Entanglement entropy in the $O(N)$ model

Max A. Metlitski,[1] Carlos A. Fuertes,[2] and Subir Sachdev[1]
[1]*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*
[2]*Instituto de Física Teórica IFT UAM/CSIC, Facultad de Ciencias C-XVI, C.U. Cantoblanco, E-28049 Madrid, Spain*
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It is generally believed that in spatial dimension $d>1$, the leading contribution to the entanglement entropy $S=-\mathrm{tr}\,\rho_A \log \rho_A$ scales as the area of the boundary of subsystem A. The coefficient of this "area law" is nonuniversal. However, in the neighborhood of a quantum critical point S is believed to possess subleading universal corrections. In the present work, we study the entanglement entropy in the quantum $O(N)$ model in $1<d<3$. We use an expansion in $\epsilon=3-d$ to evaluate (i) the universal geometric correction to S for an infinite cylinder divided along a circular boundary; (ii) the universal correction to S due to a finite correlation length. Both corrections are different at the Wilson-Fisher and Gaussian fixed points, and the $\epsilon\to 0$ limit of the Wilson-Fisher fixed point is distinct from the Gaussian fixed point. In addition, we compute the correlation length correction to the Renyi entropy $S_n=\frac{1}{1-n}\log \mathrm{tr}\,\rho_A^n$ in ϵ and large-N expansions. For $N\to\infty$, this correction generally scales as N^2 rather than the naively expected N. Moreover, the Renyi entropy has a phase transition as a function of n for d close to 3.

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I. INTRODUCTION

One of the most fascinating and counterintuitive properties of a quantum system is the entanglement of its many-body wave function. In recent years, there has been a lot of interest in using entanglement as a theoretical probe of ground-state correlations.[1] It is hoped that this viewpoint will be particularly fruitful in studying quantum critical points, which realize some of the most nonclassical entangled states of matter.

A useful measure of entanglement is given by the entanglement entropy S also known as von Neumann entropy. To compute S, we divide the system into two parts A and B and determine the reduced density matrix $\rho_A = tr_B\, \rho$, where ρ is the full density matrix of the system. Then, the entanglement entropy,

$$S_A = -\,\mathrm{tr}_A\, \rho_A \log \rho_A \tag{1.1}$$

If the system is in a pure state, then the entanglement entropy is "mutual," i.e., $S_A = S_B$.

One may ask how does the entanglement entropy behave near a quantum critical point. This question has been addressed completely for one-dimensional critical points with dynamical critical exponent $z=1$. Such critical points are described by $1+1$-dimensional conformal field theories (CFTs). In these systems, if A is chosen to be a segment of length l and B—its complement in the real line, the entanglement entropy is given by[2,3]

$$S = \frac{c}{3}\log l/a, \tag{1.2}$$

where a is the short-distance cutoff and the constant c known as the central charge, is a fundamental property of the CFT. Moreover, if the system is perturbed away from the critical point, the entanglement entropy becomes

$$S = \mathcal{A}\frac{c}{6}\log \xi/a, \tag{1.3}$$

where ξ is the correlation length and $\mathcal{A}$ is the number of boundary points of the region A. Here it is assumed that A and B are composed of intervals whose length is much larger than ξ.

The study of entanglement entropy at quantum critical points in dimension $d>1$ has received much less attention. The leading contribution to S is believed to satisfy the "area law,[4,5]"

$$S = C\frac{\mathcal{A}}{a^{d-1}}, \tag{1.4}$$

where $\mathcal{A}$ is the length/area of the boundary between the regions A and B. Physically, the area law implies that the entanglement in $d>1$ is local to the boundary even at the critical point (for a recent review of the area law, see Ref. 6). The coefficient C entering the area law is sensitive to the short-distance cutoff and is, therefore, nonuniversal. So, in contrast to the one-dimensional case, the leading term (1.4) in the entanglement entropy in higher dimensions cannot be used to characterize various critical points.

Proceeding, more generally, beyond the leading area law term, at least for Lorentz-invariant theories that we study here, it is expected that the entanglement entropy near a critical point has the scaling form,[7–11]

$$S = g_{d-1}[\mathcal{B}]a^{-(d-1)} + g_{d-2}[\mathcal{B}]a^{-(d-2)} + \ldots + g_0[\mathcal{B}]\log(L/a) + S_0(L/\xi). \tag{1.5}$$

Here L is a characteristic finite size in the problem. The coefficients of the ultraviolet divergent terms $g_i[\mathcal{B}]$ are integrals of local geometric invariants over the boundary $\mathcal{B}$ between regions A and B and scale as L^i under dilatations. In particular, the first coefficient $g_{d-1}[\mathcal{B}]$ is proportional to the area of the boundary $\mathcal{A}$. Clearly, the prefactors of extensive

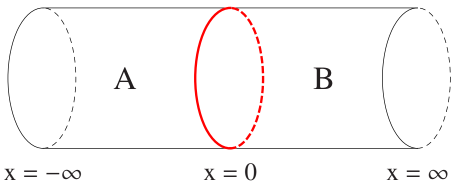

FIG. 1. (Color online) The cylindrical geometry considered in calculation of finite-size correction to the entanglement entropy.

terms $g_i[\mathcal{B}]$ with $i \geq 1$ are nonuniversal, while the coefficient of the logarithmic term $g_0[\mathcal{B}]$ is universal. The finite piece S_0 is a function of the dimensionless ratio L/ξ and encodes geometric and correlation length corrections to the entanglement entropy. It is universal up to additive dilatation invariant geometric contributions from the boundary. If such contributions, $g_0[\mathcal{B}]$, in particular, vanish, S_0 becomes completely universal. There exists some evidence[7,8,12] that this, indeed, occurs when the boundary $\mathcal{B}$ is closed and smooth and the spatial dimension $d=2$. On the other hand, if the dimension $d=3$ then $g_0[\mathcal{B}]$ is generally nonzero due to the extrinsic curvature of the boundary and S_0 contains additive nonuniversal contributions.[7,8,13] Likewise, $g_0[\mathcal{B}]$ is known to be nonzero even in $d=2$ when the boundary contains corners/end points.[9–11]

We note that the above considerations have only been verified by explicit field-theoretic calculations in free theories. These assertions were also confirmed in strongly coupled supersymmetric gauge theories using the AdS/CFT correspondence.[7,8] Recently, universal corrections were found for a special class of quantum critical points in $d=2$, which are described by dimensional reduction to a classical $d=2$ field theory.[14,15] However, such critical points are nongeneric and unstable[16,17] in physical situations to quantum critical points described by interacting field theories in three space-time dimensions.

In the present work, we compute the geometric and correlation length corrections to the entanglement entropy in the simplest generic interacting CFT in $d=2$ dimensions: the $O(N)$ model. We verify that these corrections are, indeed, universal. We perform our calculations using expansions in $\epsilon=3-d$ and $1/N$.

In the rest of this paper, we consider the following geometry. We take two semi-infinite regions A and B with a straight boundary at x=0. The boundary extends along the remaining $d-1$ spatial directions, each taken to have a length L. For technical reasons, we impose antiperiodic boundary conditions along each of these directions. We also consider more general boundary conditions with a twist by an arbitrary phase φ in a theory of $N/2$ complex scalar fields. So in the physical case $d=2$, our space is an infinite cylinder divided into regions A and B along a circle of length L (see Fig. 1). For general d, the boundary $\mathcal{B}$ between the regions A and B is a $d-1$ dimensional torus. As $\mathcal{B}$ is flat, the only geometric invariant on it is the area $\mathcal{A}=L^{d-1}$. Hence, all the subleading coefficients $g_i[\mathcal{B}]$, $0 \leq i < d-1$ in Eq. (1.5) vanish and S_0 is universal in this geometry. In particular, at the critical point S_0 becomes a universal geometric constant γ and the entanglement entropy is given by

$$S = C\frac{L^{d-1}}{a^{d-1}} + \gamma. \tag{1.6}$$

We explicitly compute the constant γ. To leading order in ϵ-expansion, we obtain

$$\gamma = -\frac{N\epsilon}{6(N+8)}\left[\log\left|\theta_1\left(\frac{\varphi(1+i)}{2\pi},i\right)\right| - \frac{\varphi^2}{4\pi} - \log\,\eta(i)\right],$$

$$d=3-\epsilon, \quad \text{Wilson-Fisher fixed point.} \tag{1.7}$$

Here θ_1 and η are Jacobi elliptic and Dedekind-eta functions and i is the square root of -1. The sign of γ depends on the value of φ: it is negative for $\varphi=\pi$ (antiperiodic boundary conditions) and positive for $\varphi \to 0$. Note that Eq. (1.7) is only valid for $\varphi \gg \epsilon^{1/2}$. For zero twist (periodic boundary conditions), we hypothesize that to leading order,

$$\gamma = -\frac{N\epsilon}{12(N+8)}\log\,\epsilon. \tag{1.8}$$

The result (1.7) should be compared to the corresponding value at the Gaussian fixed point in $d=3-\epsilon$ dimensions,

$$\gamma = -\frac{N}{6}\left[\log\left|\theta_1\left(\frac{\varphi(1+i)}{2\pi},i\right)\right| - \frac{\varphi^2}{4\pi} - \log\,\eta(i)\right],$$

$$d=3-\epsilon, \quad \text{Gaussian fixed point.} \tag{1.9}$$

We see that $|\gamma|$ is parametrically smaller at the Wilson-Fisher fixed point than at the Gaussian fixed point. Thus, entanglement entropy distinguishes these two fixed points already at leading order in ϵ-expansion.

If we perturb the system by tuning a relevant coupling t slightly away from the critical point $t=t_c$, the entanglement entropy obeys the scaling form,

$$S = C(t)\frac{L^{d-1}}{a^{d-1}} + S_0(L/\xi). \tag{1.10}$$

Here $C(t)$ is a nonuniversal, analytic function of t, while S_0 is a universal function of the dimensionless ratio L/ξ. In the limit $L/\xi \to 0$, the system is effectively critical and S_0 reduces to the geometric constant γ of Eq. (1.6). In the opposite limit $L/\xi \to \infty$, the system obeys the area law, hence,

$$S = C(t)\frac{L^{d-1}}{a^{d-1}} + r\frac{L^{d-1}}{\xi^{d-1}}, \tag{1.11}$$

where r is a universal coefficient that we compute. Note that both terms in Eq. (1.11) contribute to the t dependence of the prefactor in the area law. The contribution of the first term is analytic and so to leading order scales as $t-t_c$. On the other hand, the contribution of the second term is nonanalytic and scales as $(t-t_c)^{\nu(d-1)}$, where ν is the correlation length exponent. Since in the $O(N)$ model $\nu<1$ for $d=2$, the nonanalytic contribution from the universal term dominates.

In general, the coefficient r is tied to the specific choice for the definition of the correlation length ξ. In the $O(N)$ model, there is a very natural choice $\xi=m^{-1}$, where m is the gap to the first excitation. Note that in the present work, we only consider the phase of the $O(N)$ model with unbroken symmetry. The value of r to leading order in ϵ-expansion is found to be

$$r = -\frac{N}{144\pi}, \quad d = 3 - \epsilon, \quad \text{Wilson-Fisher fixed point.} \tag{1.12}$$

As with the finite-size correction, $|r|$ is parametrically smaller at the Wilson-Fisher fixed point than at the Gaussian fixed point, where[3]

$$r = -\frac{N}{24\pi\epsilon}, \quad d = 3 - \epsilon, \quad \text{Gaussian fixed point.} \tag{1.13}$$

We would like to note that the only corrections to the scaling forms in Eqs. (1.6) and (1.10) come from irrelevant operators and scale as L^{-p}, $p > 0$. Two operators compete for the role of the leading correction to scaling. The first of these has the usual bulk correction-to-scaling exponent $p = \omega$. The second is an operator living on the boundary $\mathcal{B}$ with $p = 2 - 1/\nu$. Numerically, $2 - 1/\nu < \omega$ for $d = 2$ and $N = 1, 2, 3$, so the corrections from the boundary operator dominate.[18]

In addition to the entanglement entropy, we study the Renyi entropy

$$S_n = \frac{1}{1-n} \log \text{tr}_A \, \rho_A^n. \tag{1.14}$$

The Renyi entropy always naturally appears in field-theoretic calculations as it is related to the partition function of the theory on an n-sheeted Riemann surface. One then obtains the entanglement entropy by taking the limit $S = \lim_{n\to 1} S_n$. At least for n close to 1, the Renyi entropy is believed to possess the same universal properties as the entanglement entropy. In particular, the finite-size and correlation length corrections are given by

$$S_n = C_n \frac{L^{d-1}}{a^{d-1}} + \gamma_n, \tag{1.15}$$

$$S_n = C_n(t) \frac{L^{d-1}}{a^{d-1}} + r_n \frac{L^{d-1}}{\xi^{d-1}}, \tag{1.16}$$

where the nonuniversal coefficient C_n of the leading area law term, as well as the universal coefficients γ_n, r_n are now n dependent. We compute r_n in ϵ and large-N expansions. A careful renormalization-group (RG) analysis demonstrates that r_n is parametrically enhanced in both of these limits. In particular, $r_n \sim O(\frac{1}{\epsilon})$ in the ϵ-expansion. However, the enhancement is most striking in the large-N expansion, where we find $r_n \sim O(N^2)$. Such scaling is in contrast with the result $r_n \sim O(N)$ that one would obtain at each order in $1/N$ for fixed correlation length ξ, implying that the limits $\xi \to \infty$ and $N \to \infty$ do not commute. As far as we know, this is the first violation of naive large-N counting in the $O(N)$ model. A common feature of the two expansions is that the leading term of r_n behaves as $r_n \sim n-1$ for $n \to 1$ and does not contribute to the entanglement entropy S. Hence, $r \sim O(N)$ in the large N limit and $r \sim O(1)$ in the ϵ-expansion.

