Galvanomagnetic Properties of the Simplest Transport Model*

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The magnetoresistance in the simple one-mobility model is exactly zero, although the curved nature of the carrier trajectories and the statistical nature of scattering both appear to be dispersive mechanisms which should produce a positive magnetoresistance. The first effect is considered separately by analyzing an artificial version of the model in which the particle is always scattered after the same time interval has elapsed. It leads to a negative magnetoresistance which is about as large as the well-known positive effects which have been calculated for isotropic semiconductors with energy-dependent scattering. Thus the zero result actually reflects a cancellation of two opposing effects. The possibility that this different point of view regarding the fundamental nature of magnetoresistance might contribute to a better understanding of the negative magnetoresistance found experimentally in several classes of materials is mentioned, but is not explored further in this paper.

I. INTRODUCTION

This paper answers a paradoxical question concerning the transverse magnetoresistance in the simplest transport model (hereafter called STM). The model referred to is the familiar one in which all carriers have the same mass m and relaxation time τ, and hence the same mobility $\mu = e\tau/m$.

As a consequence of the unique dispersionless property of the STM, the Hall coefficient R and the transverse magnetoresistance $\Delta\rho/\rho_0$ (ρ_0 is the zero-field resistance) have the special values[1]

$$R = 1/ne \tag{1}$$

and

$$\Delta\rho/\rho_0 = 0 , \tag{2}$$

where n and e are the carrier density and charge. In the realm of classical physics, these results are exact at all magnetic field strengths.

It is very common to assert that the zero magnetoresistance in the STM comes about because the Hall field can balance the Lorentz force for all of the carriers; then the carriers again move straight down the sample, *just as they did before the magnetic field was turned on.*[2]

This is clearly not true on a microscopic level. The trajectory of a classical charged particle in crossed electric and magnetic fields is cycloidal in nature, regardless of the sources of the electric field. Furthermore, because of the statistical nature of scattering, the lengths of the cycloidal segments will be randomly distributed.

The nature of the trajectory which results is sketched in Fig. 1(a). It is evident that each curved segment is longer than the corresponding straight (dashed) line between its end points, and that the total length of the zigzag succession of straight lines is greater than the length of a single straight line from the starting point r_i to some final point r_f reached after many collisions.

It therefore appears that every carrier in the STM executes a dispersive type of motion because of the combined effects of the curved trajectory and the random nature of scattering. Why does this not lead to a positive magnetoresistance?

This question will be answered by considering a version of the STM in which the scattering is completely nonrandom; i.e., the interval between scattering events is always the same time T. This assumption corresponds to the kind of motion sketched

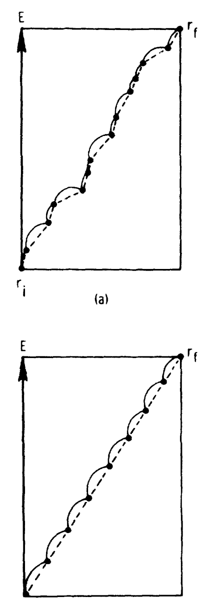

FIG. 1. Classical trajectory of a charged particle in an electric field E and a magnetic field H perpendicular to the plane of the figure. (a) random scattering, (b) nonrandom scattering.

in Fig. 1(b), and permits the effect of the curved nature of the trajectory to be considered separately from the effect of random scattering.

II. STATISTICAL AND NONSTATISTICAL TRANSPORT CALCULATIONS IN STM

The most common formal method of calculating transport properties which treats scattering as a statistical occurrence is to use the Boltzmann equation with the relaxation-time approximation.[3] This approach, or any equivalent one,[1,4] leads to Eqs. (1) and (2).

The corresponding calculation of the STM properties assuming *non*random scattering may be carried out with the aid of Fig. 2. Sketched there is the particular kind of cycloidal motion that occurs when the charged particle starts from rest under the influence of the indicated electric field E and a magnetic field H perpendicular to the plane of the figure. This is not a special assumption, however, but merely corresponds to the centroid of the thermal motion of each "shell" of carriers having a given energy.

If a charge starts from (0, 0) at $t=0$, at the time T (when it is scattered) it will have reached the point given by[5]

$$x(T)=(u/\omega)(1-\cos\omega T) \tag{3}$$

and

$$y(T)=(u/\omega)(\omega T-\sin\omega T) , \tag{4}$$

where $u=CE/H$, $\omega=eH/mC$, and C is the "compatibility factor."

