Critical Indices and Amplitudes of Classical Planar Models in Finite Field for Temperatures Greater than T_c

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Using renormalization techniques on the Englert linked-cluster expansion we have derived high-temperature series for the spin-spin correlation function of the three-dimensional, plane-rotator, and spin-infinity XY models with nearest-neighbor interactions. Analysis of our zero-field series, which are two terms longer than earlier series for these models, gives results in basic agreement with earlier work. Our finite-field series, to eighth order in the interaction and second order in the field, allow a determination of the gap index Δ; we assert $2\Delta = 3.33 \pm 0.02$. The indices are compared with scaling predictions and experimental results. Accessible critical amplitudes are determined. These are used to test predictions of two-scale-factor universality; good agreement is observed.

I. INTRODUCTION

The determination of the critical behavior of model systems via series expansions has proven useful in predicting experimental results and in testing scaling theory and the universality hypothesis.[1-3] In this paper, we present series results for several nearest-neighbor planar models, specifically the spin-infinity XY and the plane-rotator models for the fcc, bcc, and sc lattices. These models consist of spins $\vec{S}_{\vec{r}}$ at all sites $\vec{r}$ of the lattice interacting with their nearest neighbors and an external field H through a Hamiltonian of the form

$$-\beta\mathcal{H} = K\sum_{\langle\vec{r},\vec{r}\,'\rangle}(S^x_{\vec{r}}S^x_{\vec{r}\,'}+S^y_{\vec{r}}S^y_{\vec{r}\,'})/s^2 + h\sum_{\vec{r}} S^x_{\vec{r}}/s\,, \tag{1.1}$$

where $K \equiv \beta J$, J being the interaction strength, $h \equiv \beta g\mu_B H$, s is the spin quantum number, and the second sum is over all lattice sites and the first is over all nearest-neighbor pairs. For the spin-infinity XY model $\vec{S}_{\vec{r}}/s$ is a three-dimensional classical rotor of unit length, while, for the plane-rotator model, $\vec{S}_{\vec{r}}/s$ is again a classical rotor of unit length but now it is constrained to two dimensions, having no z component.

Via computer codes, which have also generated high-temperature series for the Ising and Heisenberg models,[4,5] which agree with presently extant series, we have generated high-temperature series for the correlation function

$$\Gamma(\vec{r}, \epsilon, h) = \frac{\langle S^x_0 S^x_{\vec{r}}\rangle}{s^2} - \frac{\langle S^x\rangle^2}{s^2} = \sum_{i,j=0}^{\infty} Q(\vec{r}, i, j)K^i h^{2j}\,. \tag{1.2}$$

All coefficients $Q(\vec{r}, i, 0)$ have been determined for $i \leq 10$ for the fcc lattice and for $i \leq 11$ for the bcc and sc lattices; all coefficients $Q(\vec{r}, i, 1)$ have been determined for $i \leq 8$.[6] From these series, Eq. (1.2), we are able to construct the series for the various correlation-function moments

$$\mu_{n,2m} = \sum_{i=0}^{\infty}\left[\sum_{\vec{r}} |\vec{r}|^n Q(\vec{r}, i, m)\right]K^i\,, \tag{1.3}$$

where distances $|\vec{r}|$ are measured in units of the nearest-neighbor distance. These moments are expected to have a leading singularity of the form[2]

$$\mu_{n,2m} \simeq U_{n,2m}\epsilon^{-\gamma-n\nu-2m\Delta}, \quad \epsilon \equiv 1 - K/K_c\,. \tag{1.4}$$

The analysis of these series and of the energy-density series determined from the nearest-neighbor correlation function

$$E = -J\sum_{\vec{\delta}}\Gamma(\vec{\delta}, \epsilon, h) \simeq -JU_c + \left(\frac{JA}{K_c k_B(1-\alpha)}\right)\epsilon^{-\alpha}\,, \tag{1.5}$$

where A is the specific-heat amplitude, allows the determination of the critical indices γ, ν, η, 2Δ, and α and of the critical amplitudes U_c, A, and the $U_{n,m}$.[1,2] We present the series for the energy density and several correlation-function moments in Table I. We discuss analysis determining values of T_c, γ, ν, and η from our zero-field series which are two terms longer than previous series.[7,8] These values presented in Table II are in good agreement with results from shorter series. We present and discuss analysis for determining values of α and 2Δ. This is the first reporting of a value of 2Δ for these classical systems.

