Tensor damping in metallic magnetic multilayers

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The mechanism of spin pumping, described by Tserkovnyak *et al.* [Phys. Rev. B **67**, 140404 (2003)], is formally analyzed in the general case of a magnetic multilayer consisting of two or more metallic ferromagnetic (FM) films separated by normal-metal (NM) layers. It is shown that the spin-pumping-induced dynamic coupling between FM layers modifies the linearized Gilbert equations in a way that replaces the usual local, scalar Gilbert damping constant with a nonlocal matrix of Cartesian damping tensors. As an example, explicit analytical results are obtained for a five-layer (spin valve) of form NM/FM/NM′/FM/NM. These are compared with earlier well-known results of Tserkovnyak *et al.* for the related three-layer FM/NM/FM, which are shown to have singled out the diagonal element of the local damping tensor along the axis normal to the plane of the two magnetization vectors. For spin-valve devices of technological interest, the influence of tensor damping on thermal noise fluctuations and/or spin-torque critical currents is shown to necessarily be coupled to the nonlocal tensor properties of the magnetostatic interaction as well.

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I. INTRODUCTION

For purely scientific reasons, as well as technological applications such as magnetic field sensors or dc current-tunable microwave oscillators, there is significant present interest[1] in the magnetization dynamics in current-perpendicular-to-plane (CPP) metallic multilayer devices comprising multiple ferromagnetic (FM) films separated by normal-metal (NM) spacer layers. The phenomenon of spin pumping, described earlier by Tserkovnyak *et al.*[2,3] introduces an additional source of dynamic coupling either between the magnetization of a single FM layer and its NM electronic environment or between two or more FM layers as mediated through their NM spacers. In the former case,[2] the effect can resemble an enhanced magnetic damping of an individual FM layer, which has important practical application for substantially increasing the spin-torque critical currents of CPP spin valves employed as giant-magnetoresistive (GMR) sensors for read-head applications.[4] Considered in this paper is a more general treatment in the case of two or more FM layers in a CPP stack. It will be shown in Sec. II that spin pumping modifies the linearized equations of motion in a way that replaces the local, scalar damping constant of the well-known Gilbert equations with a nonlocal matrix of Cartesian damping tensors.[5] Analytical results for the case of a five-layer spin-valve stack of the form NM/FM/NM′/FM/NM are discussed in detail in Sec. III and are in Sec. IV compared and contrasted with the early well-known results of Tserkovnyak *et al.*[3] as well as some very recent results of that author and colleagues.[6] In the case of CPP-GMR devices of technological interest, the influence of the tensor nature of the damping on the thermal magnetization fluctuations or spin-torque critical currents is shown to be linked to the additional tensor properties of the nonlocal, anisotropic magnetostatic interaction, such as is characterized by a stiffness-field tensor matrix. For the particular case of in-plane magnetized CPP-GMR devices, it is argued that the in-plane components of the damping tensor will likely always play the dominant role. This result is complimentary to the physical description used in Ref. 3, which by construction singled out the diagonal component of the damping tensor along the axis normal to the plane of the magnetization vectors.

II. SPIN-PUMPING AND TENSOR DAMPING

As discussed by Tserkovnyak *et al.*[2,3] the spin current $\boldsymbol{I}^{\text{pump}}$ flowing into the NM layer at an FM/NM interface (Fig. 1) due to the spin-pumping effect is described by the expression

$$\boldsymbol{I}^{\text{pump}} = \frac{\hbar}{4\pi}\left[\text{Re}\, g^{\uparrow\downarrow}\left(\hat{\boldsymbol{m}} \times \frac{d\hat{\boldsymbol{m}}}{dt}\right) - \text{Im}\, g^{\uparrow\downarrow}\frac{d\hat{\boldsymbol{m}}}{dt}\right], \tag{1}$$

where $g^{\uparrow\downarrow}$ is a dimensionless mixing conductance and $\hat{\boldsymbol{m}}$ is the unit magnetization vector. In this paper, $\hat{\boldsymbol{m}}$ for any FM layer is treated as a uniform macrospin. A restatement of (Eq. (1)) in terms more natural to Valet-Fert[7] form of transport equations is discussed in Appendix A. With the notational conversion $\boldsymbol{I}^{\text{pump}} \to -(\hbar/2e)A\boldsymbol{J}^{\text{pump}}$, where A is the cross-sectional area of the film stack, Eq. (1), for the case Re $g^{\uparrow\downarrow} \gg$ Im $g^{\uparrow\downarrow}$, simplifies to

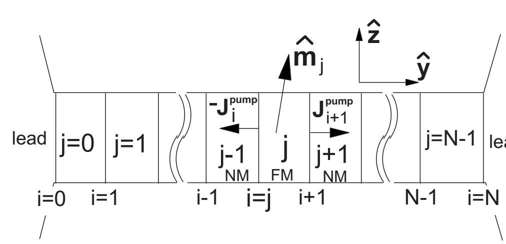

FIG. 1. Cross-section cartoon of an N-layer multilayer stack with $N-1$ interior interfaces of FM-NM or NM-FM type, such as found in CPP-GMR pillars sandwiched between conductive leads of much larger cross section. In the example shown, the jth layer is FM, sandwiched by NM layers, with spin-pumping contributions at the ith (NM/FM) and $(i+1)$st (FM/NM) interfaces located at $y=y_i$ and $y=y_{i+1}$ (with $i=j$ for the labeling scheme shown).

$$\boldsymbol{J}^{\text{pump}} \cong \mp \frac{e}{2\pi}\frac{(h/2e^2)}{r^{\uparrow\downarrow}}\left(\hat{\boldsymbol{m}} \times \frac{d\hat{\boldsymbol{m}}}{dt} + \varepsilon\frac{d\hat{\boldsymbol{m}}}{dt}\right), \quad \varepsilon \equiv \frac{\text{Im } r^{\uparrow\downarrow}}{\text{Re } r^{\uparrow\downarrow}},$$

$$\text{"}-\text{" for FM/NM interface, "}+\text{" for NM/FM interface,} \tag{2}$$

where $r^{\uparrow\downarrow}=(h/2e^2)A/|g^{\uparrow\downarrow}|$ is the inverse mixing conductance (with dimensions of resistance area) and $h/2e^2$ is the well-known inverse conductance quantum ($\cong 12.9$ kΩ). In the present notation, all spin-current densities $\boldsymbol{J}^{\text{spin}}$ have the same dimensions as electron-charge current density J_e and for conceptual simplicity are defined with a parallel (i.e., $\hat{\boldsymbol{J}}^{\text{spin}}=+\hat{\boldsymbol{m}}$) rather than antiparallel alignment with magnetization $\hat{\boldsymbol{m}}$. Positive J is defined as electrons flowing to the right (along $+\hat{\boldsymbol{y}}$ in Fig. 1).

For a FM layer sandwiched by two NM layers in which the FM layer is the jth layer ($j \geq 0$) of a multilayer film stack (as in Fig. 1), spin-pumping contributions at the ith interface, i.e., either left ($i=j$) or right ($i=j+1$) FM-NM interfaces, Eq. (2) can be expressed as

$$\boldsymbol{J}_{i=j,j+1}^{\text{pump}} = \frac{\hbar}{2e}\frac{(-1)^{i-j}}{r_i^{\uparrow\downarrow}}\left(\hat{\boldsymbol{m}}_j \times \frac{d\hat{\boldsymbol{m}}_j}{dt} + \varepsilon_i\frac{d\hat{\boldsymbol{m}}_j}{dt}\right). \tag{3}$$

