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Renormalized Perturbation Theory and Corrections to Scaling Laws

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Corrections to the scaling laws derived in Wilson's theory of critical phenomena which had been previously studied by Wegner at lowest order in ϵ are here given at higher order. They are obtained in the framework of renormalized perturbation theory.

In a recent article Wegner[1] has obtained, using Wilson's theory of critical phenomena,[2] some corrections to scaling laws in 4-ϵ dimensions to lowest order in ϵ. The same problem has been studied in the framework of renormalized perturbation theory,[3,4] and higher orders in ϵ have been calculated. The technical details have been given in these references, and the purpose of this note is to report a few consequences which might be of physical interest, as well as these higher-order contributions.

In Ref. 3 it was shown that if the ratio a/ξ of the lattice spacing to the correlation length is set equal to zero, the renormalized dimensionless coupling constant u is fixed to be the solution u_∞ of an "eigenvalue condition" at which scaling laws hold. If corrections in a/ξ are now considered, u is no longer equal to u_∞, but

$$u-u_\infty \sim (a/\xi)^\omega \tag{1}$$

and ω has been calculated[5] up to order ϵ^3.[4] Consequently, correction terms appear in the scaling laws.

For instance, for the spatial spin-spin correlation function, it has been shown[4] that, in the range $a \ll x \ll \xi$,

$$\langle \varphi(x)\,\varphi(0)\rangle \sim x^{-d+2-\eta}[1+C_1(a/x)^\omega]. \tag{2}$$

The power ω of a/x is universal, but not the constant which is in front.

Since the correlation length ξ diverges like $t^{-\nu}$, where $t=(T-T_c)/T_c$, one can derive from Eq. (1) corrections in powers of $t^{\omega\nu}$ to various temperature-dependent quantities. For instance, the magnetic susceptibility χ behaves like

$$\chi \sim t^{-\gamma}(1+C_2 t^{\omega\nu}), \tag{3}$$

the correlation length like

$$\xi \sim t^{-\nu}(1+C_3 t^{\omega\nu}), \tag{4}$$

and similarly the equation of state in scaling form becomes

$$\frac{H}{M^\delta}=f\left(\frac{t}{M^{1/\beta}}\right)(1+C_4 t^{\omega\nu}), \tag{5}$$

where in formulas (3)–(5) the three constants C_2, C_3, and C_4 in front of the correction terms vanish with the lattice spacing.

There will be also deviations from simple power law in the field magnetization relation at $T=T_c$ which read

$$H\sim M^\delta\{1+C_5[(a/\xi)^{1-(\epsilon/2)}M\xi^{1-(\epsilon/2)}]^{\omega\nu/\beta}\}. \tag{6}$$

All the exponents introduced here above can be deduced by scaling laws from γ, η, and ω. We shall give here their ϵ expansion (γ and η are quoted from Ref. 6) and add for convenience $\omega\nu$, which is present in many corrections. We have

$$\gamma=1+\frac{N+2}{2(N+8)}\epsilon+\frac{(N+2)(N^2+22N+52)}{4(N+8)^3}\epsilon^2+(N+2)\left[\frac{1}{(N+8)^5}\left(\frac{1}{8}N^4+\frac{11}{2}N^3+83N^2+312N+388\right)\right.$$
$$\left.-\frac{6(5N+22)}{(N+8)^4}\zeta(3)\right]\epsilon^3+O(\epsilon^4), \tag{7}$$

$$\eta=\frac{\epsilon^2(N+2)}{2(N+8)^2}\left\{1+\left(\frac{6(3N+14)}{(N+8)^2}-\frac{1}{4}\right)\epsilon+\left[\frac{1}{(N+8)^4}\left(-\frac{5}{16}N^4-\frac{115}{8}N^3+\frac{281}{4}N^2+1120N+2884\right)\right.\right.$$
$$\left.\left.-\frac{24(5N+22)}{(N+8)^3}\zeta(3)\right]\epsilon^2\right\}+O(\epsilon^5), \tag{8}$$

$$\omega=\epsilon-\frac{3(3N+14)}{(N+8)^2}\epsilon^2+\left(\frac{33}{4}N^2+\frac{461}{2}N+740+24(5N+22)\zeta(3)\right)\frac{\epsilon^3}{(N+8)^3}-18\frac{(3N+14)^2}{(N+8)^4}\epsilon^3+O(\epsilon^4), \tag{9}$$

$$\omega\nu = \frac{1}{2}\epsilon - \frac{-N^2+8N+68}{4(N+8)^2}\epsilon^2 + \frac{(N+2)(N+3)(N+20)}{8(N+8)^3}\epsilon^3 - \frac{9(3N+14)^2}{(N+8)^4}\epsilon^3 + \frac{1}{2(N+8)^3} \times \left[\frac{15}{4}N^2 + \frac{401}{2}N + 698 + 24(5N+22)\zeta(3)\right]\epsilon^3 + O(\epsilon^4), \tag{10}$$

in which N is the number of components of the order parameter and $\zeta(3) = 1.202\ldots$.

The numerical convergence when $\epsilon = 1$ for the exponents ω and $\omega\nu$ is extremely poor. In order to have an idea of a possible result, we have used Padé approximants and obtained the same following rough estimates:

$$\omega \simeq 0.8, \quad 2\omega\nu \simeq 0.9,$$

for the three values $N = 1$, 2, and 3.

As concluding remark, it may be necessary to point out that the corrections obtained here are all among those which can be studied in the framework of renormalizable field theories. In particular, the corrections which arise from the Φ^6, Φ^8, ... interaction terms (with cutoff) have not been considered.[7]

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[1] F. Wegner, Phys. Rev. B **5**, 4529 (1972).

[2] An extensive review is contained in K. G. Wilson and J. Kogut, Phys. Rep. (to be published).

[3] E. Brezin, J-C. Le Guillou, and J. Zinn-Justin, Phys. Rev. D 8, 434 (1973).

[4] E. Brezin, J-C. Le Guillou, and J. Zinn-Justin, Phys. Rev. D 8, 2418 (1973).

[5] The correspondance with Wegner's notation is $\omega = -\Delta_{2s}$, $\omega\nu = -y_{2s}$.

[6] E. Brezin, J-C. Le Guillou, J. Zinn-Justin, and B. G. Nickel, Phys. Lett. A **44**, 227 (1973).

[7] These terms would yield new corrections of order $(a/x)^{2+O(\epsilon)}$.