It is consistent with the universality hypothesis

TABLE I. High-temperature series.

Energy-density series XY models

fcc	bcc	sc
−0.666 666 666 6667K	−0.444 444 444 4444K	−0.333 333 333 3333K
−0.888 888 888 8889K^2	−0.717 037 037 0370K^3	−0.211 851 851 8519K^3
−1.934 814 814 8148K^3	−1.578 921 082 8364K^5	−0.228 564 709 8345K^5
−4.306 172 839 5062K^4	−4.213 025 605 2306K^7	−0.303 471 175 6856K^7
−10.120 997 732 426K^5	−12.70 683 784 7755K^9	−0.455 222 648 9979K^9
−24.999 907 505 389K^6	−41.553 506 328 525K^{11}	−0.747 817 056 1761K^{11}
−64.068 393 665 724K^7		
−169.135 013 639 65K^8		
−457.375 893 262 70K^9		
−1261.449 180 583 8K^{10}		

Plane-rotator models

fcc	bcc	sc
−1.5K	−1.0K	−0.75K
−3.0K^2	−2.875K^3	−0.656 25K^3
−8.0625K^3	−11.645 833 333 333K^5	−1.515 625K^5
−24.375K^4	−61.082 356 770 832K^7	−3.737 548 828 1249K^7
−79.0625K^5	−371.660 774 739 42K^9	−11.223 608 398 326K^9
−272.781 25K^6	−2471.742 950 0630K^{11}	−37.093 871 938 728K^{11}
−988.699 707 031 23K^7		
−3716.652 018 2290K^8		
−14 361.191 927 081K^9		
−56 709.592 106 838K^{10}		

XY models

	fcc	bcc	sc
$\mu_{0,0}$	0.333 333 333 3333	0.333 333 333 3333	0.333 333 333 3333
	1.333 333 333 3333K	0.888 888 888 8889K	0.666 666 666 6666K
	4.977 777 777 7778K^2	2.133 333 333 3333K^2	1.155 555 555 5556K^2
	18.062 222 222 222K^3	5.088 395 061 7284K^3	1.979 259 259 2593K^3
	64.430 052 910 053K^4	11.768 324 514 991K^4	3.282 962 962 9630K^4
	227.242 752 666 50K^5	27.128 432 014 781K^5	5.419 763 164 5251K^5
	795.125 171 187 81K^6	61.687 063 184 121K^6	8.819 246 941 5750K^6
	2765.786 211 1960K^7	139.984 552 140 82K^7	14.309 753 090 152K^7
	9576.891 123 8627K^8	315.270 393 008 89K^8	23.037 912 169 482K^8
	33041.371 859 582K^9	709.030 767 134 67K^9	37.017 678 860 159K^9
	113660.843 571 96K^{10}	1587.022 622 3068K^{10}	59.194 607 980 728K^{10}
		3548.437 787 0962K^{11}	94.524 494 584 240K^{11}
$\mu_{0,2}$	−0.066 666 666 6667	−0.066 666 666 6667	−0.066 666 666 6666
	−1.066 666 666 6667K	−0.711 111 111 1111K	−0.533 333 333 3333K
	−10.184 126 984 127K^2	−4.419 047 619 0476K^2	−2.425 396 825 3968K^2
	−75.757 037 037 037K^3	−21.455 238 095 238K^3	−8.533 333 333 3333K^3
	−484.655 230 032 75K^4	−89.439 599 563 282K^4	−25.668 426 303 855K^4
	−2800.269 684 72 33K^5	−336.846 206 097 25K^5	−69.587 430 587 049K^5
	−15029.546 014 957K^6	−1177.946 071 4897K^6	−174.894 80 941 539K^6
	−76276.570 633 887K^7	−3895.262 271 3870K^7	−415.182 892 011 06K^7
	−370458.303 869 20K^8	−12324.618 179 271K^8	−942.234 454 545 75K^8
$\mu_{2,0}$	1.333 333 333 3333K	0.888 888 888 8889K	0.666 666 666 6666K
	10.666 666 666 6667K^2	4.740 740 740 7407K^2	2.666 666 666 6666K^2
	61.202 962 962 963K^3	17.730 370 370 370K^3	7.312 592 592 5926K^3
	304.355 555 555 56K^4	57.647 407 407 407K^4	17.256 296 296 296K^4
	1395.062 077 7694K^5	172.314 495 114 92K^5	37.277 540 942 303K^5
	6063.704 336 1608K^6	488.602 790 142 12K^6	76.168 646 566 445K^6
	25385.751 429 221K^7	1333.374 769 8042K^7	149.516 630 747 55K^7
	103345.820 691 37K^8	3538.521 728 5087K^8	285.016 619 163 95K^8
	411692.989 813 38K^9	9185.745 625 9118K^9	530.944 109 172 44K^9
	1611852.655 816 5K^{10}	23435.452 243 359K^{10}	971.273 672 939 46K^{10}
		58937.272 744 813K^{11}	1750.365 7844 886K^{11}