Another unusual phenomenon that we find in ϵ-expansion is nonanalytic dependence of the coefficients γ_n, r_n on n. In fact, γ_n and r_n will have a discontinuity at $n = n^*$, where n^* is generally nonuniversal and lies in the range $1 < n^* \leq 1 + \frac{3}{4}\frac{N+2}{N+8}\epsilon$. The n dependence of γ_n and r_n for $n < n^*$ and $n > n^*$ is, however, universal. Thus, we have two universal branches for γ_n and r_n. We note that Eqs. (1.15) and (1.16) are understood in the limit $L \to \infty$, $\xi \to \infty$. However, there appears a new divergent length scale in the problem as $n \to n^*$, and the limits $n \to n^*$ and $L \to \infty$, $\xi \to \infty$ do not commute. In particular, if we fix the size of our regions L or the correlation length ξ, the n dependence of the Renyi entropy S_n will be completely analytic. Moreover, due to the emergence of a new length scale as $n \to n^*$, in the crossover region S_n is not entirely universal. We stress that any nonanalyticity and nonuniversality only occurs away from the point $n = 1$. In particular, the entanglement entropy $S = \lim_{n\to 1} S_n$ is well defined and universal.

The nonanalytic behavior discussed above is also found to occur in the large-N expansion in dimensions $2.74 \lesssim d < 3$. The limited range of d suggests that this phenomenon might be absent in the $O(N)$ model in the physically relevant case $d = 2$. Nevertheless, we expect that such nontrivial n dependence will occur quite generically at other quantum critical points.

This paper is organized as follows. In Sec. II, we remind the reader of the replica trick, which relates the entanglement entropy to the partition function on an n-sheeted Riemann surface. In Sec. III, we show that the coefficient of the correlation length correction to the Renyi entropy r_n is parametrically enhanced in both expansions we consider. Sections IV and V are, respectively, devoted to the evaluation of correlation length and finite-size corrections in ϵ-expansion. In Sec. VI we compute the coefficient r_n in the large-N expansion. Some concluding remarks are given in Sec. VII.

II. REPLICA TRICK

We consider the $O(N)$ model in $D = d + 1$ space-time dimensions. The action for the N-component real scalar field ϕ is given by

$$S = \int d^d x d\tau \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{t}{2}\phi^2 + \frac{u}{4}\phi^4 \right]. \tag{2.1}$$

We divide our space into two regions A and B with the boundary being a $d-1$ dimensional plane at $\mathrm{x} = 0$. We will denote the coordinates along the boundary directions by $x_\perp$. The Renyi entropy S_n may be calculated as

$$S_n = \frac{1}{1-n} \log \frac{Z_n}{Z_1^n}, \tag{2.2}$$

from which we obtain the entanglement entropy

$$S = \lim_{n\to 1} S_n. \tag{2.3}$$

Here Z_n is the partition function of the theory on an n-sheeted Riemann surface. This Riemann surface lies in the $x_\parallel = (\tau, \mathrm{x})$ plane and has a conical singularity at (τ, x)

$=(0,0)$. The surface is invariant under translations along the $x_\perp$ directions. We may use the following metric for our space time:

$$ds^2 = dr^2 + r^2 d\theta^2 + dx_\perp^2, \tag{2.4}$$

where r,θ are the polar coordinates in the (τ, x) plane. Concentrating on this plane, we see that the metric is exactly the same as for the usual Euclidean plane; the only modification is that the angular variable θ has a period $\theta \sim \theta + 2\pi n$.

III. PARAMETRIC ENHANCEMENT OF CORRELATION LENGTH CORRECTION

In this section, we show that the coefficient r_n of the correlation length correction to the Renyi entropy [Eq. (1.16)] is parametrically enhanced in both expansions that we consider. Moreover, we demonstrate that r_n can—to leading order—be extracted from the properties of the theory at the critical point.

We start with the $O(N)$ model perturbed away from the critical point $t=t_c$ by a finite $\tilde{t}=t-t_c>0$ (we drop the tilde below). To compute r_n, we need to find the dependence of the partition function Z_n on the mass gap $m=\xi^{-1}$. Here we assume that the dimensions of the boundary $L \gg \xi$, so that we can take the limit $L \to \infty$. It is useful to differentiate

$$\frac{d}{dt}\log\frac{Z_n}{Z_1^n} = -\frac{1}{2}\left(\int_{n\text{-sheets}} d^D x \langle \phi^2(x)\rangle_n - n\int_{1\text{-sheet}} d^D x\langle\phi^2(x)\rangle_1\right)$$
$$= -\frac{1}{2}L^{d-1}\int_{n\text{-sheets}} d^2 x_\parallel (\langle \phi^2(x)\rangle_n - \langle\phi^2(x)\rangle_1) \tag{3.1}$$

where we have used the fact that the contribution to the integral from each of the sheets is the same (from here on, all integrals over $d^2x_\parallel$ are understood to be over n sheets). Now, recalling, $m \sim t^\nu$, we may convert the derivative with respect to t into a derivative with respect to m,

$$m\frac{d}{dm}\log\frac{Z_n}{Z_1^n} = -\frac{1}{2\nu}L^{d-1}\int d^2 x_\parallel\, t(\langle \phi^2(x)\rangle_n - \langle\phi^2(x)\rangle_1). \tag{3.2}$$

The expression $t(\langle \phi^2(x)\rangle_n - \langle\phi^2(x)\rangle_1)$ is renormalization-group invariant.[19] Thus, we may write

$$t(\langle \phi^2(x)\rangle_n - \langle\phi^2(x)\rangle_1) = m^D f_n(mr), \tag{3.3}$$

where f_n is a universal function. The function f_n is expected to decay exponentially for $mr \gg 1$, and the integral in Eq. (3.2) converges for $r \to \infty$. The short-distance asymptotic of f_n is controlled by the critical point. From the scaling dimension of the operator $\phi^2(x)$, $[\phi^2(x)] = D-\nu^{-1}$, we conclude

$$f_n(u) \to \frac{d_n}{u^{D-1/\nu}}, \quad u \ll 1, \tag{3.4}$$

where d_n is a universal constant. So the integral in Eq. (3.2) converges for $r\to 0$ provided that $\nu^{-1} > D-2$.[20] In the $O(N)$ model in both expansions, we consider $\nu^{-1} = D-2+\nu_1$, where the correction ν_1 is given to leading order by

$$\nu_1 = \frac{6\epsilon}{N+8}, \quad D = 4-\epsilon, \tag{3.5}$$

$$\nu_1 = \frac{1}{N}\frac{8\Gamma(D)}{D\Gamma(2-D/2)\Gamma(D/2-1)^2\Gamma(D/2)},$$

$$\nu_1(D=3) = \frac{32}{3\pi^2 N}, \quad N\to\infty. \tag{3.6}$$

In particular, $\nu_1 > 0$ and ν^{-1} asymptotically approaches $D-2$ from above in both limits. With these remarks in mind, we integrate Eq. (3.2) with respect to m,

$$\log\frac{Z_n}{Z_1^n}(t) - \log\frac{Z_n}{Z_1^n}(t=0) = -\frac{\pi n}{\nu(d-1)}(mL)^{d-1}\int_0^\infty du\, u f_n(u). \tag{3.7}$$

This is as far as we can proceed in general; to make further progress, one needs the function $f_n(u)$. However, we have already noted that due to the fact $\nu^{-1}\to D-2$, the integral in Eq. (3.7) is very close to diverging in both expansions. Hence, to leading order in ϵ or $1/N$, this integral is saturated at short distances,

$$\int_0^\infty du\, u f_n(u) \to \frac{d_n}{\nu^{-1}-(D-2)} = \frac{d_n}{\nu_1}, \tag{3.8}$$

and

$$\log\frac{Z_n}{Z_1^n} \approx -\frac{\pi n}{\nu_1}d_n (mL)^{d-1}, \tag{3.9}$$

where we have dropped the constant contribution at the critical point $t=0$. So, the universal coefficient r_n of the correlation length correction [Eq. (1.16)] is given by

$$r_n \approx -\frac{\pi n}{(1-n)\nu_1}d_n. \tag{3.10}$$

Thus, to leading order the problem is reduced to evaluating the coefficient d_n in Eq. (3.4). Since this coefficient is a short-distance property, we may work directly at the critical point. Note in particular that in the large N limit, $d_n \sim O(N)$, so our result for $\log\frac{Z_n}{Z_1^n}$ scales as N^2. This is in contrast to the linear in N behavior that one would obtain at any finite order in the $1/N$ expansion for a fixed correlation length ξ.

It turns out that the leading term (3.10) behaves as $r_n \sim (n-1)$ for $n\to 1$ in both expansions and does not contribute to the entanglement entropy [Eq. (2.3)]. Thus, the correlation length correction to the entanglement entropy has the expected scaling $r \sim O(N)$. To proceed systematically beyond the leading order, one needs to use RG technology that will be developed explicitly in the context of ϵ-expansion in Sec. IV C 1.

IV. ϵ-EXPANSION: CORRELATION LENGTH CORRECTION

In this section, we compute the correlation length correction to the entanglement entropy in ϵ expansion. Recall that for the interacting $O(N)$ model, $\nu_1 = \nu^{-1} - (D-2) \sim O(\epsilon)$ in $D = 4 - \epsilon$ dimensions; hence, the argument in Sec. III can be applied. This is also true for the noninteracting (Gaussian) fixed point for $D = 4 - \epsilon$, where $\nu_1 = \epsilon$, allowing us to compare the predictions of our method to the exact calculations of Ref. 3. We first consider the Gaussian fixed point and then proceed to the Wilson-Fisher fixed point.

A. Gaussian theory

Consider the Gaussian theory,

$$L = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{t}{2}\phi^2, \tag{4.1}$$

where $t = m^2$. We need to compute the expectation value,

$$\langle \phi^2(x) \rangle_n - \langle \phi^2(x) \rangle_1, \tag{4.2}$$

at the critical point $t = 0$. To the leading order, we may work in $D = 4$. The massless propagator on an n-sheeted Riemann surface in $D = 4$ is known to be[21]

$$G_n(r, r', \theta, x_\perp) = \frac{\sinh(\eta/n)}{8\pi^2 n r r' \sinh \eta [\cosh(\eta/n) - \cos(\theta/n)]}, \tag{4.3}$$

where

$$\cosh \eta = \frac{r^2 + r'^2 + x_\perp^2}{2 r r'} \tag{4.4}$$

Hence,

$$\langle \phi^2(x) \rangle_n - \langle \phi^2(x) \rangle_1 = \frac{N}{48\pi^2 r^2}\left(\frac{1}{n^2} - 1\right). \tag{4.5}$$

So comparing to Eqs. (3.3) and (3.4), we obtain

$$d_n = \frac{N}{48\pi^2}\left(\frac{1}{n^2} - 1\right), \quad \text{Gaussian fixed point}, \quad D = 4 - \epsilon. \tag{4.6}$$

We can now use Eq. (3.10) to compute the coefficient r_n of the correlation length correction. As noted above for the Gaussian theory, $\nu_1 = \epsilon$, so

$$r_n = -\frac{N}{48\pi\epsilon}\left(1 + \frac{1}{n}\right), \tag{4.7}$$

and for the entanglement entropy proper

$$r = \lim_{n \to 1} r_n = -\frac{N}{24\pi\epsilon}. \tag{4.8}$$

This can be compared to the exact result of Ref. 3,

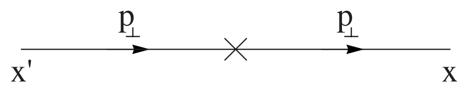

FIG. 2. Leading correction to the propagator $\delta\mathcal{G}^{1,0}$ due to the boundary perturbation. Here and below, a cross denotes an interaction vertex of c.

$$r_n = N \frac{\Gamma\left(\frac{2-D}{2}\right)}{24(4\pi)^{(D-2)/2}}\left(1 + \frac{1}{n}\right). \tag{4.9}$$

Equation (4.9) is in agreement with our result (4.7) to leading order in ϵ, which is all that the discussion in Sec. III guarantees.

B. Interacting theory

We now proceed to consider the interacting $O(N)$ model [Eq. (2.1)]. We again need to compute the expectation value (4.2). Naively, one would expect that at leading order in ϵ, one can work with the mean-field approximation $u = 0$, recovering the result (4.6). Then, one would simply substitute Eq. (4.6) into Eq. (3.10) and use the appropriate ν_1 [Eq. (3.5)], for the Wilson-Fisher fixed point. However, such reasoning turns out to be too simple minded, as it neglects "boundary perturbations." Indeed, our conical singularity will generally induce local perturbations at $r = 0$. Of these, the term with the lowest engineering dimension is

$$\delta S = \frac{c}{2}\int d^{D-2}x_\perp \phi^2(r = 0, x_\perp). \tag{4.10}$$

In the absence of the conical singularity, this perturbation is known to be irrelevant in the $O(N)$ model as the scaling dimension $[c] = \nu^{-1} - 2 < 0$.[22] However, as we will now show, the presence of the conical singularity will modify the renormalization-group flow of the coefficient c.

The engineering dimension of the coupling constant c is zero in any space-time dimension D. We wish to compute the β function $\beta(c)$. Let us perform the perturbation theory in u and c for the two-point function $\langle \phi_\alpha(x)\phi_\beta(x')\rangle = \delta_{\alpha\beta}\mathcal{G}(x, x')$. It is sufficient to work in $D = 4$ dimensions to compute the leading terms in $\beta(c)$. We use a mixed momentum/position $p_\perp$, $x_\parallel$ representation. To the first order in c and the zeroth order in u, we have the simple diagram in Fig. 2,

$$\delta^{1,0}\mathcal{G}(x_\parallel, x'_\parallel, p_\perp) = -c G_n(x_\parallel, 0, p_\perp) G_n(0, x'_\parallel, p_\perp), \tag{4.11}$$

where the superscripts on δ indicate the order in c and u. Notice that the bare propagator $G_n(x, x')$ [Eq. (4.3)] remains finite as its arguments approach the conical singularity. In fact,

$$G_n(0, x) = \frac{1}{n} G_1(x). \tag{4.12}$$

Also, $G_n(x_\parallel, x'_\parallel, p_\perp)$ is just the two-dimensional massive propagator $(-\nabla_2^2 + p_\perp^2)^{-1}$ on an n-sheeted Riemann surface. In particular, $G_n(x_\parallel, 0, p_\perp) = \frac{1}{n} K_0(p_\perp |x_\parallel|)$ [which implies that the relation (4.12) is actually correct in any dimension]. Thus, the correction (4.11) is finite.