The physical picture to now be invoked is that of *small* (thermal) fluctuations of $\hat{\boldsymbol{m}}$ about equilibrium $\hat{\boldsymbol{m}}_0$ giving rise to the $d\hat{\boldsymbol{m}}/dt$ terms in Eq. (2). Since $|\hat{\boldsymbol{m}}|\equiv 1$, the three vector components of $\hat{\boldsymbol{m}}$ and/or $d\hat{\boldsymbol{m}}/dt$ are not linearly independent. To remove this interdependency, as well as higher order terms in Eq. (3) it is useful to work in a primed coordinate system where $\hat{\boldsymbol{z}}'=\hat{\boldsymbol{m}}_0$, through use of a 3×3 Cartesian rotation matrix $\mathfrak{R}(\hat{\boldsymbol{m}}_0)$ such that $\hat{\boldsymbol{m}}=\mathfrak{R}\cdot\hat{\boldsymbol{m}}'$.[8] To *first* order in linearly *independent* quantities m'_x and m'_y, $\hat{\boldsymbol{m}}=\hat{\boldsymbol{m}}_0+\tilde{\mathfrak{R}}\cdot\boldsymbol{m}'$, where $\boldsymbol{m}'\equiv\binom{m'_{x'}}{m'_{y'}}$, and where $\tilde{\mathfrak{R}}$ denotes the 3×2 matrix from the first two (i.e., x and y) columns of $\mathfrak{R}$. Replacing $\hat{\boldsymbol{m}}\times d\hat{\boldsymbol{m}}/dt=\mathfrak{R}\cdot(\hat{\boldsymbol{m}}'\times d\boldsymbol{m}'/dt)$, $\hat{\boldsymbol{m}}'\cong\hat{\boldsymbol{m}}'_0=\hat{z}'$, and $\hat{z}'\times\_$ with matrix multiplication, the *linearized* form of Eq. (3) becomes

$$\boldsymbol{J}_{i=j,j+1}^{\text{pump}} = \frac{\hbar}{2e}\frac{(-1)^{i-j}}{r_i^{\uparrow\downarrow}}\tilde{\mathfrak{R}}_j \cdot \begin{pmatrix} \varepsilon_i & -1 \\ 1 & \varepsilon_i \end{pmatrix} \cdot \begin{pmatrix} dm'_{jx'}/dt \\ dm'_{jy'}/dt \end{pmatrix}. \tag{4}$$

Using the present sign convention, $\boldsymbol{S}_j=(M_st)_jA/\gamma\hat{\boldsymbol{m}}_j$ is the spin-angular momentum of the jth FM layer with saturation magnetization thickness product $(M_st)_j$, and $\gamma>0$ is the gyromagnetic ratio. Taking $|\boldsymbol{M}|=M_s$ as constant, it follows by angular-momentum conservation that[3]

$$\frac{(M_st)_j}{\gamma}\frac{d\hat{\boldsymbol{m}}_j}{dt} \Leftrightarrow \frac{1}{A}\frac{d\boldsymbol{S}_j}{dt} = \frac{\hbar}{2e}\sum_{i=j}^{j+1}(-1)^{i-j}\hat{\boldsymbol{m}}_j \times \boldsymbol{J}_i^{\text{NM}} \times \hat{\boldsymbol{m}}_j \tag{5}$$

is the contribution to $d\hat{\boldsymbol{m}}_j/dt$ due to the *net-transverse* spin current entering the jth FM layer (Fig. 1). In Eq. (5), $\boldsymbol{J}_i^{\text{NM}}$ denotes the spin-current density *in* the NM layer *at* the ith FM-NM interface. Taking the cross product $\hat{\boldsymbol{m}}\times$ on both sides of Eq. (5), transforming to primed coordinates by matrix multiplying by $\mathfrak{R}^{-1}=\mathfrak{R}^{\mathsf{T}}$, and employing similar linearization as to obtain Eq. (4), one finds to first order that

$$\hat{z}'_j \times \frac{1}{A}\frac{d\boldsymbol{S}'_j}{dt} = \frac{\hbar}{2e}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \tilde{\mathfrak{R}}_j^{\mathsf{T}} \cdot \left[\Delta\boldsymbol{J}_j^{\text{spin}} \equiv \sum_{i=j}^{j+1}(-1)^{i-j}\boldsymbol{J}_i^{\text{NM}}\right], \tag{6}$$

where $\tilde{\mathfrak{R}}^{\mathsf{T}}$ is the 2×3 matrix transpose of $\tilde{\mathfrak{R}}$. By definition, $\tilde{\mathfrak{R}}_j^{\mathsf{T}}\cdot\hat{\boldsymbol{m}}_{0j}=0$.

The quantities $\Delta\boldsymbol{J}_j^{\text{spin}}$ in Eq. (6) are not known *a priori* but must be determined after solution of the appropriate transport equations (e.g., Appendix B). Even in the absence of charge-current flow (i.e., $J_e=0$) as considered here, the $\Delta\boldsymbol{J}_j^{\text{spin}}$ are nonzero due to the set of $\boldsymbol{J}_i^{\text{pump}}$ in Eq. (4) which appear as source terms in the boundary conditions (Eq. (A9)) at each FM-NM interface. Given the *linear* relation of Eq. (4), one can now apply linear superposition to express

$$\Delta\boldsymbol{J}_j^{\text{spin}} = \frac{\hbar}{2e}\sum_k \frac{1}{\bar{r}_k^{\uparrow\downarrow}}\overleftrightarrow{C}_{jk}\cdot\tilde{\mathfrak{R}}_k\cdot\begin{pmatrix} \varepsilon & -1 \\ 1 & \varepsilon \end{pmatrix}\frac{d\boldsymbol{m}'_k}{dt}, \quad \frac{1}{\bar{r}_k^{\uparrow\downarrow}} \equiv \frac{1}{2}\sum_{i=k}^{k+1}\frac{1}{r_i^{\uparrow\downarrow}} \tag{7}$$

in terms of the set of three-dimensional (3D) dimensionless Cartesian tensor $\overleftrightarrow{C}_{jk}$. The $\overleftrightarrow{C}_{jk}$ are convenient for formal expressions such as Eq. (9), or for analytical work in algebraically simple cases, such as that exampled in Sec. III. However, they are also subject to methodical computation. For the kth magnetic layer, the first, second, or third columns of each $\overleftrightarrow{C}_{jk}$ are the dimensionless vectors $\Delta\boldsymbol{J}_j^{\text{spin}}$ simultaneously obtainable for all magnetic layers j from a matrix solution[9] of the Valet-Fert[7] transport equations with nonzero dimensionless spin-pump vectors $\boldsymbol{J}_{i=k,k+1}^{\text{pump}}=(-1)^{i-k}(\bar{r}_k^{\uparrow\downarrow}/r_i^{\uparrow\downarrow})$ ($\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, or $\hat{z}$).

To include spin currents via Eq. (5) into the magnetization dynamics, the conventional Gilbert equations of motion for $\hat{\boldsymbol{m}}(t)$ can be amended as

$$\frac{d\hat{\boldsymbol{m}}_j}{dt} = -\gamma(\hat{\boldsymbol{m}}_j \times \boldsymbol{H}_j^{\text{eff}}) + \alpha_j^{\text{G}}\hat{\boldsymbol{m}}_j \times \frac{d\hat{\boldsymbol{m}}_j}{dt} + \frac{\gamma}{(M_st)_j}\frac{1}{A}\frac{d\boldsymbol{S}_j}{dt}, \tag{8}$$

where α_j^{G} is the usual (scalar) Gilbert damping parameter. From Eqs. (6) and (7), one can deduce that the rightmost term in Eq. (8) will scale linearly with $d\boldsymbol{m}'/dt$, as does the conventional Gilbert damping term. Combining these terms together after applying the analogous linearization procedure to Eq. (8) as was done in going from Eq. (5) to Eq. (6), one obtains

$$\hat{z}'_j \times \frac{d\boldsymbol{m}'_j}{dt} = \gamma[\tilde{\mathfrak{R}}_j^{\mathsf{T}}\cdot\boldsymbol{H}_j^{\text{eff}} - (\hat{\boldsymbol{m}}_j\cdot\boldsymbol{H}_j^{\text{eff}})\boldsymbol{m}'_j] - \sum_k \overleftrightarrow{\alpha}'_{jk}\cdot\frac{d\boldsymbol{m}'_k}{dt},$$

$$\overleftrightarrow{\alpha}'_{jk} \equiv \begin{pmatrix} \alpha_{jk}'^{x'x'} & \alpha_{jk}'^{x'y'} \\ \alpha_{jk}'^{y'x'} & \alpha_{jk}'^{y'y'} \end{pmatrix} = \begin{pmatrix} \alpha_j^{\text{G}} & 0 \\ 0 & \alpha_j^{\text{G}} \end{pmatrix}\delta_{jk} + \overleftrightarrow{\alpha}'^{\text{pump}}_{jk},$$

