Characteristics of two-dimensional lattice models from a fermionic realization: Ising and *XYZ* models

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We develop a field theoretical approach to the classical two-dimensional (2D) models, particularly to 2D Ising model (2DIM) and *XYZ* model, which is simple to apply for calculation of various correlation functions. We calculate the partition function of 2DIM and *XY* model within the developed framework. Determinant representation of spin-spin correlation functions is derived using fermionic realization for the Boltzmann weights. The approach also allows formulation of the partition function of 2DIM in the presence of an external magnetic field.

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I. INTRODUCTION

Two-dimensional Ising model[1] (2DIM) is one of the most attractive models in physics of low dimensions that describe physical properties of real materials and admit exact solution.[2–11] Originally, 2DIM was solved by Onsager[3] in 1944, and, subsequently, had attracted a steady interest of field theorists and mathematical physicists. Many effective and interesting approaches were developed to calculate the free energy, magnetization, and correlation functions of the model at large distances and all temperatures. Behavior of the model at the critical point is governed by the conformal symmetry, and thus, can be well described by the conformal field theory, which was developed in the seminal article by Belavin *et al.*[12] All the critical indices of 2DIM were calculated within the conformal field theory approach, in full agreement with the original lattice calculations.[3,8]

Although various physical characteristics of 2DIM have been derived using different approaches, still there are open questions that need to be answered. Some of the most important characteristics of 2DIM include lattice correlation functions and form factors.[13–15] These quantities attract considerable interest in connection with the condensed-matter problems,[16–20] as well as with the problems in string theory.[21] Importance of form factors becomes especially visible when one switches on the magnetic field.[15,22] Then the system exhibits the phenomenon, known in particle physics as quark confinement,[22] observed also in spin-1/2 Heisenberg chain with frustration and dimerization.[18,23,24]

One of the effective approaches to 2DIM is based on its equivalence to the theory of two-dimensional free fermions (see Ref. 11 and references therein) due to the presence of Kac-Word sign-factor[6] in the path-integral representation of the partition function. Though many works have been dedicated to the investigation of the 2DIM problem by means of the fermionic (Grassmann) variables, none of them had linked fermionic representation with vertex *R*-matrix formulation and possible extensions to other integrable models.

One of the motivations of the present work is to fill this gap and present a systematically developed field theoretical approach (action formulation of the partition function) to the 2D Ising and *XYZ* models on a square lattice, which is based on the Grassmann fields. The developed theory utilizes the graded *R* operator formalism[25–29] and allows the generalization to other integrable models, which is demonstrated in this work by operating with rather general *R* operator.

The paper is organized as follows. In Sec. II first we introduce the partition function of the 2DIM on the square lattice and demonstrate that the *R* matrices, constructed via Boltzmann weights, satisfy Yang-Baxter equations. Then in Sec. III the description of fermionic realization for the *R* matrices is followed, with particular cases of the eight-vertex model, which is equivalent to the one-dimensional (1D) quantum *XYZ* model and 2DIM: the case of finite magnetic fields is also considered. In Sec. IV the partition function is written in the coherent-state basis in terms of scalar fermions. It is represented as a continual integral over the fermionic fields with quadratic action for the 2DIM, when magnetic field vanishes, and for the free-fermionic limit of the eight-vertex model (*XY* model). The nonlocal fermionic action is obtained in Sec. IV A for the case with nonzero magnetic field. Continuum limit of the action is derived in Sec. IV C. In Sec. IV D the classical results for the free energy and the thermal capacity are re-obtained within the developed theory.

In Sec. V we present the technique for fermionic representation of correlation functions (with details included in the Appendix). In particular, the two-point correlation functions in 2DIM are considered on the lattice and their expressions are written in the Fourier coordinate basis. In the limit of infinite lattice, large distance spin-spin correlation functions can be presented as a determinant (Sec. V), which coincides with the Toeplitz determinant, studied in Ref. 8. Section VI is devoted to the investigation of the spectrum of one-dimensional quantum chain problem, which is equivalent to the classical 2DIM. The work is supplemented with an appendix with rather detailed description of the Jordan-Wigner spin-fermion transformation on 2D lattice, which we have used in the course of the calculations.

II. BOLTZMANN WEIGHTS AND YANG-BAXTER EQUATION

(1) *Boltzmann weights.* Classical two-dimensional Ising model on the square lattice can be defined via its local Boltzmann weights

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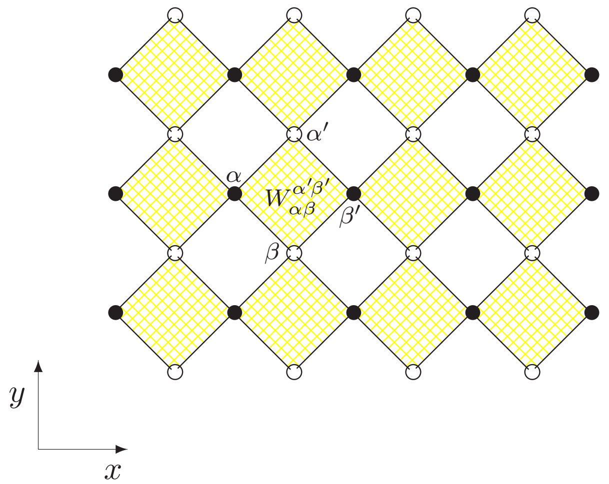

FIG. 1. (Color online) A fragment of the lattice of 2DIM: spin variables correspond to vertices, local Boltzmann weights correspond to dashed squares.

$$W_{\alpha\beta}^{\alpha'\beta'} = e^{J_1(\bar{\sigma}_\alpha\bar{\sigma}_{\alpha'}+\bar{\sigma}_\beta\bar{\sigma}_{\beta'})+J_2(\bar{\sigma}_\alpha\bar{\sigma}_\beta+\bar{\sigma}_{\alpha'}\bar{\sigma}_{\beta'})},$$

$$\alpha,\beta,\alpha',\beta' = 0,1, \tag{2.1}$$

where the two state spin variables $\bar{\sigma}_\alpha = \{\pm 1\}$ are assigned to the vertices of the lattice. The partition function

$$Z(J_1,J_2) = \sum_{\{\alpha,\beta\}} \prod W_{\alpha\beta}^{\alpha'\beta'} \tag{2.2}$$

is a sum over spin configurations of products of the Boltzmann weights $W_{\alpha\beta}^{\alpha'\beta'}$, each associated with the elementary square plaquette with vertices α, β, α', and β' and arranged in a checkerboard pattern (dashed squares in Fig. 1). There are imposed periodic boundary conditions on the spin variables.

Boltzmann weight $W_{\alpha\beta}^{\alpha'\beta'}$ in Eq. (2.1) can be regarded as a matrix,

$$W = \begin{pmatrix} e^{2(J_1+J_2)} & 1 & 1 & e^{2(-J_1+J_2)} \\ 1 & e^{2(J_1-J_2)} & e^{-2(J_1+J_2)} & 1 \\ 1 & e^{-2(J_1+J_2)} & e^{2(J_1-J_2)} & 1 \\ e^{2(-J_1+J_2)} & 1 & 1 & e^{2(J_1+J_2)} \end{pmatrix}, \tag{2.3}$$

acting as a linear operator on the direct product of two two-dimensional linear vector spaces,

$$|\alpha\rangle|\beta\rangle, \quad \alpha,\beta = 0,1; \quad \langle\beta'|\langle\alpha'|W|\alpha\rangle|\beta\rangle = W_{\alpha\beta}^{\alpha'\beta'}, \tag{2.4}$$

where $|0\rangle$ and $|1\rangle$ are orthonormalized vectors $\binom{0}{1}$ and $\binom{1}{0}$. Matrix (2.3) can be represented as a tensor product of spin operators,

$$\mathbf{W} = \frac{e^{2(J_1+J_2)}}{2}(\hat{1}\otimes\hat{1}+\sigma_z\otimes\sigma_z) + \frac{e^{2(J_1-J_2)}}{2}(\hat{1}\otimes\hat{1}-\sigma_z\otimes\sigma_z) + \frac{e^{2(-J_1+J_2)}}{2}(\sigma_1\otimes\sigma_1-\sigma_2\otimes\sigma_2) + \frac{e^{-2(J_1+J_2)}}{2}(\sigma_1\otimes\sigma_1 + \sigma_2\otimes\sigma_2) + (\hat{1}\otimes\sigma_1+\sigma_1\otimes\hat{1}), \tag{2.5}$$

where $\sigma_\alpha(\alpha=1,2,z)$ are Pauli matrices and $\hat{1}$ is the two-dimensional identity operator.

By use of a unitary transformation, one can represent the matrix W in the form of the R matrix, corresponding to the eight-vertex model. Let us define the unitary matrix $U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, such that $U\sigma_z U^{-1} = \sigma_1$, $U\sigma_1 U^{-1} = \sigma_z$, and $U\sigma_2 U^{-1} = -\sigma_2$. Then the action of these unitary transformations on the linear spaces, which are assigned to every site of the two-dimensional square lattice (Fig. 1), yields

$$\mathbf{R} = (U^{-1}\otimes U^{-1})\mathbf{W}(U\otimes U) = \frac{e^{2(J_1+J_2)}}{2}(\hat{1}\otimes\hat{1}+\sigma_1\otimes\sigma_1) + \frac{e^{2(J_1-J_2)}}{2}(\hat{1}\otimes\hat{1}-\sigma_1\otimes\sigma_1) + \frac{e^{2(J_2-J_1)}}{2}(\sigma_z\otimes\sigma_z - \sigma_2\otimes\sigma_2) + \frac{e^{-2(J_1+J_2)}}{2}(\sigma_z\otimes\sigma_z + \sigma_2 \otimes\sigma_2) + (\hat{1}\otimes\sigma_z + \sigma_z\otimes\hat{1}). \tag{2.6}$$

Partition function (2.2) of the model can be expressed via new weights [Eq. (2.6)], as

$$Z = \sum_{\{\alpha,\beta\}} \prod R_{\alpha\beta}^{\alpha'\beta'}, \tag{2.7}$$

where R operator has the following matrix form:

$$R = 2\begin{pmatrix} \cosh[2J_1]\cosh[2J_2]+1 & 0 & 0 & \cosh[2J_1]\sinh[2J_2] \\ 0 & \sinh[2J_1]\cosh[2J_2] & \sinh[2J_1]\sinh[2J_2] & 0 \\ 0 & \sinh[2J_1]\sinh[2J_2] & \sinh[2J_1]\cosh[2J_2] & 0 \\ \cosh[2J_1]\sinh[2J_2] & 0 & 0 & \cosh[2J_1]\cosh[2J_2]-1 \end{pmatrix}. \tag{2.8}$$

From Eq. (2.8) it is apparent that $R_{\alpha\beta}^{\alpha'\beta'}$ has the form of R matrix corresponding to the XY model. It fulfills the "free-fermionic" condition of the XY model,

$$R_{00}^{00}R_{11}^{11} - R_{00}^{11}R_{11}^{00} = R_{01}^{01}R_{10}^{10} - R_{01}^{10}R_{10}^{01}. \tag{2.9}$$

(2) *Yang-Baxter equations.* In this section we examine whether matrix (2.8) is a solution of Yang-Baxter equations. We shall verify this by using Baxter's transformation,[9]

$$e^{\pm 2J_2} = \mathrm{cn}[iu,k] \mp i\mathrm{sn}[iu,k],$$

$$e^{\pm 2J_1} = i(\mathrm{dn}[iu,k] \pm 1)/(k\mathrm{sn}[iu,k]). \tag{2.10}$$

It has been proven in Ref. 9 that for fixed k parameter, two transfer matrices with different parameters u commute. The case of $k=1$ corresponds to the point of phase transition.

Now we rewrite R matrix (2.8) in terms of functions (2.10),

$$R(u,k) = \begin{pmatrix} 1 + i\dfrac{\mathrm{cn}[iu,k]\mathrm{dn}[iu,k]}{k\mathrm{sn}[iu,k]} & 0 & 0 & \dfrac{\mathrm{dn}[iu,k]}{k} \\ 0 & i\dfrac{\mathrm{cn}[iu,k]}{k\mathrm{sn}[iu,k]} & \dfrac{1}{k} & 0 \\ 0 & \dfrac{1}{k} & i\dfrac{\mathrm{cn}[iu,k]}{k\mathrm{sn}[iu,k]} & 0 \\ \dfrac{\mathrm{dn}[iu,k]}{k} & 0 & 0 & -1 + i\dfrac{\mathrm{cn}[iu,k]\mathrm{dn}[iu,k]}{k\mathrm{sn}[iu,k]} \end{pmatrix}. \tag{2.11}$$

Let us multiply matrix (2.11) by $-ik\mathrm{sn}(iu,k)$ and define the matrix $\mathbf{r}(u,k)$ as

$$\mathbf{r}(u,k) = -ik\mathrm{sn}[iu,k]R(u,k). \tag{2.12}$$

It is straightforward to verify that $\mathbf{r}(0,k)=I$, where $I=\hat{1}\otimes\hat{1}$ is the identity matrix. Importantly, it takes place the relation

$$\mathbf{r}(-u,k) = \mathbf{r}^{-1}(u,k). \tag{2.13}$$

Using the properties of the Jacobi elliptic functions, one can verify that $\mathbf{r}(u,k)$ satisfies the Yang-Baxter equation

$$\sum_{\beta_1,\beta_2,\beta_2'} \mathbf{r}_{\alpha_1\alpha_2}^{\beta_1\beta_2}(u-v,k)\mathbf{r}_{\beta_2\alpha_3}^{\beta_2'\gamma_3}(u,k)\mathbf{r}_{\beta_1\beta_2'}^{\gamma_1\gamma_2}(v,k) = \sum_{\beta_2,\beta_2',\beta_3} \mathbf{r}_{\alpha_2\alpha_3}^{\beta_2\beta_3}(v,k)\mathbf{r}_{\alpha_1\beta_2}^{\gamma_1\beta_2'}(u,k)\mathbf{r}_{\beta_2'\beta_3}^{\gamma_2\gamma_3}(u-v,k). \tag{2.14}$$

