations, we have

$$u_n P_{\pm}(\omega) u_n^{-1} = P_{\pm}(t_n \omega). \quad (15)$$

Therefore, according to proposition 1, C_{\pm} are integers that do not change under continuous deformations as long as $P_{\pm}(\omega)$ remains exponentially localized. As we shall see, the exponential localization is guaranteed by the existence of the energy gap for $H(\omega)$ and by the spectral gap of $P(\omega)\hat{s}_zP(\omega)$. To compute the spin-Chern number explicitly, we can turn off the disorder and the Rashba interaction, in which case the model [Eq. (12)] reduces to two decoupled Haldane models,²² for which $C_{\pm} = \pm 1$ if $\lambda_{SO} > 0$ and $C_{\pm} = \mp 1$ if $\lambda_{SO} < 0$.

We now start the proof of the exponential localization of the projectors $P_{\pm}(\omega)$. To ease the notation we drop ω . First, let us show that the gap Δ_1 of the operator $P\hat{s}_zP$, viewed as an operator on the space of occupied states $\mathcal{K} = P\mathcal{H}$, remains open and clean when the Rashba term is turned on, as opposed to immediately closing or filling with additional spectrum due to some instability. For this, we notice first that the projector P in the occupied space is analytic in λ_R . This property is protected by the insulating gap Δ_2 of the Hamiltonian $H(\omega)$. We then show that, at least for small λ_R , $P\hat{s}_zP - \zeta I_{\mathcal{K}}$ is invertible for ζ in the vicinity of 0 (here $I_{\mathcal{K}}$ is the identity operator for the space \mathcal{K}). One could try to work with the expression

$$(Ps_zP - \zeta I_{\mathcal{K}})^{-1} = P(Ps_zP - \zeta I)^{-1} \quad (16)$$

and use the continuity of P , but this expression has a problem when $\zeta = 0$ since $(P\hat{s}_zP - \zeta I)^{-1}$ diverges because we include the unoccupied states where $P\hat{s}_zP$ is zero.

Here is an alternative approach inspired from Ref. 23. Let $R(\zeta) = P(\hat{s}_z - \zeta)^{-1}P$, with ζ in a neighborhood of zero. We have successively

$$\begin{aligned} R(\zeta)(P\hat{s}_zP - \zeta I_{\mathcal{K}}) &= P(\hat{s}_z - \zeta)^{-1}\{(\hat{s}_z - \zeta)P + [P, \hat{s}_z]\}P \\ &= P + P(\hat{s}_z - \zeta)^{-1}[P, \hat{s}_z]P \equiv I_{\mathcal{K}} + Q. \end{aligned} \quad (17)$$

Q is small, at least for small λ_R , since

$$Q = P(\hat{s}_z - \zeta)^{-1}[P - P_0, \hat{s}_z]P, \quad (18)$$

where P_0 is the projector onto the occupied states for $\lambda_R=0$. The small factor comes from $P - P_0$, which is proportional to λ_R . Above, we used the fact that \hat{s}_z commutes with P_0 even in the presence of disorder. In this case, the operator $I_{\mathcal{K}} + Q$ has an inverse given by the convergent series $\sum_{n=0}^{\infty} (-Q)^n$ and we find

$$(I_{\mathcal{K}} + Q)^{-1}R(\zeta)(P\hat{s}_zP - \zeta I_{\mathcal{K}}) = I_{\mathcal{K}}, \quad (19)$$

$$(P\hat{s}_zP - \zeta I_{\mathcal{K}})^{-1} = (I_{\mathcal{K}} + Q)^{-1}P(\hat{s}_z - \zeta)^{-1}P. \quad (20)$$

Thus we showed that resolvent of $P\hat{s}_zP$ is finite for ζ near the origin, which excludes the existence of any eigenvalues in this region.