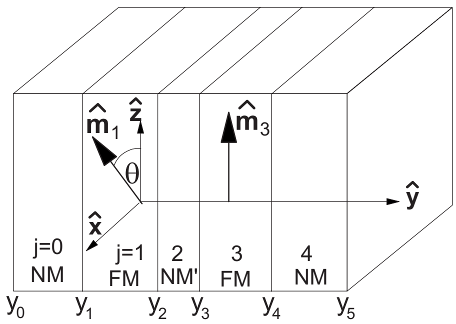

FIG. 2. Cartoon of a prototypical five-layer CPP-GMR stack (leads not shown) with two FM layers (1 and 3) sandwiching a central NM spacer layer (2) and with outer NM cap layers (0 and 4). For discussion purposes described in the text, the magnetization vectors $\hat{\boldsymbol{m}}_1$ and $\hat{\boldsymbol{m}}_3$ can be considered to lie in the film plane (x-z plane).

$$\overleftrightarrow{\alpha}'^{\text{pump}}_{jk} = \frac{\hbar\gamma}{(4\pi M_s t)_j}\frac{h/2e^2}{\tilde{r}_k^{\uparrow\downarrow}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\cdot\tilde{\mathfrak{R}}_j^{\mathsf{T}}\cdot\overleftrightarrow{C}_{jk}\cdot\tilde{\mathfrak{R}}_k\cdot\begin{pmatrix}\varepsilon & -1\\ 1 & \varepsilon\end{pmatrix}, \tag{9}$$

where Kronecker delta $\delta_{jk}=1$ if $j=k$, and $\delta_{jk}=0$ if $j\neq k$.

In Eq. (9), $\overleftrightarrow{\alpha}'_{jj}$ is a two-dimensional Cartesian "damping tensor" expressed in a coordinate system where $\hat{\boldsymbol{m}}'_{0j}=\hat{z}'_j$, while $\overleftrightarrow{\alpha}'_{jk\neq j}$ is a "nonlocal tensor" spanning two such coordinate systems. This formalism follows naturally from the linearization of the equations of motion for noncollinear macrospins and is particularly useful for describing the influence of "tensor damping" on the thermal fluctuations and/or spin-torque critical currents of such multilayer film structures (e.g., as described further in Sec. IV). Due to the spin-pumping contribution $\overleftrightarrow{\alpha}'^{\text{pump}}_{jk}$, the four individual $\alpha'^{u'v'}_{jk}$ (with u', $v'=x'$, or y') are in general nonzero with $\alpha'^{x'x'}_{jk}\neq\alpha'^{y'y'}_{jk}$, reflecting the true tensor nature of the damping in this circumstance, which is additionally nonlocal between magnetic layers (i.e., $\alpha'^{u'v'}_{jk\neq j}\neq 0$). The $\alpha'^{u'v'}_{jk}$ are somewhat arbitrary to the extent that one may replace $\tilde{\mathfrak{R}}\leftrightarrow\tilde{\mathfrak{R}}\cdot\mathfrak{R}_2$ in Eq. (9), where $\mathfrak{R}_2$ is the 2×2 matrix representation of any rotation about the $\hat{z}'$ axis.

It is perhaps tempting to contemplate an "inverse linearization" of Eq. (9) to obtain a 3D nonlinear Gilbert equation with a fully 3D damping tensor $\overleftrightarrow{\alpha}_{jk}=\mathfrak{R}_j\cdot\overleftrightarrow{\alpha}'_{jk}\cdot\mathfrak{R}_k^{\mathsf{T}}$. However, Eq. (9) has a null $\hat{z}'$ component and contains no information regarding the heretofore undefined quantities $\alpha'^{u'z'}_{jk}$ or $\alpha'^{z'z'}_{jk}$. For local, isotropic/scalar Gilbert damping, one can independently argue on spatial symmetry grounds that $\alpha'^{z'z'}_{\text{G}}=\alpha'^{u'u'}_{\text{G}}=\alpha_{\text{G}}$. However, the analogous extension is not so obviously available for $\overleftrightarrow{\alpha}'^{\text{pump}}_{jk}$, given the intrinsically nonlocal, anisotropic nature of spin pumping. The proper general equation remains that of Eq. (8), with the rightmost term given by that in Eq. (5) or its equivalent.

III. EXAMPLE: FIVE-LAYER SYSTEM

Figure 2 shows a five-layer system with two FM layers resembling a CPP-GMR spin-valve to be used as a prototype. Although the full generalization is straightforward, the material properties and layer thickness will be assumed symmetric about the central NM′ spacer layer 2, which will additionally be taken to have a large spin-diffusion length $l_2\gg t_2$ (with t_j the thickness of the jth layer), such that the "ballistic" approximation (B3) applies. The inverse mixing conductances $r^{\uparrow\downarrow}_{i=1-4}$ will also be assumed to be real. Referring to either of the two the outer boundary conditions described by Eq. (B5) of Appendix B, one finds for the FM-NM interfaces at $y=y_1$ and y_4, that

$$\boldsymbol{J}^{\text{NM}}_{i=1}=J_1^{\text{FM}}\hat{\boldsymbol{m}}_{j=1}+\frac{r_1^{\uparrow\downarrow}}{r_1'^{\uparrow\downarrow}}\boldsymbol{J}_1^{\text{pump}},\quad \boldsymbol{J}^{\text{NM}}_4=\boldsymbol{J}_4^{\text{FM}}\hat{\boldsymbol{m}}_3+\frac{r_1^{\uparrow\downarrow}}{r_1'^{\uparrow\downarrow}}\boldsymbol{J}_4^{\text{pump}}, \tag{10a}$$

$$\frac{1}{2}\Delta V^{\text{FM}}_{i=1}=+r_1'J_1^{\text{FM}},\quad \frac{1}{2}\Delta V_4^{\text{FM}}=-r_1'J_4^{\text{FM}}, \tag{10b}$$

$$r_1'\equiv r_1+[\rho l\ \text{hypb}(t/l)]_{\text{NM}},\quad r_1'^{\uparrow\downarrow}\equiv r_1^{\uparrow\downarrow}+[\rho l\ \text{hypb}(t/l)]_{\text{NM}}, \tag{10c}$$

where $r_1=r_4$ and $r_1^{\uparrow\downarrow}=r_4^{\uparrow\downarrow}$(by assumed symmetry), hypb $\equiv$tanh or coth (depending on boundary condition), and subscript "NM" refers to either outer layer 0 or 4. In Eq. (10) and below, $\hat{\boldsymbol{m}}_j\leftrightarrow\hat{\boldsymbol{m}}_{0j}$ are used interchangeably. Inside FM layer 3, Eqs. (B1) and (B2) of Appendix B have solution

$$\Delta V_3(y_3\leq y\leq y_4)$$
$$=2A_3\sinh[(y-y_3)/l_{\text{FM}}]+2B_3\cosh[(y-y_3)/l_{\text{FM}}],$$

$$J_3^{\text{spin}}(y)=1/(\rho l)_{\text{FM}}\{A_3\cosh[(y-y_3)/l_{\text{FM}}]+B_3\sinh[(y-y_3)/l_{\text{FM}}]\},$$

$$B_3=-A_3[\{r_1'+[\rho l\tanh(t/l)\}_{\text{FM}}]/[(\rho l)_{\text{FM}}+r_1'\tanh(t/l)_{\text{FM}}], \tag{11}$$