Note that there is also another R matrix corresponding to 2DIM. It is known, that the classical 2DIM is a special case of the eight-vertex model.[9] In general, the R matrix of the eight-vertex (or XYZ) model can be parameterized by two model parameters, k and λ, as

$$r_{xyz} = \begin{pmatrix} \dfrac{\mathrm{sn}\left[i\dfrac{\lambda-u}{2},k\right]}{\mathrm{sn}[i\lambda,k]} & 0 & 0 & -k\mathrm{sn}\left[i\dfrac{\lambda+u}{2},k\right]\mathrm{sn}\left[i\dfrac{\lambda-u}{2},k\right] \\ 0 & 1 & \dfrac{\mathrm{sn}\left[i\dfrac{\lambda+u}{2},k\right]}{\mathrm{sn}[i\lambda,k]} & 0 \\ 0 & \dfrac{\mathrm{sn}\left[i\dfrac{\lambda+u}{2},k\right]}{\mathrm{sn}[i\lambda,k]} & 1 & 0 \\ -k\mathrm{sn}\left[i\dfrac{\lambda+u}{2},k\right]\mathrm{sn}\left[i\dfrac{\lambda-u}{2},k\right] & 0 & 0 & \dfrac{\mathrm{sn}\left[i\dfrac{\lambda-u}{2},k\right]}{\mathrm{sn}[i\lambda,k]} \end{pmatrix}. \tag{2.15}$$

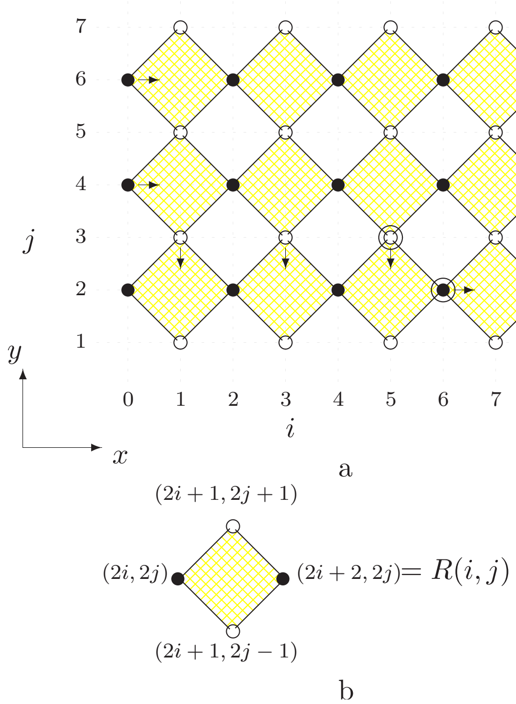

FIG. 2. (Color online) (a) Bold lines and circled vertices represent a fragment of the lattice of the model in coordinate plane (dotted lines). (b) Local R operator.

The "Ising" limit corresponds to the choice of $\lambda = \frac{1}{2}I'$, where I' is one of two half-periods of the elliptic functions.[9] In this case the eight-vertex model, defined on a rectangular lattice, splits into two independent Ising models defined on the two sublattices.

(3) *The transfer matrix and the Hamiltonian.* For convenience we denote the coordinates of the lattice sites by even-even $(2i,2j)$ (black circles on Figs. 1 and 2), and odd-odd $(2i+1,2j+1)$ (white circles on Figs. 1 and 2) numbers, and assign two-dimensional linear spaces of quantum states of spins $|\alpha_{2i,2j}\rangle$ and $|\alpha_{2i+1,2j+1}\rangle$ to each of these spaces. Periodic boundary conditions imply

$$|\alpha_{0,2j}\rangle = |\alpha_{2N,2j}\rangle \quad \text{and} \quad |\alpha_{2i+1,1}\rangle = |\alpha_{2i+1,2N+1}\rangle. \tag{2.16}$$

Local $R(i,j)$ operators (2.8) are acting linearly on the product of spaces $|\alpha_{2i,2j}\rangle|\alpha_{2i+1,2j-1}\rangle$ at the sites $(2i,2j)$ and $(2i+1,2j-1)$, moving them onto the sites $(2i+1,2j+1)$ and $(2i+2,2j)$,

$$R(i,j){:}|\alpha_{2i,2j}\rangle|\alpha_{2i+1,2j-1}\rangle \Rightarrow |\alpha_{2i+1,2j+1}\rangle|\alpha_{2i+2,2j}\rangle. \tag{2.17}$$

The product of R matrices on each chain along x direction on the lattice (see Fig. 2) in the formulas of partition function (2.7), after summation over the boundary states, constitutes a transfer matrix, τ. Taking into account conditions (2.16), we obtain the following representation for the transfer matrices:

$$\tau_j = \mathrm{tr}_1 \prod_{i=N-1}^{0} R(i,j) \equiv \sum_{\alpha_{0,2j}} \langle \alpha_{0,2j}| \prod_{i=N-1}^{0} R(i,j)|\alpha_{0,2j}\rangle. \tag{2.18}$$

They act on the states of spins at sites $\{2i+1,2j-1\}_{i=0,N-1}$,

$$|\Sigma_j\rangle = |\alpha_{1,2j-1}\rangle|\alpha_{3,2j-1}\rangle \cdots |\alpha_{2N-1,2j-1}\rangle,$$

and map them onto the states at sites $\{2i+1,2j-1\}_{i=0,N-1}$,

$$|\Sigma_{j+1}\rangle = |\alpha_{1,2j+1}\rangle|\alpha_{3,2j+1}\rangle \cdots |\alpha_{2N-1,2j+1}\rangle.$$

One can interpret the, y direction, marked by integers j, as a time direction, while the transfer matrix τ_j will be the evolution operator for discrete lattice time. In terms of the transfer matrices, partition function (2.7) acquires the following form:

$$Z = \mathrm{tr}_2 \prod_{j=N}^{1} \tau_j \equiv \sum_{\{\alpha_{2i+1,1}\}_{i=0,N-1}} \langle \Sigma_1| \prod_{j=N}^{1} \tau_j |\Sigma_1\rangle. \tag{2.19}$$

In Eqs. (2.18) and (2.19), tr_1 and tr_2 represent sums of the states at the boundaries with (even, even) and (odd, odd) coordinates (dark and light circles in the figures), respectively. Borrowing the usual terminology of the transfer-matrix theory, we can refer the states marked by white circles on the lattice as "quantum" states, while the states marked with black circles as "auxiliary" states.

(4) *2D classical model as a* $(1+1)D$ quantum theory. Following to Ref. 30, one can introduce the limit

$$J_1 \sim J\Delta t, \quad e^{-J_2} \sim h\Delta t, \quad \Delta t \ll 1 \tag{2.20}$$

for the continuous time, in order to establish connection between two-dimensional classical Ising model and a *quantum* one-dimensional model. In this limit R-matrix (2.8) becomes

$$R = \frac{I}{h\Delta t} + \begin{pmatrix} 1 & 0 & 0 & J/h \\ 0 & 0 & J/h & 0 \\ 0 & J/h & 0 & 0 \\ J/h & 0 & 0 & -1 \end{pmatrix} + \frac{h\Delta t}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.21}$$

It acquires the following operator form:

$$\mathbf{R} = \frac{1}{h\Delta t}\{\hat{1} \otimes \hat{1} + 2\Delta t[J\sigma_1 \otimes \sigma_1 + h(1 \otimes \sigma_z + \sigma_z \otimes 1)]\}. \tag{2.22}$$

The coefficient of Δt in $\log \tau$, constructed with R-matrices (2.22), defines 1D quantum Hamiltonian for the Ising model on a chain in a *transverse* magnetic field h,[10]

$$\mathcal{H} = \sum_i [J\sigma_1(i)\sigma_1(i+1) + h\sigma_z(i)]. \tag{2.23}$$

(5) *Finite magnetic field.* The Boltzmann weights of the classical 2DIM in a uniformly applied magnetic field B have the following matrix representation: $W_{B\,\alpha\beta}^{\alpha'\beta'} = e^{J_1(\bar{\sigma}_\alpha\bar{\sigma}_{\alpha'}+\bar{\sigma}_\beta\bar{\sigma}_{\beta'})+J_2(\bar{\sigma}_\alpha\bar{\sigma}_\beta+\bar{\sigma}_{\alpha'}\bar{\sigma}_{\beta'})+(B/2)(\bar{\sigma}_\alpha+\bar{\sigma}_\beta+\bar{\sigma}_{\alpha'}+\bar{\sigma}_{\beta'})}$. Operator representation of W_B after unitary transformation gets

$$\mathbf{R}_B = U^{-1} \otimes U^{-1}\mathbf{W}_B U \otimes U$$
$$= \frac{e^{2(J_1+J_2)}}{2}\cosh[2B](\hat{1} \otimes \hat{1} + \sigma_1 \otimes \sigma_1) + \frac{e^{2(J_1-J_2)}}{2}$$
$$\times(\hat{1} \otimes \hat{1} - \sigma_1 \otimes \sigma_1) + \frac{e^{2(J_2-J_1)}}{2}$$
$$\times(\sigma_z \otimes \sigma_z - \sigma_2 \otimes \sigma_2) + \frac{e^{-2(J_1+J_2)}}{2}(\sigma_z \otimes \sigma_z + \sigma_2 \otimes \sigma_2)$$
$$+ \frac{e^{2(J_1+J_2)}}{2}\sinh[2B](\hat{1} \otimes \sigma_1 + \sigma_1 \otimes \hat{1}) + \sinh[B]$$
$$\times(\sigma_1 \otimes \sigma_z + \sigma_z \otimes \sigma_1) + \cosh[B](\hat{1} \otimes \sigma_z + \sigma_z \otimes \hat{1}). \tag{2.24}$$

In the limit

$$J_1 \sim J\Delta t, \quad e^{-J_2} \sim h\Delta t, \quad B \sim \mathcal{B}\Delta t, \quad \Delta t \ll 1, \tag{2.25}$$

operator R_B obtains the form

$$\mathbf{R}_B = \frac{1}{h\Delta t}\left\{\hat{1} \otimes \hat{1} + 2\Delta t\left[J\sigma_1 \otimes \sigma_1 + h(\hat{1} \otimes \sigma_z + \sigma_z \otimes \hat{1}) + \frac{\mathcal{B}}{2}(\hat{1} \otimes \sigma_1 + \sigma_1 \otimes \hat{1})\right]\right\}, \tag{2.26}$$

which establishes an equivalence with the quantum 1DIM in the magnetic field $\mathcal{B}$. In this case the operator $\frac{1}{\Delta t}\log \tau$, when $\Delta t \to 0$, defines the following structure of Hamiltonian operator:

$$\mathcal{H}_B = \sum_i \left(J\sigma_1(i)\sigma_1(i+1) + h\sigma_z(i) + \frac{\mathcal{B}}{2}\sigma_1(i)\right).$$

III. FERMIONIC REALIZATION OF BOLTZMANN WEIGHTS

Besides the matrix formulation for Boltzmann weights (2.5) and (2.6), one can think of alternative representations. Examples include representation of the R matrix in the Fock space of the scalar fermions and in a space with the basis of fermionic coherent states, developed in Refs. 25–27 and 29. These reformulations are fully equivalent, and they allow developing a field theory corresponding to the model. The latter simplifies calculations of physical quantities (particularly, the free energy and the magnetization) of the model, and can be extended for the computation of form factors too, which are problematic in the standard scheme.

Let us now consider the graded Fock space of scalar fermions $c^+(i,j)$ and $c(i,j)$, on the lattice, $([c^+(i,j),c(i,j)]_+ = \delta_{ik}\delta_{jr})$, identifying the two-dimensional basis at each site, labeled by $\{i,j\}$, with $|0\rangle_{i,j}$ $(c(i,j)|0\rangle_{i,j}=0)$; and $|1\rangle_{i,j} = c^+(i,j)|0\rangle_{i,j}$.

Then it is not hard to construct fermionic representation of the R operator. Note, that the R operator defined in the previous section "permutes" the arrangement of the spaces (as it is a "check" R matrix), $R:|\alpha'\rangle_1|\beta'\rangle_2 \Rightarrow |\beta\rangle_2|\alpha\rangle_1$, so for graded spaces we have

$$\mathcal{R} = R_{\alpha\beta}^{\beta'\alpha'}|\beta\rangle_2|\alpha\rangle_{12}\langle\beta'|_1\langle\alpha'|$$
$$= R_{\alpha\beta}^{\beta'\alpha'}|\beta\rangle_2\langle\beta'||\alpha\rangle_1\langle\alpha'|(-1)^{p(\beta')p(\alpha)}, \tag{3.1}$$

where $p(\alpha)=\alpha$ is the parity of the space $|\alpha\rangle$. Operators $|\alpha\rangle_i\langle\alpha'|$ act on the Fock space with the basis $\{|0\rangle_i,|1\rangle_i\}$, as

$$|0\rangle_i\langle 0| = 1 - c_i^+c_i, \quad |1\rangle_i\langle 1| = c_i^+c_i, \quad |0\rangle_i\langle 1| = c_i, \quad |1\rangle_i\langle 0| = c_i^+.$$

This means, that in terms of two scalar fermions $\{c_i^+,c_j\} = \delta_{ij}$ and $\{c_i,c_j\}=\{c_i^+,c_j^+\}=0$, $i,j=1,2$, the R-perator in the zero-field limit, $B=0$, reads

$$\mathcal{R}(c_1^+,c_1;c_2^+,c_2) = R_{00}^{00} + R_{01}^{01}c_1^+c_2 + R_{10}^{10}c_2^+c_1 + (R_{10}^{01} - R_{00}^{00})c_1^+c_1 + (R_{01}^{10} - R_{00}^{00})c_2^+c_2 + R_{00}^{11}c_2^+c_1^+ + R_{11}^{00}c_2c_1 + (R_{00}^{00} - R_{10}^{01} - R_{01}^{10} - R_{11}^{11})c_1^+c_1c_2^+c_2, \tag{3.2}$$

where R_{ij}^{kr} are the matrix elements of the R matrix.