TABLE I. (*Continued*)

Plane-rotator models			
	fcc	bcc	sc
$\mu_{0,0}$	0.5	0.5	0.5
	3.0K	2.0K	1.5K
	16.5K^2	7.0K^2	3.75K^2
	87.375K^3	24.25K^3	9.1875K^3
	453.0K^4	81.0K^4	21.75K^4
	2317.093 75K^5	269.229 166 66666K^5	51.171 875K^5
	11 742.679 6875K^6	881.588 541 66666K^6	118.527 343 75K^6
	59 110.856 445312K^7	2878.516 276 0416K^7	273.473 144 531 25K^7
	296 031.611 65364K^8	9323.895 182 2916K^8	626.002 441 406 24K^8
	1476 556.389 7949K^9	30 142.589 941 405K^9	1429.408 764 6482K^9
	7340 722.316 7912K^{10}	96 965.248 920 352K^{10}	3248.075 703 9377K^{10}
		311 485.772 423 55K^{11}	7367.687 320 6967K^{11}
$\mu_{0,2}$	−0.1875	−0.1875	−0.1875
	−4.5K	−3.0K	−2.25K
	−63.843 75K^2	−27.5625K^2	−15.046 875K^2
	−700.875K^3	−196.5K^3	−77.0625K^3
	−6585.832 03125K^4	−1196.679 6875K^4	−335.326 171 875K^4
	−55 703.343 75K^5	−6561.3125K^5	−1309.851 5625K^5
	−436 585.100 830 07K^6	−33 319.651 855 469K^6	−4730.152 954 1015K^6
	−3229 669.823 2421K^7	−159 703.636 718 75K^7	−16 100.564 941 406K^7
	−22 831 488.242 724K^8	−731 363.484 895 82K^8	−52 310.525 903 319K^8
		−3228 774.96 430 65K^9	
$\mu_{2,0}$	3.0K	2.0K	1.5K
	36.0K^2	16.0K^2	9.0K^2
	306.375K^3	88.25K^3	36.1875K^3
	2243.25K^4	420.0K^4	123.75K^4
	15 068.968 75K^5	1829.229 166 6667K^5	385.296 875K^5
	95 688.156 25K^6	7532.979 166 6667K^6	1130.671 875K^6
	583 946.856 445 31K^7	29 792.016 276 042K^7	3180.332 519 5312K^7
	3459 600.292 3177K^8	114 389.177 083 33K^8	8671.888 671 8750K^8
	20 031 441.019 026K^9	429 115.892 024 73K^9	23 079.210 522 461K^9
	113 878 974.706 68K^{10}	1580 539.358 8758K^{10}	60 257.659 651 689K^{10}
		5734 156.830 1925K^{11}	154 873.212 516 00K^{11}

TABLE II. Values of the indices for several planar systems.

