Renormalization-Group Verification of Crossover with Respect to the Lattice Anisotropy Parameter

T. S. Chang and H. Eugene Stanley
Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
(Received 20 March 1973; revised manuscript received 30 May 1973)

A renormalization-group demonstration of the relation $\varphi = \gamma$ is sketched for crossover with respect to the anisotropy parameter R in systems with lattice anisotropy. Our work extends the existing knowledge of the crossover phenomena to noninteger lattice dimensions.

Wilson's renormalization-group procedure[1] has not yet been applied to systems with lattice anisotropy. The reason is that the recursion relations[1] were derived by treating all lattice dimensions equivalently. To understand the crossover phenomena in systems with lattice anisotropy, we must observe how the weakly interacting dimensions manifest themselves as the renormalization transformation is taking place in the strongly interacting dimensions. Following this procedure, we are able to demonstrate that there is scaling with respect to the anisotropy parameter R and to prove[2] that $\varphi = \gamma$ for all lattice dimensions including noninteger dimensions. These ideas represent the first concrete results on systems with lattice anisotropy using the renormalization-group approach.

Consider a system of d-dimensional "hyperplanes" of ferromagnetically interacting Ising spins which form a $\bar{d}$-dimensional lattice by interacting with adjacent hyperplanes. For example, the case of $d = 2$, $\bar{d} = 3$ is described by the Ising Hamiltonian

$$\mathcal{H} = -J\sum_{\langle ij\rangle}^{xy} S_i S_j - RJ\sum_{\langle ij\rangle}^{z} S_i S_j \ . \tag{1}$$

Here R is the ratio of nearest-neighbor (nn) coupling strengths between planes and within a plane, and $J > 0$. The first sum is over all nn pairs within a plane, while the second sum is over all pairs on adjacent planes.

To study the crossover phenomena for general lattice anisotropy in a $\bar{d}$-dimensional lattice, we apply Wilson's renormalization scheme[1] in each d-dimensional hyperplane and observe how the weak-coupling terms between the hyperplanes grow with the iteration procedure. We have, therefore, an effective reduced Hamiltonian after l steps of iteration in each hyperplane,

$$H_l = -\sum_m \int d\vec{x}\,[\tfrac{1}{2}\,|\nabla s_m(\vec{x})|^2 + Q_l(s_m(\vec{x}))]$$

$$-R_l \sum_m \int d\vec{x}\; s_m(\vec{x})\, s_{m+i}(\vec{x}) \ , \tag{2}$$

where m labels the hyperplane and the integration is carried out over the dimensionality d of each of the strongly coupled hyperplanes.

It is known that[1]

$$s_m \cong 2^{(2-d-\eta)/2}\, s'_m \tag{3}$$

in each step of the iteration. Here η describes the behavior of the d-dimensional correlation function. Thus, the last term in (2) grows as

$$R2^{(2-\eta)l}\sum_m \int d\vec{x}\, s_m(\vec{x})\, s_{m+i}(\vec{x})$$

after l iterations, where R is the anisotropy parameter. This term must be counted when it is of similar magnitude to the other terms in (2), or when $R2^{(2-\eta)l} \sim$ constant or

$$R \sim 2^{(\eta-2)l} \ . \tag{4}$$

But the correlation length in each $\bar{d}$-dimensional hyperplane is $\xi \sim 2^l$. Thus, we have

$$R \sim \xi^{(\eta-2)} \ . \tag{5}$$

Consider the critical temperature $T_c(R)$. We have by definition

$$T - T_c(0) \sim \xi^{-1/\nu} \ , \tag{6}$$

where ν is the exponent for the correlation length associated with each strongly coupled hyperplane. Combining (5), (6), and the fact that $(2-\eta)\nu = \gamma$ from the renormalization-group scheme, we have for $T = T_c(R)$,

$$T_c(R) - T_c(0) \sim R^{1/\gamma} \ . \tag{7}$$

If we define a crossover exponent[3] through

$$T_c(R) - T_c(0) \sim R^{1/\varphi} \ , \tag{8}$$

then we have $\varphi = \gamma$.[2] This result was previously obtained independently of the renormalization-group arguments for *integer* lattice dimensions.[2] The present result is also valid for noninteger dimensions and may be generalized to arbitrary dimensional spins.

The authors are indebted to Professor K. G. Wilson for advice, discussions, and assistance in the derivation of the result $\varphi = \gamma$. This work was supported by NSF, ONR, and AFOSR.

[1]K. G. Wilson, Phys. Rev. B **4**, 3174 (1971); Phys. Rev. B **4**, 3184 (1971); unpublished lectures.

[2]R. Abe, Prog. Theor. Phys. **44**, 339 (1970); M. Suzuki, Prog. Theor. Phys. **46**, 1054 (1971). This result was made rigorous in L. L. Liu and H. E. Stanley, Phys. Rev. Lett. **29**, 927 (1972); Phys. Rev. B **8**, 2279 (1973); this proof was independently obtained by M. Grover (private communication).

[3]E. Riedel and F. Wegner, Z. Phys. **225**, 195 (1969).