(1) The fermionic representation [Eq. (3.2)] of the Ising model's R-matrix (2.8) has the following matrix elements:

$$R_{00}^{00} = 2(\cosh[2J_1]\cosh[2J_2] + 1),$$
$$R_{11}^{11} = 2(\cosh[2J_1]\cosh[2J_2] - 1),$$
$$R_{01}^{01} = R_{10}^{10} = 2\,\sinh[2J_1]\cosh[2J_2], \tag{3.3}$$
$$R_{10}^{01} = R_{01}^{10} = 2\,\sinh[2J_1]\sinh[2J_2],$$
$$R_{00}^{11} = R_{11}^{00} = 2\,\cosh[2J_1]\sinh[2J_2].$$

In the following we shall operate in the coherent-state basis. For that purpose we need to represent $\mathcal{R}$ operator in the normal ordered form. For *zero magnetic field* the operator $\mathcal{R}$ can be expressed as an exponent of a quadratic form [a consequence of property (2.9)],

$$\mathcal{R} = R_{00}^{00}{:}\exp\,\mathcal{A}(c_1^+,c_2^+,c_1,c_2){:}, \tag{3.4}$$

where

$$\mathcal{A}(c_1^+,c_2^+,c_1,c_2) = (c-1)(c_1^+c_1 + c_2^+c_2) + b(c_1^+c_2 + c_2^+c_1) + d(c_2^+c_1^+ + c_2c_1), \tag{3.5}$$

$$a = \frac{R_{00}^{00}}{2}, \quad b = R_{01}^{01}/R_{00}^{00},$$
$$c = R_{01}^{10}/R_{00}^{00}, \quad d = R_{00}^{11}/R_{00}^{00}. \tag{3.6}$$

For a *finite magnetic field* B the fermionic realization $\mathcal{R}_B$ of the R_B-operator (2.24) can be obtained in the same way, substituting the corresponding matrix elements in formulas (3.1). Then the normal ordered form of the $\mathcal{R}_B$ can be represented as follows:

$$\mathcal{R}_B = r_B{:}e^{\mathcal{A}_B(c_1^+,c_2^+,c_1,c_2)}{:}, \tag{3.7}$$

$$\mathcal{A}_B(c_1^+,c_2^+,c_1,c_2) = c_B(c_1^+c_1 + c_2^+c_2) + b_B(c_1^+c_2 + c_2^+c_1) + d_B(c_2^+c_1^+ + c_2c_1) + h_B(c_1 + c_2 + c_1^+ + c_2^+) + (\eta_1c_1 + \eta_2c_1^+)c_2^+c_2 + (\eta_2c_2 + \eta_1c_2^+)c_1^+c_1, \quad (3.8)$$

with

$$r_B = 2\cosh[B] + 2\cosh[2J_1]\cosh[2J_2] + e^{2(J_1+J_2)}(\sinh[B])^2,$$

$$c_B = -\frac{2}{r_B}(\cosh[B] + \cosh[2(J_1 - J_2)]),$$

$$b_B = \frac{1}{2r_B}(e^{2(J_1+J_2)}(\sinh[B])^2 + 2\cosh[2J_2]\sinh[2J_1]),$$

$$d_B = \frac{1}{2r_B}(e^{2(J_1+J_2)}(\sinh[B])^2 + 2\cosh[2J_1]\sinh[2J_2]),$$

$$h_B = \frac{1}{r_B}(\sinh[B] + \sinh[2B]e^{2(J_1+J_2)}),$$

$$\eta_1 = \frac{4}{r_B^2}(e^{2J_1}\cosh[B] + \cosh[2J_2])\sinh[B]\sinh[2J_1],$$

$$\eta_2 = -\frac{4}{r_B^2}\sinh[B]\sinh[2J_1]\sinh[2J_2]. \quad (3.9)$$

(2) *General R* matrix and XYZ model. The fermionic representation (3.2) is justified as well for arbitrary R matrix, which has form

$$R = \begin{pmatrix} R_{00}^{00} & 0 & 0 & R_{00}^{11} \\ 0 & R_{01}^{01} & R_{01}^{10} & 0 \\ 0 & R_{10}^{01} & R_{10}^{10} & 0 \\ R_{11}^{00} & 0 & 0 & R_{11}^{11} \end{pmatrix}. \quad (3.10)$$

Now the $\mathcal{A}(c_1^+,c_2^+,c_1,c_2)$ in the normal ordered form (3.4) has a quartic term also,

$$\mathcal{A}(c_1^+,c_2^+,c_1,c_2) = (c-1)c_1^+c_1 + (c'-1)c_2^+c_2 + bc_1^+c_2 + b'c_2^+c_1 + dc_2^+c_1^+ + d'c_2c_1 + \Delta c_1^+c_1c_2^+c_2, \quad (3.11)$$

$$a = \frac{R_{00}^{00}}{2}, \quad b = \frac{R_{01}^{01}}{R_{00}^{00}}, \quad b' = \frac{R_{10}^{10}}{R_{00}^{00}}, \quad c = \frac{R_{10}^{01}}{R_{00}^{00}}, \quad c' = \frac{R_{01}^{10}}{R_{00}^{00}},$$

$$d = \frac{R_{11}^{00}}{R_{00}^{00}}, \quad d' = \frac{R_{00}^{11}}{R_{00}^{00}},$$

$$\Delta = \frac{R_{01}^{01}R_{10}^{10} + R_{11}^{00}R_{00}^{11} - R_{10}^{01}R_{01}^{10} - R_{00}^{00}R_{11}^{11}}{R_{00}^{00^2}}. \quad (3.12)$$

For the *XYZ* model's general R matrix, given in Eq. (2.15), the Δ parameter writes as

$$\Delta = 2\frac{\mathrm{sn}\left[i\frac{\lambda+u}{2},k\right]\mathrm{cn}[i\lambda,k]\mathrm{dn}[i\lambda,k]}{\mathrm{sn}\left[i\frac{\lambda-u}{2},k\right](\mathrm{sn}[i\lambda,k])^2}. \quad (3.13)$$

The limit $\Delta=0$ corresponds to the free-fermionic *XY* model. It fulfills when

$$\mathrm{cn}[i\lambda,k]\mathrm{dn}[i\lambda,k] = 0. \quad (3.14)$$

Possible solutions are $\mathrm{cn}[I,k]=0$ and $\mathrm{dn}[I+iI'k]=0$. Here I,I' are the half-periods of the elliptic functions. Note that the Ising limit derived in the Ref. 9 corresponds to the values

$$\lambda = I'/2, \quad \Delta = 2\frac{k^{1/2}(1+k)\mathrm{sn}\left[i\frac{\lambda+u}{2},k\right]}{\mathrm{sn}\left[i\frac{\lambda-u}{2},k\right]}. \quad (3.15)$$

IV. QUANTUM FIELD THEORY REPRESENTATION ON THE LATTICE: WITH GENERAL *R* MATRIX AND 2DIM

In this section we will introduce fermionic fields $\bar{\psi}(i,j)$ and $\psi(i,j)$, corresponding to the coherent states of scalar fermions $c^+(i,j)$ and $c(i,j)$. By definition, coherent states are the eigenstates of annihilation operators of scalar fermions. For a set of the scalar fermions c_i and c_i^+, they are defined by the following relations:

$$c_i|\psi_i\rangle = \psi_i|\psi_i\rangle, \quad \langle\bar{\psi}_i|c_i^+ = \langle\bar{\psi}_i|\bar{\psi}_i. \quad (4.1)$$

Because of the Fermi statistics, namely, anticommutation relations, these eigenvalues are Grassmann variables denoted in Eq. (4.1) by ψ_i and $\bar{\psi}_i$. They fulfill (i) orthonormality and (ii) completeness relations,

$$\langle\bar{\psi}_i|\psi_j\rangle = \delta_{ij}e^{\bar{\psi}_i\psi_i}, \quad \int d\bar{\psi}_i d\psi_i e^{-\bar{\psi}_i\psi_i}|\psi_i\rangle\langle\bar{\psi}_i| = I. \quad (4.2)$$

The *kernel* of any normal ordered operator $K(\{c_i^+,c_j\})$ in terms of coherent states can be obtained simply by replacing creation-annihilation operators c_i,c_i^+ by their eigenvalues and multiplying by $\exp(\Sigma_i\bar{\psi}_i\psi_i)$,

$$\mathcal{K}(\{\bar{\psi}_i,\psi_j\}) \equiv \left\langle\prod\bar{\psi}_i\middle|K(\{c_i^+,c_j\})\middle|\prod\psi_j\right\rangle = \exp\left(\sum_i\bar{\psi}_i\psi_i\right)K(\{\bar{\psi}_i,\psi_j\}). \quad (4.3)$$

The trace of the operator $K(\{c_i^+,c_j\})$ in coherent states is an integral over the Grassmann variables,

$$\mathrm{tr}\,K(\{c_i^+,c_j\}) = \int D\psi D\bar{\psi}\exp\left(\sum_i\bar{\psi}_i\psi_i\right)\mathcal{K}(\{\bar{\psi}_i,\psi_j\}),$$

$$D\psi D\bar{\psi} = \prod_i d\psi_i d\bar{\psi}_i. \quad (4.4)$$

In order to obtain the form of the partition function Z [Eq. (2.7)] in the basis of coherent states, let us at each circled vertex of the lattice, between the $R(i,j)$ operators (see Fig. 2), insert the following identity operators:

$$I=\int d\bar{\psi}(i,j)d\psi(i,j)e^{-\bar{\psi}(i,j)\psi(i,j)}|\psi(i,j)\rangle\langle\bar{\psi}(i,j)|.$$

With the properties of coherent states represented above, we can easily calculate the matrix elements of the $\mathcal{R}$-operator (3.2), using the normal ordered form representations (3.4) and (3.11), in terms of Grassmann fields,

$$\begin{aligned}&\langle\bar{\psi}(2i+1,2j+1)|\langle\bar{\psi}(2i+2,2j)|\\&\quad\times\mathcal{R}(i,j)|\psi(2i,2j)\rangle|\psi(2i+1,2j-1)\rangle\\&\quad=R_{00}^{00}\exp\{\mathcal{A}[\bar{\psi}(2i+2,2j),\bar{\psi}(2i+1,2j+1),\psi(2i,2j),\psi(2i\\&\quad+1,2j-1)]+\bar{\psi}(2i+2,2j)\psi(2i,2j)+\bar{\psi}(2i+1,2j\\&\quad+1)\psi(2i+1,2j-1)\}.\end{aligned}\tag{4.5}$$

Then the partition function Z for large N can be written as a path integral,

$$Z=\mathrm{tr}\prod_{j=1}^{N}\prod_{i=N-1}^{0}\mathcal{R}(i,j)=(R_{00}^{00})^{N^2}\int D\bar{\psi}D\psi e^{-A(\bar{\psi},\psi)},\tag{4.6}$$

with action $A(\bar{\psi},\psi)$,

$$\begin{aligned}-A(\bar{\psi},\psi)=&\sum_{i,j}\{b\bar{\psi}(2i+2,2j)\psi(2i+1,2j-1)+b'\bar{\psi}(2i\\&+1,2j+1)\psi(2i,2j)+c\bar{\psi}(2i+2,2j)\psi(2i,2j)\\&+c'\bar{\psi}(2i+1,2j+1)\psi(2i+1,2j-1))+d\bar{\psi}(2i\\&+2,2j)\bar{\psi}(2i+1,2j+1)+d'\psi(2i,2j)\psi(2i+1,2j\\&-1)+\Delta\bar{\psi}(2i+2,2j)\psi(2i+1,2j-1)\bar{\psi}(2i+1,2j\\&+1)\psi(2i,2j)-\bar{\psi}(2i,2j)\psi(2i,2j)-\bar{\psi}(2i+1,2j\\&+1)\psi(2i+1,2j+1)\}+\sum_{j}\bar{\psi}(2N,2j)\psi(0,2j)\\&+\sum_{i}\bar{\psi}(2i+1,2N+1)\psi(2i+1,1).\end{aligned}\tag{4.7}$$

In sum (4.7) the last two terms come from the trace. So, the partition function of a model, defined by R-matrix (3.10), has fermionic path-integral representation with local action (4.7) on the two-dimensional lattice.

As it is apparent, the 2DIM has fermionic representation with local quadratic action [see Eqs. (3.5) and (3.3)]. It is true also for the partition function of the XY model, defined with the Eqs. (3.14), as well, since the Gaussian quadratic form is a consequence of the "free-fermionic" property, given by Eq. (2.9), and the formula of the coefficient at quartic term (3.12). On the other hand, the Ising limit of the eight-vertex (XYZ) model does not correspond to a quadratic action, as it is followed from the Eqs. (3.15). However it is well known that the two limits of the XYZ model-Ising limit (XZ) and free-fermionic limit (XY) are equivalent and can be brought one to another by redefinition of the model parameters.

A. Path integral representation of partition function for the case of finite magnetic field

For construction of partition function of 2DIM in a nonzero magnetic field we make use of fermionic expression (3.7). In order to take into account the graded character of the $\mathcal{R}(B)$ operators (recall, that they have no definite parity), we are led to include nonlocal operators,

$$Z(B)=\mathrm{tr}\prod_{j=1}^{N}\prod_{i=N-1}^{0}\{\mathcal{R}_B^{(\mathrm{even})}(i,j)+\mathcal{R}_B^{(\mathrm{odd})}(i,j)J(i,j)\}.\tag{4.8}$$

In Eq. (4.8) operators $\mathcal{R}_B^{(\mathrm{even})}$ and $\mathcal{R}_B^{(\mathrm{odd})}$ represent the parts of the $\mathcal{R}(B)$-operator (3.7) that have even and odd gradings (or their series expansions consist of even/odd powers of B or fermionic operators), correspondingly. Operator $J(i,j)=\Pi(1-2n)$ is the Jordan-Wigner nonlocal operator [see Eq. (5.5) and the Appendix].