We now show that, as long as the gaps Δ_1 and Δ_2 of $P\hat{s}_zP$ and $H(\omega)$ remain opened, the spectral projectors P_{\pm} are exponentially localized. By exponential localization of an operator T we mean the existence of a strictly positive α such that

$$|\langle \mathbf{m}, \sigma | T | \mathbf{n}, \sigma' \rangle| \leq \text{ct} \cdot e^{-\alpha|\mathbf{m}-\mathbf{n}|}. \quad (21)$$

We will use the following simple observation: if U_q denotes the nonunitary transformation,

$$U_q | \mathbf{n}, \sigma \rangle = e^{q \cdot \mathbf{n}} | \mathbf{n}, \sigma \rangle, \quad (22)$$

then the following is true.²⁴

Proposition 2. If T is exponentially localized, then $T_q \equiv U_q T U_q^{-1}$ is a bounded operator for all orientations of \mathbf{q} , provided $|q|$ is smaller than α , and conversely if T_q is bounded for any orientation of \mathbf{q} and $|q| < \alpha$, then T is exponentially localized with a localization exponent equal or larger than α .

Also, it is a fact that if T is exponentially localized, then ($\| \cdot \|$ denotes the operator norm)

$$\|T - T_q\| \rightarrow 0 \quad \text{as } q \rightarrow 0, \quad (23)$$

i.e., the difference between T and T_q is small for q small.

Since

$$P_{\pm} = \frac{i}{2\pi} \int_{C_{\pm}} P(P\hat{s}_zP - \zeta I)^{-1} d\zeta, \quad (24)$$

it is enough to show the exponential localization of $P(P\hat{s}_zP - \zeta I)^{-1}$ for all ζ on the contours C_{\pm} shown in Fig. 1. Now the projector P itself is exponentially localized and a general proof of this statement can be found in Ref. 25. Then P_q is bounded and we have successively

$$U_q P(P\hat{s}_zP - \zeta I)^{-1} U_q = P_q (P_q \hat{s}_z P_q - \zeta I)^{-1} \quad (25)$$

and

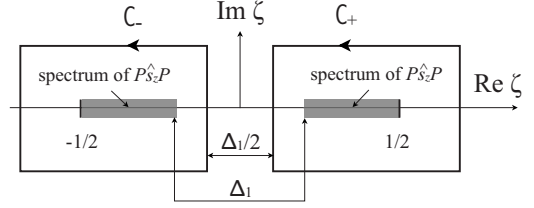


FIG. 1. Illustration of the spectrum of $P\hat{s}_zP$ for small Rashba coupling of the gap Δ_1 and of the integration contours C_{\pm} used in the main text.

$$P_q \hat{s}_z P_q - \zeta I = P \hat{s}_z P - \zeta I + P_q \hat{s}_z P_q - P \hat{s}_z P, \quad (26)$$

thus

$$\|P_q \hat{s}_z P_q - \zeta I\| \geq \frac{\Delta}{2} - \|P_q \hat{s}_z P_q - P \hat{s}_z P\|, \quad (27)$$

which is strictly positive for small q due to Eq. (23). In other words, $P_q \hat{s}_z P_q - \zeta I$ is invertible, at least for small q , which shows that all operators appearing in the right-hand side of Eq. (25) are bounded. Consequently, $P_{\pm}(\omega)$ are exponentially localized.

V. DISCUSSION

Our results show that the spin-Chern number is a topological invariant that is well defined in the thermodynamic limit. It is now clear that the robustness is due to the existence of two spectral gaps: the insulating gap of the Hamiltonian and the spectral gap of the operator $P\hat{s}_zP$. Contrary to many folkloric beliefs, not all topological invariants are robust against disorder. In fact, to date, a rigorous proof for such statement exists only for the classic Chern number. The proof makes heavy use of the particular expression of the Chern number, thus it cannot be automatically extended to other topological invariants. Therefore, it is remarkable that we can add the spin-Chern number to the list of topological invariants that are robust to disorder.

The robustness of the spin-Chern number has physical consequences. We recall that, according to Refs. 26 and 27, generically for dimension smaller than 2, all states are exponentially localized at any strength of disorder; for dimension greater than 2, there exist extended states for low disorder and for dimension 2, all the states are localized except for states corresponding to isolated critical energies at which the localization length diverges. The Chern number in IQHE not only has a classifying role but its quantization in the presence of disorder implies the existence of such critical energy regions below the Fermi level.¹² The quantization of the spin-Chern number leads to the same conclusion, which explains the finite spin-Hall conductance in the presence of strong disorder.⁴ This remarkable property is solely related to the spin because the spin-Chern number is robust against perturbations that violate the time reversal symmetry. Indeed, none of our constructions or proofs use the time reversal symmetry as an argument.