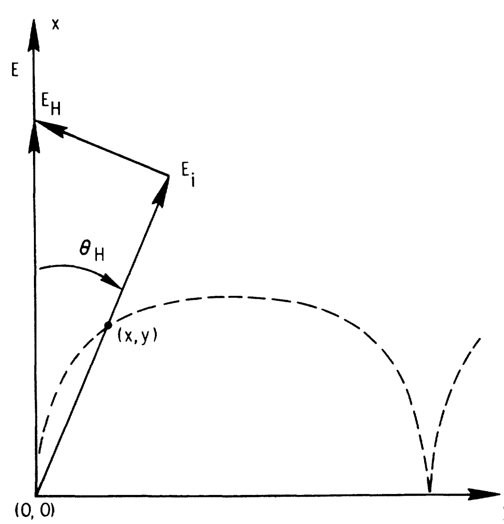

FIG. 2. Cycloidal trajectory of a charged particle, subject to an electric field $E=E_x$ and $H=H_z$, starting from rest at the point (0, 0) and scattered at (x,y). Θ_H is the Hall angle, E_H and E_i are the Hall field and the component responsible for the current.

Since this motion repeats itself endlessly, and since all carriers have the same properties, the steady-state current density is simply

$$i_H=nev=ner(T)/T , \tag{5}$$

where $r=(x^2+y^2)^{1/2}$. The Hall angle Θ_H, identified in Fig. 2, is given by $\cos\Theta_H=x(T)/r(T)$, so that the Hall field E_H and the electric field E_i that produces the current are the components of E perpendicular and parallel to the Θ_H direction.

Note that this "forces-in-fluxes-out" approach corresponds to that used in the Boltzmann equation. But since the model is isotropic, the *relative* directions and magnitudes of i, E, and H are a property of the medium, not of the boundary conditions. Hence there is no need to invert and rotate the solution. The Θ_H direction, whatever it may be, is simply assumed to coincide with the sample orientation.

The resistivity in the magnetic field is E_i/i_H, and this must be compared with the corresponding ratio in the same direction when E_i is the only force present. The current density in that case is

$$i_0=ne(eE\cos\Theta_H\, T^2/2m)/T . \tag{6}$$

Consequently,

$$\frac{\rho_H}{\rho_0}=\frac{i_0}{i_H}=\frac{(z^2/2)(1-\cos z)}{z^2-2\cos z-2z\sin z+2} , \tag{7}$$

where $z=\omega T$. In weak magnetic fields (i.e., to the lowest order in z),

$$\rho_H/\rho_0=1-\tfrac{1}{36}z^2 , \tag{8}$$

so that

$$\Delta\rho/\rho_0=-\tfrac{1}{36}z^2 . \tag{9}$$

Hence the weak-field magnetoresistance in the STM with nonrandom scattering is *negative*.

The definitions σ_0 (zero-field conductivity)$=i_0/E_i$, $\mu=\sigma_0/ne$, $R=E_H/i_0H$, and μ_H (the Hall mobility) $=R\sigma_0$, lead in a straightforward manner to the results summarized in Table I, including an alternative form for $\Delta\rho/\rho_0$, in which the factor z^2 is replaced by $(\mu_H H/C)^2$. The corresponding results for the standard, random-scattering STM are also shown for comparison. Note that the expressions for R, μ_H, and $\Delta\rho/\rho_0$ are weak-field approximations in the nonrandom model, but hold at all magnetic fields in the statistical treatment.

The nonrandom-scattering results may be transformed into the usual ones for the STM simply by computing average values for $x(T)$ and $y(T)$ for a large number of random values of T. Suppose there are N cycloidal arcs having durations T which are distributed in accordance with a constant probability of scattering, $1/\tau$. If these arcs are all transposed in time, so that they begin at $t=0$, then the number dN which ends between t and $t+dt$ is

TABLE I. Transport properties of the STM with random and nonrandom scattering.

	Random	Nonrandom
Conductivity σ_0	$ne^2\tau/m$	$\frac{1}{2}(ne^2T/m)$
Mobility μ	$e\tau/m$	$\frac{1}{2}(eT/m)$
Hall coefficient R	$1/ne$	$\frac{2}{3}(1/ne)$
Hall mobility μ_H	$e\tau/m$	$\frac{1}{3}(eT/m)$
Magnetoresistance $\Delta\rho/\rho_0$	0	$-\frac{1}{4}(\mu_H H/C)^2$

$$dN=(N/\tau)e^{-t/\tau}dt\ . \tag{10}$$

The total displacements in the x and y directions from the N cycloidal arcs are $\int x dN$ and $\int y dN$. To obtain the corresponding components of average drift velocity, the displacements must be divided by the total time required to execute the N orbits, viz., $\int t dN = N\tau$.