System	Ref.	γ	ν	η	α	2Δ
spin-∞ XY	7	1.312 ± 0.006			$0 \leq \alpha \leq \frac{1}{32}$	
spin-∞ XY and plane rotator	8	1.32 ± 0.01	0.670 ± 0.007		0.0 ± 0.1	
	this work	1.318 ± 0.010	0.670 ± 0.006	0.04 ± 0.01	−0.02 ± 0.03	3.33 ± 0.02
spin-1/2 XY	10	1.35 ± 0.02				
	11					3.3 ± 0.1
Superfluid transition	13		0.674 ± 0.001			
	14				−0.01 ± 0.01	
	15				$-0.06 \leq \alpha \leq 0.0$	
Renormalization group $n = 2$	16	1.30		0.039		
Lieb model	12	1.31 ± 0.03			0.00 ± 0.05	

TABLE III. Values of K_c^{-1} and several indices for the planar models considered here.

XY model	$T_c = K_c^{-1}$	γ	ν	$\eta\nu = 2\nu - \gamma$
fcc	3.3417 ± 0.0007	1.318 ± 0.010	0.669 ± 0.005	0.025 ± 0.005
bcc	2.175 ± 0.003	1.32 ± 0.03	0.668 ± 0.010	0.027 ± 0.008
sc	1.552 ± 0.003	1.32 ± 0.04	0.668 ± 0.010	0.025 ± 0.012
Plane-rotator model				
fcc	4.820 ± 0.003	1.323 ± 0.015	0.670 ± 0.007	0.028 ± 0.005
bcc	3.121 ± 0.005	1.32 ± 0.03	0.673 ± 0.010	0.027 ± 0.010
sc	2.203 ± 0.006	1.32 ± 0.05	0.675 ± 0.015	0.03 ± 0.02

to expect that all planar systems, those with a two-dimensional order parameter, will have the same critical properties. This includes the planar models discussed here and elsewhere and the superfluid transition in liquid helium.[9] Comparing our results with those from other planar systems, we find negligible differences consistent with universality.[7,8,10–16] We also find good agreement with the predictions of scaling theory for the indices.[2]

Values for the common correlation-function amplitudes, consistent with our values for the indices and the critical temperature, are presented and discussed within the framework of two-scale-factor universality.[17–21] These investigations are especially interesting since most previous tests involved lattice dependence only.

II. CRITICAL INDICES

A. Determination of K_c^{-1}, γ, ν, and $\eta\nu$

Using straight ratio and log-derivative ratio methods, we determined sequences for T_c from the moment series $\mu_{n,0}$, $-1 \leq n \leq 2$.[22] We then used a Neville table to extrapolate these sequences.[23] We feel the index-independent tests, especially the log-derivative tests, are the most reliable in finding the value of K_c^{-1}, in that they do not contain the obvious built-in bias of the index-dependent tests.[4,5,24] Our analysis indicates the values shown in Table III. These values are identical, to well within the quoted uncertainty, with the results of index-dependent tests using previously determined values of γ and ν.

The values we find for γ, ν, and $\eta\nu \equiv 2\nu - \gamma$, shown in Table III, are consistent with earlier results[7,8], but because our series are longer, we feel our results are more reliable. Straight ratio and log-derivative ratio methods enabled us to form sequences for γ, ν, and $\eta\nu$ from moment series and various products of moment series using the above values for K_c^{-1}. Neville table extrapolations of these sequences indicate the values presented in Table III; the quoted uncertainties reflect both our uncertainty in reading the Neville table and the uncertainty in K_c^{-1}. We also formed ratio series which allowed a K_c-independent determination of ν and $\eta\nu$: The values indicated by this analysis are consistent with the values from the K_c-dependent analysis. The values of each of the indices γ, ν, and $\eta\nu$ for all these planar systems are identical to within the uncertainties suggesting the universal values presented in Table II.