The operator \hat{s}_z is not special. In fact, C_s 's remain invariant to deformations of the Hamiltonian and of $P\hat{s}_zP$ as long

as the spectral gaps of the two operators remain open. Closing any of the two gaps can result in jumps for the spin-Chern number. The jumps are always by an even number. The argument against the spin-Chern number in Refs. 7 and 8 was that, using rotations in the spin sector, one can connect the Hamiltonians with $+\lambda_{SO}$ and with $-\lambda_{SO}$, without changing the insulating gap. But at the end of such rotation, C_s changes sign and it was concluded that C_s is not a well-defined topological invariant. The change in sign was attributed to the closing of the gap of the Hamiltonian with the spin boundary conditions. We now can give an alternative explanation: during the rotation in the spin sector, the gap of $P\hat{s}_zP$ closes and then opens again. This problem can be easily fixed, the solution being to deform not only the Hamiltonian but also the operator $P\hat{s}_zP$. For the case of spin rotations, this can be easily accomplished by applying the spin rotations to \hat{s}_z inside $P\hat{s}_zP$. If there is such easy fix, then the old question resurfaces: does the spin-Chern number contain more information than the Z_2 invariant? The answer is no. After the continuous rotation mentioned above, $P\hat{s}_zP \rightarrow -P\hat{s}_zP$ and $P_{\pm} \rightarrow P_{\mp}$. Thus, there is no canonical way to choose the projectors P_{\pm} and, since this choice determines the sign of C_s , the sign contains no additional information.

The construction can be easily generalized. For more complex insulators, it is very probable that one can replace \hat{s}_z by other nontrivial operators \hat{w} by combining spin and/or point group symmetry operators. Using such operators one might discover nontrivial topological structures in seemingly trivial insulators. For example, it was recently pointed out²⁸ that certain surface states in ordinary semiconductors can have topological origins. The nonzero Chern numbers for the spectral sectors of $P\hat{w}P$ will then lead to a \hat{w} -Hall effect. For other models, we can find operators $P\hat{w}P$ that have n islands of isolated spectrum with exponentially localized projectors. If \hat{w} commutes with the translations of the unit cell, we can define a Chern number for each spectral island of $P\hat{w}P$. A simple example of this type is the model of Eq. (12) with spin- $\frac{3}{2}$ particles. For $\lambda_R=0$, \hat{s}_z commutes with the Hamiltonian and $P\mathcal{H}$ splits into four sectors, corresponding to $s_z = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2},$ and $\frac{3}{2}$. Turning on the disorder will not affect these eigenvalues. For each sector we can define a Chern number, which takes the values (assuming $\lambda_{SO}>0$) $C_{-3/2} = C_{-1/2} = -1$ and $C_{1/2} = C_{3/2} = 1$. When the Rashba interaction is turned on, the spectrum of $P\hat{s}_zP$ spreads, but it is still contained into four isolated spectral islands. Thus, we can still split $P\mathcal{H}$ into four sectors and our analysis shows that the Chern numbers for each sector will be conserved as long as the insulating gap remains open and the spectral islands of $P\hat{s}_zP$ remain isolated. The spin-Chern number can be defined in various ways, depending how we group the sectors. If we repeat the construction for spin $\frac{1}{2}$ and put the negative s_z sectors into \mathcal{K}_- and the positive s_z sectors into \mathcal{K}_+ , C_s be-

comes 2. Other possible groupings gives $C_s=0$ and $C_s=-2$.

The existence of the nontrivial Chern numbers for different sectors does not automatically imply the existence of chiral edge modes. Generically, only the insulators with odd C_s display such edge modes.⁷ The discussion given here hardly touches the problem of classification and of the edge modes. For these two problems, one has to explore how the occupied states relate (connect) to the unoccupied states. This has to be done on solvable models. It is at this point where our analysis becomes relevant because now we have a guiding principle which tells when the realistic models, which should include disorder and will generally not be solvable, can be deformed into smooth solvable models without changing the topological invariants. The robustness of the edge modes was also recently investigated in Ref. 29, which introduced a quantized edge index. Although this edge index was also constructed via a splitting, the connection between the edge index and C_s is not clear to us at this moment.