where the expression for B_3 follows from Eq. (10b). Subscript "FM" refers to either layer 1 or 3. The boundary conditions (A5) and (A9) applied to the FM/NM boundary at $y=y_3$ can be expressed in combination as

$$\frac{1}{2}(2B_3-\boldsymbol{\Delta V}_2)=(r_2-r_2^{\uparrow\downarrow})[A_3/(\rho l)_{\text{FM}}=\boldsymbol{J}_2^{\text{spin}}\cdot\hat{\boldsymbol{m}}_3]\hat{\boldsymbol{m}}_3+r_2^{\uparrow\downarrow}(\boldsymbol{J}_2^{\text{spin}}-\boldsymbol{J}_3^{\text{pump}}), \tag{12}$$

where $r_2=r_3$ and $r_2^{\uparrow\downarrow}=r_3^{\uparrow\downarrow}$. The ballistic values $\boldsymbol{\Delta V}_2$ and $\boldsymbol{J}_2^{\text{spin}}$ are constant inside central layer 2. Using Eq. (11) to eliminate coefficient B_3 in Eq. (12), the latter may be rewritten as

$$-\frac{1}{2}\boldsymbol{\Delta V}_2=r_2^{\uparrow\downarrow}[(\overleftrightarrow{1}+2q\hat{\boldsymbol{m}}_3\cdot\hat{\boldsymbol{m}}_3^{\mathsf{T}})\cdot\boldsymbol{J}_2^{\text{spin}}-\boldsymbol{J}_3^{\text{pump}}],$$

$$q\equiv\frac{1}{2r_2^{\uparrow\downarrow}}\left\{r_2-r_2^{\uparrow\downarrow}+\frac{r_1'+[\rho l\tanh(t/l)]_{\text{FM}}}{1+r_1'[\tanh(t/l)/(\rho l)]_{\text{FM}}}\right\}, \tag{13}$$

where $\overleftrightarrow{1}$ is the 3D identity tensor and $\hat{\boldsymbol{m}}_3\cdot\hat{\boldsymbol{m}}_3^{\mathsf{T}}$ denotes the 3D tensor formed from the vector *outer* product of $\hat{\boldsymbol{m}}_3$ with itself.

Working through the equivalent computations applied now to the NM/FM interface at $y=y_2$, one finds the analogous result

$$+\frac{1}{2}\boldsymbol{\Delta V}_2 = r_2^{\uparrow\downarrow}[(\overleftrightarrow{1}+2q\hat{\boldsymbol{m}}_1\cdot\hat{\boldsymbol{m}}_1^{\mathsf{T}})\cdot\boldsymbol{J}_2^{\text{spin}}-\boldsymbol{J}_2^{\text{pump}}]. \quad (14)$$

Eliminating $\boldsymbol{\Delta V}_2$ between Eqs. (13) and (14) provides the remaining needed result for $\boldsymbol{J}_2^{\text{spin}}$

$$\boldsymbol{J}_2^{\text{NM}} = \boldsymbol{J}_3^{\text{NM}} = \boldsymbol{J}_2^{\text{spin}} = \frac{1}{2}\overleftrightarrow{Q}\cdot(\boldsymbol{J}_2^{\text{pump}}+\boldsymbol{J}_3^{\text{pump}}),$$

$$\overleftrightarrow{Q} \equiv [\overleftrightarrow{1}+q(\hat{\boldsymbol{m}}_1\cdot\hat{\boldsymbol{m}}_1^{\mathsf{T}}+\hat{\boldsymbol{m}}_3\cdot\hat{\boldsymbol{m}}_3^{\mathsf{T}})]^{-1} \quad (15)$$

treating tensor $\overleftrightarrow{Q}$ as the 3×3 matrix inverse of the []-bracketed tensor in Eq. (15). Using Eqs. (10a) and (15) to compute $J_{i=1-4}^{\text{NM}}$, then additional use of Eqs. (4) and (6), allow computation of the $\overleftrightarrow{C}_{jk}$ defined in Eq. (7)

$$\overleftrightarrow{C}_{11} = \overleftrightarrow{C}_{33} = a\overleftrightarrow{1} + b\overleftrightarrow{Q}, \quad \overleftrightarrow{C}_{13} = \overleftrightarrow{C}_{31} = -b\overleftrightarrow{Q},$$

$$a \equiv \bar{r}^{\uparrow\downarrow}/r_1'^{\uparrow\downarrow}, \quad b \equiv \bar{r}^{\uparrow\downarrow}/2r_2^{\uparrow\downarrow}; \quad 1/\bar{r}^{\uparrow\downarrow} = \tfrac{1}{2}(1/r_1^{\uparrow\downarrow}+1/r_2^{\uparrow\downarrow}). \quad (16)$$

For explicit evaluation of $\overleftrightarrow{\alpha}_{jk}'^{\text{pump}}$, it is convenient to assume a choice of $\tilde{\mathfrak{R}}_{j=1,3}$ for which $\hat{\boldsymbol{y}}_1'=\hat{\boldsymbol{y}}_3'$, such that $\hat{\boldsymbol{m}}_{03}$ and $\hat{\boldsymbol{m}}_{01}$ lie in the x'-z' plane. To simplify the intermediate algebra to obtain $\overleftrightarrow{Q}$ from Eq. (15), one can consider "in-plane" magnetizations (Fig. 2), taking $\hat{\boldsymbol{m}}_{03}=\hat{\boldsymbol{z}}$, and $\hat{\boldsymbol{m}}_{01}$ in the x-z plane ($\hat{\boldsymbol{m}}_{01}\cdot\hat{\boldsymbol{z}}=\cos\theta$). This allows a particularly easy determination of $\tilde{\mathfrak{R}}_j$ for which $\hat{\boldsymbol{y}}_1'=\hat{\boldsymbol{y}}_3'=\hat{\boldsymbol{y}}$

$$\tilde{\mathfrak{R}}_{j=1,3}^{\mathsf{T}} = \begin{pmatrix}\cos\theta_j & 0 & -\sin\theta_j\\ 0 & 1 & 0\end{pmatrix}; \quad \theta_1=\theta, \quad \theta_3=0. \quad (17)$$

Using Eqs. (16) and (17) with Eq. (9) allows explicit solution for the $\overleftrightarrow{\alpha}_{jk}'^{\text{pump}}$

$$\overleftrightarrow{\alpha}_{jk}'^{\text{pump}} = \frac{\hbar\gamma}{(4\pi M_s t)_j}\frac{h/2e^2}{\bar{r}_j^{\uparrow\downarrow}}\begin{pmatrix}a\delta_{jk}+b(2\delta_{jk}-1) & 0\\ 0 & a\delta_{jk}+bd_{jk}\end{pmatrix},$$

$$d_{11}=d_{33}=\frac{1+q+q\cos^2\theta}{1+2q+q^2\sin^2\theta},$$

$$d_{13}=d_{31}=\frac{-(1+2q)\cos\theta}{1+2q+q^2\sin^2\theta}. \quad (18)$$

Taking $\cos\theta=\hat{\boldsymbol{m}}_{01}\cdot\hat{\boldsymbol{m}}_{03}$, Eq. (18) holds for arbitrary orientation of $\hat{\boldsymbol{m}}_{01}$ and $\hat{\boldsymbol{m}}_{03}$, provided the flexibility in choosing the $\tilde{\mathfrak{R}}_{j=1,3}$ is used to maintain $\hat{\boldsymbol{y}}_1'=\hat{\boldsymbol{y}}_3'$. However, for multilayer film stacks with three or more magnetic layers with magnetizations $\hat{\boldsymbol{m}}_{0j}$ that do not all lie in a single plane, it will generally be the case that some of the off-diagonal elements of the $\overleftrightarrow{\alpha}_{jk}'^{\text{pump}}$ will be nonzero.