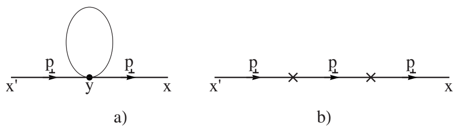

FIG. 3. Corrections to the propagator; (a) $\delta\mathcal{G}^{0,1}$ and (b) $\delta\mathcal{G}^{2,0}$. Here and below, a dot denotes an interaction vertex of u.

We next consider the Hartree-Fock (first order in u) correction to the propagator [Fig. 3(a)],

$$\delta^{0,1}\mathcal{G}(x_\parallel,x'_\parallel,p_\perp) = -(N+2)u\int d^2y_\parallel G_n(x_\parallel,y_\parallel,p_\perp)G_n(y_\parallel,x'_\parallel,p_\perp) \times[G_n(y,y)-G_1(y,y)]. \tag{4.13}$$

We have already evaluated $G_n(y,y)-G_1(y,y)\sim\frac{1}{y_\parallel^2}$ [Eq. (4.5)]. Thus, the integral (4.13) has an ultraviolet divergence in the region $y_\parallel\to 0$,

$$\delta^{0,1}\mathcal{G}(x_\parallel,x'_\parallel,p_\perp) \overset{UV}{=} \frac{(N+2)u}{24\pi}\left(n-\frac{1}{n}\right) \times G_n(x_\parallel,0,p_\perp)G_n(0,x'_\parallel,p_\perp)\log(\Lambda). \tag{4.14}$$

Notice that this divergence is local to the conical singularity and, as is evident from Eq. (4.11), can be canceled by an additive renormalization of the coupling constant c. Hence, the perturbation (4.10) will be automatically induced by the presence of the conical singularity.

We also consider the second-order contribution in c to the propagator [Fig. 3(b)],

$$\delta^{2,0}\mathcal{G}(x_\parallel,x'_\parallel,p_\perp) = c^2G_n(x_\parallel,0,p_\perp)G_n(0,x'_\parallel,p_\perp)G_n(0,0,p_\perp). \tag{4.15}$$

The quantity $G_n(0,0,p_\perp)$ is UV singular,

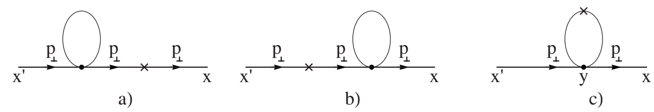

FIG. 4. Corrections to the propagator $\delta\mathcal{G}^{1,1}$.

$$G_n(0,0,p_\perp) = \int d^2y_\perp G_n(0,0,y_\perp)e^{-ip_\perp y_\perp} = \frac{1}{4\pi^2 n}\int d^2y_\perp\frac{1}{y_\perp^2}e^{ip_\perp y_\perp} \overset{UV}{=} \frac{1}{2\pi n}\log(\Lambda/p_\perp), \tag{4.16}$$

so

$$\delta^{2,0}\mathcal{G}(x_\parallel,x'_\parallel,p_\perp) \overset{UV}{=} \frac{c^2}{2\pi n}G_n(x_\parallel,0,p_\perp)G_n(0,x'_\parallel,p_\perp)\log(\Lambda). \tag{4.17}$$

The divergence of Eq. (4.17) is a manifestation of the well-known fact that the two-dimensional δ-function potential requires regularization. Again, from Eq. (4.11), we observe that the divergence can be eliminated by a renormalization of the coefficient c.

Finally, we consider corrections which are bilinear in c and u (Fig. 4). For c small, these corrections are generally subleading compared to $\delta^{0,1}\mathcal{G}$ [Fig. 3(a)]. However, for $n\to 1$, $\delta^{0,1}\mathcal{G}$ vanishes, and the diagram in Fig. 4(c) becomes important. On the other hand, the diagrams in Figs. 4(a) and 4(b) can be ignored to leading order for all n since they also vanish at $n=1$.[23] With this in mind, we only need to evaluate Fig. 4(c) at $n=1$. We recognize that this is just the diagram corresponding to the usual multiplicative renormalization of the ϕ^2 operator. Explicitly,

$$\delta^{1,1}\mathcal{G}(x_\parallel,x'_\parallel,p_\perp) \overset{n=1}{=} (N+2)uc\int d^2y_\parallel G_1(x_\parallel,y_\parallel,p_\perp)G_1(y_\parallel,x'_\parallel,p_\perp)\int d^2z_\perp G_1(y_\parallel,z_\perp)^2$$
$$= (N+2)uc\int d^2y_\parallel G_1(x_\parallel,y_\parallel,p_\perp)G_1(y_\parallel,x'_\parallel,p_\perp)\frac{1}{16\pi^3 y_\parallel^2} \overset{UV}{=} \frac{(N+2)uc}{8\pi^2}G_1(x_\parallel,0,p_\perp)G_1(0,x'_\parallel,p_\perp)\log\Lambda. \tag{4.18}$$

We can now introduce counterterms to cancel the divergences considered above,

$$c = c_r + \left[\frac{(N+2)u_r}{24\pi}\left(n-\frac{1}{n}\right)+\frac{(N+2)u_rc_r}{8\pi^2}+\frac{c_r^2}{2\pi n}\right]\log(\Lambda/\mu), \tag{4.19}$$

where c_r and u_r are the renormalized coupling constants and μ is the renormalization scale. Note that the coefficient of the u_rc_r term has been only computed at $n=1$. So,

$$\beta(c_r) = \mu\left.\frac{\partial}{\partial\mu}c_r\right|_{c,u} = \frac{(N+2)u_r}{24\pi}\left(n-\frac{1}{n}\right)+\frac{(N+2)u_rc_r}{8\pi^2} + \frac{c_r^2}{2\pi n}. \tag{4.20}$$

Note that the RG flow of u is not affected by the boundary perturbation or by the presence of the conical singularity,

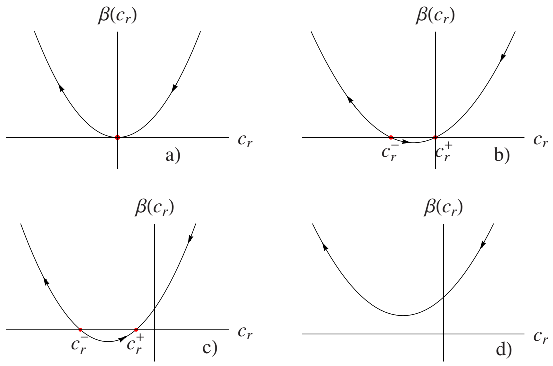

FIG. 5. (Color online) β function of the boundary coupling c_r for (a) noninteracting theory ($u=0$), (b) interacting theory $n=1$, (c) interacting theory $n<n_c$, and (d) interacting theory $n>n_c$.

$$\beta(u_r) = -\epsilon u_r + \frac{N+8}{8\pi^2}u_r^2, \tag{4.21}$$

and we have the usual Wilson-Fisher fixed point $u^* = \frac{8\pi^2\epsilon}{N+8}$.

We now discuss the RG flow of c_r in detail. Let us start with the noninteracting theory $u=0$, which corresponds to the well-studied problem of a particle in a two-dimensional δ-function potential. Then, $\beta(c_r) = \frac{1}{2\pi n}c_r^2$. As demonstrated in Fig. 5(a), the coupling constant c_r flows logarithmically to zero for $c_r>0$ and runs away to $-\infty$ for $c_r<0$, signaling the formation of a bound state.

Next, consider turning on the interaction u, in the absence of conical singularity ($n=1$). Then, $\beta(c_r) = -\eta_2(u_r)c_r + \frac{c_r^2}{2\pi}$, where η_2 is just the usual anomalous dimension of the ϕ^2 operator, ($[\phi^2] = D-2-\eta_2$),

$$\eta_2(u_r) = -\frac{(N+2)u_r}{8\pi^2}. \tag{4.22}$$

The RG flow of c is sketched in Fig. 5(b). We find two fixed points: $c_r^+=0$ and $c_r^- = -\frac{N+2}{N+8}(2\pi\epsilon)$. The first fixed point $c_r^+=0$ is stable, due to $\beta'(c_r=0) = -\eta_2(u^*)>0$, which implies that for c small, the perturbation (4.10) is irrelevant.[22] This conclusion can be immediately reached by consideration of scaling dimensions at the interacting fixed point since $[c]=D-2-[\phi^2]=\eta_2<0$.

The second fixed point c_r^- is unstable, and for $c_r<c_r^-$ the RG flow runs away to $c_r=-\infty$. Naively, such a flow may be interpreted as a tendency of ϕ to condense in the vicinity of $r=0$. However, this would result in a condensate that is effectively $D-2<2$ dimensional, which, at least for $N\geq 2$ and $t>0$, is prohibited by the Mermin-Wagner theorem. Exactly at the critical point, long-range forces could, in principle, stabilize the condensate. However, as we will discuss in Sec. VI, large-N expansion suggests that no such condensation occurs even at $t=0$, and the flow actually terminates at a scale invariant fixed point, which is inaccessible in our perturbative expansion. However, this fixed point can likely be interpreted in terms of a fluctuating "boundary" order parameter.

Finally, we proceed to the interacting case in the presence of a conical singularity. For $n<n_c\approx 1+\frac{3}{4}\frac{N+2}{N+8}\epsilon$, we again obtain two fixed points [Fig. 5(c)],

$$c_r^{\pm} = \pi\left[-\frac{N+2}{N+8}n\epsilon \pm \sqrt{\left(\frac{N+2}{N+8}\right)^2 n^2\epsilon^2 - \frac{2}{3}\frac{N+2}{N+8}(n^2-1)\epsilon}\right]. \tag{4.23}$$

The fixed point c_r^+ is stable, while c_r^- is unstable. In the limit $n\to 1$, which is relevant for the computation of entanglement entropy, c_r^+ smoothly evolves to the $c_r^+=0$ stable fixed point, which we obtained in the absence of the conical singularity. Moreover, for $n\to 1$, we expect the starting point of the RG flow $c_r\to 0$. Hence, for n close to 1, the RG flow will terminate at the fixed point c_r^+. The stability exponent of this fixed point $\beta'(c_r^+)\to[\phi^2]-(D-2)=2-1/\nu$ as $n\to 1$. This boundary exponent, as well as the usual bulk exponent $\omega=\beta'(u_r^*)$ will control corrections to scaling for the entanglement entropy S.

Thus, the main effect of the conical singularity is to shift c_r^+ away from 0. The parametric magnitude of this shift depends on whether $1-n\gg\epsilon$ or $|1-n|\ll\epsilon$,

$$c_r^+ \approx \pi\sqrt{\frac{2}{3}\frac{N+2}{N+8}(1-n^2)\epsilon}, \quad 1-n\gg\epsilon, \tag{4.24}$$

$$c_r^+ \approx -\frac{2\pi}{3}(n-1) - \frac{2\pi}{9}\frac{N+8}{N+2}\frac{(n-1)^2}{\epsilon}, \quad |1-n|\ll\epsilon. \tag{4.25}$$

Thus, for $1-n\gg\epsilon$, $c_r^+\sim O(\sqrt{\epsilon})$: this is the regime in which the $u_r c_r$ term in the β function (4.20) can be ignored. On the other hand, for $|n-1|\ll\epsilon$, $c_r^+\sim(n-1)\ll\epsilon$ and the $u_r c_r$ term in $\beta(c_r)$ becomes important. Note that in both regimes, c_r^+ is parametrically small and the perturbative expansion in c_r is justified.

For $n>n_c$, both fixed points disappear and the RG flow runs away to $c_r=-\infty$ [Fig. 5(d)]. As discussed above for the case $n=1$, large N analysis suggests that the flow is toward another fixed point (which itself evolves as a function of n). Now there are two possibilities. If as n increases from 1 to n_c, the initial value of c_r determined by the microscopic details of the theory satisfies $c_r(n)>c_r^-(n)$ then the runoff to the $c_r=-\infty$ fixed point will occur precisely at $n=n^*=n_c$. On the other hand, if the initial value of the coupling $c_r(n)<c_r^-(n)$ for $n>n^*$, where $1<n^*<n_c$, the runaway to $c_r=-\infty$ will occur before n reaches n_c. Note that the value of n^* is generally nonuniversal. In either case, the long-distance physics is controlled by the c_r^+ fixed point for $n<n^*$ and the $c_r=-\infty$ fixed point for $n>n^*$. Thus, the constants γ_n, r_n, Eqs. (1.15) and (1.16) will always have a discontinuity at some $n=n^*$, $1<n^*\leq n_c$. Note that Eqs. (1.15) and (1.16) are understood in the limit when the size of the regions whose entanglement entropy we are computing and the correlation length ξ tend to infinity. However, as $n\to n^*$ a new divergent length scale emerges in the problem. In fact, we can think of the point $n=n^*$, $t=0$ as a multicritical point. Thus, the limits L, $\xi\to\infty$, and $n\to n^*$ do not commute. In particular, if we fix L or ξ, the dependence of the Renyi entropy on n will be com-

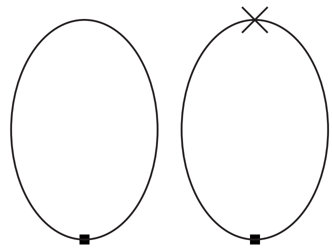

FIG. 6. Leading contributions to $\langle\phi^2(x)\rangle_n$ (denoted by a black square here and below): (a) mean-field result; (b) correction due to the boundary perturbation.

pletely analytic. Moreover, the emergence of a new length scale as $n\to n^*$ implies that the Renyi entropy in the crossover region is not entirely universal.