We want to make a final remark about the edge modes. The existence of nontrivial topological sectors can indicate reach edge and surface physics. For example, for the model of spin- $\frac{3}{2}$ particles, when cutting an edge we can be sure that four (which become two) edge bands shoot out of the bulk spectrum. The bands hybridize and return back into the same part of the bulk spectrum where they originated. Nevertheless, these bands lead to (gapped) edge states, which can still be useful for practical applications. In the presence of disorder, these edge bands will localize, but if the localization length is large, these edge states, for example, can efficiently trap light and thus be useful in photovoltaic devices.

VI. CONCLUSIONS

We have demonstrated that the spin-Chern number can be defined in the thermodynamic limit without the need of any boundary conditions, therefore showing that the spin-Chern number describes the bulk and not the edges as it is currently believed. By making a connection with the noncommutative theory of Chern number, we were able to demonstrate the robustness of the spin-Chern number against disorder and smooth deformations of the models. The robustness, which is solely related to the spin and has nothing to do with the time reversal symmetry, has physical consequences: it implies the existence of critical energies below the Fermi level where the localization length diverges. We have also theorized possible generalizations of our construction, which could aid the search and discovery of new topological phases.

ACKNOWLEDGMENT

This work was supported by a grant from the Research Corporation.

- ¹C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 226801 (2005).
- ²B. A. Bernevig, T. L. Hughes, and S. C. Zhang, Science **314**, 1757 (2006).
- ³M. König, S. Wiedmann, C. Brune, A. Roth, H. Buhmann, L. W. Molenkamp, X. L. Qi, and S. C. Zhang, Science **318**, 766 (2007).
- ⁴L. Sheng, D. N. Sheng, C. S. Ting, and F. D. M. Haldane, Phys. Rev. Lett. **95**, 136602 (2005).
- ⁵C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802 (2005).
- ⁶D. N. Sheng, Z. Weng, L. Sheng, and F. D. M. Haldane, Phys. Rev. Lett. **97**, 036808 (2006).
- ⁷L. Fu and C. L. Kane, Phys. Rev. B **74**, 195312 (2006).
- ⁸T. Fukui and Y. Hatsugai, Phys. Rev. B **75**, 121403(R) (2007).
- ⁹T. Fukui and Y. Hatsugai, J. Phys. Soc. Jpn. **76**, 053702 (2007).
- ¹⁰A. M. Essin and J. E. Moore, Phys. Rev. B **76**, 165307 (2007).
- ¹¹L. Fu and C. L. Kane, Phys. Rev. B **76**, 045302 (2007).
- ¹²J. Bellissard, A. V. Elst, and H. Schulz-Baldes, J. Math. Phys. **35**, 5373 (1994).
- ¹³S. Eilenberg and S. MacLane, Ann. Math. **58**, 55 (1953).
- ¹⁴S. Eilenberg and S. MacLane, Ann. Math. **60**, 49 (1954).
- ¹⁵S. Eilenberg and S. MacLane, Ann. Math. **60**, 513 (1954).
- ¹⁶J. Milnor, Ann. Math. **63**, 272 (1956).
- ¹⁷J. Milnor, Ann. Math. **63**, 430 (1956).
- ¹⁸G. Panati, Ann. Henri Poincaré **8**, 995 (2007).
- ¹⁹W. Kohn, Phys. Rev. **115**, 809 (1959).
- ²⁰F. Des Cloizeaux, Phys. Rev. **135**, A685 (1964).
- ²¹F. Des Cloizeaux, Phys. Rev. **135**, A698 (1964).
- ²²F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).
- ²³A. Nenciu and G. Nenciu, Commun. Math. Phys. **190**, 541 (1998).
- ²⁴E. Prodan, S. Garcia, and M. Putinar, J. Phys. A **39**, 389 (2006).
- ²⁵E. Prodan, J. Math. Phys. **50**, 083517 (2009).
- ²⁶E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, Phys. Rev. Lett. **42**, 673 (1979).
- ²⁷P. W. Anderson, D. J. Thouless, E. Abrahams, and D. S. Fisher, Phys. Rev. B **22**, 3519 (1980).
- ²⁸Y. Kuge, I. Maruyama, and Y. Hatsugai, arXiv:0802.2425 (unpublished).
- ²⁹E. Prodan, J. Phys. A: Math. Theor. **42**, 082001 (2009).