Let us introduce formal definitions,

$$\begin{aligned}\mathcal{R}_B'^{(\mathrm{even})/(\mathrm{odd})}(i,j)&=\mathcal{R}_B^{(\mathrm{even})/(\mathrm{odd})}(i,j)[1-2n(2i,2j)],\\\mathcal{R}_B''^{(\mathrm{even})/(\mathrm{odd})}(i,j)&=[1-2n(2i+1,2j+1)]\mathcal{R}_B^{(\mathrm{even})/(\mathrm{odd})}(i,j).\end{aligned}\tag{4.9}$$

Then we can expand the product in Eq. (4.8) and rewrite it as

$$Z(B)=\mathrm{tr}\sum_{\mathcal{C}_{\{k,r\}}}\prod_{i,j}\mathbb{R}_B^{\mathcal{C}_{kr}}(i,j),\tag{4.10}$$

where the sum goes over all lattice sites denoted by $\mathcal{C}_{\{k,r\}}$. Operator $\mathbb{R}_B^{\mathcal{C}_{kr}}(i,j)$ is attached to the square (i,j) (see Fig. 2). It is equal to $\mathcal{R}_B^{(\mathrm{odd})}(i,j)$ or $\mathcal{R}_B'^{/\prime\prime(\mathrm{odd})}(i,j)$, if $\{i,j\}=\{k,r\}$, and is equal to $\mathcal{R}_B^{(\mathrm{even})}(i,j)$ or $\mathcal{R}_B'^{/\prime\prime(\mathrm{even})}(i,j)$ otherwise.

Each summand in Eq. (4.10) can be written in the basis of coherent states in the same way as it was done in previous subsection. Finally we find

$$\begin{aligned}Z(B)=\int D\bar{\psi}D\psi\exp\Big[&-\sum_{i,j}\bar{\psi}(i,j)\psi(i,j)\\&+\sum_{j}\bar{\psi}(2N,2j)\psi(0,2j)\\&+\sum_{i}\bar{\psi}(2i+1,2N+1)\psi(2i+1,1)+\mathcal{I}_B(\bar{\psi},\psi)\Big],\end{aligned}$$

$$\begin{aligned}\mathcal{I}_B(\bar{\psi},\psi)=\ln\sum_{\mathcal{C}_{\{k,r\}}}\prod_{i,j}&\langle\bar{\psi}(2i+1,2j+1)|\langle\bar{\psi}(2i+2,2j)|\mathbb{R}_B^{\mathcal{C}_{kr}}(i,j)\\&\times|\psi(2i,2j)\rangle|\psi(2i+1,2j-1)\rangle.\end{aligned}\tag{4.11}$$

Because we operate with local fermionic R_B matrices, form (4.11) of the partition function will be held for the case of inhomogeneous magnetic field as well.

B. Partition function

For calculation of partition function (4.6) in the "free-fermionic" case $\Delta=0$, we need to diagonalize the action $A(\bar{\psi},\psi)$. Taking into account antiperiodic boundary conditions imposed on the Grassmann fields,

$$\bar{\psi}(2N,2j) = -\bar{\psi}(0,2j), \quad \psi(2N,2j) = -\psi(0,2j),$$

$$\bar{\psi}(2N+1,2j+1) = -\bar{\psi}(1,2j+1),$$

$$\psi(2N+1,2j+1) = -\psi(1,2j+1),$$

$$\bar{\psi}(2i,2N) = -\bar{\psi}(2i,0), \quad \psi(2i,2N) = -\psi(2i,0),$$

$$\bar{\psi}(2i+1,2N+1) = -\bar{\psi}(2i+1,1),$$

$$\psi(2i+1,2N+1) = \psi(2i+1,1), \tag{4.12}$$

we can perform the Fourier transformation with odd momenta,

$$\psi(r,k) = \frac{1}{N}\sum_{n_r,n_k=0}^{N-1} e^{-(i\pi/2N)[(2n_r+1)r+(2n_k+1)k]} \times \psi_\alpha\left[\frac{\pi}{2N}(2n_r+1), \frac{\pi}{2N}(2n_k+1)\right]. \tag{4.13}$$

Here $\alpha=1$ for even coordinates (r,k), and $\alpha=2$ for odd coordinates. After defining new Grassmann fields $\psi_3(p,q), \psi_4(p,q)$ as

$$\psi_1(\pi-p,\pi-q) \equiv -\bar{\psi}_3(p,q),$$

$$\psi_2(\pi-p,\pi-q) \equiv -\bar{\psi}_4(p,q),$$

$$\bar{\psi}_1(\pi-p,\pi-q) \equiv \psi_3(p,q), \quad \bar{\psi}_2(\pi-p,\pi-q) \equiv \psi_4(p,q), \tag{4.14}$$

we will come to the following simple form for the action A (4.7) in the momentum space

$$-A(\bar{\psi},\psi) = \sum_{p,q}\sum_{k,r} \mathrm{A}_{kr}(p,q)\bar{\psi}_k(p,q)\psi_r(p,q). \tag{4.15}$$

In Eq. (4.15) we have introduced the notations

$$\mathrm{A}(p,q) = \begin{pmatrix} ce^{i2p}-1 & be^{i(p+q)} & 0 & -de^{i(p-q)} \\ b'e^{i(p+q)} & c'e^{2iq}-1 & de^{-i(p-q)} & 0 \\ 0 & d'e^{-i(p-q)} & ce^{-2ip}-1 & b'e^{-i(p+q)} \\ -d'e^{i(p-q)} & 0 & be^{-i(p+q)} & c'e^{-2iq}-1 \end{pmatrix}, \tag{4.16}$$

and

$$p = \frac{\pi}{2N}(2n_p+1), \quad q = \frac{\pi}{2N}(2n_q+1),$$

$$n_p = 1, \ldots, N/2-1, \quad n_q = 1, \ldots, N-1. \tag{4.17}$$

Then the partition function acquires form of a product of determinants

$$Z = (R^{00}_{00})^{N^2} \prod_{n_p,n_q=0}^{(N/2)-1,N-1} \mathrm{Det}\left[\mathrm{A}\left(\pi\frac{(2n_p+1)}{2N}, \pi\frac{(2n_q+1)}{2N}\right)\right]. \tag{4.18}$$

The determinants are found as

$$\begin{aligned}\mathrm{Det}[\mathrm{A}(p,q)] = \Bigg\{ & 1 + 2\left(\frac{R^{10}_{01}}{R^{00}_{00}}\right)^2 + \left(\frac{R^{11}_{11}}{R^{00}_{00}}\right)^2 \\ & - 2\frac{R^{10}_{01}(R^{00}_{00}-R^{11}_{11})}{(R^{00}_{00})^2}(\cos[2p]+\cos[2q]) \\ & + 4\left(\frac{R^{10}_{01}}{R^{00}_{00}}\right)^2\cos[2p]\cos[2q] - 2\left(\frac{R^{01}_{01}}{R^{00}_{00}}\right)^2 \\ & \times\cos[2(p+q)] - 2\left(\frac{R^{11}_{00}}{R^{00}_{00}}\right)^2\cos[2(p-q)]\Bigg\}. \end{aligned} \tag{4.19}$$

Here we have assumed that $R^{10}_{01}=R^{01}_{10}$ and $R^{01}_{01}=R^{10}_{10}$.

Ising model. Substituting the matrix elements given in Eqs. (3.3), we are arriving at

$$\begin{aligned}\mathrm{Det}[\mathrm{A}(p,q)]\{1+\cosh[2J_1]\cosh[2J_2]\}^2 &= 2\{1+(\cosh[2J_1]\cosh[2J_2])^2+(\sinh[2J_1]\sinh[2J_2])^2 \\ &- 2\sinh[2J_1]\sinh[2J_2](\cos[2p]+\cos[2q]) \\ &- (\sinh[2J_1])^2\cos[2(p+q)] - (\sinh[2J_2])^2 \\ &\times\cos[2(p-q)]\}. \end{aligned} \tag{4.20}$$

In the limit $p\to 0$, $q\to 0$, it takes place $\mathrm{Det}[\mathrm{A}(p,q)] \to (\mathrm{Det}[\mathrm{A}_0])^2$, with

$$\mathrm{Det}[\mathrm{A}_0] = 2\left(\frac{1-\sinh[2J_1]\sinh[2J_2]}{1+\cosh[2J_1]\cosh[2J_2]}\right). \tag{4.21}$$

Equations (4.20) and (4.21) suggest that $\mathrm{Det}[\mathrm{A}(p,q)]>0$ everywhere if $\{p,q\}\neq\{0,0\}$, and only exactly at $\{p,q\}=\{0,0\}$ and $\{1-\sinh[2J_{1c}]\sinh[2J_{2c}]=0\}$, we have $\mathrm{Det}[\mathrm{A}_0]=0$, corresponding to the point of the second-order phase transition.

XY model. Free fermionic limit of the eight-vertex model corresponds to relation (3.14). The solutions of that relations are (to within the periods of the elliptic functions)

$$i\lambda = I \quad \text{or} \quad i\lambda = I + iI'. \tag{4.22}$$

When $i\lambda=I$, then the expression in Eq. (4.19) writes as

$$\begin{aligned}\mathrm{Det}[\mathrm{A}_{xy}(p,q)] = 2\Bigg\{ & 1 + \left(\frac{\mathrm{sn}[iu',k]\mathrm{dn}[iu,k]}{\mathrm{cn}[iu',k]}\right)^2 \\ & \times(1+2\cos[2p]\cos[2q]) - \frac{\mathrm{dn}[iu',k]^2}{\mathrm{cn}[iu',k]^2} \\ & \times\cos[2(p+q)] - (k\,\mathrm{sn}[iu',k])^2 \\ & \times\cos[2(p-q)]\Bigg\}. \end{aligned} \tag{4.23}$$

Here there is redefinition of the parameter u of the Eq. (2.15), $u'=\frac{\lambda+u}{2}$. When $k=0$ XY model goes to the XX model,

$$\mathrm{Det}[\mathrm{A}_{xx}(p,q)] = \frac{2}{(\cos[u'])^2}\{1 + 2(\sin[u'])^2 \times\cos[2p]\cos[2q] - \cos[2(p+q)]\}. \quad (4.24)$$

As we can see this expression goes to the value 0, when $p=-q=\frac{\pi}{4}$, for all the values of parameter u', which is a hint of the known fact,[9] that the region $-1\leq\Delta\leq1$ corresponds to the critical line of the *XYZ* model.

C. Continuum limit: IM

At the critical line, which is described by the parameters J_{1c} and J_{2c}, correlation length of the system goes to infinity. All the relevant distances become large at criticality and it is natural at that limit to be interested in large distances compared to the lattice constant. It is well known that in the continuum limit, at the point of the second-order phase transition, 2DIM is described by free massless Majorana fermions. Below, for achieving it, we are going to expand the action near the point $J_1=J_2=J_c$ (considering for simplicity homogeneous case) with small values of momenta p,q.

Diagonalization of matrix (4.16) brings the action to the form

$$-A(\bar{\psi},\psi) = \sum_{p,q} E_k(p,q)\bar{\psi}'_k(p,q)\psi'_k(p,q), \quad (4.25)$$

with the eigenvalues $E_k(p,q)$ of $\mathrm{A}(p,q)$ being

$$E_1(p,q) = \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \quad E_2(p,q) = \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3,$$

$$E_3(p,q) = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_4, \quad E_4(p,q) = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4,$$

$$\mathbf{e}_1 = \frac{c}{2}(\cos[2p] + \cos[2q]) - 1,$$

$$\mathbf{e}_2 = \left(\frac{c^2}{4}(\cos[2p] - \cos[2q])^2 - (d\sin[p-q])^2 - (b\sin[p+q])^2\right)^{1/2},$$

$$\mathbf{e}_3 = \left(\frac{c^2}{4}(\cos[4p] + \cos[4q] - 2) + (d\cos[p-q])^2 + (b\cos[p+q])^2\right) - c(\cos[2p] + \cos[2q])\mathbf{e}_2)^{1/2},$$

$$\mathbf{e}_4 = \left(\frac{c^2}{4}(\cos[4p] + \cos[4q] - 2) + (d\cos[p-q])^2 + (b\cos[p+q])^2\right) + c(\cos[2p] + \cos[2q])\mathbf{e}_2)^{1/2}. \quad (4.26)$$

We see from Eq. (4.26) that at the critical value of coupling J, $J=J_c$, and at the momenta $\{p,q\}=\{0,0\}$ (or $\{p,q\}=\{0,\pi\}$) two eigenvalues $E_2(p,q)$ and $E_4(p,q)$ become 0, whereas the remaining two eigenvalues $E_1(p,q)$ and $E_3(p,q)$ take the value $-\frac{4}{3}$. As we have mentioned, taking the continuum limit at the point of second-order phase transition is justified, as the lattice constant can be neglected compared to the correlation length, and the latter is proportional to the inverse of mass. Thus, we expand the action for the massless fermions $\bar{\psi}'_k(p,q)$ and $\psi'_k(p,q)$, $k=2,4$, at the critical point. Expansion of the eigenvalues gives

$$E_2(p,q) = \sqrt{2}(J-J_c) - \sqrt{-p^2-q^2},$$

$$E_4(p,q) = \sqrt{2}(J-J_c) + \sqrt{-p^2-q^2}. \quad (4.27)$$

After a linear transformation of the field variables $\psi'_2(p,q)$ and $\psi'_4(p,q)$, the sum

$$(\sqrt{2}(J-J_c) - \sqrt{-p^2-q^2})\bar{\psi}'_2(p,q)\psi'_2(p,q) + (\sqrt{2}(J-J_c) + \sqrt{-p^2-q^2})\bar{\psi}'_4(p,q)\psi'_4(p,q)$$

takes the form

$$(\bar{\psi}_+(p,q),\bar{\psi}_-(p,q))\begin{pmatrix} m & iq-p \\ iq+p & m \end{pmatrix}\begin{pmatrix}\psi_+(p,q)\\ \psi_-(p,q)\end{pmatrix},$$

$$m = \sqrt{2}(J-J_c). \quad (4.28)$$

Continuum action of 2DIM can be conveniently written upon introducing two-dimensional gamma matrices $\gamma^0=\sigma_1$ and $\gamma^1=i\sigma_2$, as

$$-A(\bar{\psi},\psi) = \int \bar{\psi}(p,q)(m - i\gamma^\mu p_\mu)\psi(p,q). \quad (4.29)$$

Here $\psi(p,q)=\binom{\psi_+(p,q)}{\psi_-(p,q)}$, $\bar{\psi}(p,q)=(\bar{\psi}_+(p,q)\ \ \bar{\psi}_-(p,q))$, $p_0=iq$, and $p_1=p$.