Having discussed the nontrivial n dependence of the Renyi entropy that occurs for n away from 1, we come back to the range $n<n_c$ and concentrate on the c_r^+ fixed point. We will from here on denote c_r^+ as c_r^*. Let us now compute the value of $\langle\phi^2(x)\rangle$ at this fixed point. The leading correction to the mean-field result [Fig. 6(a)] Eq. (4.5) is given by the diagram in Fig. 6(b),

$$\delta^{1,0}\langle\phi^2(x)\rangle = -Nc_r\int d^{D-2}y_\perp G_n^2(x,y) = -\frac{Nc_r}{16\pi^3 n^2}\frac{1}{r^2}. \tag{4.26}$$

Since to leading order we still have $t=m^2$, from Eqs. (3.3) and (3.4),

$$d_n \approx N\left[\frac{1}{48\pi^2}\left(\frac{1}{n^2}-1\right) - \frac{c_r^+}{16\pi^3 n^2}\right], \tag{4.27}$$

and from Eqs. (3.5) and (3.10), the coefficient of the correlation length correction to the Renyi entropy is

$$r_n \approx -\frac{\pi n(N+8)}{6\epsilon(1-n)}d_n. \tag{4.28}$$

As we see, in the regime $1-n \gg \epsilon$, taking the boundary perturbation into account only weakly modifies the mean-field result for d_n [Eq. (4.6)] by a term of order $\sqrt{\epsilon}$. Note that r_n is still strongly modified due to a different value of ν_1.

However, in the regime $|1-n| \ll \epsilon$,

$$d_n \approx \frac{N(N+8)}{(N+2)}\frac{(n-1)^2}{72\pi^2\epsilon}, \quad |1-n| \ll \epsilon, \tag{4.29}$$

$$r_n \approx \frac{N(N+8)^2}{N+2}\frac{n-1}{432\pi\epsilon^2}, \quad |1-n| \ll \epsilon. \tag{4.30}$$

Thus, for $n\to 1$, the behavior of d_n at the Wilson-Fisher is drastically different from the mean-field result [Eq. (4.6)]. In particular, notice that to the present order in ϵ, the correction due to the boundary perturbation precisely cancels the term linear in $n-1$ coming from Eq. (4.5). The technical reason for this remarkable cancellation is as follows. For $n\to 1$, we expect $c_r \sim O(n-1)$, and we can work just to first order in c. Then, in considering the corrections to the propagator, we can drop the diagram in Fig. 3(b), keeping only Figs. 3(a) and 4(c). These diagrams are, essentially, Hartree-Fock corrections to the propagator, and the "Hartree-Fock potential" at y is just $\langle\phi^2(y)\rangle_n - \langle\phi^2(y)\rangle_1 \sim 1/y_\parallel^2$. As a result, the diagrams diverge for $y_\parallel \to 0$. The β function for the coupling constant c_r vanishes precisely when this divergence is absent, i.e., $\langle\phi^2(y)\rangle_n - \langle\phi^2(y)\rangle_1 = 0$.

The crucial consequence of Eq. (4.29) is that to this order, the correction to entanglement entropy proper $r=\lim_{n\to 1} r_n = 0$. Thus,

$$r \sim O(1), \quad D = 4-\epsilon. \tag{4.31}$$

We conclude that the correlation length-dependent contribution to the entanglement entropy at the Wilson-Fisher fixed point is parametrically smaller than at the Gaussian fixed point in $D=4-\epsilon$ [Eq. (4.8)]. As a result, we have to proceed to the higher order in ϵ to evaluate it. This will be done in the next section.

Before we perform the higher-order computation, let us ask how do the correlation functions of the field $\phi(x)$ behave as x approaches the conical singularity. This question is connected to the effective boundary conditions on the field ϕ that are generated at the singularity. In accordance with the general theory of boundary critical phenomena,[24] we expect the field ϕ to satisfy the operator product expansion (OPE),

$$\phi(x_\parallel, x_\perp) \sim r^\alpha \phi(0, x_\perp), \quad r \to 0, \tag{4.32}$$

where $\phi(0,x_\perp)$ is an operator living on the conical singularity. The exponent α can be extracted from the two-point function $\mathcal{G}(x,x')$. Combining the free propagator with the boundary correction [Eq. (4.11)],

$$\mathcal{G}(x_\parallel, x'_\parallel, p_\perp) \overset{x_\parallel\to 0}{=} \left[1 + \frac{c_r}{2\pi n}\log(p_\perp r)\right] G_n(0, x'_\parallel, p_\perp), \tag{4.33}$$

from which we conclude,

$$\alpha = \frac{c_r^*}{2\pi n}. \tag{4.34}$$

Note that from Eq. (4.23), the exponent α is positive for $n<1$, implying effective Dirichlet boundary conditions on $\phi(x)$ at the conical singularity. On the other hand, α is negative for $1<n<n_c$ and correlation functions of $\phi(x)$ exhibit a power-law divergence as $x_\parallel$ approaches the origin.

C. Beyond the leading order in ϵ

1. Inhomogeneous renormalization-group equation

At leading order in ϵ, our calculation has relied on the integral in Eq. (3.7) being saturated at short distances $u=mr\to 0$, allowing us to work directly at the critical point. However, we saw that the coefficient d_n of the short-distance asymptotic of f_n [Eq. (3.4)] behaved as $d_n \sim (n-1)^2/\epsilon$ for $n\to 1$, giving no contribution to the entanglement entropy. We expect that to the next order in ϵ, d_n will acquire a term linear in $n-1$, $d_n \sim \epsilon(n-1)$, which by Eq. (3.9) will give a contribution of $O(1)$ to S. Notice that this is of the same order as the contribution of the long distance $u\to\infty$, part of the integral (3.7), which now has to be taken into account. Thus, we need to compute the long-distance part of f_n to leading order in ϵ and the short-distance part to subleading order. Although the separation between short- and long-distance contributions is unambiguous to the present order, it is convenient to introduce a formalism that allows one to consistently treat the problem order by the order in ϵ.[25]

Let us define,

$$\Phi(p) = n\int_{1-\text{sheet}} d^2x_{\parallel}(\langle[\phi^2(x)]_r\rangle_n - \langle[\phi^2(x)]_r\rangle_1)e^{-i\vec{p}\vec{x}}. \tag{4.35}$$

Here, we have introduced the usual renormalization of the ϕ^2 operator,

$$[\phi^2(x)]_r = \frac{Z_2}{Z}\phi^2(x), \quad t_r = \left(\frac{Z_2}{Z}\right)^{-1}t. \tag{4.36}$$

We are considering Φ at a finite momentum p in order to make Φ well defined even at the critical point $t=0$. We are actually interested in computing Φ at $p=0$ in the gapped phase $t\neq 0$, as from Eq. (3.1),

$$t_r\frac{\partial}{\partial t_r}\log\frac{Z_n}{Z^n} = -\frac{1}{2}t_r\Phi(p=0)L^{D-2}. \tag{4.37}$$

As already observed in Sec. III, although the integrand in Eq. (4.35) is finite, the integral diverges logarithmically for $|x|\to 0$ at each order in u. Thus, $\Phi(p)$ requires an additive renormalization,

$$\Phi(p) = \Phi_r(p) + C(u_r,c_r,\mu/\Lambda)\mu^{-\epsilon}, \tag{4.38}$$

where C is a renormalization constant. We will use dimensional regularization below, so that C is, in fact, just a function of u_r and c_r. Then Φ_r satisfies the inhomogeneous renormalization-group equation,

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(u_r)\frac{\partial}{\partial u_r} + \beta(c_r)\frac{\partial}{\partial c_r} - \eta_2(u_r)\left(1 + t_r\frac{\partial}{\partial t_r}\right)\right]\Phi_r = B(u_r,c_r)\mu^{-\epsilon}, \tag{4.39}$$

with

$$B(u_r,c_r) = -\left\{\beta(u_r)\frac{\partial}{\partial u_r} + \beta(c_r)\frac{\partial}{\partial c_r} - [\eta_2(u_r) + \epsilon]\right\}C(u_r,c_r), \tag{4.40}$$

where as usual,

$$\eta_2(u_r) = \mu\left.\frac{\partial}{\partial\mu}\right|_u \log\frac{Z_2}{Z}. \tag{4.41}$$

Note that B must be finite, as the left-hand side of Eq. (4.39) is finite. The solution to Eq. (4.39) can be represented as a sum of the solution to the homogeneous RG equation and a particular solution. In the scaling limit $t_r\to 0$,

$$\Phi_r(p=0) = A_s\mu^{-\epsilon}\left(\frac{t_r}{\mu^2}\right)^{-(\epsilon+\eta_2)/(2+\eta_2)} + A_{ns}(u_r,c_r)\mu^{-\epsilon}, \tag{4.42}$$

where the coefficient of the particular solution A_{ns} satisfies,

$$\left\{\beta(u_r)\frac{\partial}{\partial u_r} + \beta(c_r)\frac{\partial}{\partial c_r} - [\eta_2(u_r) + \epsilon]\right\}A_{ns}(u_r,c_r) = B(u_r,c_r). \tag{4.43}$$

Hence, at the critical point,

$$A_{ns}(u_r^*,c_r^*) = -\frac{1}{\eta_2+\epsilon}B_* = -\frac{1}{\nu_1}B_*, \tag{4.44}$$

where we recall our definition in Sec. III, $\nu_1=\nu^{-1}-(D-2)$ and $\nu^{-1}=2+\eta_2$.

Thus, from Eq. (4.37),

$$\log\frac{Z_n}{Z^n} = -\frac{A_s}{2\nu(D-2)}\left[\mu\left(\frac{t_r}{\mu^2}\right)^{\nu}\right]^{D-2}L^{D-2}, \tag{4.45}$$

where we have dropped terms analytic in t_r. Note that the mass gap m is related to $\mu(\frac{t_r}{\mu^2})^{\nu}$ via a finite proportionality constant, which at leading order in ϵ is just 1. So to leading order,

$$r_n \approx -\frac{A_s}{2(1-n)}. \tag{4.46}$$

Hence, we must compute A_s. To do so, we perturbatively calculate $\Phi_r(p=0)$ and $B(u_r,c_r)$. A_s can then be determined by matching the perturbative expansion with the solution to the RG equation (4.42) at the critical point, where the corrections to scaling vanish. Notice that we always need to compute B to one higher order in ϵ than $\Phi_r(p=0)$ due to the factor ν_1 in the denominator of Eq. (4.44). Moreover, since Φ_r is finite for $\epsilon\to 0$, while $A_{ns}=-B_*/\nu_1$ behaves as $1/\epsilon$, to leading order $A_s=-A_{ns}=B_*/\nu_1$. Precisely, this fact was utilized in Sec. III, and we identify to leading order $B_* = 2\pi n d_n$.

2. Regularization

For the purpose of computing the entanglement entropy S, we can work to linear order in $n-1$. Since the fixed-point value $c_*\sim O(n-1)$, we also work to linear order in c. Therefore, all diagrams that include an insertion of c can be evaluated at $n=1$. In addition, power counting indicates that if we work to linear order in c, all diagrams will be finite for $D<4$ [by contrast, higher-order diagrams in c, such as Fig. 3(b) diverge even for $D<4$]. Thus, we use dimensional regularization and minimal subtraction below. We remind the reader that in dimensional regularization, the bare coupling constant $u=\mu^{\epsilon}u_r Z_u/Z^2$. We list below the renormalization constants in the minimal subtraction scheme to the order that they will be needed in our calculation,

$$\frac{Z_u}{Z^2} = 1 + \frac{(N+8)}{\epsilon}\frac{u_r}{8\pi^2}, \tag{4.47}$$

$$\frac{Z_2}{Z} = 1 + \frac{(N+2)}{\epsilon}\frac{u_r}{8\pi^2} + \frac{(N+2)(N+5)}{\epsilon^2}\left(\frac{u_r}{8\pi^2}\right)^2 - \frac{5(N+2)}{4\epsilon}\left(\frac{u_r}{8\pi^2}\right)^2. \quad (4.48)$$

Correspondingly,

$$\beta(u_r) = -\epsilon u_r + \frac{(N+8)u_r^2}{8\pi^2}, \quad (4.49)$$

$$\eta_2(u_r) = -(N+2)\frac{u_r}{8\pi^2}\left(1 - \frac{5}{2}\frac{u_r}{8\pi^2}\right). \quad (4.50)$$

As we saw, the boundary coupling constant c will also require renormalization. To linear order in c,

$$c = D(u_r) + \frac{Z_2}{Z}c_r, \quad (4.51)$$

where we observe that the multiplicative renormalization of c to the zeroth order in $(n-1)$ is just Z_2/Z. On the other hand, the additive renormalization, which behaves as $D(u_r) \sim (n-1)$ for $n \to 1$, needs to be computed explicitly. So the β function,

$$\beta(c_r) = -\left(\frac{Z_2}{Z}\right)^{-1}\beta(u_r)\frac{\partial D}{\partial u_r} - \eta_2(u_r)c_r. \quad (4.52)$$

3. Entanglement entropy to $O(1)$

To calculate the entanglement entropy to $O(1)$ in ϵ, we need to find the finite part of $\Phi(p=0)$ [Eq. (4.35)] at $t \neq 0$ to $O(1)$ in u and the divergent part of $\Phi(p)$, which determines B [Eq. (4.40)] to $O(u)$.

$\Phi(p)$ to $O(1)$ in u is given by the two diagrams in Fig. 6. Figure 6(a) is just the mean-field contribution computed in Ref. 3,

$$\begin{aligned}\Phi(p=0)_{MF} &= N\int d^2x_{\parallel}[G_n(x,x) - G_1(x,x)] \\ &= N\int \frac{d^{D-2}k_{\perp}}{(2\pi)^{D-2}}\int d^2x_{\parallel}[G_n^{D=2}(x,x;k_{\perp}^2+m^2) - n \to 1] \\ &= -\frac{N}{12}\left(n - \frac{1}{n}\right)\int \frac{d^{D-2}k_{\perp}}{(2\pi)^{D-2}}\frac{1}{k_{\perp}^2+m^2} \\ &= -\frac{N}{12}\left(n - \frac{1}{n}\right)\frac{\Gamma(2-D/2)}{(4\pi)^{D/2-1}}m^{D-4}, \end{aligned} \quad (4.53)$$

where $G_n^{D=2}(x,x';M^2)$ is the two-dimensional massive propagator on the n-sheeted Riemann surface, and we have used the result proved in Ref. 3,

$$\int d^2x_{\parallel}[G_n^{D=2}(x,x;M^2) - G_1^{D=2}(x,x;M^2)] = -\frac{1}{12}\left(n - \frac{1}{n}\right)\frac{1}{M^2}. \quad (4.54)$$

The diagram in Fig. 6(b) is the boundary correction,

$$\delta^{1,0}\Phi(p=0) = -Nc_r\int d^2x_{\parallel}\int d^{D-2}y_{\perp}G_1^2(x_{\parallel},y_{\perp}) = -Nc_r\frac{\Gamma(2-D/2)}{(4\pi)^{D/2}}m^{D-4}. \quad (4.55)$$

Combining Eqs. (4.53) and (4.55),

$$\Phi(p=0) \overset{O(1)}{=} -N\left(\frac{n-1}{12\pi} + \frac{c_r}{8\pi^2}\right)\left[\frac{1}{\epsilon} + \frac{1}{2}\log 4\pi - \frac{\gamma}{2} - \log(m/\mu)\right]\mu^{-\epsilon}, \quad (4.56)$$

where we keep only terms linear in $n-1$.