B. Determination of α

Using the energy-density series in Table I, we formed specific-heat series

$$C = \frac{\partial E}{\partial T} = \sum_{n=2}^{\infty} a_n K^n \simeq A\epsilon^{-\alpha} + a \,. \qquad (2.1)$$

From these and the values of K_c^{-1}, we formed the standard ratio sequence for the index, in this case α,[22]

$$l_n^0 = nK_c a_n / a_{n-1} - n + 1 \,. \qquad (2.2)$$

Knowing only the first few terms in this sequence, we attempted the extrapolate to its limit using the Neville table, the two-dimensional array[23]

$$l_n^i = [n l_n^{i-1} - (n-1) l_{n-1}^{i-1}] / i \,. \qquad (2.3)$$

In this array, the l_n^1 are the linear extrapolants of l_n^0, the l_n^2 the quadratic extrapolants, and so forth. The Neville table for the series (2.1) is presented in Table IV. Although these extrapolations are not particularly smooth or well behaved, they are better behaved than specific-heat extrapolations for other models.[5,10] Also presented in Table IV are those values for α which feel are indicated by these extrapolations. The uncertainties quoted reflect our estimated uncertainty in reading these tables; the effect of the uncertainty in K_c is negligible due to the irregularity of these tables. Other methods of analysis give consistent results. It should be noted that these tables favor a slightly negative value for α and that the values of the index for all these planar systems are identical to within the uncertainties suggesting the universal value $\alpha = -0.02 \pm 0.03$.

TABLE IV. Neville table extrapolations of specific-heat series for the index α.

	XY model			Plane-rotator model		
	fcc, $\alpha = -0.02 \pm 0.03$			fcc, $\alpha = -0.02 \pm 0.04$		
n/i	0	1	2	0	1	2
5	0.6641			0.5089		
6	0.5167	−0.0727		0.3647	−0.2121	
7	0.4351	0.0269	0.2262	0.2949	−0.0545	0.2607
8	0.3683	−0.0322	−0.1800	0.2638	0.0776	0.4079
9	0.3200	−0.0185	0.0227	0.2392	0.0670	0.0353
10	0.2831	−0.0116	0.0124	0.2150	0.0209	−0.1407
11	0.2534	−0.0142	−0.0248	0.1925	−0.0092	−0.1294
	bcc, $\alpha = -0.02 \pm 0.04$			bcc, $\alpha = -0.04 \pm 0.07$		
n/i	0	1	2	0	1	2
6	0.5511			0.3862		
8	0.3683	−0.1803		0.2616	−0.1124	
10	0.2779	−0.0836	0.0615	0.2125	0.0165	0.2097
12	0.2232	−0.0505	0.0157	0.1724	−0.0281	−0.1172
	sc, $\alpha = 0.0 \pm 0.3$			sc, $\alpha = 0.0 \pm 0.2$		
n/i	0	1	2	0	1	2
6	0.4924			0.5870		
8	0.3141	−0.2207		0.1351	−1.2206	
10	0.2014	−0.2495	−0.2927	0.1836	0.3777	2.7752
12	0.1660	−0.0109	0.4663	0.1635	0.0630	−0.5665

TABLE V. Neville table extrapolations of μ_{02}/μ_{00} for the index 2Δ.