D. Thermal capacity

Determinant representation of partition function (4.18) leads to the following expression for free energy, $F=-(T\ln Z)/N^2$, per site:

$$F = -T/N^2\sum_{p,q}\ln 2\{1 + (\cosh[2J_1]\cosh[2J_2])^2 + (\sinh[2J_1]\sinh[2J_2])^2 - 2\sinh[2J_1]\sinh[2J_2](\cos[2p] + \cos[2q]) - (\sinh[2J_1])^2\cos[2(p+q)] - (\sinh[2J_2])^2\cos[2(p-q)]\}. \quad (4.30)$$

The thermal capacity is related to the second derivative of the free energy with respect to the temperature as follows:

$$C = -T\frac{\partial^2 F}{\partial T^2}. \quad (4.31)$$

In order to obtain the temperature dependence of the free energy, one has to replace parameters (J_1,J_2) with $(J_1/T,J_2/T)$. Then the result for thermal capacity C follows upon performing this replacement in Eq. (4.31) and substituting F into Eq. (4.31). The result has a simple form in homogeneous case, $J_1=J_2=J$, and reads

$$C = 4\frac{1}{N^2}\left(\frac{J}{T}\right)^2 \sum_{p,q}^{N,N/2} \left\{ 2\frac{\cosh\left[4\frac{J}{T}\right] + \cosh\left[8\frac{J}{T}\right] - 4(\cos[p]\cos[q])^2\cosh\left[4\frac{J}{T}\right]}{\left(\cosh\left[2\frac{J}{T}\right]\right)^4 - 4\left(\cos[p]\cos[q]\sinh\left[2\frac{J}{T}\right]\right)^2} - \left(\frac{\left(1+\cosh\left[4\frac{J}{T}\right] - 4(\cos[p]\cos[q])^2\right)\sinh\left[4\frac{J}{T}\right]}{\left(\cosh\left[2\frac{J}{T}\right]\right)^4 - 4(\cos[p]\cos[q])^2}\right)^2 \right\}. \tag{4.32}$$

In the thermodynamic limit, $N\to\infty$, the sum in Eq. (4.32) should be replaced by the integral as

$$\frac{1}{N^2}\sum_{p,q}^{N/2,N} \to \frac{1}{\pi^2}\int_0^{\pi/2}\int_0^{\pi} dpdq. \tag{4.33}$$

Then, after performing the integration, we obtain

$$C = \frac{4}{\pi}\left(\frac{J}{T}\mathrm{csch}\left[4\frac{J}{T}\right]\right)^2\left\{-4\left(\cosh\left[2\frac{J}{T}\right]\right)^2\left(\pi + \left(1+\cosh\left[4\frac{J}{T}\right]\right)\mathbf{E}\left[4\left(\mathrm{sech}\left[2\frac{J}{T}\right]\tanh\left[2\frac{J}{T}\right]\right)^2\right]\right) + \left(15+\cosh\left[8\frac{J}{T}\right]\right)\mathbf{K}\left[4\left(\mathrm{sech}\left[\frac{2J}{T}\right]\tanh\left[\frac{2J}{T}\right]\right)^2\right]\right\}. \tag{4.34}$$

Here the functions **E** and **K** are the complete elliptic integrals of the second and the first kinds. Equation (4.34) reproduces the expression for the thermal capacity obtained from Onsager's solution.[3,7,8] The consequence of the factorization property of the determinants in expression Eq. (4.18) for the partition function,

$$\begin{aligned}\mathrm{Det}[\mathrm{A}(p,q)](1+\cosh[2J_1]\cosh[2J_2])^2/4 &= (\cosh[2J_1]\cosh[2J_2] - \cos[p+q]\sinh[2J_1] - \cos[p-q]\sinh[2J_2]) \times (\cosh[2J_1]\cosh[2J_2] + \cos[p+q]\sinh[2J_1] + \cos[p-q]\sinh[2J_2]),\end{aligned} \tag{4.35}$$

demonstrates the link to Onsager's solution.[3] Note, that the first and the second terms in the product on right-hand side of Eq. (4.35) differ only by shifts $\pi-\bar{p}$ and $\pi-\bar{q}$, where $\bar{p}=p+q$ and $\bar{q}=p-q$. Therefore, expression (4.18) for the partition function can be written as a product of the first terms in Eq. (4.35) only, where $\bar{p}=p+q$ and $\bar{q}=p-q$ take values in the interval from zero to π.

V. CORRELATION FUNCTIONS: IM, $B=0$

Fermionic approach formulated above is very convenient for calculation of correlation functions and spontaneous magnetization. Let us first analyze vacuum expectation value of the spin variable, $\bar{\sigma}_\alpha$,

$$\langle\bar{\sigma}_\alpha(i,j)\rangle = \frac{1}{Z}\sum_{\{\bar{\sigma}\}}\left\{\bar{\sigma}_\alpha(i,j)\prod_{k,r} W_{\alpha'\beta'}^{\alpha''\beta''}(k,r)\right\}. \tag{5.1}$$

Here, as in the beginning, $\bar{\sigma}_\alpha(i,j)$ are classical spin variables attached to the vertex (i,j).

Our recipe for further evaluation is simple. For calculation of the average of any quantity, say $\bar{g}(\{\bar{\sigma}_\alpha(i,j)\})$, first we represent it in the spin operator form (as it was done for the Boltzmann weights in Sec. I) as a function of Pauli operators, $g(\{\sigma_k(i,j)\})$. Then we determine corresponding fermionic realization of g in the normal ordered form $N[g^f(\{c^+(i,j),c(i,j)\})]$. Average $\langle\bar{g}(\{\bar{\sigma}_\alpha(i,j)\})\rangle$ then will be equivalent to the Green's function $\langle N[g^f(\{\bar{\psi}(i,j),\psi(i,j)\})]\rangle$ in the corresponding fermionic field theory with local quadratic action (4.7) on the lattice.

The average of a spin variable in the Eq. (5.1) can be expressed via operator forms of Boltzmann weights (2.5) and R-matrices (2.6),

$$\begin{aligned}\langle\bar{\sigma}_\alpha(i,j)\rangle &= \frac{1}{Z}\mathrm{tr}\prod_{\{k,r>j\}} W(k,r)\prod_{\{k>i\}} W(k,j)\sigma_z(i,j) \\ &\quad\times\prod_{\{k,r\le j\}} W(k,r)\prod_{\{k\le i\}} W(k,j) \\ &= \frac{1}{Z}\mathrm{tr}\prod_{\{k,r>j\}} R(k,r)\prod_{\{k>i\}} R(k,j)\sigma_1(i,j) \\ &\quad\times\prod_{\{k,r\le j\}} R(k,r)\prod_{\{k\le i\}} R(k,j).\end{aligned} \tag{5.2}$$

Here the trace is understood as the composition of tr_a defined in Eqs. (2.18) and (2.19): $\mathrm{tr}=\mathrm{tr}_2\,\mathrm{tr}_1$. By taking into account Jordan-Wigner nonlocal operator, $J=\Pi(1-2n)$ (for details, see the Appendix), we can represent the single spin operators on the lattice via fermionic creation-annihilation operators,

$$\sigma_1(2i,2j)\rangle = [c(2i,2j) + c^+(2i,2j)]J(i,j), \tag{5.3}$$

$$\sigma_1(2i+1,2j+1)\rangle = [c(2i+1,2j+1) + c^+(2i+1,2j+1)]J(i,j), \quad (5.4)$$

$$J(i,j) = \prod_{k<i}[1-2n(2k+1,2j+1)] \cdot \prod_{r>j}[1-2n(0,2r+2)]. \quad (5.5)$$

For a finite lattice the expectation value given by Eq. (5.2) always acquires value 0 due to the $\mathcal{Z}_2$ symmetry of the model. In fermionic approach this is quite apparent, as it corresponds to an integration of a polynomial over odd Grassmann variables [see Eqs. (5.3) and (5.4)], while the integration goes by even number of variables, Eq. (4.4). The case of infinite lattice will be specified in the next section.

Now it is convenient to rewrite operators $(1-2n)$ as

$$1-2n = (c^+ + c)(c^+ - c), \quad (5.6)$$

which brings expressions (5.3) and (5.4) to the form $(c^+ + c)[\Pi(c^+ + c)(c^+ - c)]$. Then we insert the resulting formulas of Eqs. (5.3) and (5.4) into Eq. (5.2). In the previous section we included fermionic fields for each R operator *locally* (or for each dashed square in the lattice on Fig. 2), later represented them in the normal ordered form and finally switched to the coherent basis. In order to escape complications in the further calculations, we shall always attach "even-even" $[c^+(2i,2j) \pm c(2i,2j)]$ fermionic operators to $R(i,j)$ matrix, [Fig. 3(a)], and the "odd-odd" fermionic operators $[c^+(2k+1,2r+1) \pm c(2k+1,2r+1)]$ to $R(k,r)$ matrix [Fig. 3(b)]. In Fig. 3 operators $(c^+ \pm c)$ are shown by large circles on the vertices. This choice, which of course will not affect the result of the calculation of expectation values, has a simple explanation.

Let us consider normal ordered forms of the operators $\mathcal{R}(i,j)[c^+(2i,2j) \pm c(2i,2j)]$ and $[c^+(2k+1,2r+1) \pm c(2k+1,2r+1)]\mathcal{R}(i,j)$, where $\mathcal{R}$ is the fermionic R operator given by Eq. (3.2). They are particular cases of a general expression

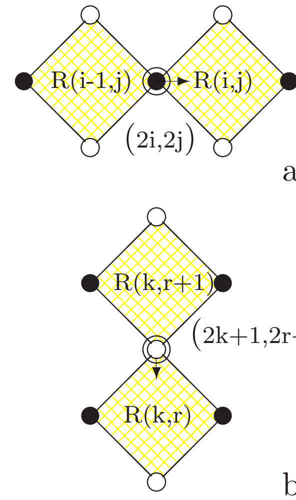

FIG. 3. (Color online) (a) $\mathcal{R}(i,j)[c^+(2i,2j) \pm c(2i,2j)]$, (b) $[c^+(2k+1,2r+1) \pm c(2k+1,2r+1)]\mathcal{R}(i,j)$.

$$R_{00}^{00}:[x_1c_1 + x_2c_2 + x_3c_1^+ + x_4c_2^+]e^{\mathcal{A}(c_1^+,c_2^+,c_1,c_2)}:, \quad (5.7)$$

with different choice of x_1,x_2,x_3, and x_4 depending on parameters b,c,d [see Eq. (3.6)]. $\mathcal{A}$ is defined by Eq. (3.5).

While for operators $[c^+(2i,2j) \pm c(2i,2j)]\mathcal{R}(i-1,j)$ and $\mathcal{R}(i,j+1)[c^+(2k+1,2r+1) \pm c(2k+1,2r+1)]$, fermionic normal ordered form belongs to the following general expression:

$$R_{00}^{00}:[x(c_1 + c_1^+ + c_2 + c_2^+) + (x_1c_1 + x_2c_1^+)c_2^+c_2 + (x_3c_2 + x_4c_2^+)c_1^+c_1]e^{\mathcal{A}(c_1^+,c_2^+,c_1,c_2)}:. \quad (5.8)$$

Equations (5.7) and (5.8) show that in the latter case we have additional powers of Grassmann fields.