Subtracting the pole, we obtain for the additive renormalization constant C [Eq. (4.38)],

$$C \overset{O(1)}{=} -N\left(\frac{n-1}{12\pi} + \frac{c_r}{8\pi^2}\right)\frac{1}{\epsilon}, \quad (4.57)$$

and consequently from Eq. (4.40),

$$B \overset{O(1)}{=} \epsilon C = -N\left(\frac{n-1}{12\pi} + \frac{c_r}{8\pi^2}\right), \quad (4.58)$$

and

$$\Phi_r(p=0) \overset{O(1)}{=} -N\left(\frac{n-1}{12\pi} + \frac{c_r}{8\pi^2}\right)\left[\frac{1}{2}\log 4\pi - \frac{\gamma}{2} - \log(m/\mu)\right]\mu^{-\epsilon}. \quad (4.59)$$

In particular, at the critical point, by Eq. (4.25),

$$c_r^* \overset{O(1)}{=} -\frac{2\pi}{3}(n-1), \quad (4.60)$$

and

$$\Phi_r^*(p=0) = O(\epsilon), \quad B_* = O(\epsilon). \quad (4.61)$$

Thus, in the minimal subtraction scheme $\Phi_r^*(p=0)$ vanishes at the critical point to $O(1)$ in ϵ. The fact that $B_* = 2\pi n d_n$ vanishes to $O(1)$ in ϵ has already been observed in Sec. IV B. Thus, from Eqs. (4.42) and (4.44),

$$A_s \overset{O(1)}{=} \frac{B_*}{\nu_1}. \quad (4.62)$$

We now proceed to evaluate B to $O(\epsilon)$. To do this, we compute $\Phi(p)$ at the critical point. We first evaluate $\langle[\phi^2]_r\rangle_n - \langle[\phi^2]_r\rangle_1$ and use it to determine the renormalization of the coupling c in dimensional regularization. We then perform the Fourier transform [Eq. (4.35)] to find the subtraction constant C and hence B. To leading order, we have the two familiar diagrams in Fig. 6,

$$\langle\phi^2(x)\rangle_n - \langle\phi^2(x)\rangle_1 \overset{O(1)}{=} N\left[J(D) - c_r\frac{\Gamma(D/2-1)^3}{16\pi^{D/2+1}\Gamma(D-2)}\right]\frac{1}{r^{D-2}}, \quad (4.63)$$

where we have defined,

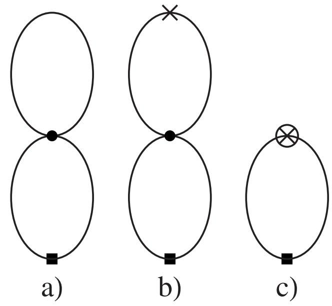

FIG. 7. Contributions to $\langle\phi^2(x)\rangle_n - \langle\phi^2(x)\rangle_1$ at order u. The counterterm $\delta^1 c$ is denoted by a circled cross here and below.

$$G_n(x,x) - G_1(x,x) = \frac{J(D)}{r^{D-2}}. \tag{4.64}$$

Note that in dimensional regularization $\langle\phi^2\rangle_1 = NG_1(x,x) = 0$ at the critical point. We will show in Sec. IV C 4 that to linear order in $n-1$,

$$J(D) = (n-1)\frac{\Gamma(D/2)^3}{4\pi^{D/2}(1-D/2)\Gamma(D)}. \tag{4.65}$$

In particular, $J(D=4) = -\frac{n-1}{24\pi^2}$ in agreement with Eq. (4.5). We note that the diagrams that contain the tadpole (4.64) can effectively be evaluated with $n=1$. The computation is simplest in position space, where one uses,

$$G_1(x,x') = \frac{\Gamma(D/2-1)}{4\pi^{D/2}|x-x'|^{D-2}}. \tag{4.66}$$

At order u, $\langle\phi^2\rangle_n - \langle\phi^2\rangle_1$ receives contributions from the diagrams in Fig. 7. Note that the diagram (c) is the renormalization of the coupling constant $c_0 = c_r + \delta^1 c + \ldots$. Taking the multiplicative renormalization of the operator ϕ^2 into account, we obtain

$$\begin{aligned}\langle[\phi^2]_r\rangle_n - \langle[\phi^2]_r\rangle_1 \\ \stackrel{O(u)}{=} N\left(\frac{Z_2}{Z}\frac{1}{r^{D-2}} - \frac{\Gamma(D/2-1)\Gamma(2-D/2)^2}{16\pi^{D/2}(D-3)\Gamma(4-D)}\frac{(N+2)u_r\mu^\epsilon}{r^{2(D-3)}}\right) \\ \times\left(J - \frac{\Gamma(D/2-1)^3}{16\pi^{D/2+1}\Gamma(D-2)}c_r\right) - N\frac{\Gamma(D/2-1)^3}{16\pi^{D/2+1}\Gamma(D-2)}\frac{\delta^1 c}{r^{D-2}}.\end{aligned} \tag{4.67}$$

Performing minimal subtraction,

$$\delta^1 c = \frac{(N+2)u_r}{\epsilon}\left(\frac{n-1}{12\pi} + \frac{c_r}{8\pi^2}\right) \tag{4.68}$$

Notice that the coefficient of the multiplicative renormalization is precisely Z_2/Z as expected. We also obtain the additive renormalization constant [Eq. (4.51)],

$$D(u_r) = \frac{(N+2)u_r}{\epsilon}\frac{n-1}{12\pi}. \tag{4.69}$$

Hence, from Eq. (4.52), to first order in u,

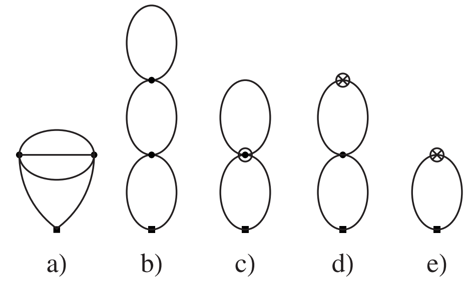

FIG. 8. Contributions to $\langle\phi^2(x)\rangle_n - \langle\phi^2(x)\rangle_1$ at order u^2 (diagrams involving insertions of c_r are not shown). The counterterm for the coupling u is shown as a circled dot.

$$\beta(c_r) \stackrel{O(u)}{=} (N+2)u_r\left(\frac{n-1}{12\pi} + \frac{c_r}{8\pi^2}\right), \tag{4.70}$$

in agreement with the expression (4.20) obtained earlier using cut-off regularization.

By Fourier transforming Eq. (4.67), we can compute $\Phi(p)$ at the critical point to order u. From the divergent part, we obtain the additive renormalization constant C (4.38),

$$C(u_r, c_r) = -N\left(\frac{1}{\epsilon} + \frac{N+2}{\epsilon^2}\frac{u_r}{8\pi^2}\right)\left(\frac{n-1}{12\pi} + \frac{c_r}{8\pi^2}\right), \tag{4.71}$$

which gives the $O(u)$ correction to our previous result (4.57). Substituting into Eq. (4.40), we obtain

$$B \stackrel{O(u)}{=} -N\left(\frac{n-1}{12\pi} + \frac{c_r}{8\pi^2}\right). \tag{4.72}$$

Comparing the above result to Eq. (4.58), we observe that B receives no additional contributions at $O(u)$. Thus, from Eq. (4.62),

$$A_s \stackrel{O(1)}{=} -\frac{N(N+8)}{6\epsilon}\left(\frac{n-1}{12\pi} + \frac{c_r^*}{8\pi^2}\right), \tag{4.73}$$

which, upon determination of c_r^* to order ϵ would yield the entanglement entropy [Eq. (4.46)].

4. $\beta(c_r)$ to order u^2

To complete our calculation, we need the value of the fixed-point coupling c_r^* to order ϵ. This requires the knowledge of $\beta(c_r)$ to order u^2. As before, we will determine the renormalization of c by computing the expectation value $\langle[\phi^2]_r\rangle_n - \langle[\phi^2]_r\rangle_1$. As explained in Sec. IV C 2, we need to find only the additive renormalization of c. Hence, we ignore all diagrams with vertices proportional to c_r. At order u^2, we obtain the graphs shown in Fig. 8.

Now we are faced with a new technical difficulty. Up to this point, to linear order in $n-1$, the conical singularity entered our calculations through the tadpole term $G_n(x,x) - G_1(x,x)$, whose form was fixed by dimensional analysis [Eq. (4.64)], up to an overall constant $J(D)$. Moreover, the renormalization constants only depended on $J(D=4)$, which could be extracted from the explicit form of the propagator [Eq. (4.3)]. However, at the present order, we are faced with the diagram in Fig. 8(a), which requires the full position dependence of the propagator $G_n(x,x')$. Yet, as far as we know, there is no simple expression for $G_n(x,x')$ in arbitrary dimension and, even in $D=4$, Eq. (4.3) is rather awkward to work with.

To address this problem, we expand the propagator $G_n(x,x')$ to linear order in $n-1$ in terms of the usual propagators $G_1(x,x')$ [Eq. (4.66)]. The simplest way to do this is to consider the $O(N)$ model in the presence of an arbitrary metric $g_{\mu\nu}$,

$$S=\int d^Dx\sqrt{\det g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\frac{t}{2}\phi^2+\frac{u}{4}\phi^4\right). \quad (4.74)$$

It is convenient to parametrize the n-sheeted Riemann surface using rescaled variables,

$$\tilde{r}=\sqrt{n}r, \quad \varphi=\theta/n. \quad (4.75)$$

Then, the angular variable $\varphi\sim\varphi+2\pi$. We may also define,

$$\tilde{\tau}=\tilde{r}\cos\varphi, \quad \tilde{x}=\tilde{r}\sin\varphi. \quad (4.76)$$

The coordinates $(\tilde{\tau},\tilde{x})$ form the usual two-dimensional Euclidean plane and uniquely specify each point on the Riemann surface. With this choice of variables, the metric (2.4) in the $x_\parallel$ plane becomes

$$g_{\alpha\beta}=n\delta_{\alpha\beta}+\left(\frac{1}{n}-n\right)\frac{\tilde{x}_\alpha\tilde{x}_\beta}{\tilde{x}^2}, \quad (4.77)$$

where α, β run over $\tilde{\tau},\tilde{x}$. Note that we have chosen to rescale r in such a way that

$$\det g=1. \quad (4.78)$$

Moreover, expanding g in powers of $n-1$, $g_{\alpha\beta}=\delta_{\alpha\beta}+\delta g_{\alpha\beta}$,

$$\delta g_{\alpha\beta}\approx(n-1)\left(\delta_{\alpha\beta}-\frac{2\tilde{x}_\alpha\tilde{x}_\beta}{\tilde{x}^2}\right). \quad (4.79)$$

We drop the tildes on variables τ, x in what follows. We can now obtain the usual Feynman graph expansion for the theory (4.74), treating $\delta g_{\alpha\beta}$ as a perturbation. Note that all the integrals in the resulting expansion are over the usual D-dimensional Euclidean space. In particular, note that the bare propagator becomes

$$G_n(x,x')\approx G_1(x,x')+\delta G_n(x,x'), \quad (4.80)$$

$$\delta G_n(x,x')=(n-1)\int d^Dy\left(\delta_{\alpha\beta}-\frac{2y_\alpha y_\beta}{y_\parallel^2}\right)\times\partial_\alpha G_1(x-y)\partial_\beta G_1(x'-y). \quad (4.81)$$

By performing the integral, we immediately obtain Eq. (4.65) for $G_n(x,x)-G_1(x,x)$.

Using the expansion (4.81), we compute the divergent part of the diagrams in Fig. 8 to linear order in $n-1$. After accounting for the multiplicative renormalization of the ϕ^2 operator [Eq. (4.48)], we extract the additive renormalization of the coupling constant c [Eq. (4.51)] to $O(u^2)$,

$$D(u_r)=\frac{n-1}{12\pi}\left(\frac{(N+2)u_r}{\epsilon}+\frac{(N+2)(N+5)}{\epsilon^2}\frac{u_r^2}{8\pi^2}-\frac{7(N+2)}{4\epsilon}\frac{u_r^2}{8\pi^2}\right), \quad (4.82)$$

and from Eq. (4.52),

$$\beta(c_r)=(N+2)u_r\left(1-\frac{7}{2}\frac{u_r}{8\pi^2}\right)\frac{n-1}{12\pi}+(N+2)\frac{u_r}{8\pi^2}\left(1-\frac{5}{2}\frac{u_r}{8\pi^2}\right)c_r. \quad (4.83)$$

Hence,

$$c_r^*=-\frac{2\pi}{3}\left(1-\frac{u_r^*}{8\pi^2}\right)(n-1)=-\frac{2\pi}{3}\left(1-\frac{\epsilon}{N+8}\right)(n-1), \quad (4.84)$$

and from Eq. (4.73),

$$A_s=-\frac{N}{72\pi}(n-1), \quad (4.85)$$

which by Eq. (4.46) finally yields the coefficient of the correlation length correction to the entanglement entropy,

$$r=-\frac{N}{144\pi}. \quad (4.86)$$

V. ϵ-EXPANSION: FINITE-SIZE CORRECTION

In this section, we compute the geometric corrections γ, γ_n to the entanglement entropy and the Renyi entropy [Eqs. (1.6) and (1.15)] at the critical point.