Using Eqs. (3) and (4) for x and y then leads to[5]

$$\overline{v_x}=(e\tau/m)[1/(1+\omega^2\tau^2)]E \tag{11}$$

and

$$\overline{v_y}=(e\tau/m)[\omega\tau/(1+\omega^2\tau^2)]E\ , \tag{12}$$

so that

$$\overline{v}=(e\tau/m)\ E\ , \tag{13}$$

i.e., the magnetoresistance vanishes.

III. DISCUSSION

First, it must be emphasized that this paper examines two versions of the STM, a model in which all carriers have the same mobility. This is to be distinguished from all other models in which carriers of different mobilities contribute. In the latter cases, different Hall angles are involved, and the carriers will disperse or fan out about the average Hall angle or overall direction of the total current.

For example, in the case of isotropic semiconductor models with energy-dependent scattering, it may be shown that[6,7]

$$R>1/ne \tag{14}$$

and

$$\Delta\rho/\rho_0>0\ . \tag{15}$$

The question raised in Sec. I was concerned specifically with magnetoresistance, and the result of main interest here is the fact that it turns out to be negative when nonrandom scattering is assumed. This surprising result may be seen from a better perspective if preceded by a brief discussion of the other results listed in Table I.

The presence of the numerical coefficient $\frac{1}{2}$ in the expressions for σ_0 and μ is of course to be anticipated from the most elementary considerations: If a particle, starting from rest, undergoes an acceleration a for a time T, its average velocity will be $\frac{1}{2}aT$. The fact that $v=a\tau$, when τ is properly defined as the reciprocal of a scattering probability, is discussed at some length by Shockley.[8]

The rather curious value $\frac{2}{3}(1/ne)$ for the weak-field Hall coefficient obtained in the nonrandom-scattering STM may be found in textbook derivations dating from 1914 through 1969.[9-11] Surprisingly, there is little or no discussion of this result. In particular, it is not pointed out that the coefficients $\frac{2}{3}$ and 1 correspond to nonrandom- and random-scattering assumptions within the framework of the STM. In fact, the only comment made in some of those references is that a more rigorous calculation will lead to a somewhat different result. This remark can be misleading if the reader assumes that a more realistic band model is required, i.e., one in which there is some mobility variation among the contributing carriers.

A number of nonstatistical calculations of magnetoresistance in the STM have been published.[9,12-15] They produced a variety of results, some positive and some negative, and there was considerable discussion of which sign was to be preferred. Some of those publications[9,12] contain a particularly serious error: The magnetoresistance was determined from the ratio of the *total* electric field to the *component* of the current in the direction of E. One possible result of such an erroneous assumption is sketched in Fig. 3.

Shown there is the current density i_0 resulting from the field E alone. If there is no magnetoresistance, application of a transverse magnetic field causes the current to rotate through the angle Θ_H and to be reduced to $i_H=i_0\cos\Theta_H$. A longer current vector i'_H in the same direction corresponds to a negative magnetoresistance. But its component in the E direction, $i'_H\cos\Theta_H$, can still be less than i_0, and in the case cited[12] was incorrectly interpreted as indicating a positive effect.

Since the original version of this paper was prepared, a 1924 paper by Page[15] was discovered in which Eq. (9) was actually obtained for the weak-field magnetoresistance. However, Page wrote this paper to correct an earlier calculation of the Hall effect by Eldridge.[16] The negative magnetoresistance which Page found led him to conclude that his own model was also inadequate, since this result disagreed with the positive effects found experimentally.

The first comment to be made about the negative

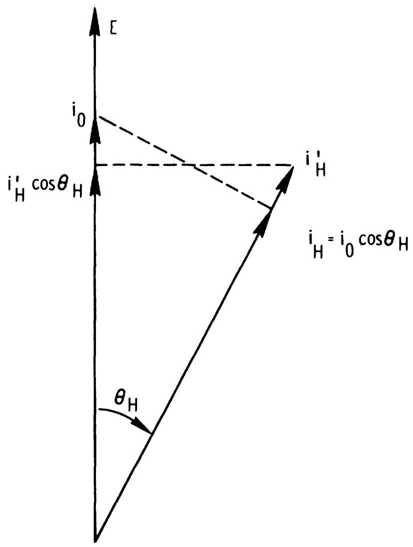

FIG. 3. Vector diagram illustrating zero magnetoresistance ($i_H = i_0 \cos\Theta_H$), a negative magnetoresistance ($i_H' > i_H$), and an erroneous calculation of an apparently positive magnetoresistance ($i_H' \cos\Theta_H < i_0$). i_H and i_H' are the current densities in the magnetic field, i_0 the zero-field current density, and Θ_H the Hall angle.