In expressions of the two-point operators $\sigma_1(i,j)\sigma_1(k,r)$ some of $(1-2n)$ operators coincide and cancel each other due to Eq. (A5). The remaining operators between two points, (i,j) and (k,r), form a path, which can be deformed using feature (A16). For example, if $i_1 > i_2$, $j_2 > j_1$, we have

$$\sigma_1(2i_2+1,2j_2+1)\sigma_1(2i_1,2j_1) = [c(2i_2+1,2j_2+1) + c^+(2i_2+1,2j_2+1)][1-2n(2i_2+1,2j_2+1)]\prod_{r=j_1+1}^{j_2}[1-2n(2i_2+2,2r)]\prod_{k=i_2+1}^{i_1-1}[1-2n(2k+1,2j_1+1)][c(2i_1,2j_1) + c^+(2i_1,2j_1)]$$
$$= [c^+(2i_2+1,2j_2+1) - c(2i_2+1,2j_2+1)]\prod_{r=j_1+1}^{j_2}[1-2n(2i_2+2,2r)]\prod_{k=i_2+1}^{i_1-1}[1-2n(2k+1,2j_1+1)][c(2i_1,2j_1) + c^+(2i_1,2j_1)]. \quad (5.9)$$

Making use of expression (5.6), we bring the correlation functions $\langle\sigma_1(i_1,j_1)\sigma_1(i_2,j_2)\rangle$ to the form

$$\langle[c^+ - c]\Big[\prod (c^+ + c)(c^+ - c)\Big][c^+ + c]\rangle. \quad (5.10)$$

With the help of the Wick's theorem, we can represent average (5.10) in terms of the Pfaffian form with elements

$$\langle[c^+(i,j) \pm c(i,j)][c^+(k,r) \pm c(k,r)]\rangle.$$

Let us consider in details the cases when two spins are arranged along a direct line on the lattice, in horizontal or in vertical directions. Since there is a translational invariance in both directions we shall restrict ourselves by two cases: $\langle\sigma_1(0,0)\sigma_1(0,2k)\rangle$ and $\langle\sigma_1(2k+1,1)\sigma_1(1,1)\rangle$. For the vertically arranged spins we have

$$\begin{aligned} G(k) &\equiv \langle\sigma_1(0,2k)\sigma_1(0,0)\rangle \\ &= \Bigg\langle [c^+(0,2k)+c(0,2k)] \\ &\quad\times\prod_{r=0}^{k-1}[1-2n(0,2r)][c^+(0,0)+c(0,0)]\Bigg\rangle \\ &= \Big\langle [c^+(0,2k)+c(0,2k)] \\ &\quad\times\left(\textstyle\prod_{r=1}^{k-1}[c(0,2r)-c^+(0,2r)][c^+(0,2r)+c(0,2r)]\right) \\ &\quad\times[c(0,0)-c^+(0,0)]\Big\rangle. \end{aligned} \tag{5.11}$$

Wick's rules allow representing the last expression in Eq. (5.11) as a square root of a determinant, and hence $G(k)$ has the following representation by means of a Gaussian path integral:

$$G(k)=\int D\chi\,\exp\left[\frac{1}{2}\sum_{i,j=1}^{2k}\mathcal{G}_{ij}\chi(i)\chi(j)-\sum_{i=0}^{2k-1}\chi(2i+1)\chi(2i)\right]. \tag{5.12}$$

Here $\chi(i)$'s are Grassmann variables. Antisymmetric matrix elements $\mathcal{G}_{ij}$ are defined as

$$\mathcal{G}_{2i+12j+1}=\langle[c^+(0,2i)-c(0,2i)][c^+(0,2j)-c(0,2j)]\rangle,$$

$$\mathcal{G}_{2i2j}=\langle[c^+(0,2i)+c(0,2i)][c^+(0,2j)+c(0,2j)]\rangle,$$

$$\mathcal{G}_{2i2j+1}=\langle[c^+(0,2i)+c(0,2i)][c(0,2i)-c^+(0,2j)]\rangle. \tag{5.13}$$

All the expressions in Eq. (5.13) can be easily derived in the basis of coherent states (4.1). As it was stated earlier, the normal ordered form of the operator $\mathcal{R}(i,j)[c^+(2i,2j)\pm c(2i,2j)]$ has form (5.7). The parameters in this case are given by

$$\{x_1,x_2,x_3,x_4\}=\{\pm 1,d,c,b\}. \tag{5.14}$$

Let $\{x_1',x_2',x_3',x_4'\}$ and $\{x_1'',x_2'',x_3'',x_4''\}$ be the parameters corresponding to the operator $(c^+\pm c)$ at $(0,2i)$ and $(0,2j)$ points, respectively. Then we can rewrite all the expressions in Eq. (5.13) as a general function of these parameters, namely, $\mathcal{G}(i,j,\{x'\},\{x''\})$, which has the following integral form in coherent-state basis:

$$\begin{aligned} \mathcal{G}(i,j,\{x'\},\{x''\}) &= \frac{(R_{00}^{00})^{N^2}}{Z}\int D\bar{\psi}D\psi e^{A(\bar{\psi},\psi)}\times[x_1'\psi(0,2i) \\ &\quad+x_2'\psi(1,2i-1)+x_3'\bar{\psi}(2,2i)+x_4'\bar{\psi}(1,2i \\ &\quad+1)]\times[x_1''\psi(0,2j)+x_2''\psi(1,2j-1) \\ &\quad+x_3''\bar{\psi}(2,2j)+x_4''\bar{\psi}(1,2j+1)], \end{aligned} \tag{5.15}$$

where A and Z are defined in Eqs. (4.7) and (4.6).

The second sum $[-\Sigma_{i=0}^{k-1}\chi(2i+1)\chi(2i)]$ in Eq. (5.12) and hence the additional unity elements with $\mathcal{G}_{2i+12i}(-\mathcal{G}_{2i,2i+1})$ are conditioned by the normal ordered version of relation $1-2n=(c^++c)(c^+-c)$, i.e., $:1-2n:=1+:(c^++c)(c^+-c):$.

Then straightforward calculations lead to the following expression for $\mathcal{G}(r,j,\{x'\},\{x''\})$:

$$\begin{aligned} \mathcal{G}(r,j,\{x'\},\{x''\}) &= 1/N^2\sum_{n_1=1,n_2=1}^{N/2,N}\Bigg\{\left(\frac{K_1}{k_1}x_3'x_4''+\frac{k}{K_1}x_1'x_2''-\frac{k}{K_1}x_4'x_3''-\frac{K_1}{k_1}x_2'x_1''\right)(\mathrm{A}^{-1})_{14}+\left(K_1x_3'x_2''-\frac{kk_1}{K_1}x_1'x_4''+K_1x_4'x_1''-\frac{kk_1}{K_1}x_2'x_3''\right) \\ &\quad\times(\mathrm{A}^{-1})_{12}+\left(\frac{K_1}{k}x_4'x_3''+\frac{k_1}{K_1}x_2'x_1''-\frac{k_1}{K_1}x_3'x_4''-\frac{K_1}{k}x_1'x_2''\right)(\mathrm{A}^{-1})_{23}+\left(\frac{1}{K_1}x_3'x_2''-\frac{K_1}{kk_1}x_1'x_4''+\frac{1}{K_1}x_4'x_1''-\frac{K_1}{kk_1}x_2'x_3''\right)(\mathrm{A}^{-1})_{34} \\ &\quad+\left(K-\frac{1}{K}\right)((x_3'x_3''-x_1'x_1'')(\mathrm{A}^{-1})_{13}+(x_4'x_4''-x_2'x_2'')(\mathrm{A}^{-1})_{24})+k\left(Kx_3'x_1''-\frac{x_1'x_3''}{K}\right)(\mathrm{A}^{-1})_{11}+k_1\left(Kx_4'x_2''-\frac{x_2'x_4''}{K}\right) \\ &\quad\times(\mathrm{A}^{-1})_{22}+\frac{1}{k}\left(\frac{x_1''x_3'}{K}-Kx_1'x_3''\right)(\mathrm{A}^{-1})_{33}+\frac{1}{k_1}\left(\frac{x_4'x_2''}{K}-Kx_2'x_4''\right)(\mathrm{A}^{-1})_{44}\Bigg\}\Bigg|_{p=2\pi 2n_1+1/2N,q=2\pi 2n_2+1/2N}. \end{aligned} \tag{5.16}$$

Here A is the 4×4 matrix defined by Eq. (4.16), and

$$K=e^{i(2rp+2jq)},\quad K_1=e^{i[(2r+1)p+(2j+1)q]},$$

$$k=e^{i2p},\quad k_1=e^{i2q},\quad r,j=1,\ldots,N. \tag{5.17}$$

Similar expressions can be obtained for the horizontally arranged spins too,

$$G'(k)\equiv\langle\sigma_1(2k+1,1)\sigma_1(1,1)\rangle=\left\langle[c^+(2k+1,1)-c(2k+1,1)]\prod_{r=1}^{k-1}[1-2n(2r+1,1)][c^+(1,1)+c(1,1)]\right\rangle, \tag{5.18}$$

which also admits integral representation (5.12), in this case with the matrix elements

$$\mathcal{G}_{2i+1,2j+1} = \langle [c^+(2i+1,1) - c(2i+1,1)][c^+(2j+1,1) - c(2j+1,1)] \rangle,$$

$$\mathcal{G}_{2i,2j} = \langle [c^+(2i+1,1) + c(2i+1,1)][c^+(2j+1,1) + c(2j+1,1)] \rangle,$$

$$\mathcal{G}_{2i,2j+1} = \langle [c^+(2i+1,1) - c(2i+1,1)][c^+(2j+1,1) + c(2j+1,1)] \rangle. \tag{5.19}$$

The elements given by Eq. (5.19) can also be expressed by the function $\mathcal{G}(i,j,\{x'\},\{x''\})$ [Eqs. (5.15) and (5.16)], but here the parameters defined by the normal ordered form (5.7) of the operator $[c^+(2i+1,2j+1) \pm c(2i+1,2j+1)]\mathcal{R}(i,j)$ read

$$\{x_1,x_2,x_3,x_4\} = \{b,c,d,\pm 1\}. \tag{5.20}$$

The above relations enable us to find correlation functions for all the statistical models with weights, which can be written in the matrix form (3.10), with condition (2.9), letting $R_{01}^{10}=R_{10}^{01}, R_{01}^{01}=R_{10}^{10}$.

2DIM. Inserting the parameters $\{x'\},\{x''\}$ and the elements of the inverse matrix A^{-1} defined for the 2DIM (3.3) into Eq. (5.16), we find the following expression for Eq. (5.19):

$$\mathcal{G}_{2i2j} = \mathcal{G}_{2i+12j+1} = \frac{-2}{N^2} \sum_{n_1=1}^{N/2-1} \sum_{n_2=1}^{N-1} \frac{\sin[2(i-j)p]}{a^2\,\mathrm{Det}[A(p,q)]} \{\cosh[2J_2]\sin[2(p-q)] - \cosh[2J_1]\sin[2(p+q)] + 2\sin[2q](\cos[2p] - 2\sinh[J_1]\sinh[J_2])\},$$

$$\mathcal{G}_{2k2r+1} = -\mathcal{G}_{2r+12k} = \frac{2}{N^2} \sum_{n_1=1}^{N/2-1} \sum_{n_2=1}^{N-1} \frac{1}{a^2\,\mathrm{Det}[A(p,q)]} \{\cos[2(r-k)p](3 + \cosh[2J_1]\cosh[2J_2] - 4\cosh[J_1]\cosh[J_2] + 2\cos[2p]\cos[2q] - \cos[2(p+q)]\cosh[2J_1] - \cos[2(p-q)]\cosh[2J_2] - 4(\cos[2p] + \cos[2q])\sinh[J_1]\sinh[J_2]) + \cos[2(r-k-1)p]\sinh[2J_1]\sinh[2J_2]\},$$

$$p = \frac{\pi}{2N}(2n_1+1), \quad q = \frac{\pi}{2N}(2n_2+1). \tag{5.21}$$

The elements $\mathcal{G}_{ij}$ in Eq. (5.13) for the vertical case can be obtained from expressions (5.21) simply by interchanging the coupling constants J_1 and J_2.

In the homogeneous case $J_1=J_2=J$, we have $\mathcal{G}_{2i2j}=0$ and $\mathcal{G}_{2i+12j+1}=0$, and the expression for $G(i)[G'(i)]$ simplifies to the determinant

$$G(i) = \mathrm{Det}\begin{pmatrix} \bar{\mathcal{G}}_{i0} & \bar{\mathcal{G}}_{i1} & \cdots & \bar{\mathcal{G}}_{ii-1} \\ \bar{\mathcal{G}}_{i-10} & \bar{\mathcal{G}}_{i-11} & \cdots & \bar{\mathcal{G}}_{i-1i-1} - 1 \\ \vdots & & & \vdots \\ \bar{\mathcal{G}}_{10} & \bar{\mathcal{G}}_{11} - 1 & \cdots & \bar{\mathcal{G}}_{1i-1} \end{pmatrix}, \tag{5.22}$$

$$\bar{\mathcal{G}}_{k+ik} \equiv \bar{\mathcal{G}}_i \equiv \mathcal{G}_{2(k+i)2k+1}. \tag{5.23}$$

Therefore we can rewrite the Gaussian integral representation (5.12) in the following way:

$$G(i) = \int D\bar{\chi}' D\chi \exp\left(\sum_{k=1,r=0}^{i,i-1} \bar{\mathcal{G}}_{k-r} \bar{\chi}_k' \chi_r - \sum_{k=1}^{i-1} \bar{\chi}_k' \chi_k \right). \tag{5.24}$$

After the replacement $\bar{\chi}_k' = \bar{\chi}_{k-1}$, Eq. (5.24) reads

$$G(i) = \int D\bar{\chi} D\chi \exp\left(\sum_{k,r=0}^{i-1} \bar{\mathcal{G}}_{k-r+1} \bar{\chi}_k \chi_r - \sum_{k=1}^{i-1} \bar{\chi}_{k-1} \chi_k \right). \tag{5.25}$$

Of course, the expressions for correlation functions can be caught as well from the logarithmic derivatives of the partition function $Z(B)$ [Eqs. (4.8) and (4.11)] with respect to inhomogeneous field $B(i,j)$, taken at $B(i,j)=0$.