	XY model				Plane-rotator model			
	fcc, $2\Delta = 3.33 \pm 0.01$				fcc, $2\Delta = 3.34 \pm 0.02$			
n/i	0	1	2	3	0	1	2	3
3	3.433				3.529			
4	3.408	3.305			3.488	3.324		
5	3.391	3.308	3.314		3.461	3.326	3.329	
6	3.380	3.312	3.320	3.328	3.442	3.330	3.340	3.353
7	3.372	3.315	3.324	3.331	3.429	3.334	3.347	3.360
8	3.366	3.317	3.325	3.327	3.419	3.337	3.346	3.344
	bcc, $2\Delta = 3.33 \pm 0.03$				bcc, $2\Delta = 3.34 \pm 0.04$			
n/i	0	1	2		0	1	2	
1	3.678				3.845			
3	3.457	3.237			3.558	3.271		
5	3.403	3.295	3.324		3.473	3.302	3.317	
7	3.380	3.310	3.325		3.435	3.323	3.344	
9					3.415	3.332	3.347	
2	3.431				3.539			
4	3.396	3.343			3.473	3.373		
6	3.376	3.327	3.314		3.436	3.343	3.320	
8	3.365	3.327	3.328		3.415	3.344	3.347	
	sc, $2\Delta = 3.33 \pm 0.03$				sc, $2\Delta = 3.33 \pm 0.04$			
n/i	0	1	2		0	1	2	
1	3.865				4.086			
3	3.492	3.119			3.593	3.101		
5	3.431	3.310	3.406		3.504	3.326	3.438	
7	3.403	3.318	3.326		3.460	3.326	3.325	
2	3.491				3.616			
4	3.424	3.325			3.504	3.335		
6	3.401	3.342	3.356		3.463	3.363	3.383	
8	3.386	3.335	3.326		3.438	3.350	3.335	

C. Determination of 2Δ

Using the series

$$\frac{\mu_{n,2}}{\mu_{n,0}} = \sum_{j=0}^{\infty} C_{j,n} K^j \simeq \frac{U_{n,2}}{U_{n,0}} \epsilon^{-2\Delta}, \qquad (2.4)$$

we employed standard ratio methods to determine the index 2Δ. Presented in Table V are the Neville table extrapolations, Eq. (2.3), of sequence (2.2) using the series $\mu_{0,2}/\mu_{0,0}$ as well as the values for 2Δ which we feel are indicated by the analysis. The uncertainties quoted here reflect not only our uncertainty in reading these extrapolations but also the effect of the uncertainty in K_c on these extrapolations. Other analyses, the results of which are consistent with those presented here, include log-derivative tests and K_c-independent ratio-series tests. Consistently with our analysis, we suggest the universal value $2\Delta = 3.33 \pm 0.02$ for all these planar systems.

D. Scaling, Universality, and the Superfluid Transition

How do the values we have determined for the indices agree with scaling predictions, and what predictions does scaling enable us to make about undetermined indices? According to scaling the following quantities should be zero[2]; using our values of the indices, as presented in Table II, we find

$$2-\alpha-3\nu = 0.01 \pm 0.05, \qquad 3\nu+\gamma-2\Delta = 0.01 \pm 0.05, \qquad (2.5)$$

so that scaling predictions are obeyed to within sizable uncertainties, which are large enough to mask violations greater than the apparent violations observed for the Ising model.[4,24,25] Because the indices obey scaling to well within the uncertainties in the indices, we can use scaling relations to predict values for undetermined indices such as β and δ:

$$\beta = \Delta - \gamma = 0.347 \pm 0.020, \qquad \delta = \frac{\Delta}{\Delta-\gamma} = 4.8 \pm 0.3\ ; \qquad (2.6)$$

$$\beta = \tfrac{1}{2}(3\nu-\gamma) = 0.346 \pm 0.014, \quad \delta = \frac{6\nu}{3\nu-\gamma} = 4.8 \pm 0.3;$$

having confidence that the true values will agree with these to within the uncertainties.

It is the contention of the universality hypothesis that all systems with the same symmetry, in this case all planar systems, should have the same set of indices.[3] The values presented in Tables II–V support this contention to within the uncertainties with which the indices have been determined.

The attractive fractional values $\gamma = \frac{4}{3}$, $\nu = \frac{2}{3}$, $\alpha = 0$, and $\eta = 0$ have been suggested as the universal values for these planar systems.[8,17] These fractional values are not excluded by the data, but the series analysis prefers the somewhat different values shown in Table II. Even recent experimental results for the superfluid transition in liquid helium opt for a higher value of ν and a slightly negative value of α.[13–15]