magnetoresistance in the nonrandom STM is that it is not a small effect. The coefficient $(-\frac{1}{4})$ in the table is almost as large in magnitude as the well-known coefficient $(4/\pi) - 1$ ($\simeq 0.275$), which occurs in the isotropic semiconductor with a constant m and $\tau \propto \epsilon^{-1/2}$ (ϵ = carrier energy).

The second comment is that the negative result is not confined to weak magnetic fields. If the magnetoresistance becomes positive at higher H, then Eq. (7) must equal unity at some intermediate value. Some algebra shows that this does occur when

$$\tfrac{1}{2} z = \tan(\tfrac{1}{2} z) \; . \qquad (16)$$

Thus there is only one such point for each cycloidal arch, i.e., within each interval $2\pi N \leq z \leq 2\pi \times (N+1)$. In the first interval, the only solution is the uninteresting one, $z=0$ (i.e., $H=0$). Furthermore, the remaining solutions do not separate regions of negative and positive magnetoresistance. Between each pair of successive solutions of Eq. (16), the cycloidal path touches the y axis. At every such point, the magnetoresistance is clearly negative, since those points can never be reached when $E = E_x$ and $H = 0$.

It turns out that the negative magnetoresistance discussed in the present paper is actually not an isolated effect. It is analogous to the famous Brachistochrone problem, which was posed and solved by Bernoulli in 1696.[17] For the present purposes, that problem may be stated in the following form:

A particle is dropped from rest in a uniform gravitational field. Along what path must it be guided (by frictionless constraints) so as to arrive in the shortest possible time at a given final point not directly beneath the original one? Bernoulli showed that the cycloid through the two points which has a cusp at the first one is the fastest of all possible trajectories.

The gravitational field and the frictionless constraints correspond to the electric and magnetic forces of the present problem. Since the cycloidal path is quicker than all other paths connecting the two points, it must in particular be faster than the straight line path with which it is compared in a magnetoresistance calculation.

As outlined briefly at the end of the previous section, the second and final step in the calculation of the magnetoresistance in the STM is to introduce the effect of the statistical distribution of scattering events. This brings the calculation into coincidence with the formal, real-space version of the statistical treatment of the STM.[4] Hence it must, and does, lead to the usual zero magnetoresistance.

IV. CONCLUDING REMARKS

This paper has answered a question which, although elementary in nature and pertaining to the simplest imaginable model, does not seem to have been discussed before. The analysis revealed a layer of physics which is concealed by the usual statistical calculations of galvanomagnetic properties.

The significance of the present work is that it provides a somewhat altered viewpoint regarding the fundamental nature of magnetoresistance. Formerly, it appeared that for any model in which τ and m were not functions of H, all relevant aspects of the model either contributed to a positive magnetoresistance or had no effect one way or the other. In the case of the STM, only the latter effect was present, so that in that case, and in that case alone, there was no magnetoresistance.

It now appears that the magnetoresistance phenomenon is a delicate balance between the inherently negative effect of trajectory curvature and a positive contribution due to the statistical nature of scattering. In the STM, the effects just cancel, but in other models, the additional dispersion that results from the variation of carrier mobilities leads to a positive effect.

This paper has been restricted to a discussion of models for which a scattering or relaxation time may be defined. But the negative magnetoresistance factor should be present in a larger class of models for which the concept of a trajectory is meaningful.

There are several classes of more or less exotic materials in which a negative magnetoresistance has been found. Theories have been proposed to account for these results, but they cannot be said to have provided a generally satisfactory explanation of the phenomenon.

Would a better understanding follow if some of the proposed theories were modified or reinterpreted in terms of the notion that the scattering process is, in one sense or another, nonrandom? This speculative question will not be taken up in the present paper, but it does seem worth mentioning in connection with the present results.

ACKNOWLEDGMENTS

The question asked and answered in this paper has bothered me for a long while, and I have, from time to time, discussed it with many scientists. I appreciate those stimulating discussions, but I don't think I convinced anyone that there was any paradox concerning the STM which needed to be resolved. One exception was Professor J. W. McClure, to whom I am also grateful for pointing out a factor-of-2 error in my original derivation of Eq. (9).

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