Limit of an infinite lattice and large distances. Magnetization

In the limit of an infinite lattice, $N \to \infty$, one can replace the sums in Eq. (5.21) by integrals in accordance with Eq. (4.33). After evaluation of the integral over q, one will obtain

$$\mathcal{G}_{2i2j+1} = \delta_{i,j} + \frac{2}{\pi}\int_0^{\pi/2} \frac{\cos[2(i-j-1)p]\sinh[J_1]\sinh[J_2] - \cos[2(i-j)p]}{\sqrt{1+(\sinh[2J_1]\sinh[2J_2])^2 - 2\cos[2p]\sinh[2J_1]\sinh[2J_2]}}dp,$$

$$\mathcal{G}_{2i2j} = \mathcal{G}_{2i+12j+1} = 0. \tag{5.26}$$

The last expression in Eq. (5.26) shows that as in the homogeneous case, on the infinite lattice in the inhomogeneous case $J_1 \neq J_2$ also we can use determinant representation (5.22) instead of Eq. (5.12),

$$G(i) = \int D\bar{\chi}D\chi \exp\left(\sum_{k,r=0}^{i-1,i-1} \bar{\mathcal{G}}'_{k-r+1}\bar{\chi}_k\chi_r\right) = \mathrm{Det}[\mathcal{G}'(i)],$$

$$[\mathcal{G}'(i)]_{k,r} \equiv \bar{\mathcal{G}}'_{k+1-r}, \quad k,r = 0,i-1, \tag{5.27}$$

with

$$\bar{\mathcal{G}}'_{k-r+1} = \frac{2}{\pi}\int_0^{\pi/2} \frac{\cos[2(k-r-1)p]\sinh[J_1]\sinh[J_2] - \cos[2(k-r)p]}{\sqrt{1+(\sinh[2J_1]\sinh[2J_2])^2 - 2\cos[2p]\sinh[2J_1]\sinh[2J_2]}}dp. \tag{5.28}$$

It is easy to see that the integral for the matrix elements in Eq. (5.28) can be transformed into the form

$$\begin{aligned}
\bar{\mathcal{G}}'_{n+1} &= \frac{2}{\pi}\int_0^{\pi/2} \frac{\cos[2(n-1)p]\sinh[2J_1]\sinh[2J_2] - \cos[2np]}{\sqrt{1+(\sinh[2J_1]\sinh[2J_2])^2 - 2\cos[2p]\sinh[2J_1]\sinh[2J_2]}}dp \\
&= \frac{1}{\pi}\int_0^{\pi} \frac{\cos[(n-1)p]\sinh[2J_1]\sinh[2J_2] - \cos[np]}{\sqrt{1+(\sinh[2J_1]\sinh[2J_2])^2 - 2\cos[p]\sinh[2J_1]\sinh[2J_2]}}dp \\
&= \frac{1}{2\pi}\int_0^{\pi} \frac{e^{inp}(e^{-ip}\sinh[2J_1]\sinh[2J_2]-1) + e^{-inp}(e^{ip}\sinh[2J_1]\sinh[2J_2]-1)}{\sqrt{(e^{ip}\sinh[2J_1]\sinh[2J_2]-1)(e^{-ip}\sinh[2J_1]\sinh[2J_2]-1)}}dp \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(np)}\frac{\sqrt{e^{-ip}\sinh[2J_1]\sinh[2J_2]-1}}{\sqrt{e^{ip}\sinh[2J_1]\sinh[2J_2]-1}}dp.
\end{aligned} \tag{5.29}$$

It is well known that one can investigate the magnetization $\langle\sigma_1(i,j)\rangle$ by analyzing the large distance asymptotes of two spin-correlation function on an infinite lattice. Namely,

$$\begin{aligned}
[\langle\bar{\sigma}_\alpha(i,j)\rangle]^2 &= \lim_{K\to\infty}[\lim_{N\to\infty}\langle\bar{\sigma}_\alpha(0,0)\bar{\sigma}_{\alpha'}(K,K)\rangle] \\
&= \lim_{K\to\infty}[\lim_{N\to\infty}\langle\bar{\sigma}_\alpha(0,0)\bar{\sigma}_{\alpha'}(0,K)\rangle],
\end{aligned} \tag{5.30}$$

where N is the linear size of the square lattice.

In Ref. 8 it was shown that spin-spin correlation functions $\langle\bar{\sigma}_\alpha(0,0)\bar{\sigma}_{\alpha'}(i,i)\rangle$ and $\langle\bar{\sigma}_\alpha(0,0)\bar{\sigma}_{\alpha'}(0,i)\rangle$ (for $T<T_c$) have a determinant representation. These correlation functions have been represented as a determinant of an $i\times i$ matrix, $\mathcal{C}_i$, of the Toeplitz type,

$$\mathcal{C}_i = \begin{pmatrix} c_0 & c_{-1} & \cdots & c_{-i+1} \\ c_1 & c_0 & \cdots & c_{-i+2} \\ \vdots & \vdots & \vdots & \vdots \\ c_{i-1} & c_{i-2} & \cdots & c_0 \end{pmatrix}. \tag{5.31}$$

In Eq. (5.31) the matrix elements are given by

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-in\theta}\mathrm{C}(e^{i\theta}),$$

$$\mathrm{C}(e^{i\theta}) = \left(\frac{(1-\alpha_1 e^{i\theta})(1-\alpha_2 e^{-i\theta})}{(1-\alpha_1 e^{-i\theta})(1-\alpha_2 e^{i\theta})}\right)^{1/2}. \tag{5.32}$$

For the case

$$\lim_{N\to\infty}\langle\bar{\sigma}_\alpha(0,0)\bar{\sigma}_{\alpha'}(i,i)\rangle = \mathrm{Det}[\mathcal{C}_i], \tag{5.33}$$

(α_i)'s are defined as follows:

$$\alpha_1 = 0, \quad \alpha_2 = (\sinh[2J_1]\sinh[2J_2])^{-1}. \tag{5.34}$$

Careful analysis of the matrix $\mathcal{G}'(i)$, given by Eq. (5.27), shows that due to Eqs. (5.28) and (5.29) and after some rearrangement of its rows, which leave the determinant invariant, $\mathcal{G}'(i)$ coincides with $\mathcal{C}_i$. Note that in our notations the coordinate plane on the lattice is 45° rotated with respect to the coordinate plane in Ref. 8, so the correlation function of the spins arranged in the horizontal or vertical lines in our case coincide with $\langle\bar{\sigma}_\alpha(0,0)\bar{\sigma}_{\alpha'}(i,i)\rangle$ derived in Ref. 8.

Now one can follow the technique developed in Ref. 8, based on the Szegö's theorem, and find the solution for the magnetization. The theorem can be applied when $T<T_c$ and directly reproduces the known result for the magnetization,[3,5,9] originally derived by Yang in article:[5]

$$\lim_{i\to\infty}\langle\bar{\sigma}_\alpha(0,0)\bar{\sigma}_{\alpha'}(0,i)\rangle = \lim_{i\to\infty}\langle\bar{\sigma}_\alpha(0,0)\bar{\sigma}_{\alpha'}(i,i)\rangle = \left(\frac{(1-\alpha_2^2)(1-\alpha_1^2)}{(1-\alpha_1\alpha_2)^2}\right)^{1/4} = (1-(\sinh[2J_1]\sinh[2J_2])^{-2})^{1/4}. \tag{5.35}$$

VI. RELATED ONE-DIMENSIONAL QUANTUM PROBLEM

It is possible to connect the partition function of the quantum 1DIM to the partition function of two-dimensional classical system (2DIM) using limit (2.20) and the Trotter formula, see Ref. 30.

As it was stated in the second section, the transfer matrix of two-dimensional model, which is defined as a product of R matrices, plays a role of the discrete time evolution operator defined on a 1D chain.

In this section we investigate the transfer matrix given by Eq. (2.18) and express it via one-dimensional fermionic fields defined on a chain. By the convention, the trace of the transfer matrix can be connected with the partition function of the quantum chain model, defined by Hamiltonian operator $\mathcal{H}$,

$$\mathrm{tr}\ \tau = \mathrm{tr}\ e^{-\mathcal{H}}.$$

The trace in the definition of the transfer matrix, $\tau_j = \mathrm{tr}_1\Pi_i R(i,j)$, in Eq. (2.18) is taken over the variables which have even-even lattice coordinates (denoted by black circles on the figures); $R(i,j)$ matrices are arranged along the horizontal chain with N vertices (white circles on the figures). In the following we shall omit the coordinate indices and will use only indices denoting the vertices on the chain. Using R operators represented in terms of fermionic creation-annihilation operators, $\mathcal{R}(\mathbf{c}_1^+,\mathbf{c}_1;c_2^+,c_2)$ [Eqs. (3.2) and (3.11)], one easily comes to the transfer matrix

$$\tau(\{c_n^+,c_n\}) = \mathrm{tr}_1\prod_{i=1}^{N}\mathcal{R}(\mathbf{c}_1^+,\mathbf{c}_1;c_i^+,c_i). \tag{6.1}$$

We can evaluate the trace in Eq. (6.1) passing to the coherent basis with Grassmann variables for the fermionic operators $\mathbf{c}_1^+,\mathbf{c}_1$ and $\{c_i^+,c_i\}$. After integration by the variables corresponding to the operators $\mathbf{c}_1^+,\mathbf{c}_1$, we shall arrive at [we have chosen the homogeneous case $b=b'$, $c=c'$, $d=d'$, Eq. (3.11)]

$$t(\{\bar{\psi},\psi\}) = \prod_i\langle\bar{\psi}_i|\tau(\{c_n^+,c_n\})\prod_i|\psi_i\rangle = (R_{00}^{00^N}+R_{01}^{10^N})e^{-\mathbb{H}(\bar{\psi}_i,\psi_i)}, \tag{6.2}$$

$$-\mathbb{H}(\bar{\psi}_i,\psi_i) = \sum_{k=1}^{N}\frac{(-1)^k c^{N-k}\Delta^k}{(1+c^N)^k}\sum_{i_1<\cdots<i_k} n_{i_1}\cdots n_{i_k} + c\sum_{i=1}^{N} n_i + \frac{1}{1+c^N}\left(1+\sum_{k=1}^{N}\frac{(-1)^k}{(1+c^N)^k}\left[\prod_{i=1}^{N}(c-\Delta n_i)-c^N\right]^k\right) \times\sum_{i,j}(b\bar{\psi}_i-d\psi_i)K_{i,j}(b\psi_{j-1}+d\bar{\psi}_{j-1}) \tag{6.3}$$

$$K_{i,j} = \begin{cases} 1, & i=j\\ (c-\Delta n_j)\cdots(c-\Delta n_{i-1}), & i>j\\ -(c-\Delta n_1)\cdots(c-\Delta n_{i-1})(c-\Delta n_j)\cdots(c-\Delta n_N), & i<j. \end{cases} \tag{6.4}$$

Here $n_i=\bar{\psi}_i\psi_i$. Correspondingly the normal ordered expression of $\tau(\{c_n^+,c_n\})$ is

$$\tau(\{c_n^+,c_n\}) = (R_{00}^{00^N}+R_{01}^{10^N}){:}\exp\left[-\mathbb{H}(c_i^+,c_i)-\sum_i c_i^+c_i\right]{:}. \tag{6.5}$$

The expression of $\mathbb{H}$ in Eq. (6.3) simplifies if $c=0$. In case of $\Delta=0$ the function $\mathbb{H}$ is a quadratic function and admits diagonalization by means of Fourier transformation.

IM, XY. Here we are presenting the transfer matrices, which correspond to the free-fermionic cases: $\Delta=0$ in Eq. (3.11), i.e., IM [Eq. (2.11)] and XY model [Eqs. (2.15) and (4.22)]. Now logarithm of Eq. (6.1) is a quadratic function over N pairs of fermion operators, $\{c_n^+,c_n\}$, due to the Eqs. (3.4) and (3.5). After performing Fourier transformation for operators c_n^+,c_n in Eq. (6.3), the transfer matrix takes the form

$$\tau = (R_{00}^{00})^N(1-c^N){:}\exp\left\{\sum_{p=0}^{N/2-1}\mathbb{H}(p)\right\}{:}; \tag{6.6}$$

$$\mathbb{H}(p)=\left(c-1+\frac{b^2e^{i\pi 2p+1/N}}{1-ce^{i\pi 2p+1/N}}+\frac{d^2e^{-i\pi 2p+1/N}}{1-ce^{-i\pi 2p+1/N}}\right)c_p^+c_p+\left(c-1+\frac{b^2e^{-i\pi 2p+1/N}}{1-ce^{-i\pi 2p+1/N}}+\frac{d^2e^{i\pi 2p+1/N}}{1-ce^{i\pi 2p+1/N}}\right)c_{N-p-1}^+c_{N-p-1}$$
$$+\frac{2ibd\sin\left[\pi\frac{2p+1}{N}\right]}{1+c^2-2c\cos\left[\pi\frac{2p+1}{N}\right]}(c_p^+c_{N-p-1}^++c_pc_{N-p-1}). \tag{6.7}$$

In the course of calculation of the partition function in Sec. III we have diagonalized this type of quadratic expression by a simple change of basis (4.14). Recall that here c_p^+,c_p are not Grassmann variables but rather fermionic operators and *any transformation must keep anticommutation relations*. So we distinguish two kind of fermion fields, defined as $c_{\alpha p}$, $\alpha=1,2$,

$$c_{1p}^+=c_p^+,\quad c_{1p}=c_p,\quad c_{2p}^+=c_{N-p-1},\quad c_{2p}=c_{N-p-1}^+. \tag{6.8}$$

These replacements bring the operator $\Sigma_{p=0}^{N/2-1}\mathbb{H}(p)$ to the form

$$\sum_{p=0}^{N/2-1}\sum_{\alpha,\beta=1,2}\mathbb{H}'_{\alpha\beta}(p)c_{\alpha p}^+c_{\beta p}. \tag{6.9}$$

The task now is to diagonalize the matrix

$$\mathbb{H}'(p)=\begin{pmatrix} r_1(p) & r_2(p)\\ -r_2(p) & -r_1(-p)\end{pmatrix}, \tag{6.10}$$

where

$$r_1(p)=c-1+\frac{b^2e^{i\pi 2p+1/N}}{1-ce^{i\pi 2p+1/N}}+\frac{d^2e^{-i\pi 2p+1/N}}{1-ce^{-i\pi 2p+1/N}},$$

$$r_2(p)=\frac{2ibd\sin\left[\pi\frac{2p+1}{N}\right]}{1+c^2-2c\cos\left[\pi\frac{2p+1}{N}\right]}. \tag{6.11}$$

We can represent the transfer matrix given by Eq. (6.6) in the following diagonal form:

$$\tau\approx\exp\left[\sum_{p=0}^{N/2-1}(a'_+(p)c_{1p}'^+c'_{1p}+a'_-(p)c_{2p}'^+c'_{2p})\right], \tag{6.12}$$

with the eigenvalues of matrix (6.10),

$$a'_\pm(p)=\frac{1}{2}(r_1(p)-r_1(-p)\pm\sqrt{[r_1(p)+r_1(-p)]^2-4[r_2(p)]^2}). \tag{6.13}$$

Thus, we arrive at a 1D quantum system defined with Hamiltonian operator

$$\mathcal{H}=-\sum_{p=0}^{N/2-1}(a'_+(p)c_{1p}'^+c'_{1p}+a'_-(p)c_{2p}'^+c'_{2p}). \tag{6.14}$$

Particularly, for the IM [where b,c,d are defined as in Eq. (3.6)], in the homogeneous case, $J_1=J_2=J$, we have $r_1(p)=r_1(-p)$ and eigenvalues (6.13) acquire the form

$$a'_\pm(p)=\pm\sqrt{|r_1(p)|^2+|r_2(p)|^2}. \tag{6.15}$$

The ground state of the system is composed by the negative-energy modes. In the thermodynamic limit, $N\to\infty$, the gap between two spectral curves, $a'_\pm(p)$, is found at the Fermi points with momenta $0,\pi$ and is equal to

$$[a'_+(0)-a'_-(0)]\equiv 2r_1(0)=2\left(1-\sinh\left[\frac{2J}{T}\right]\right)\times\left(1+\sinh\left[\frac{2J}{T}\right]\right). \tag{6.16}$$

We see that $r_1(0)$ vanishes at the critical temperature T_c of 2DIM, given by $\sinh[2J/T_c]=1$, as

$$[a'_+(0)-a'_-(0)]\sim(T-T_c), \tag{6.17}$$

demonstrating that at $T=T_c$ the 1D system is gapless and has no massive excitations. Behavior (6.17) holds true for the inhomogeneous case $J_1\neq J_2$ also.