As before, we consider two semi-infinite regions A and B with a boundary at x=0. However, we now take the remaining $D-2$ spatial directions to have a finite length L. In order to avoid dealing with the zero mode, we use twisted boundary conditions along these directions,

$$\phi(x+L\hat{n}_i)=e^{i\varphi_i}\phi(x), \quad (5.1)$$

where $\hat{n}_i$ are unit vectors along the boundary. If the fields ϕ are real, then $\varphi_i=0$ or π. On the other hand, in an $O(N)$ model with N even, we can group our fields into $N/2$ complex pairs—then, an arbitrary twist is allowed [however, this breaks the $O(N)$ symmetry down to $U(1)\times SU(N/2)$]. We note that when accessing $D=3$ via ϵ-expansion, we will choose all φ_i's to be equal. Thus, the boundary between regions A and B is a $D-2$ dimensional torus. Since this manifold is smooth, we expect the constants γ, γ_n to be universal. Moreover, we do not have to take into account divergences, which appear as $D\to4$ when the boundary has a finite curvature,[8,13] since this manifold is flat.

A. Gaussian theory

Let us begin with the free theory. We wish to compute,

$$\log\frac{Z_n}{Z^n}=-\frac{N}{2}[\mathrm{Tr}\log(-\partial^2)_n-n\,\mathrm{Tr}\log(-\partial^2)_1], \quad (5.2)$$

$$= -\frac{N}{2}\sum_{\vec{k}_\perp}[\mathrm{Tr}_\parallel \log(-\partial_\parallel^2+\vec{k}_\perp^2)_n - n\,\mathrm{Tr}_\parallel \log(-\partial_\parallel^2+\vec{k}_\perp^2)_1], \tag{5.3}$$

where $k_\perp^i=\frac{2\pi n_i+\varphi_i}{L}$ and n_i are integers. We leave the regularization of Eq. (5.3) implicit for now (we will later use dimensional regularization). Equation (5.3) involves the partition function of the two-dimensional massive Gaussian theory evaluated in Ref. 3,

$$\log\left.\frac{Z_n}{Z^n}\right|_{D=2} = -\frac{1}{2}[\mathrm{Tr}_\parallel \log(-\partial_\parallel^2+m^2) - n\,\mathrm{Tr}_\parallel \log(-\partial_\parallel^2+m^2)_1]$$
$$= \frac{1}{24}\left(n-\frac{1}{n}\right)\log(m^2). \tag{5.4}$$

Thus,

$$\log\frac{Z_n}{Z^n} = \frac{N}{24}\left(n-\frac{1}{n}\right)\sum_{\vec{k}_\perp}\log(\vec{k}_\perp^2) = -N\frac{\pi}{6}\left(n-\frac{1}{n}\right)L^{D-2}G_1^L(x,x). \tag{5.5}$$

Here, $G_n^L(x,x)$ is the free propagator on an n-sheeted Riemann surface, which incorporates the finite-size effects in the transverse direction. Explicitly,

$$G_n^L(x,x') = \frac{1}{L^{D-2}}\sum_{\vec{k}_\perp} G_n^{D=2}(x_\parallel,x_\parallel';k_\perp^2)e^{i\vec{k}_\perp\cdot(\vec{x}_\perp-\vec{x}_\perp')}. \tag{5.6}$$

In particular, for $n=1$,

$$G_1^L(x,x') = \frac{1}{L^{D-2}}\sum_{\vec{k}_\perp}\int\frac{d^2k_\parallel}{(2\pi)^2}\frac{1}{k_\parallel^2+k_\perp^2}e^{ik(x-x')}, \tag{5.7}$$

justifying the last step in Eq. (5.5).

An alternative representation for the propagator (5.6) on the torus can be obtained by Poisson resumming $\vec{k}_\perp$, which is equivalent to "periodizing" the infinite volume propagator,

$$G_n^L(x,x') = \sum_{\vec{l}} e^{i\vec{l}\vec{\varphi}}G_n(x+\vec{l}L,x'), \tag{5.8}$$

where $\vec{l}$ is a vector of $D-2$ integers in the plane parallel to the boundary. Note that when $x=x'$, only the $l=0$ term in Eq. (5.8) is ultraviolet divergent and $G_n^L(x,x)-G_n(x,x)$ is finite. Moreover, since the $l=0$ term, $G_1(x,x)\sim\Lambda^{D-2}$, is L independent, it gives a nonuniversal contribution to $\log(Z_n/Z^n)$ [Eq. (5.5)] proportional to the area of the boundary. Concentrating on the universal constant term,

$$\log\frac{Z_n}{Z^n} = -N\frac{\pi}{6}\left(n-\frac{1}{n}\right)L^{D-2}[G_1^L(x,x)-G_1(x,x)], \tag{5.9}$$

where from Eqs. (4.66) and (5.8),

$$L^{D-2}[G_1^L(0)-G_1(0)] = \frac{\Gamma(D/2-1)}{4\pi^{D/2}}\sum_{\vec{l}\neq 0}\frac{e^{i\vec{l}\vec{\varphi}}}{|\vec{l}|^{D-2}}. \tag{5.10}$$

Here and below, we abbreviate $G_1^L(x,x)$ by $G_1^L(0)$.

We can now explicitly evaluate the universal constant contribution γ_n to the entanglement entropy for $D=3$ and $D=4$,

$$\gamma_n = -\frac{N}{12}\left(1+\frac{1}{n}\right)\log(2|\sin\varphi/2|), \quad D=3, \tag{5.11}$$

$$\gamma = -\frac{N}{6}\log(2|\sin\varphi/2|), \quad D=3. \tag{5.12}$$

For $D=4$, we note that the sum

$$\sum_{\vec{l}\neq 0}\frac{e^{i\vec{l}\vec{\varphi}}}{\vec{l}^2} = (2\pi)^2G^{D=2}(\vec{\varphi}), \tag{5.13}$$

where $G^{D=2}(\vec{\varphi})$ is the massless two-dimensional propagator (with the zero mode removed) on a torus with side length 2π. This propagator can be expressed in terms of the Jacobi-theta function θ_1,

$$G^{D=2}(\vec{\varphi}) = -\frac{1}{2\pi}\left[\log\left|\theta_1\left(\frac{\varphi_1+i\varphi_2}{2\pi},i\right)\right| - \frac{\varphi_2^2}{4\pi} - \log\,\eta(i)\right], \tag{5.14}$$

where η is the Dedekind-eta function.

Thus,

$$\gamma_n = \frac{\pi N}{6}\left(1+\frac{1}{n}\right)G^{D=2}(\vec{\varphi}), \quad D=4, \tag{5.15}$$

$$\gamma = \frac{\pi N}{3}G^{D=2}(\vec{\varphi}), \quad D=4. \tag{5.16}$$

B. ϵ expansion

We now compute the universal finite-size correction to leading order in ϵ-expansion. The leading correction to the free theory behavior comes from the boundary perturbation (4.10), as at the fixed point $c_r^*\sim\sqrt{\epsilon}$ for $1-n\gg\epsilon$ and $c_r^*\sim(n-1)$ for $|1-n|\ll\epsilon$. Thus,

$$\delta^{1,0}\log\frac{Z_n}{Z^n} = -\frac{c_r}{2}\int d^{D-2}x_\perp\langle\phi^2(r=0)\rangle_n$$
$$= -\frac{Nc_r}{2}L^{D-2}G_n^L(r=r'=0) = -\frac{Nc_r}{2n}L^{D-2}G_1^L(x,x), \tag{5.17}$$

where in the last step we have used Eqs. (4.12) and (5.8). Again, subtracting the nonuniversal area law piece $\sim L^{D-2}G_1(0)$ and combining Eq. (5.17) with the free theory result (5.9),

$$\log\frac{Z_n}{Z^n} = -N\left[\frac{\pi}{6}\left(n-\frac{1}{n}\right)+\frac{c_r}{2n}\right]L^{D-2}[G_1^L(0)-G_1(0)]. \tag{5.18}$$

Now replacing c_r by its fixed-point value and taking $D\to 4$,

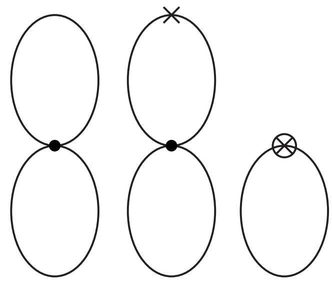

FIG. 9. Contributions to the partition function at order u.

$$\gamma_n = N\left[\frac{\pi}{6}\left(1+\frac{1}{n}\right)+\frac{c_r^*}{2n(n-1)}\right]G^{D=2}(\varphi,\varphi). \quad (5.19)$$

Here we have set all the twists φ_i equal. For $1-n \gg \epsilon$ [Eq. (4.24)], the c^* term gives a correction of order $\sqrt{\epsilon}$ to the free theory result. However, in the limit $|1-n| \ll \epsilon$, Eq. (4.25), the correction due to the boundary perturbation cancels with the free theory result to leading order in $n-1$, leaving,

$$\gamma_n \overset{n\to 1}{\approx} -\frac{\pi N(N+8)}{9(N+2)}\frac{n-1}{\epsilon}G^{D=2}(\varphi,\varphi). \quad (5.20)$$

This implies that at the Wilson-Fisher fixed point, the universal finite-size correction to the entanglement entropy

$$\gamma \sim O(\epsilon) \quad (5.21)$$

parametrically smaller than at the Gaussian fixed point in $D=4-\epsilon$.

C. Beyond the leading order in ϵ

We now evaluate the universal finite-size correction to the entanglement entropy γ to order ϵ. As before, we only work to leading order in $n-1$. To order u, the partition function receives contributions from the diagrams in Fig. 9. The diagram in Fig. 9(a) is given by

$$\begin{aligned}\delta^{0,1}\log\frac{Z_n}{Z^n} &= -\frac{N(N+2)u_r\mu^\epsilon}{4}\int d^Dx[G_n^L(x,x)-G_1^L(x,x)] \\ &\quad\times\{[G_n^L(x,x)-G_1(x,x)]+[G_1^L(x,x)-G_1(x,x)]\} \\ &\overset{n\to 1}{\approx} -\frac{N(N+2)u_r\mu^\epsilon}{2}(G_1^L(0)-G_1(0)) \\ &\quad\times\sum_{\vec{k}_\perp}\int d^2x[G_n^{D=2}(x,x;k_\perp^2)-G_1^{D=2}(x,x;k_\perp^2)] \\ &= \frac{N(N+2)(n-1)u_r\mu^\epsilon}{12}[G_1^L(0)-G_1(0)]\sum_{\vec{k}_\perp}\frac{1}{\vec{k}_\perp^2},\end{aligned} \quad (5.22)$$

where in the last step we have used Eq. (4.54).

The diagram in Fig. 9(b) can be evaluated with $n=1$ propagators,

$$\begin{aligned}\delta^{1,1}\log\frac{Z_n}{Z^n} &= \frac{N(N+2)u_r c_r\mu^\epsilon}{2}[G_1^L(0)-G_1(0)] \\ &\quad\times\int d^{D-2}x_\perp\int d^Dx' G_1^L(x_\perp,x')^2 \\ &= \frac{N(N+2)u_r c_r\mu^\epsilon}{8\pi}[G_1^L(0)-G_1(0)]\sum_{\vec{k}_\perp}\frac{1}{\vec{k}_\perp^2}.\end{aligned} \quad (5.23)$$

Finally, the diagram in Fig. 9(c) can be obtained from Eq. (5.17) by substituting the counterterm for c [Eq. (4.68)]. Combining all the diagrams in Fig. 9 with the $O(1)$ result [Eq. (5.18)],

$$\begin{aligned}\log\frac{Z_n}{Z^n} &= -\frac{N}{2}\left[\frac{2\pi}{3}(n-1)+c_r\right]L^{D-2}[G_1^L(0)-G_1(0)] \\ &\quad\times\left\{1-\frac{(N+2)u_r}{4\pi}\left[(\mu L)^\epsilon\sum_{\vec{k}_\perp}\frac{1}{(L\vec{k}_\perp)^2}-\frac{1}{2\pi\epsilon}\right]\right\}.\end{aligned} \quad (5.24)$$

Applying the usual technique for analytically continuing sums over D-dimensional vectors,

$$\sum_{\vec{k}_\perp}\frac{1}{(Lk_\perp)^2} = \int_0^\infty ds T(s)^{D-2}, \quad (5.25)$$

where

$$T(s) = \sum_n e^{-s(2\pi n+\varphi)^2}. \quad (5.26)$$

The function $T(s)$ has the following asymptotics:

$$T(s) \to \frac{1}{\sqrt{4\pi s}}, \quad s\to 0, \quad (5.27)$$

$$T(s) \to e^{-s\varphi^2}, \quad s\to\infty. \quad (5.28)$$

Hence, for finite φ the integral in Eq. (5.25) converges in the $s\to\infty$ region. Moreover, the $s\to 0$ region contributes a pole for $D\to 4$,

$$\sum_{\vec{k}_\perp}\frac{1}{(L\vec{k}_\perp)^2} \to \frac{1}{2\pi\epsilon} + \text{finite terms}. \quad (5.29)$$

As expected, this pole precisely cancels with the c counterterms, so that the expression (5.24) is finite. Moreover, setting c_r to its fixed-point value [Eq. (4.84)], the prefactor in Eq. (5.24) is already $O(\epsilon)$, so that we can neglect the $O(u)$ terms in the square brackets. Thus,

$$\log\frac{Z_n}{Z^n} = -\frac{N\pi\epsilon(n-1)}{3(N+8)}L^{D-2}[G_1^L(0)-G_1(0)], \quad (5.30)$$

and

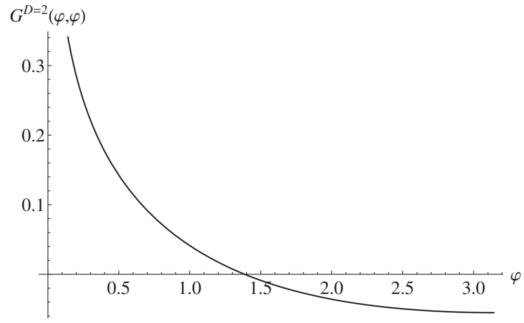

FIG. 10. The function $G^{D=2}(\varphi,\varphi)$ determining the dependence of γ on the twist φ [Eq. (5.31)].

$$\gamma = \frac{N\pi\epsilon}{3(N+8)} G^{D=2}(\varphi,\varphi). \tag{5.31}$$

Note that the result (5.31) is of $O(1)$ in N for $N\to\infty$, instead of the naively expected $O(N)$. It is not clear if this is an artifact of working to leading order in ϵ.

