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Antiferromagnetic Spherical Model and the Infinite-Spin-Dimensionality Limit

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It is shown that the antiferromagnetic spherical model reveals its high-spin-dimensionality character much more directly than the usual ferromagnetic model. The introduction of a staggered ordering field requires a second spherical field. The phase boundary of the model is compared with that expected for antiferromagnetic isotropic n-vector systems. The results indicate that a tricritical point is unlikely to exist in spherical models.

I. INTRODUCTION

The spherical model (SM) was introduced by Berlin and Kac[1] as an exactly soluble model for a ferromagnet. The spins of the model are real scalars that can take all values subject to an overall spherical constraint. Historically, the scalar character of the spins caused the SM to be regarded as a model for an Ising-type system. Only recently has it been realized[2,3] that the SM can in fact be considered as the limit of infinite-spin dimensionality ($n=\infty$). For this reason the SM might well be regarded as closer to a (classical) Heisenberg model ($n=3$) than to an Ising model ($n=1$). For a one-dimensional lattice this can be explicitly demonstrated.[4] The differences between the various ferromagnetic n-vector models are mainly of a quantitative nature. However, in the antiferromagnetic case there is a qualitative difference between the scalar Ising model and the other higher-n isotropic models. This has its origin in the fact that in the presence of a uniform magnetic field the spontaneous sublattice magnetization tends to set itself orthogonal to this field as soon as $n\geq 2$. This is the so-called "spin-flip" phenomenon. We expect, therefore, to find more direct evidence of the high-spin-dimensionality character in the antiferromagnetic SM than in the corresponding ferromagnetic model.

The antiferromagnetic SM has been studied[5,6] in the literature, and Mazo[5] reported that at $T=0$ the value of the critical field was twice as high as to be expected for the corresponding Ising model. This paper appeared long before the $n=\infty$ character of the SM was recognized and no satisfactory explanation was given. We find that this doubling of the critical field value at $T=0$ is, in fact, quite common to all the $n\geq 2$ vector models.

Even more anomalous behavior is found when one introduces a staggered field H'. This is a field with alternating signs on the two sublattices, and is the field which is conjugate to the spontaneous (staggered) magnetization in an antiferromagnet.[7] For a true scalar model one would expect: (i) The transition is destroyed as soon as $H'\neq 0$. (ii) Below the critical temperature the staggered magnetization tends, in the limit $H'\to 0$, to a "spontaneous" value different from zero. We will find that neither of these expectations is borne out for the antiferromagnetic spherical model.

The outline of the paper is as follows. In Sec. II we calculate the phase boundary in the (H, H', T) space of an antiferromagnetic SM and show explicitly the validity of the statements just made. However, we demonstrate that the spin-spin correlation functions do exhibit a long-range order below the critical temperature and a corresponding order parameter is identified. This order parameter has in general, however, no relationship to a spontaneous magnetization. Various critical exponents, describing the behavior of the system near the phase boundary, are obtained. In particular, it is found that the susceptibilities (with respect to H and H') remain finite at the phase boundary. We discuss this fact in connection with the long-range order of the correlation functions.

In Sec. III we compare the shape of the phase boundary for the antiferromagnetic SM with that predicted by a mean-field approximation to the n-vector models. It is found that these boundaries are quite similar for $n\geq 2$ and become equal in the limit $n\to\infty$. The main distinction that then remains between the ∞-vector model and the antiferromagnetic SM is that the former has a spontaneous magnetization in a direction orthogonal to the field, whereas the latter has, as a consequence of the scalar character forced upon it, no spontaneous magnetization.

In Sec. IV, following Riedel and Wegner,[8] we generalize the SM and allow the spins to be vectors in a plane. The phase boundary and all averages of spin components along the field are exactly the same as found for the scalar SM but, as anticipated, the order parameter can now be identified with the spontaneous (staggered) magnetization orthogonal to the field.