VII. SUMMARY

In this work we have presented an approach to the investigation of two-dimensional statistical models, basing on the fermionic formulation of the vertex R matrices (Boltzmann weights). If the operator form of the R matrix in terms of scalar fermionic creation and annihilation operators has definite even grading [for *XYZ* model and 2DIM see Eq. (3.2)], then fermionic representation of $R(i,j)$ on the lattice acquires local character. If the operators have indefinite grading [models in the presence of an external magnetic field, see Eq. (3.7)], then one must take into account Jordan-Wigner nonlocal operator, as in Eq. (4.8), which is discussed in details in the Appendix.

For the models under consideration we derive partition functions as continual integrals with corresponding field theoretical actions on the square lattice: Eq. (4.7) gives the fermionic action corresponding to the general eight-vertex model, which includes both *XYZ* model and two-dimensional

Ising model. Although there is a correspondence between 2DIM and *XZ* models, we straightforwardly presented the R matrix of the 2DIM in Eq. (2.22) as a solution of Yang-Baxter equation which ensures the integrability of the model. For the free-fermionic case the direct calculation of the partition function and correlation functions is performed [Eqs. (4.18) and (5.16)]. In case of the 2DIM the continuum limit of the two-dimensional action is presented in Eq. (4.29) and the known thermodynamic and magnetic characteristics are reproduced [see Eqs. (4.20), (4.34), and (5.35)]. We also consider 2DIM in the presence of a finite magnetic field and corresponding nonlocal fermionic action is evaluated [Eq. (4.11)].

In light of correspondence of two-dimensional classical statistical models and one-dimensional quantum models, we obtain one dimensional quantum fermionic Hamiltonian operator (6.3) for eight-vertex model. For free-fermionic cases the Hamiltonian operators are brought to the diagonal form (6.14), the spectral analysis of which reflects the critical behavior of the underling models.

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APPENDIX

Jordan-Wigner transformation. Fermionic representation of spin states naturally introduces grading for both states and operators. $\bar{\sigma}_\alpha, \alpha=0,1$ spin states can be represented by $|0\rangle,|1\rangle$ fermionic states with zero and one fermions. Single fermion states are anticommuting at different points of the lattice. The same property takes place for the odd operators in terms of fermionic creation and annihilation operators. This property does not hold for spin states and operators. Therefore, if one would like to represent the action of odd number of spin operators $\sigma_1^{(k)}$ defined in the space of spins (nongraded space)

$$\{\hat{1}^{(1)} \otimes \hat{1}^{(2)} \cdots \otimes \sigma_1^{(k)} \otimes \cdots \otimes \hat{1}^{(n)}\}:|\alpha_1\rangle|\alpha_2\rangle\cdots|\alpha_n\rangle, \tag{A1}$$

in terms of fermionic operators $(c+c^+)^{(k)}$, which act on graded states $|\alpha_k\rangle$, one has to take into account the graded behavior of all states $|\alpha_i\rangle$, $i<k$, placed before the state $|\alpha_k\rangle$. This can be done with the help of the operator $1-2n$, action of which on the state $|\alpha\rangle$ depends on the parity, $p(\alpha)=\alpha$, as follows:

$$(1-2n)|\alpha\rangle = (-1)^{p(\alpha)}|\alpha\rangle. \tag{A2}$$

Using these operators, one can represent the action of a spin operator $\sigma_1^{(k)}$, as

$$\{\hat{1}^{(1)} \otimes \hat{1}^{(2)} \cdots \otimes \sigma_1^{(k)} \otimes \cdots \otimes \hat{1}^{(n)}\}$$
$$\Rightarrow (c+c^+)^{(k)}(1-2c^+c)^{(1)}\cdots(1-2c^+c)^{(k-1)}. \tag{A3}$$

This expression constitutes the inverse Jordan-Wigner spin-fermion nonlocal transformation.

It is clear that for the product of two odd operators at different points one needs to take into account only the states between them,

$$\{\cdots\hat{1}^{(i-1)} \otimes \sigma_1^{(i)} \otimes \hat{1}^{(i+1)}\cdots\hat{1}^{(k-1)} \otimes \sigma_1^{(k)} \otimes \hat{1}^{(k+1)}\cdots\}$$
$$\Rightarrow (c+c^+)^{(i)}\prod_{r=i}^{k-1}(1-2c^+c)^{(r)}(c+c^+)^{(k)}, \tag{A4}$$

which is a consequence of the property

$$(1-2c^+c)^{(i)}(1-2c^+c)^{(i)} = 1. \tag{A5}$$

Note that operator $(1-2c^+c)$ is the fermionic form corresponding to the Pauli matrix σ_z. This means that if we place the operators $\sigma_z^{(i)}$ instead of unity $1^{(i)}$ in Eq. (A3) for all $i<k$, we shall have

$$\{\sigma_z^{(1)} \otimes \sigma_z^{(2)} \cdots \otimes \sigma_1^{(k)} \otimes \cdots \otimes \hat{1}^{(n)}\} \Rightarrow (c+c^+)^{(k)}. \tag{A6}$$

Similarly, we have

$$\{\ldots\hat{1}^{(i-1)} \otimes (\sigma_1\sigma_z)^{(i)} \otimes \sigma_z^{(i+1)}\ldots\sigma_z^{(k-1)} \otimes \sigma_1^{(k)} \otimes \hat{1}^{(k+1)}\ldots\}$$
$$\Rightarrow (c+c^+)^{(i)}(c+c^+)^{(k)}. \tag{A7}$$

Jordan-Wigner spin-fermion transformation on the two-dimensional lattice. In Sec. II the partition function Eq. (2.19) was defined as an expectation value of the products of R operators. These products can be rewritten as

$$Z = \sum_{\{\alpha_{2i+1,1}\}_{i=0,N-1}} \sum_{\{\alpha_{0,2j}\}_{j=1,N}} \langle\Sigma|\prod_{j=N}^{1}\prod_{i=N-1}^{0} R(i,j)|\Sigma\rangle, \tag{A8}$$

where the trace is taken over both "auxiliary" and "quantum" states,

$$|\Sigma\rangle = |\alpha_{0,2N}\rangle\cdots|\alpha_{0,4}\rangle|\alpha_{0,2}\rangle|\alpha_{1,1}\rangle|\alpha_{3,1}\rangle\cdots|\alpha_{2N-1,1}\rangle.$$

In fermionic representation described in Sec. III, the states $|\alpha_{i,j}\rangle$ acquire grading and the arrangement in $|\Sigma\rangle$ becomes significant. Fermionic R operator given by Eq. (3.2) has zero parity, which ensures the local "fermionization" of the partition function: each R operator in Eq. (A8) can be replaced with its fermionic counterpart without any "tail." But the formulas of spin-spin correlation functions contain the spin operator $\sigma_1(k,r)$, which in the fermionic formulation has odd parity. From the inverse Jordan-Wigner transformation in Eq. (A4) it follows that the fermionic operator corresponding to $\sigma_1(k,r)$ should contain nonlocal operator $\Pi[1-2n(i,j)]$, where the product runs over sites (i,j), arranged before the site (k,r).

Recall, that the mean value of the operator $\sigma_1(2k,2r)$ is defined by

$$\langle\sigma_1(2k,2r)\rangle = \frac{1}{Z}\sum\langle\Sigma|(R\cdots R(k,r)\sigma_1(2k,2r)$$
$$\times R(k-1,r)\cdots R)|\Sigma\rangle. \tag{A9}$$

Moreover due to the conventions adopted in the previous sections, the $R(i,j)$ operator acts as

$$R(i,j)|\alpha_{2i,2j}\rangle|\alpha_{2i+1,2j-1}\rangle = R^{\alpha_{2i+1,2j+1}\alpha_{2i+2,2j+2}}_{\alpha_{2i,2j}\alpha_{2i+1,2j-1}}|\alpha_{2i+1,2j+1}\rangle \times|\alpha_{2i+2,2j+2}\rangle, \qquad \text{(A10)}$$

with the matrix elements defined by Eq. (3.3). Then one can notice that the action of R operators, placed on the right side of $\sigma_1(2k,2r)$ in the right-hand side of Eq. (A9), on the state $|\Sigma\rangle$, transforms it to the following state:

$$\prod_{i=k-1}^{0} R(i,r)\prod_{j=r}^{1}\prod_{i=N-1}^{0} R(i,j)|\Sigma\rangle \Rightarrow |\alpha_{0,2N}\rangle\cdots|\alpha_{0,2r+2}\rangle|\alpha_{1,2r+1}\rangle\cdots|\alpha_{2k-1,2r+1}\rangle|\alpha_{2k,2r}\rangle \times|\alpha_{2k+1,2r-1}\rangle\cdots|\alpha_{2N-1,2r-1}\rangle. \qquad \text{(A11)}$$

Hence, according to Eq. (A3), operator $\sigma_1(2i,2j)$ in its fermionic formulation reads

$$[c(2k,2r)+c^{+}(2k,2r)]\prod_{i=k-1}^{0}[1-2n(2i+1,2r+1)] \times\prod_{j=r+1}^{N}[1-2n(0,2j)]. \qquad \text{(A12)}$$

Similarly, in expression for the vacuum average value of spin operators $\sigma_1(2k+1,2r+1)$, defined at odd-odd sites,

$$\langle\sigma_1(2k+1,2r+1)\rangle = \frac{1}{Z}\sum\langle\Sigma|[R\cdots R(k+1,r)\sigma_1(2k+1,2r+1)R(k,r)\cdots R]|\Sigma\rangle, \qquad \text{(A13)}$$

the R operators on the right-hand side of $\sigma_1(2k+1,2r+1)$ transform the state $|\Sigma\rangle$ into

$$\prod_{i=k}^{0} R(i,r)\prod_{j=r}^{1}\prod_{i=N-1}^{0} R(i,j)|\Sigma\rangle \Rightarrow |\alpha_{0,2N}\rangle\cdots|\alpha_{0,2r+2}\rangle|\alpha_{1,2r+1}\rangle\cdots|\alpha_{2k+1,2r+1}\rangle \times|\alpha_{2k+2,2r}\rangle|\alpha_{2k+3,2r-1}\rangle\cdots|\alpha_{2N-1,2r-1}\rangle, \qquad \text{(A14)}$$

which means that in its fermionic formulation $\sigma_1(2k+1,2r+1)$ is equipped with the same nonlocal operator as Eq. (A12),

$$[c(2k+1,2r+1)+c^{+}(2k+,2r+1)] \times\prod_{i=k-1}^{0}[1-2n(2i+1,2r+1)]\prod_{j=r+1}^{N}[1-2n(0,2j)]. \qquad \text{(A15)}$$

As an example, consider the spin operators on the vertices (5,3) and (6,2) in Fig. 2. There the positions of the $1-2n$ operators are marked by arrows at the corresponding sites. If spin operators are placed on the edges of the lattice $\sigma_1(2i+1,1)$ and $\sigma_1(0,2j)$, they immediately act on $|\Sigma\rangle$ and can be replaced by the fermionic operators,

$$[c(2i+1,1)+c^{+}(2i+1,1)] \times\prod_{r=1}^{N}[1-2n(0,2r)]\prod_{k=1}^{i-1}[1-2n(2k+1,1)]$$

and

$$[c(0,2j)+c^{+}(0,2j)]\prod_{r=j+1}^{N}[1-2n(0,2r)],$$

respectively, in accordance with general expressions in Eqs. (A12) and (A15).

In order to calculate the correlation function $\langle\sigma_1(i,j)\sigma_1(k,r)\rangle$ in the fermionic operator form, it is necessary to replace $\sigma_1(i,j)$ by corresponding fermionic operators (A12) and (A15). As it is shown in the first part of this section, coinciding operators $1-2n$ in fermionic counterparts of $\sigma_1(i,j),\sigma_1(k,r)$ operators cancel each other and only operators placed on a path, which connects points (i,j) and (k,r), will be left. The choice of the path is arbitrary, which is a result of property (A5), $1-2n=\sigma_z$ and

$$(\sigma_z\otimes\sigma_z)\mathbf{R}(\sigma_z\otimes\sigma_z)=\mathbf{R}, \qquad \text{(A16)}$$

with $\mathbf{R}$ operator defined in Eq. (2.6).

More precisely, a fermionic realization for the product of two spin operators, $\sigma_1(i,j)$ and $\sigma_1(k,r)$, when $i<k, j<k$, has the form presented in Eq. (5.9). It looks as if one inserts into the vertices on a path between (i,j) and (k,r) points, operators $\sigma_z(=1-2n)$, instead of unity operators in the spin representation, and vice versa: it is a simple task to derive the correlation function $\langle\sigma_1(i,j)[\prod\sigma_z(i',j')]\sigma_1(k,r)\rangle$ on the two-dimensional lattice, where the operators σ_z are placed on a path of vertices connecting points (i,j) and (k,r). It can be done by replacing operators σ_1 by $(c+c^{+})$ and finding corresponding Green's functions [see Eq. (A7)].

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