IV. DISCUSSION

It is perhaps instructive to compare and contrast the results of Eqs. (9) and (18) with the prior results in Ref. 3. The latter are for a trilayer stack, corresponding most directly to taking $\rho_{\text{NM}}\to\infty$ in the present model, whereby $\boldsymbol{J}_{i=1,4}^{\text{pump}}=\boldsymbol{J}_{i=1,4}^{\text{NM}}=0$. It is also effectively equivalent to the five-layer case with insulating outer boundaries in the limit $(t/l)_{\text{NM}}\to0$, whereby $\boldsymbol{J}_{i=1,4}^{\text{pump}}\neq0$ but $\boldsymbol{J}_{i=1,4}^{\text{NM}}\to0$ due to perfect cancellation by the spin current reflected from the $y_{i=0,5}$ boundaries without intervening spin-flip scattering. Either way, it corresponds to r_1', $r_1'^{\uparrow\downarrow}\to\infty$ in Eq. (10) and $a\to0$ in Eqs. (16) and (18).

However, a more interesting difference is that Ref. 3 treats $\hat{\boldsymbol{m}}_3$ as stationary (hence $\boldsymbol{J}_3^{\text{pump}}=0$) and $\hat{\boldsymbol{m}}_1$ as undergoing a perfectly *circular* precession about $\hat{\boldsymbol{m}}_3$ with a possibly large cone angle θ. By contrast, the present analysis treats $\hat{\boldsymbol{m}}_1$ and $\hat{\boldsymbol{m}}_3$ equally as quasistationary vectors which undergo small but otherwise random fluctuations about their equilibrium positions $\hat{\boldsymbol{m}}_{01}$ and $\hat{\boldsymbol{m}}_{03}$ with $\hat{\boldsymbol{m}}_{03}\cdot\hat{\boldsymbol{m}}_{03}=\cos\theta$. To further elucidate this distinction, one can assume the aforementioned physical model of Ref. 3 and reanalyze that situation in terms of the present formalism. With $d\boldsymbol{m}_3/dt=0=\boldsymbol{J}_3^{\text{pump}}$ and by explicitly inserting the condition [e.g., from Eq. (3)] that $\boldsymbol{J}_2^{\text{pump}}\cdot\hat{\boldsymbol{m}}_1\equiv0$, an explicit solution of Eq. (15) can be expressed in the form

$$\boldsymbol{J}_2^{\text{NM}} = \frac{1}{2}\left[\boldsymbol{J}_2^{\text{pump}} + \frac{q^2\cos\theta\hat{\boldsymbol{m}}_1 - q(q+1)\hat{\boldsymbol{m}}_3}{(1+q)^2-q^2\cos^2\theta}\boldsymbol{J}_2^{\text{pump}}\cdot\hat{\boldsymbol{m}}_3\right]. \quad (19)$$

Combining Eq. (19) with the earlier result from Eq. (5) and then Eq. (3) (with $\varepsilon=0$), it is readily found that

$$\begin{aligned}\hat{\boldsymbol{m}}_1\times\frac{d\hat{\boldsymbol{m}}_1}{dt} &\Leftrightarrow \frac{\gamma}{(M_s t)_1}\hat{\boldsymbol{m}}_1\times\frac{1}{A}\frac{d\boldsymbol{S}_1}{dt} = -\frac{\hbar}{2e}\frac{\gamma}{(M_s t)_1}\hat{\boldsymbol{m}}_1\times J_2^{\text{NM}}\\ &= -\frac{\hbar\gamma/4e}{(M_s t)_1}\left(\hat{\boldsymbol{m}}_1\times\boldsymbol{J}_2^{\text{pump}} + \frac{q(q+1)(\hat{\boldsymbol{m}}_3\cdot\boldsymbol{J}_2^{\text{pump}})}{(1+q)^2-q^2\cos^2\theta}\hat{\boldsymbol{m}}_3\times\hat{\boldsymbol{m}}_1\right)\\ &= -\left[\frac{\hbar\gamma}{(8\pi M_s t)_1}\frac{h/2e^2}{r_2^{\uparrow\downarrow}}\left(1-\frac{q(q+1)\sin^2\theta}{(1+q)^2-q^2\cos^2\theta}\right)\right]\frac{d\hat{\boldsymbol{m}}_1}{dt}.\end{aligned} \quad (20)$$

The last result in Eq. (20) uses $\boldsymbol{J}_2^{\text{pump}}$ from Eq. (3), and the fact that $|\hat{\boldsymbol{m}}_3\times\hat{\boldsymbol{m}}_1|=\sin\theta$, and that $d\hat{\boldsymbol{m}}_1/dt$ and $\hat{\boldsymbol{m}}_3\times\hat{\boldsymbol{m}}_1$ are *parallel* vectors in the case of steady *circular* precession of $\hat{\boldsymbol{m}}_1$ about a fixed $\hat{\boldsymbol{m}}_3$. It is the direct equivalent of Eq. (9) of Ref. 3 with the identification $\nu\Leftrightarrow q/(q+1)$.

Although the final expression in Eq. (20) is azimuthally invariant with vector orientation of $\hat{\boldsymbol{m}}_1$, it is most convenient to compare it with Eq. (18) at that instant where $\hat{\boldsymbol{m}}_1$ is "in plane" as shown in Fig. 2. At that orientation, $d\hat{\boldsymbol{m}}_1/dt\to dm_{1y}/dt=dm_{1y}'/dt$, and it is immediately confirmed from Eqs. (9) and (18) (with $a\to0$) that the []-term in Eq. (20) is simply the tensor element $\alpha_{11}'^{y'y'}$ of $\overleftrightarrow{\alpha}_{11}'^{\text{pump}}$. It is now seen that the analysis of Ref. 3 happens to mask the tensor nature of the spin-pump damping by its restricting attention a specific form of the motion of the magnetization vectors, which in this case singles out the single diagonal element of the $\overleftrightarrow{\alpha}_{11}'^{\text{pump}}$ tensor along the axis perpendicular to the plane formed by vectors $\hat{\boldsymbol{m}}_1$ and $\hat{\boldsymbol{m}}_3$. The very recent results of Ref. 6 do address this deficiency of generality, and reveal the tensor nature of $\overleftrightarrow{\alpha}'^{\text{pump}}_{11}$ with specific results for $\theta=0$, $\pi/2$, and π. The present Sec. III additionally includes the nonlocal tensors $\overleftrightarrow{\alpha}'^{\text{pump}}_{13}=\overleftrightarrow{\alpha}'^{\text{pump}}_{31}$ as well as diagonal terms $a\delta_{jk}$ in Eq. (18) (and the variation in parameter q) when it is *not* the case that $r_{\text{NM-FM}} \ll (\rho l)_{NM}\text{hyp}(t_{\text{NM}}/l_{\text{NM}})$ in boundary condition (B4). The latter condition will likely apply in the case of the technological important example of CPP-GMR spin valves.

Speaking of such, two important practical issues for these devices involve thermal magnetic noise and spin-torque-induced oscillations. As described previously,[8] an explicit linearization of the $\boldsymbol{H}^{\text{eff}}$ term in Eq. (9) about equilibrium state $\hat{\boldsymbol{m}}_0$ that is a minimum of the free energy E leads to the following matrix form of the linearized Gilbert equation including spin pumping (with $J_e=0$):

$$\sum_k (\overleftrightarrow{G}'_{jk}+\overleftrightarrow{D}'_{jk})\cdot\frac{d\boldsymbol{m}'_k}{dt}+\sum_k \overleftrightarrow{H}'_{jk}\cdot\boldsymbol{m}'_k\cdot=\boldsymbol{h}'_j(t)\equiv p_j\tilde{\mathfrak{R}}^{\mathsf{T}}_j\cdot\boldsymbol{h}_j(t),$$

$$\overleftrightarrow{G}'_{jk}\equiv\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\frac{p_j}{\gamma}\delta_{jk}+\frac{p_j\overleftrightarrow{\alpha}'_{jk}-p_k\overleftrightarrow{\alpha}'_{kj}}{2\gamma},\quad p_j\equiv\frac{(M_stA)_j}{\Delta m}$$