II. MODEL

The spins of the model will be localized on a cubic lattice; as usual for an antiferromagnet, it is convenient to consider this lattice as the union of two sublattices A and B, every A site being surrounded by neighboring B sites and vice versa. The energy of a given spin configuration $\{\sigma_{\vec{j}}\}$ is given by

$$E\{\sigma_{\vec{j}}\} = \sum_{\vec{i},\vec{j}} J_{\vec{i},\vec{j}}\sigma_{\vec{i}}\sigma_{\vec{j}} - H\sum_{\vec{j}}\sigma_{\vec{j}} - H'\left(\sum_{\vec{j}\in A}\sigma_{\vec{j}} - \sum_{\vec{j}\in B}\sigma_{\vec{j}}\right) , \quad (2.1)$$

where H is the uniform magnetic field and H' represents the staggered field which is[7] the ordering field for the antiferromagnetic transition. In the case of nearest-neighbor interactions, $J_{\vec{i},\vec{j}}$ differs from zero only if $\vec{i}\in A$ and $\vec{j}$ is one of the B sites neighboring $\vec{i}$. We will find it convenient to suppose more generally

$$J_{\vec{i},\vec{j}} \equiv J_{\vec{i}-\vec{j}} \geq 0 \quad \text{if } \vec{i}\in A \text{ and } \vec{j}\in B,$$
$$J_{\vec{i},\vec{j}} = 0 \quad \text{otherwise.} \quad (2.2)$$

Note that our sign convention is the opposite of that normally adopted; thus positive values of J represent antiferromagnetic coupling.

The spherical model replaces the N constraints $\sigma_{\vec{j}} = \pm 1$ of the corresponding Ising model by an over-all "spherical" constraint

$$\sum_{\vec{j}} \sigma_{\vec{j}}^2 = N. \quad (2.3)$$

The properties of the SM can, when sufficient care is taken,[9] also be obtained from the so-called "mean spherical model" (MSM) which is a grand canonical version of the SM first presented by Lewis and Wannier.[10] In the MSM a Gaussian weight function with a spherical field z is introduced and z is determined so that

$$\left\langle \sum_{\vec{j}} \sigma_{\vec{j}}^2 \right\rangle = N. \quad (2.4)$$

In the case of a translational-invariant interaction this implies $\langle\sigma_{\vec{j}}^2\rangle = 1$ for all sites $\vec{j}$. This translational invariance is the main reason why the introduction[11] of the so-called "m-spherical models," in which the spherical constraint is replaced by m independent spherical constraints on m sublattices, does not alter the properties of the model. In the present case, however, the translational symmetry is broken by the staggered field H' and if we still want to require that the mean-square spin at every site is unity, we must introduce two spherical constraints, namely,

$$\sum_{\vec{j}\in A} \sigma_{\vec{j}}^2 = \tfrac{1}{2}N, \quad \sum_{\vec{j}\in B} \sigma_{\vec{j}}^2 = \tfrac{1}{2}N. \quad (2.5)$$

In the MSM this corresponds to the introduction of two spherical fields z_A and z_B determined by the two conditions

$$\left\langle \sum_{\vec{j}\in A} \sigma_{\vec{j}}^2 \right\rangle = \tfrac{1}{2}N, \quad \left\langle \sum_{\vec{j}\in B} \sigma_{\vec{j}}^2 \right\rangle = \tfrac{1}{2}N. \quad (2.6)$$

The equivalence of A sites, resp B sites, among themselves leads again to the conclusion $\langle\sigma_{\vec{j}}^2\rangle = 1$ for all $\vec{j}$. As is to be expected, we will find $z_A = z_B$ when $H' = 0$.

The free-energy density of the model is now given by

$$f = -\frac{k_BT}{N} \ln\left[\int \prod_{\vec{j}} d\sigma_{\vec{j}} \exp\left(-\beta E\{\sigma_{\vec{j}}\} - z_A \sum_{\vec{j}\in A}\sigma_{\vec{j}}^2 - z_B \sum_{\vec{j}\in B}\sigma_{\vec{j}}^2\right)\right] , \quad (2.7)$$

in which z_A and z_B are to be determined by (2.6) and $\beta = 1/k_BT$. As usual we introduce Fourier-transformed variables

$$S_{\vec{k}} \equiv N^{1/2} \sum_{\vec{j}} e^{i\vec{k}\cdot\vec{j}}\sigma_{\vec{j}},$$
$$(k_x, k_y, k_z) = 2\pi N^{-1/3}(l_x, l_y, l_z). \quad (2.8)$$