The function $G^{D=2}(\varphi,\varphi)$, which determines the φ dependence of γ, is shown in Fig. 10. We observe that γ is a monotonically decreasing function of φ for $0<\varphi<\pi$. In particular, for $\varphi=\pi$,

$$\gamma = -\frac{N\epsilon}{12(N+8)}\log 2. \tag{5.32}$$

Thus, γ is negative for antiperiodic boundary conditions. On the other hand, for $\varphi\to 0$,

$$\gamma \approx -\frac{N\epsilon}{6(N+8)}\log \varphi, \quad \varphi\to 0, \tag{5.33}$$

suggesting that γ is positive for periodic boundary conditions. Note that our expression for γ becomes invalid for φ sufficiently small. The value of φ, where the breakdown of direct perturbative expansion occurs, can be estimated as follows. Let us separate out the quasizero mode ϕ_0 of the field ϕ,

$$\phi(x) = \frac{1}{L^{(D-2)/2}}\phi_0(x_\parallel)e^{i\vec{\varphi}\cdot\vec{x}_\perp/L} + \tilde{\phi}(x), \tag{5.34}$$

where $\tilde{\phi}(x)$ has the $\vec{k}_\perp = \frac{\vec{\varphi}}{L}$ mode omitted. At the mean-field level, the effective action for ϕ_0 is a two-dimensional ϕ^4 field theory, with an effective mass $m_{2D}^2 \sim \frac{\varphi^2}{L^2}$ and quartic coupling $u_{2D}\sim\frac{u}{L^{D-2}}$. We know that the perturbative expansion in a two-dimensional theory is valid for $u_{2D}/m_{2D}^2 \ll 1$. Thus, setting $D=4$ and $u=u^*$, we obtain,

$$\varphi^2 \gg \epsilon, \tag{5.35}$$

as the domain of validity of perturbation theory. For smaller values of φ, the zero mode must be treated separately and nonperturbatively. This result can be checked in the $1/N$ expansion, where one obtains a slightly stronger condition $\varphi^2 \gg \epsilon\log\varphi$. Cutting off the logarithmic divergence of Eq. (5.33) at the value of φ, where perturbation theory breaks down, we obtain

$$\gamma \approx -\frac{N\epsilon}{12(N+8)}\log\epsilon. \tag{5.36}$$

We conjecture that Eq. (5.36) is the leading-order result for the case of zero twist (periodic boundary conditions).

VI. LARGE N LIMIT

In this section, we compute the correlation length correction to the Renyi entropy S_n [Eq. (1.16)] in the large N limit. Although we are mainly interested in the physical case $D=3$, we will keep the dimension of space-time arbitrary in our discussion in order to compare the results of the large-N and ϵ-expansions.

When working in the large-N limit, it is more convenient to use the nonlinear σ-model version of the $O(N)$ model (2.1), where the quartic interaction is replaced by a local constraint $\phi^2(x)=\frac{1}{g}$. Enforcing this constraint with the help of the Lagrange multiplier $\lambda(x)$, the action takes the form,

$$S = \int d^Dx\left[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}i\lambda\left(\phi^2 - \frac{1}{g}\right)\right]. \tag{6.1}$$

Our discussion in Sec. III is then directly transcribed into the present case with the replacement, $t\to -(\frac{1}{g}-\frac{1}{g_c})$, $\phi^2\to i\lambda$. In particular, to determine the coefficient r_n of the correlation length correction to leading order in $1/N$, we need to find the behavior of $\langle i\lambda(x)\rangle$ at the critical point.

We tune the $O(N)$ model to criticality $g=g_c$. At $N=\infty$, the problem is reduced to finding the saddle-point value of the Lagrange multiplier $\langle i\lambda(x)\rangle_n$ such that the gap equation

$$G_n(x,x) = \frac{1}{N}\langle\phi^2(x)\rangle_n = \frac{1}{Ng_c} \tag{6.2}$$

is satisfied. Here $G_n(x,x')$ is the Green's function of the operator $-\partial^2+\langle i\lambda(x)\rangle_n$ on the n-sheeted Riemann surface. The quantity $G_n(x,x)$ requires regularization; we will implicitly use point splitting regularization. It is convenient to rewrite the gap equation as

$$G_n(x,x) - G_1(x,x) = 0. \tag{6.3}$$

We note that at $N=\infty$, the scaling dimension of $\lambda(x)$ is 2, so

$$\langle i\lambda(x)\rangle_n = \frac{a_n}{r^2}. \tag{6.4}$$

From Eq. (3.3), with the appropriate replacement $\phi^2\to i\lambda$, $t\to g_c^{-1}-g^{-1}$, the constant a_n is related to the constant d_n (3.4) as

$$d_n = \frac{1}{m^{D-2}}\left(\frac{1}{g_c}-\frac{1}{g}\right)a_n. \tag{6.5}$$

Now from the gap equation at finite m,

$$\frac{1}{Ng}-\frac{1}{Ng_c}=\int\frac{d^Dp}{(2\pi)^D}\left(\frac{1}{p^2+m^2}-\frac{1}{p^2}\right)$$
$$=\frac{1}{(4\pi)^{D/2}}\Gamma(1-D/2)m^{D-2} \quad (6.6)$$

and

$$d_n=-\frac{N}{(4\pi)^{D/2}}\Gamma(1-D/2)a_n. \quad (6.7)$$

In particular in $D=3$, $d_n=\frac{N}{4\pi}a_n$. Thus, the problem of computing the entanglement entropy at $N=\infty$ reduces to finding the constants a_n.

We now need to find the Green's function G_n. The main observation is that the angular harmonics on an n-sheeted Riemann surface are $\frac{1}{\sqrt{2\pi n}}e^{il\theta/n}$, where l is an integer. Hence,

$$G_n(x,x')=\int\frac{d^{D-2}k_\perp}{(2\pi)^{D-2}}e^{ik_\perp(x_\perp-x'_\perp)}G_n^{D=2}(r,r',\theta;k_\perp^2), \quad (6.8)$$

where the two-dimensional massive propagator on an n-sheeted Riemann surface is given by

$$G_n^{D=2}(r,r',\theta;m^2)=\sum_l\frac{e^{il(\theta-\theta')/n}}{2\pi n}g_l(r,r';m^2). \quad (6.9)$$

Here,

$$\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{(l/n)^2+a_n}{r^2}+m^2\right]g_l(r,r';m^2)=\frac{1}{r}\delta(r-r'). \quad (6.10)$$

We use spectral decomposition for g_l,

$$g_l(r,r';m^2)=\int dE\frac{1}{E+m^2}\phi_{l,E}(r)\phi_{l,E}^*(r'), \quad (6.11)$$

where

$$\left[-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{(l/n)^2+a_n}{r^2}\right]\phi_{l,E}=E\phi_{l,E} \quad (6.12)$$

and $\phi_{l,E}$ are normalized to

$$\int drr\phi_{l,E}^*(r)\phi_{l,E'}(r)=\delta(E-E'). \quad (6.13)$$

The constant a_n must be positive in order to avoid the presence of negative-energy states, which would render our saddle point unstable. Let us call the quantity $l^2/n^2+a_n=\nu^2$. Equation (6.12) admits two linearly independent solutions,

$$\phi(r)=\frac{1}{\sqrt{2}}J_{|\nu|}(\sqrt{E}r), \quad (6.14)$$

$$\phi(r)=\frac{1}{\sqrt{2}}J_{-|\nu|}(\sqrt{E}r). \quad (6.15)$$

We recall that

$$J_\nu(x)\sim|x|^\nu, \quad x\to0. \quad (6.16)$$

When working in free space [in the absence of conical singularity and potential (6.4)], one chooses only the solutions with a positive index $|\nu|=|l|$, so that $\phi_E(r)$ is finite and differentiable at $r=0$. However, in the present problem there is no *a priori* physical reason why the solutions (and hence the propagator) have to remain finite as $r\to0$.

In fact, a particle in a $1/r^2$ potential is a famous problem known as conformal quantum mechanics. Note that the potential (6.4) is highly singular and requires regularization at short distances. Such regularization will automatically appear in the linear $O(N)$ model, which can be obtained from (6.1) by adding a term $\lambda^2/4u$ to the Lagrangian. In that case, Eq. (6.4) only holds for $ur^{4-D}\gg1$ and the saddle-point value $\langle i\lambda\rangle$ is modified at short distances. We note that even after this regularization, the $l\neq0$ states still experience an l^2/r^2 centrifugal barrier and we must choose positive index solutions [Eq. (6.14)] for them. We now concentrate on the $l=0$ sector. For simplicity, imagine cutting the $1/r^2$ divergence off at some radius $r=r_0$ and replacing it by a finite potential. Generally, the resulting scattering states will approach the positive index solutions [Eq. (6.14)] for $\sqrt{E}r_0\to0$. However, nontrivial behavior can occur if the potential is close to developing a bound state. In that case, for $|\nu|<1$, one "dynamically" generates a length scale ξ and the scattering solutions become linear combinations of Eqs. (6.14) and (6.15) with coefficients (and, thus, the phase shifts) depending on $\sqrt{E}\xi$. Since we are looking for a scale invariant solution to the gap equation, we need $\xi\to\infty$, i.e., the system is exactly at the threshold of bound-state formation. At this threshold, for $\sqrt{E}r_0\to0$ one obtains negative index solutions [Eq. (6.15)]. Note that this behavior is special to the range $|\nu|<1$ and does not occur for $|\nu|>1$. This fact could be anticipated as the negative index solutions are square integrable at short distances for $|\nu|<1$ but not for $|\nu|>1$.

Thus, applying RG terminology to the simple quantum mechanics problem (6.12), we conclude that there are two fixed points: one stable (6.14) and one unstable (6.15). However, we are allowed to choose the unstable fixed-point solutions as we are fine tuning both the long- and short-distance parts of $\langle i\lambda\rangle$ to solve the gap equation.

With these remarks in mind,

$$g_l(r,r';m^2)=\int_0^\infty kdk\frac{1}{k^2+m^2}J_{\nu_l}(kr)J_{\nu_l}(kr'), \quad (6.17)$$

where $\nu_l=\alpha$ for $l=0$ and $\nu_l=\sqrt{l^2/n^2+\alpha^2}$ for $|l|>0$, with $a_n=\alpha^2$. The constant α can be either positive or negative. We note that as discussed in Ref. 26, α enters the operator product expansion of the field $\phi(x)$ as x approaches the conical singularity,

$$\phi(x_{\parallel},x_{\perp}) \sim r^{\alpha}\phi(0,x_{\perp}), \quad r \to 0. \tag{6.18}$$

Combining Eqs. (6.8), (6.9), and (6.17) and performing the integrals over $k_{\perp}$, k, we obtain

$$G_n(r=r',\theta,x_{\perp}=x'_{\perp}) = \frac{\Gamma[(3-D)/2]}{2\pi n(4\pi)^{(D-1)/2}r^{D-2}}\sum_l \frac{\Gamma(D/2-1+\nu_l)}{\Gamma(2-D/2+\nu_l)}e^{il\theta/n}. \tag{6.19}$$

Since we are mostly interested in $G_n(x=x')$, we have set $r=r'$, $x_{\perp}=x'_{\perp}$ in Eq. (6.19); we have left $\theta\neq 0$ as a regulator.

As an aside that will be of some interest later, we note that Eq. (6.19) is meaningful only for $\alpha > -(D/2-1)$. For $\alpha \leq -(D/2-1)$, one obtains an infrared divergence in the $k_{\perp},k \to 0$ region of integrals (6.8) and (6.17). We note that at $\alpha = -(D/2-1)$, Eq. (6.12) has a zero energy solution,

$$\phi(r) = \frac{1}{r^{D/2-1}}. \tag{6.20}$$

The solution (6.20) could, in principle, correspond to a saddle point with a nonzero expectation value $\langle\phi(x)\rangle$. Note that the r dependence of Eq. (6.20) is consistent with the scaling dimension $[\phi(x)]=D/2-1$ in the $N\to\infty$ limit. Alternatively, observe that the scaling dimension of the "boundary" operator, $[\phi(0,x_{\perp})]=D/2-1+\alpha\to 0$ as $\alpha\to -(D/2-1)$, indicating a tendency to condense. However, the infrared divergence of the propagator (6.19) indicates that condensation of $\phi(x)$ at the conical singularity is unstable to fluctuations. This is not unexpected, as the condensate would be $D-2<2$ dimensional. Such a condensate certainly cannot exist for any $g>g_c$, as it would violate the Mermin-Wagner theorem. Long-range interactions could potentially stabilize the condensate exactly at the critical point; however, the above discussion shows that this does not occur (at least in the large-N limit).