$$\overleftrightarrow{D}'_{jk}\equiv\frac{p_j\overleftrightarrow{\alpha}'_{jk}+p_k\overleftrightarrow{\alpha}'_{kj}}{2\gamma},\quad \boldsymbol{H}^{\text{eff}}_{0j}\equiv\frac{-1}{\Delta m}\frac{\partial E(\hat{\boldsymbol{m}}_0)}{\partial\hat{\boldsymbol{m}}_j},$$

$$\overleftrightarrow{H}_{jk}\equiv(\hat{\boldsymbol{m}}_{0j}\cdot\boldsymbol{H}^{\text{eff}}_{0j})\overleftrightarrow{1}\delta_{jk}-\frac{\partial\boldsymbol{H}^{\text{eff}}_{0j}}{\partial\hat{\boldsymbol{m}}_k},\quad \overleftrightarrow{H}'_{jk}\equiv\tilde{\mathfrak{R}}^{\mathsf{T}}_j\cdot\overleftrightarrow{H}_{jk}\cdot\tilde{\mathfrak{R}}_k,\tag{21}$$

where the $\boldsymbol{h}_j(t)$ are small perturbation fields. The form of $\overleftrightarrow{D}'_{jk}$ and $\overleftrightarrow{G}'_{jk}$ in Eq. (21) is chosen so that they retain the original delineation[8] as symmetric and antisymmetic tensors regardless of the symmetry of $\overleftrightarrow{\alpha}'_{jk}$. By use of a fixed "reference moment" Δm in the definition of $\boldsymbol{H}^{\text{eff}}_j$, the "stiffness-field" tensor matrix $H'^{u'v'}_{jk}\propto\partial E/\partial m'_{ju'}\partial m'_{kv'}$ is symmetric positive definite and $\delta E=-A\Sigma_j(M_st)_j\boldsymbol{h}_j\cdot\delta\boldsymbol{m}_j=-\Delta m\Sigma_j\boldsymbol{h}'_j\cdot\boldsymbol{m}'_j$ has the proper conjugate form so that Eq. (21) are now ready to directly apply fluctuation-dissipation expressions specifically suited to such linear-matrix equations of motion.[8] Treating the fields $\boldsymbol{h}'_j(t)$ now as thermal fluctuation fields driving the $\boldsymbol{m}'_j(t)$ fluctuations

$$\langle h'_{ju'}(\tau)h'_{kv'}(0)\rangle=\frac{2k_BT}{\Delta m}D'^{u'v'}_{jk}\,\delta(\tau),$$

$$S'_{h'_{ju'},h'_{kv'}}(\omega)=\frac{2k_BT}{\Delta m}D'^{u'v'}_{jk}\tag{22}$$

are the time-correlation or cross-power spectral-density (PSD) Fourier transform pairs. Through their relationship described in Eq. (21), the nonlocal, tensor nature of the spin-pumping contribution $\overleftrightarrow{\alpha}'^{\text{pump}}_{jk}$ to $\overleftrightarrow{\alpha}'_{jk}$ is directly translated into those of the $2N_{\text{FM}}\times 2N_{\text{FM}}$ system "damping tensor matrix" $\overleftrightarrow{\boldsymbol{D}}'\leftrightarrow\overleftrightarrow{D}'^{u'v'}_{jk}$, where N_{FM} is the number of FM layers in the multilayer film stack. The cross-PSD tensor matrix $\overleftrightarrow{\boldsymbol{S}}'_{\boldsymbol{m}'\boldsymbol{m}'}(\omega)\leftrightarrow S'_{m'_{ju'},m'_{kv'}}(\omega)$ for the $\boldsymbol{m}'$ fluctuations can then be expressed as[8]

$$\overleftrightarrow{\boldsymbol{S}}'_{\boldsymbol{m}'\boldsymbol{m}'}(\omega)=\frac{k_BT}{i\omega\Delta m}[\overleftrightarrow{\boldsymbol{\chi}}'(\omega)-\overleftrightarrow{\boldsymbol{\chi}}'^{\dagger}(\omega)]$$
$$\rightarrow\overleftrightarrow{\boldsymbol{\chi}}'^{\dagger}(\omega)\cdot\overleftrightarrow{\boldsymbol{S}}'_{\boldsymbol{h}'\boldsymbol{h}'}\cdot\overleftrightarrow{\boldsymbol{\chi}}'^{\dagger}(\omega),$$

$$\overleftrightarrow{\boldsymbol{\chi}}'(\omega)\equiv[\overleftrightarrow{\boldsymbol{H}}'-i\omega(\overleftrightarrow{\boldsymbol{G}}'+\overleftrightarrow{\boldsymbol{D}}')]^{-1},\tag{23}$$

where $\overleftrightarrow{\boldsymbol{\chi}}'(\omega)$ is the complex susceptibility tensor matrix for the $\{\boldsymbol{m}',\boldsymbol{h}'\}$ system and $\overleftrightarrow{\boldsymbol{\chi}}'^{\dagger}(\omega)$ its Hermitian transpose. It has been theoretically argued[10] that Eq. (22), and thus the *second* expression in Eq. (23), remain valid when $J_e\neq 0$, despite spin-torque contributions to $\boldsymbol{H}^{\text{eff}}_j$ resulting in an *asymmetric* $\overleftrightarrow{\boldsymbol{H}}'$ [e.g., see Eq. (25)] that violates the condition of thermal equilibrium implicitly assumed for the fluctuation-dissipation relations.

Since $\overleftrightarrow{\boldsymbol{H}}'$ is in general fully nonlocal with anisotropic/tensor character, any additional tensor nature of $\overleftrightarrow{\boldsymbol{D}}$ will likely be altered or muted as to the influence on the detectable $\boldsymbol{m}'$ fluctuations. As an example, one can again consider the situation depicted in Fig. 2, applied to the case of a CPP-GMR spin valve with typical *in-plane* magnetization. The device's output noise PSD will reflect fluctuations in $\hat{\boldsymbol{m}}_1\cdot\hat{\boldsymbol{m}}_3$. Taking $\hat{\boldsymbol{m}}_3$ to again play the simplifying role of an ideal fixed (or pinned) reference layer (i.e., $d\hat{\boldsymbol{m}}_3/dt\rightarrow 0$), the PSD will be proportional to $\sin^2\theta\, S'_{m'_{1x},m'_{1x}}(\omega)$. As was also shown previously,[11] it follows from Eq. (23) (and assuming the symmetry $H'^{x'y'}_{11}=H'^{y'x'}_{11}=0$) that

$$S'_{m'_{1x},m'_{1x}}(\omega)\cong\frac{2k_BT\gamma}{(M_stA)_1}\frac{\alpha'^{x'x'}_{11}(H'^{y'y'}_{11}/H'^{x'x'}_{11})\omega_0^2+\alpha'^{y'y'}_{11}\omega^2}{(\omega^2-\omega_0^2)^2+(\omega\Delta\omega)^2},$$

$$\omega_0=\gamma\sqrt{H'^{y'y'}_{11}H'^{x'x'}_{11}}\quad \Delta\omega=\gamma(\alpha'^{x'x'}_{11}H'^{y'y'}_{11}+\alpha'^{y'y'}_{11}H'^{x'x'}_{11})\tag{24}$$

treating $\alpha'^{x'x'}_{11}\alpha'^{y'y'}_{11}\lll 1$. The tensor influence of the $\alpha'^{u'u'}_{11}$ is seen to be weighted by the relative size of the stiffness-field matrix elements $H'^{v'v'}_{11}$. For the thin-film geometries with $t\ll\sqrt{A}$ typical of such devices, out-of-plane demagnetization field contribution typically result in $H'^{y'y'}_{11}$ that are an order of magnitude larger than $H'^{x'x'}_{11}$. Since $\alpha'^{y'y'}_{11}\leq\alpha'^{x'x'}_{11}$ from Eq. (18), it follows that the linewidth $\Delta\omega$ and the PSD $S'_{m'_{1x},m'_{1x}}(\omega\leq\omega_0)$ in the spectral range of practical interest will both be expected to be determined primarily by $\alpha'^{x'x'}_{11}$.