III. UNIVERSALITY OF THE CRITICAL AMPLITUDES

In investigations of critical phenomena, most attention has been paid to the indices. In part, this is due to the relatively clear universality of the indices. In the past few years, it has become increasingly evident that the amplitudes should also be universal, in a somewhat restricted sense.[17,26] The most explicit formulation of the universality of the critical amplitudes is two-scale-factor universality.[18,21] It asserts that, near the critical point of any system, the free energy per site $F(\epsilon, h)$ and the correlation function can be written in scaling form[2] as follows:

$$F(\epsilon, h) = (g\epsilon)^{d\nu} f(nh/(g\epsilon)^{\Delta})/n,$$
$$\Gamma(\vec{r}, \epsilon, h) = \frac{D((g\epsilon)^{\nu} r/l,\ nh/(g\epsilon)^{\Delta})}{(r/l)^{d-2+\eta}}, \qquad (3.1)$$

where $\rho^s l^d/n$ is universal.[27] In these expressions the indices as well as the functions f and D are universal, i.e., the same for all systems with the same symmetry, while n, l, and g are system-dependent (nonuniversal) scale factors, and ρ^s is the particle density (in the case of lattice systems the site density). Equations (3.1) make an explicit separation between the universal and nonuniversal parts of these functions; they also allow us to make explicit predictions regarding the critical amplitudes. Specifically, within a given symmetry class—in our case the set of all planar systems—they allow us to write the amplitudes for any member of the class in terms of the amplitudes of some reference system in the class, the scale factors for the member in question, and ρ^s and ρ^s_r, the particle density for the member and the reference system, respectively. Several examples follow[18]: for the susceptibility

$$\mu_{0,0} \simeq U_{0,0}\epsilon^{-\gamma}, \qquad \mu_{0,0} = ng^{-\gamma}(U_{0,0})_r\ ; \qquad (3.2a)$$

for the correlation length

$$\xi = \kappa^{-1} \simeq 1/\kappa_0 \epsilon^{\nu}, \qquad \kappa_0 = (\kappa_0)_r g^{\nu}/l\ ; \qquad (3.2b)$$

for the specific heat

$$C \simeq A\epsilon^{-\alpha} + a, \qquad A = (g)^{d\nu}(A)_r/n\ ; \qquad (3.2c)$$

for the $d=3$, $h=0$ correlation function in the Orn-

TABLE VI. Accessible critical amplitudes as defined in Eqs. (3.2).

	XY model			Plane-rotator model		
	fcc	bcc	sc	fcc	bcc	sc
$\kappa_0 a$	2.5390 ± 0.0013	2.470 ± 0.002	2.276 ± 0.003	2.302 ± 0.004	2.23 ± 0.02	2.057 ± 0.007
$U_{0,0}$	0.2681 ± 0.003	0.2761 ± 0.0005	0.304 ± 0.001	0.4577 ± 0.0010	0.466 ± 0.002	0.5135 ± 0.0025
A	-27.6 ± 0.1	-27.8 ± 0.2	-27.4 ± 0.3	-20.76 ± 0.12	-20.72 ± 0.12	-20.1 ± 1.2
$\mathfrak{A}$	0.0937 ± 0.0002	0.0993 ± 0.0003	0.1211 ± 0.0006	0.1320 ± 0.0008	0.1373 ± 0.0030	0.168 ± 0.002
D_c	0.0854 ± 0.0001	0.0906 ± 0.0002	0.1098 ± 0.0006	0.1209 ± 0.0001	0.1273 ± 0.0003	0.153 ± 0.0001

TABLE VII. Tests of two-scale-factor universality.