We use contour integration to write Eq. (6.19) in a somewhat more convenient form,

$$G_n(r=r',\theta,x_{\perp}=x'_{\perp}) = \frac{1}{4\pi^{D/2}\Gamma(2-D/2)r^{D-2}} \times\left[\int_0^{\infty} d\nu \frac{\nu}{\sqrt{\nu^2+\alpha^2}}U_{\sqrt{\nu^2+\alpha^2}}(\theta)R(\nu) + \theta(-\alpha)\frac{iR(i\alpha)}{n}\right], \tag{6.21}$$

with

$$U_{\nu}(\theta) = \frac{\cosh[\nu(\pi n - |\theta|)]}{\sinh(\pi n \nu)}, \tag{6.22}$$

$$R(\nu) = -i\Gamma(3-D)\left[\frac{\Gamma(-i\nu+D/2-1)}{\Gamma(-i\nu+2-D/2)} - \frac{\Gamma(i\nu+D/2-1)}{\Gamma(i\nu+2-D/2)}\right], \tag{6.23}$$

$$= \frac{2\pi\Gamma(3-D)\sin[\pi(3-D)/2]\sinh(\pi\nu)}{\cosh^2 \pi\nu - \sin^2[\pi(3-D)/2]}\frac{1}{|\Gamma(i\nu+2-D/2)|^2}. \tag{6.24}$$

In particular, for $D=3$, $R(\nu)=\pi\tanh(\pi\nu)$. We note that despite the presence of the $\theta(-\alpha)$ term in Eq. (6.21), $G_n(r=r',\theta,x_{\perp}=x'_{\perp})$ is analytic at $\alpha=0$ as is evident from Eq. (6.19). Thus, the gap Eq. (6.3) takes the form,

$$(G_n - G_1)|_{x=x'} = \frac{1}{4\pi^{D/2}\Gamma(2-D/2)r^{D-2}}\left\{\int_0^{\infty} d\nu\left[\frac{\nu}{\sqrt{\nu^2+\alpha^2}}\coth(\pi n\sqrt{\nu^2+\alpha^2}) - \coth(\pi\nu)\right]R(\nu) + \theta(-\alpha)\frac{i}{n}R(i\alpha)\right\} = 0. \tag{6.25}$$

The function $R(\nu)$ is positive for real values of ν. So the left-hand side of the gap equation goes to $-\infty$ as $\alpha\to\infty$ and to ∞ as $\alpha\to -(D/2-1)^{+}$. Hence, the gap equation always has at least one solution and, more generally, an odd number of solutions. Numerically, we find that the gap equation has a unique solution for all n for $D<D_c$, $D_c\approx 3.74$. For $D>D_c$, there are one or three solutions depending on the value of n, as we will discuss below.

As we are mainly interested in the entanglement entropy, let us consider the limit $n\to 1$. Then we expect $\alpha\to 0$. The integral in Eq. (6.25) is nonanalytic at $\alpha=0$, due to singular behavior in the $\nu\to 0$ region. Noting that $R(\nu)\approx R'(0)\nu$, as $\nu\to 0$, we obtain to leading order in α,

$$\begin{aligned}(G_n - G_1)|_{x=x'} - (G_n - G_1)|_{x=x',\alpha=0} &\approx \frac{R'(0)}{4n\pi^{D/2}\Gamma(2-D/2)r^{D-2}}\left[\frac{1}{\pi}\int_0^{\infty} d\nu\left(\frac{\nu^2}{\nu^2+\alpha^2}-1\right) - \theta(-\alpha)\alpha\right] \\ &= \frac{R'(0)}{4n\pi^{D/2}\Gamma(2-D/2)r^{D-2}}\left[-\frac{1}{2}|\alpha| - \theta(-\alpha)\alpha\right] = -\frac{\Gamma(D/2-1)^2\Gamma(D/2)}{4\pi^{D/2}\Gamma(D-1)r^{D-2}}\frac{\alpha}{n},\end{aligned} \tag{6.26}$$

where the contributions from the integral and the θ function have combined to produce a result analytic in α. Now using Eqs. (4.64) and (4.65) for $(G_n - G_1)|_{x=x',\alpha=0}$,

$$\alpha \approx -\frac{D-2}{2(D-1)}(n-1), \quad n \to 1. \qquad (6.27)$$

Note that the exponent α controlling the OPE (6.18) of the field $\phi(x)$ at the conical singularity is positive for $n<1$ and negative for $n>1$. Now, from Eq. (6.27),

$$a_n = \frac{(D-2)^2}{4(D-1)^2}(n-1)^2, \quad n \to 1. \qquad (6.28)$$

Therefore, combining Eqs. (3.10) and (6.5), we find that

$$r_n \propto \frac{a_n}{1-n} \propto n-1, \quad n \to 1, \qquad (6.29)$$

and the correlation length correction to the entanglement entropy proper vanishes at leading order in N,

$$r = \lim_{n\to 1} r_n = 0. \qquad (6.30)$$

Thus, for all dimensions $2<D<4$,

$$r \sim O(N), \qquad (6.31)$$

even though $r_n \sim O(N^2)$ for all $n \neq 1$.

So far we have concentrated on the solution to the gap equation in the $n\to 1$ limit for arbitrary dimension. However, we can also obtain an analytic solution for arbitrary n in the limit $D=4-\epsilon$. Such a solution is useful for comparison to the results of the ϵ-expansion presented in Sec. IV.

When $D=4-\epsilon$, the function $R(\nu) = -2\nu^{1-\epsilon}\Gamma(-1+\epsilon)$. The divergence of the Γ function is not important here as it is just an overall factor in the gap equation [which anyway cancels with $\Gamma(2-D/2)$ in Eq. (6.21)]. However, the integral (6.25) now diverges for $\nu\to\infty$ if $\epsilon=0$. Hence, for generic n and $D=4-\epsilon$, the leading α-dependent contribution to the gap equation comes from the region $\nu \gg 1$ and is of order $\frac{1}{\epsilon}\alpha^2$. This suggest that α will be at most of order $\epsilon^{1/2}$. However, for α very small (i.e., $n\to 1$), we already know from the previous discussion that the leading contribution to the integral scales as $|\alpha|$ and comes from the $\nu\to 0$ region. Keeping these two contributions (one nonanalytic in α and the other analytic, but with a diverging coefficient) and setting $\alpha=0$ in the rest of the integral, we reduce the gap equation to

$$\frac{1}{\pi n}\int_0^\infty d\nu\left(\frac{\nu^2}{\nu^2+\alpha^2}-1\right) + \int_0^\infty d\nu\,\nu\left(\frac{\cosh(\pi n\nu)}{\sinh(\pi n\nu)} - \frac{\cosh(\pi\nu)}{\sinh(\pi\nu)}\right) - \frac{1}{2}\alpha^2\int_{\nu\gg 1} d\nu\,\nu^{-\epsilon-1} - \theta(-\alpha)\frac{\alpha}{n} = 0, \qquad (6.32)$$

$$\frac{\alpha}{n} + \frac{\alpha^2}{\epsilon} - \frac{1}{6}\left(\frac{1}{n^2}-1\right) = 0. \qquad (6.33)$$

The quadratic has two solutions,

$$\alpha_\pm = -\frac{\epsilon}{2n} \pm \frac{1}{2n}\sqrt{\epsilon^2 + \frac{2\epsilon}{3}(1-n^2)}, \qquad (6.34)$$

and the corresponding values of d_n [Eq. (6.7)] are

$$d_n^\pm = \frac{N}{8\pi^2}\left[\frac{1}{6}\left(\frac{1}{n^2}-1\right) + \frac{\epsilon \mp \sqrt{\epsilon^2 + \frac{2}{3}\epsilon(1-n^2)}}{2n^2}\right]. \qquad (6.35)$$

Equation (6.35) is in agreement with the result of the ϵ-expansion [Eq. (4.27)], and we can identify the $\alpha_\pm$ saddle points with the $c_r^\pm$ fixed points. Moreover, we see that the predictions of the large-N (6.34) and ϵ-expansion (4.34) for the OPE exponent α also agree. Note that both saddle points [Eq. (6.34)] disappear for $n>n_c\approx 1+3\epsilon/4$. This coincides with the value of n at which runaway of RG flow is observed in the ϵ-expansion. However, as we noted earlier, the gap equation always has an odd number of solutions. Thus, we have missed a solution in our discussion above. This solution has $\alpha \approx -(D/2-1)\to -1$, i.e., α is not small. Its existence is possible due to a cancellation of $1/\epsilon$ divergences between the large ν part of the integral and the $\theta(-\alpha)$ term in Eq. (6.25). Keeping these two contributions to the gap equation, we obtain in the $\alpha\to -(D/2-1)$ limit,

$$\frac{\alpha^2}{\epsilon} - \frac{\epsilon}{n}\frac{1}{\alpha + D/2 - 1} = 0. \qquad (6.36)$$

So,

$$\alpha = -1 + \frac{1}{2}\epsilon + \frac{1}{n}\epsilon^2. \qquad (6.37)$$

Equations (6.34) and (6.37) comprise the three solutions to the gap equation for $1<n<n_c$, and Eq. (6.37) is the only solution for $n>n_c$. We speculate that the runaway of the RG flow observed in ϵ-expansion for $n>n_c$ is toward the fixed point (6.37). As we noted above, the value $\alpha=-(D/2-1)$ corresponds to the would be condensation of the ϕ field at the conical singularity. Thus, for $\epsilon\to 0$, the saddle-point (6.37) is proximate to such condensation. This is consistent with our interpretation of the RG flow $c\to -\infty$ as the tendency to the formation of $\langle\phi(x)\rangle \neq 0$. However, the large-$N$ analysis demonstrates that no true spontaneous symmetry breaking at the conical singularity occurs for $D<4$.

To our knowledge, no such nontrivial n dependence has been previously observed in any theories. Still, in the large-N expansion, such behavior is only present for $D>D_c\approx 3.74$ and its relevance to the physical case $D=3$ is doubtful. Moreover, the nonanalyticity occurs away from $n=1$ and, thus, is unimportant for computing the entanglement entropy proper. Indeed, the behavior of the theory for $n\to 1$ Eq. (6.27) is found to evolve smoothly as the dimension D increases from 2 to 4.

We now come back to the physical case $D=3$, where the solution to the gap equation is unique. The numerical solution for the first few integers n is listed in Table I. Then, from Eqs. (3.9) and (6.5),

TABLE I. Solution to the gap equation in the large-N limit for $D=3$.

n	α_n
2	−0.16515
3	−0.26594
4	−0.32905
5	−0.36743

$$r_n = \frac{3\pi^2 N^2}{128}\frac{n\alpha_n^2}{n-1}, \quad D=3. \tag{6.38}$$

The coefficient (6.38) can be, in principle, obtained numerically by performing classical Monte Carlo simulations of the $O(N)$ model in the spirit of Refs. 27 and 28.

So far our large-N computation has been confined to the correlation length correction to the Renyi entropy. At leading order, the calculation was technically fairly simple, as utilizing the discussion in Sec. III, we could work at the critical point. In particular, the form of the Lagrange multiplier $\langle i\lambda(r)\rangle$ was fixed by scale invariance up to an overall constant. To proceed beyond the leading order, as is required for the calculation of the correlation length correction to the entanglement entropy proper, we would have to work in the gapped phase. The Lagrange multiplier $\langle i\lambda(r)\rangle$ would now be a nontrivial function of r with a length scale determined by the correlation length $\xi=m^{-1}$. Similarly, if we wish to compute the finite-size correction γ to the entanglement entropy, $\langle i\lambda(r)\rangle$ will again vary nontrivially with a length scale determined by the size L of the compact direction. In both cases, we have to solve the gap equation for a whole function $\langle i\lambda(r)\rangle$ rather than a single number a_n. In principle, this problem can be addressed numerically. It would be particularly interesting to check whether $\gamma \sim O(1)$ for $N\to\infty$ as suggested by the ϵ-expansion [Eq. (5.31)].

VII. CONCLUSION: FUTURE DIRECTIONS

In the present work, we have computed the universal finite-size and correlation length corrections to the entanglement entropy and the Renyi entropy for the $O(N)$ model. The evaluation of this entropy required a study of the $O(N)$ field theory on a n-sheeted Riemann surface for general n and an understanding of the nature of the $n\to 1$ limit. For $n\neq 1$, there is a conical singularity at the origin of the Riemann surface and we have presented a detailed analysis of the structure of the "boundary" excitations of the $O(N)$ CFT at this singularity. (A closely related CFT with vortex boundary conditions was studied in Ref. 26 with a very different physical motivation.) In particular, we showed that in the context of $\epsilon=3-d$ expansion, the RG flow of the boundary coupling c in Eq. (4.10) was the key to a determination of the entanglement entropy. The RG flow of c had two possible structures shown in Figs. 5(c) and 5(d). For n greater than a critical n_c, we had flow in the infrared to $c=-\infty$ as in Fig. 5(d). In contrast for $n<n_c$, we had three possible fixed points, and the $n\to 1$ limit was controlled by the nonzero fixed point $c=c_r^+$, at which all strong hyperscaling assumptions were obeyed. All our computations in the ϵ and $1/N$ expansions were consistent with this RG flow and fixed-point structure. One crucial consequence of the boundary perturbation and the subtle limit $n\to 1$ is that the finite-size and correlation length corrections to the entanglement entropy are different at the Wilson-Fisher and Gaussian fixed points already at leading order in ϵ-expansion.

In this paper, we have considered a geometry with a smooth straight boundary between regions A and B. Therefore, we can make no strong claims regarding the possible presence of nonuniversal terms in the entanglement entropy associated with the curvature of the boundary. Nevertheless, we generally expect such terms to be absent in spatial dimension $d=2$. Indeed, as we discussed in the introduction, any nonuniversal contributions to the entanglement entropy must involve integrals of local geometric quantities over the length of the boundary. The simplest geometric object for a one-dimensional boundary is the curvature vector $\vec{\kappa}$. Assuming that the integrand is analytic in $\vec{\kappa}$, the leading correction due to the boundary curvature that we can construct is

$$\Delta S \sim \int_{\mathcal{B}} ds\,\vec{\kappa}^2, \tag{7.1}$$

which scales as $1/L$ under dilatations. Such a behavior is subleading not only to the universal terms in the entanglement entropy but also to corrections to scaling coming from irrelevant operators.

One possible extension of our work is to consider boundaries with sharp corners. In such geometries, it is expected that the entanglement entropy will contain a universal logarithmically divergent term.[9–11,14,15] Moreover, we have only studied the correlation length correction to the entanglement entropy in the symmetry unbroken region $t>0$. It would be interesting to extend our treatment to the symmetry broken phase $t<0$.

While our paper was being completed, we learned of the numerical study of entanglement entropy in the $d=2$ quantum Ising model in Ref. 29. At the quantum critical point, the authors of Ref. 29 find evidence for a finite-size correction γ as in Eq. (1.6) in the case when the boundary between regions A and B is smooth. We note that the geometry studied in Ref. 29 is an $L\times L$ torus divided into two equal cylinders rather than the infinite cylinder cut in half that we have considered here. Thus, the two results cannot be compared directly. Nevertheless, the value of γ in Ref. 29 is found to be positive, as in our conjecture in Eq. (1.8) for the case of periodic boundary conditions along the cylinder.

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