A similar circumstance also applies to the important problem of critical currents for spin-torque magnetization excitation in CPP-GMR spin valves with $J_e\neq 0$. Consider the same example as above, again treating $\hat{\boldsymbol{m}}_3$ as stationary and seeking nontrivial solutions of Eq. (21) (with $\boldsymbol{h}'(t)=0$) of the form $\boldsymbol{m}'_1(t)\propto e^{-st}$. Summarizing results obtainable from Eqs. (5), (8), and (21)

$$\boldsymbol{H}_1^{\text{eff}} = \boldsymbol{H}_1^{\text{eff}}|_{J_e=0} + \frac{\hbar/2e}{(M_s t)_1}\boldsymbol{J}_2^{\text{NM}} \times \hat{\boldsymbol{m}}_1,$$

$$\det\begin{pmatrix} H_{11}'^{x'x'} - s'\alpha_{11}'^{x'x'} & H_{11}'^{x'y'} + s' \\ H_{11}'^{y'x'} - s' & H_{11}'^{y'y'} - s'\alpha_{11}'^{y'y'} \end{pmatrix} = 0, \quad (25)$$

where $s' = s/\gamma$, $\alpha_{11}'^{u'v'}$ as in Eq. (18) and where $\boldsymbol{J}_2^{\text{NM}} \propto J_e$ in Eq. (25) is now the solution of the transport equations with $\boldsymbol{J}^{\text{pump}}=0$ but $J_e \neq 0$. The cross product form of the spin-torque contribution to $\boldsymbol{H}_1^{\text{eff}}$ explicitly yields an asymmetric/nonreciprocal contribution to $\overleftrightarrow{\boldsymbol{H}}'$, i.e., $H_{11}'^{x'y'} - H_{11}'^{y'x'} \propto J_e$. The critical-current density is that value of J_e where Re s becomes negative. Given the basic stability criterion that $\det \overleftrightarrow{H}_{11}' > 0$, the spin-torque critical condition from Eq. (25) can be expressed as

$$\alpha_{11}'^{y'y'} H_{11}'^{x'x'} + \alpha_{11}'^{x'x'} H_{11}'^{y'y'} = H_{11}'^{x'y'} - H_{11}'^{y'x'}. \quad (26)$$

Like for thermal noise, the spin-torque critical point should again be determined primarily by $\alpha_{11}'^{x'x'}$ for in-plane magnetized CPP-GMR spin valves with typical $H_{11}'^{y'y'} \gg H_{11}'^{x'x'}$. This simply reflects the fact that the (quasiuniform) modes of thermal fluctuation or critical-point spin-torque oscillation tend to exhibit rather "elliptical," mostly in-plane motion when $H_{11}'^{y'y'} \gg H_{11}'^{x'x'}$. This is obviously different than the steady, pure circular precession described in Ref. 3, which contrastingly highlights the influence of $\alpha_{11}'^{y'y'}$, along with its interesting, additional θ dependence.

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APPENDIX A: INTERFACE BOUNDARY CONDITIONS

The well-known "circuit theory" formulation[12] of the boundary conditions for the electron-charge current density J_e and the (dimensionally equivalent) spin-current density $\boldsymbol{J}_{\text{NM}}^{\text{spin}}$ at a FM/NM interface can (taking $\Delta \boldsymbol{V}_{\text{FM}} = \Delta V_{\text{FM}}\hat{\boldsymbol{m}}$) be expressed as

$$J_e = (G^{\uparrow} + G^{\downarrow})(\bar{V}_{\text{NM}} - \bar{V}_{\text{FM}}) + \frac{1}{2}(G^{\uparrow} - G^{\downarrow})(\boldsymbol{\Delta V}_{\text{NM}} \cdot \hat{\boldsymbol{m}} - \Delta V_{\text{FM}}), \quad (\text{A1})$$

$$\boldsymbol{J}_{\text{NM}}^{\text{spin}} = \left[(G^{\uparrow} - G^{\downarrow})(\bar{V}_{\text{NM}} - \bar{V}_{\text{FM}}) + \frac{1}{2}(G^{\uparrow} + G^{\downarrow}) \times (\boldsymbol{\Delta V}_{\text{NM}} \cdot \hat{\boldsymbol{m}} - \Delta V_{\text{FM}}) \right]\hat{\boldsymbol{m}} + \text{Re } G^{\uparrow\downarrow}(\hat{\boldsymbol{m}} \times \boldsymbol{\Delta V}_{\text{NM}} \times \hat{\boldsymbol{m}}) + \text{Im } G^{\uparrow\downarrow}(\boldsymbol{\Delta V}_{\text{NM}} \times \hat{\boldsymbol{m}}) \quad (\text{A2})$$

in terms of spin-independent electric potential $\bar{V}$ and accumulation $\boldsymbol{\Delta V}$ ($=e\boldsymbol{\Delta \mu}$). Setting $J_e=0$ in Eq. (A1) and substituting into Eq. (A2), one obtains in the limit Im $G^{\uparrow\downarrow} \to 0$ the result

$$\boldsymbol{J}_{\text{NM}}^{\text{spin}}|_{J_e=0} = \frac{2G^{\uparrow}G^{\downarrow}}{G^{\uparrow} + G^{\downarrow}}(\boldsymbol{\Delta V}_{\text{NM}} \cdot \hat{\boldsymbol{m}} - \Delta V_{\text{FM}})\hat{\boldsymbol{m}} + G^{\uparrow\downarrow}(\hat{\boldsymbol{m}} \times \boldsymbol{\Delta V}_{\text{NM}} \times \hat{\boldsymbol{m}}). \quad (\text{A3})$$

Comparing with Eq. (4) of Tserkovnyak *et al.*[3] (with $\boldsymbol{\Delta V} \Leftrightarrow \boldsymbol{\mu}_s$) and remembering the present conversion of $\boldsymbol{J}_{\text{NM}}^{\text{spin}} \leftrightarrow -(A\hbar/2e)^{-1}\boldsymbol{I}_{\text{NM}}^{\text{spin}}$, one immediately makes the identification

$$g^{\uparrow\downarrow} = 2A(h/2e^2)G^{\uparrow\downarrow} \quad (\text{A4})$$

relating dimensionless $g^{\uparrow\downarrow}$ in Eq. (1) to $G^{\uparrow\downarrow}$, the conventional mixing conductance (per area).

The common approximations that $\boldsymbol{J}_{\text{FM}}^{\text{spin}} = J_{\text{FM}}^{\text{spin}}\hat{\boldsymbol{m}}$ inside all FM layers, and that *longitudinal* spin-current density is conserved at FM/NM interfaces, yields the usual interface-boundary condition

$$\boldsymbol{J}_{\text{NM}}^{\text{spin}} \cdot \hat{\boldsymbol{m}} = J_{\text{FM}}^{\text{spin}}. \quad (\text{A5})$$

Solving for $\boldsymbol{J}_{\text{NM}}^{\text{spin}} \cdot \hat{\boldsymbol{m}}$ from Eq. (A2) then leads [with Eq. (A1)] to a second-scalar boundary condition

$$\bar{V}_{\text{NM}} - \bar{V}_{\text{FM}} = \frac{G^{\uparrow} + G^{\downarrow}}{4G^{\uparrow}G^{\downarrow}}J_e - \frac{G^{\uparrow} - G^{\downarrow}}{4G^{\uparrow}G^{\downarrow}}J_{\text{FM}}^{\text{spin}}. \quad (\text{A6})$$

Equation (A6) is identical in form with the standard (collinear) Valet-Fert model[7] and immediately yields the following identifications:

$$r = \frac{G^{\uparrow} + G^{\downarrow}}{4G^{\uparrow}G^{\downarrow}}, \quad \gamma = \frac{G^{\uparrow} - G^{\downarrow}}{G^{\uparrow} + G^{\downarrow}} \quad (\text{A7})$$

for the conventional Valet-Fert interface parameters r and y.