	XY model			Plane rotator model		
	fcc	bcc	sc	fcc	bcc	sc
$100\alpha\rho^s A/(\kappa_0 a)^3$	4.77 ± 0.04	4.79 ± 0.05	4.65 ± 0.07	4.81 ± 0.07	4.85 ± 0.15	4.6 ± 0.3
$g = \left[\left(\frac{\kappa_0 a}{(\kappa_0 a)_r}\right)^3 \Big/ \left(\frac{U_{00}\rho_r^s}{\rho^s (U_{00})_r}\right)\right]^{1/(3\nu-\gamma)}$	1	1.047 ± 0.010	1.23 ± 0.02	1.42 ± 0.02	1.43 ± 0.07	1.69 ± 0.05
$g = \left(\frac{A}{A_r}\frac{U_{00}}{(U_{00})_r}\right)^{1/(2-\alpha-\gamma)}$	1	1.053 ± 0.020	1.18 ± 0.03	1.43 ± 0.04	1.46 ± 0.03	1.61 ± 0.16
$g = \left[\left(\frac{D_c}{(D_c)_r}\right)^{1/(1+\eta)} \frac{\kappa_0 a}{(\kappa_0 a)_r}\right]^{1/\nu}$	1	1.044 ± 0.007	1.218 ± 0.013	1.423 ± 0.008	1.46 ± 0.03	1.68 ± 0.03
$l = \left(\frac{D_c}{(D_c)_r}\right)^{1/(1+\eta)}$	1	1.058 ± 0.003	1.273 ± 0.007	1.397 ± 0.002	1.468 ± 0.006	1.75 ± 0.01
$l = (\mathfrak{A}/\mathfrak{A}_r)^{1/(1+\eta)}$	1	1.057 ± 0.004	1.28 ± 0.01	1.39 ± 0.01	1.45 ± 0.04	1.75 ± 0.02
$l = \left(\frac{\rho_r^s}{\rho^s}\right)^{(1/3)} \left[\left(\frac{U_{00}}{(U_{00})_r}\right)^{2-\alpha} \left(\frac{A}{A_r}\right)^{\gamma}\right]^{1/3(2-\alpha-\gamma)}$	1	1.063 ± 0.010	1.260 ± 0.018	1.398 ± 0.017	1.461 ± 0.016	1.72 ± 0.07

stein-Zernike limit $\Gamma(r,\ \epsilon,\ 0) = \mathfrak{A}\kappa^{\eta} e^{-\kappa r}/r$,

$$\mathfrak{A} = l^{1+\eta}(\mathfrak{A})_r\,; \tag{3.2d}$$

for the $d=3$, $\epsilon=0$, $h=0$ correlation function $\Gamma(r,\ 0,\ 0) = D_c/r^{d-2+\eta}$,

$$D_c = l^{1+\eta}(D_c)_r\,. \tag{3.2e}$$

Our values for the amplitude D_c were determined using the "lattice Green's-function" result of Ref. 28, $D_c = \Gamma(\vec{\delta},\ 0,\ 0)(1-\epsilon')$, where $\vec{\delta}$ is the nearest-neighbor displacement and ϵ' is 0.020, 0.065, and 0.075 for the fcc, bcc, and sc lattices, respectively. Values for $\Gamma(\vec{\delta},\ 0,\ 0)$ and the other amplitudes for our planar systems were determined using standard ratio analysis for the critical amplitudes as discussed at length in Ref. 18. The results of this analysis is shown in Table VI. The uncertainties quoted here reflect only our uncertainty in reading the appropriate Neville table extrapolations and do not reflect our uncertainty in the indices and in K_c. We quote these uncertainties because we believe it is most important to know the amplitudes associated with a given set of indices and a given K_c. However, it is also important to have some idea how the amplitudes vary with values of K_c^{-1} and the index within the range of our uncertainty. We find, for these series, that an increase of 0.01 in the index effects a decrease of roughly 3% in the amplitude and that an increase of 0.1% in K_c^{-1} effects a decrease of roughly 2% in the amplitude. These estimates are only approximate because of the unsatisfactory behavior of the Neville table extrapolations for these extreme values of the parameters.

Using the amplitudes shown in Table VI, we test predictions of two-scale-factor universality, Eqs. (3.1) and (3.2). First, we evaluate $100\alpha\rho^s A/\kappa_0^3$, which is postulated to be universal.[18–20] The values, presented in Table VII, are universal to within our uncertainty in the amplitudes. Second, we have used Eqs. (3.2) to make three independent determinations of the scale factors l and g. As shown in Table VII, these independent determinations give the same value for each scale factor to within the inherent uncertainty. This is the first determination of the scale factor relating systems which differ in more than their lattice structure.

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