The three vector terms on the right of Eq. (A2) are mutually orthogonal. Working in a rotated (primed) coordinate system where $\hat{z}' = \hat{\boldsymbol{m}}'$, Eqs. (A1) and (A2) can be similarly inverted to solve for the three components of the vector $(\boldsymbol{\Delta V}_{\text{NM}}' - \Delta V_{\text{FM}}\hat{\boldsymbol{m}}')$ in terms of $\boldsymbol{J}_{\text{NM}}'^{\text{spin}}$, $J_{\text{FM}}^{\text{spin}}$, and J_e. A final transformation back to the original (unprimed) coordinates yields the vector interface-boundary condition

$$\frac{1}{2}(\boldsymbol{\Delta V}_{\text{NM}} - \Delta V_{\text{FM}}\hat{\boldsymbol{m}}) = [(r - \text{Re } r^{\uparrow\downarrow})J_{\text{FM}}^{\text{spin}} - r\gamma J_e]\hat{\boldsymbol{m}} + \text{Re } r^{\uparrow\downarrow}\boldsymbol{J}_{\text{NM}}^{\text{spin}} + \text{Im } r^{\uparrow\downarrow}\hat{\boldsymbol{m}} \times \boldsymbol{J}_{\text{NM}}^{\text{spin}},$$
$$r^{\uparrow\downarrow} \equiv 1/(2G^{\uparrow\downarrow}) = (h/2e^2)/(g^{\uparrow\downarrow}/A). \quad (\text{A8})$$

Combined with Eq. (A4), the last relation in Eq. (A8) yields Eq. (2). Equation (A8) is a generalization of Valet-Fert to the noncollinear case.

As noted by Tserkovnyak *et al.*,[3] boundary conditions (A3) do not directly include spin-pumping terms but instead involve only "backflow" terms $\boldsymbol{J}_{\text{NM}}^{\text{spin}} \leftrightarrow \boldsymbol{J}_{\text{NM}}^{\text{back}}$ in the NM layer. With spin-pumping physically present, $\boldsymbol{J}_{\text{NM}}^{\text{back}}$ arises as the response to the spin accumulation $\boldsymbol{\Delta V}_{\text{NM}}$ created by $\boldsymbol{J}^{\text{pump}}$. It follows that $\boldsymbol{J}_{\text{NM}}^{\text{back}} = \boldsymbol{J}_{\text{NM}}^{\text{spin}} - \boldsymbol{J}^{\text{pump}}$, where $\boldsymbol{J}_{\text{NM}}^{\text{spin}}$ is henceforth the *total* spin current in the NM layer. Thus, including spin pumping in Valet-Fert transport equations is then a matter of replacing $\boldsymbol{J}_{\text{NM}}^{\text{spin}} \to \boldsymbol{J}_{\text{NM}}^{\text{spin}} - \boldsymbol{J}^{\text{pump}}$ in Eq. (A8). The modified form of Eq. (A8), for a FM/NM interface, becomes

$$\frac{1}{2}(\boldsymbol{\Delta V}_{\mathrm{NM}} - \Delta V_{\mathrm{FM}}\hat{\boldsymbol{m}}) = [(r - \mathrm{Re}\, r^{\uparrow\downarrow})J^{\mathrm{spin}}_{\mathrm{FM}} - r\gamma J_e]\hat{\boldsymbol{m}} + \mathrm{Re}\, r^{\uparrow\downarrow}(\boldsymbol{J}^{\mathrm{spin}}_{\mathrm{NM}} - \boldsymbol{J}^{\mathrm{pump}}) + \mathrm{Im}\, r^{\uparrow\downarrow}\hat{\boldsymbol{m}} \times (\boldsymbol{J}^{\mathrm{spin}}_{\mathrm{NM}} - \boldsymbol{J}^{\mathrm{pump}}). \tag{A9}$$

For an NM/FM interface, the sign is flipped on the left sides of Eqs. (A6) and (A9).

APPENDIX B: 1D TRANSPORT EQUATIONS

For one-dimensional transport (flow along the y axis), the quasistatic Valet-Fert[7] (drift diffusion, quasistatic) transport equations can be written as[9]

$$\frac{\partial^2 \boldsymbol{\Delta V}}{\partial y^2} = \frac{\boldsymbol{\Delta V}}{l^2}, \quad \frac{\partial}{\partial y}\left[J_e = \frac{1}{\rho}\left(\frac{\partial \bar{V}}{\partial y} + \frac{1}{2}\beta\hat{\boldsymbol{m}}\cdot\frac{\partial \boldsymbol{\Delta V}}{\partial y}\right)\right] = 0,$$

along with

$$\boldsymbol{J}^{\mathrm{spin}} = \frac{1}{\rho}\left(\beta\frac{\partial \bar{V}}{\partial y}\hat{\boldsymbol{m}} + \frac{1}{2}\frac{\partial \boldsymbol{\Delta V}}{\partial y}\right), \tag{B1}$$

where ρ=bulk resistivity,[13] l=spin diffusion length, and β =bulk/equilibrium spin-current polarization in FM layers ($\beta \equiv 0$ in NM layers). The solution for any one layer has the form

$$\bar{V} = \rho J_e y + C - \frac{1}{2}\beta\boldsymbol{\Delta V}\cdot\hat{\boldsymbol{m}}, \quad \boldsymbol{\Delta V} = \boldsymbol{A}e^{y/l} + \boldsymbol{B}e^{-y/l}. \tag{B2}$$

For FM layers, $\boldsymbol{A}=A\hat{\boldsymbol{m}}$, $\boldsymbol{B}=B\hat{\boldsymbol{m}}$. In the case where $l \ggg$ film thickness, one may employ an alternative ballistic approximation:

$$\boldsymbol{\Delta V} = \boldsymbol{A}, \quad \boldsymbol{J}^{\mathrm{spin}} = \boldsymbol{B}, \quad \bar{V} = C. \tag{B3}$$

It is not necessary to solve for $\bar{V}$ and/or the C coefficients using Eq. (A6) if only $\boldsymbol{\Delta V}$ and $\boldsymbol{J}^{\mathrm{spin}}$ are required. The remaining coefficients are determined by the interface-boundary conditions Eq. (A5), (A6), (7), and (A9), and external boundary conditions at the outer two surfaces of the film stack.

Regarding the latter, one approximation is to treat the external "leads" (with quasi-infinite cross section) as equilibrium reservoirs and set $\boldsymbol{\Delta V}(y=y_{i=0,N}) \to 0$ at the outermost ($i=0,N$) lead-stack interfaces of an N-layer stack (Fig. 1). The complimentary approximation is of an insulating boundary, with. $\boldsymbol{J}^{\mathrm{spin}}(y=y_{i=0,N}) \to 0$. For the case (such as in Sec. III) where the outer ($j=0,N-1$) layers are NM, and the adjacent inner ($j=1,N-2$) layers are FM, it is readily found using Eqs. (B1) and (B2) that

$$\boldsymbol{\Delta V}^{\mathrm{NM}}_{i=1,N-1} = \pm 2(\rho l)_{j=0,N-1}\mathrm{hypb}(t_j/l_j)\boldsymbol{J}^{\mathrm{NM}}_i, \tag{B4}$$

where hypb( )=tanh( ) or coth( ) for equipotential, or insulating boundaries, respectively. Combining Eq. (B4) with Eq. (A9), and neglecting Im $r^{\uparrow\downarrow}$, one finds for $J_e=0$ that

$$\pm\frac{1}{2}\Delta V^{\mathrm{FM}}_{i=1,N-1} = [r_i + (\rho l)_{j=0,N-1}\mathrm{hypb}(t_j/l_j)]J^{\mathrm{FM}}_i,$$

$$\boldsymbol{J}^{\mathrm{NM}}_i = J^{\mathrm{FM}}_i\hat{\boldsymbol{m}} + \frac{r^{\uparrow\downarrow}_i\boldsymbol{J}^{\mathrm{pump}}_i}{r^{\uparrow\downarrow}_i + (\rho l)_j\mathrm{hypb}(t_j/l_j)}. \tag{B5}$$

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