When we invoke periodic boundary conditions, the argument of the exponent in (2.7) takes the form

$$\sum_{\vec{k}} [-\beta\hat{J}(\vec{k}) - \tfrac{1}{2}(z_A + z_B)] S_{\vec{k}} S_{\vec{k}}^* - \tfrac{1}{2}(z_A - z_B) S_{\vec{k}} S_{\vec{\pi}-\vec{k}}^* + N^{1/2}\beta H S_0 + N^{1/2}\beta H' S_{\vec{\pi}}. \quad (2.9)$$

In this expression

$$\hat{J}(\vec{k}) \equiv \frac{1}{2}\sum_{\vec{j}} e^{-i\vec{k}\cdot\vec{j}} J_{\vec{j}} \quad (2.10)$$

and $\vec{\pi} = (\pi, \pi, \pi)$ is the wave vector that minimizes $\hat{J}(\vec{k})$. Note that in view of (2.2) we have

$$\hat{J}(\vec{k}) = -\hat{J}(\vec{\pi} - \vec{k}). \quad (2.11)$$

The calculation of the free energy is now essentially reduced to the diagonalization of the 2×2 quadratic forms constituting (2.9). When, in the thermodynamic limit, we replace the resulting sum over $\vec{k}$ by an integral we find

$$f = \frac{\tfrac{1}{2}k_BT}{(2\pi)^3} \int d\vec{k} \ln\{\pi^{-2}[z_A z_B - \beta^2\hat{J}^2(\vec{k})]\} + \beta[4z_A z_B - 4\beta^2\hat{J}^2(0)]^{-1}\{H^2[\beta\hat{J}(0) - \tfrac{1}{2}(z_A + z_B)] + HH'(z_A - z_B) + H'^2[-\beta\hat{J}(0) - \tfrac{1}{2}(z_A + z_B)]\}, \quad (2.12)$$

in which the integration over $\vec{k}$ is to be performed over half a Brillouin zone.

The spherical constraints (2.6) are obtained by differentiating the sum over $\vec{k}$, leading to (2.12) in the thermodynamic limit, with respect to z_A and z_B. For large N we again replace the resulting sum by an integral omitting, as usual,[8] the term representing the lowest eigenvalue since it may make a macroscopic contribution. This yields asymptotically

$$1 \simeq \frac{1}{(2\pi)^3} \int d\vec{k}\, \frac{z_B}{z_A z_B - \beta^2 \hat{J}^2(\vec{k})} + \frac{1}{2N} \frac{z_B}{z_A z_B - \beta^2 \hat{J}^2(0)} + (M+M')^2, \quad (2.13)$$

and

$$1 \simeq \frac{1}{(2\pi)^3} \int d\vec{k}\, \frac{z_A}{z_A z_B - \beta^2 \hat{J}^2(\vec{k})} + \frac{1}{2N} \frac{z_A}{z_A z_B - \beta^2 \hat{J}^2(0)} + (M-M')^2. \quad (2.14)$$

Here M and M' are, respectively, the uniform and staggered magnetization defined by

$$M \equiv N^{-1/2}\langle S_0\rangle = -\left(\frac{\partial f}{\partial H}\right)_{\beta, z_A, z_B}, \quad (2.15)$$

$$M' \equiv N^{-1/2}\langle S_{\vec{\pi}}\rangle = -\left(\frac{\partial f}{\partial H'}\right)_{\beta, z_A, z_B}. \quad (2.16)$$

These quantities could, of course, be expressed in terms of z_A, z_B, H, and H'. It turns out, however, that it is more convenient to consider M and M' as independent variables. It is easily verified that H and H' are then given by

$$\beta H = [2\beta \hat{J}(0) + z_A + z_B] M + (z_A - z_B) M', \quad (2.17)$$

$$\beta H' = (z_A - z_B) M + [z_A + z_B - 2\beta \hat{J}(0)] M'. \quad (2.18)$$

The integrals in (2.13) and (2.14) can be expressed in terms of the basic integral

$$W_3(y) \equiv \frac{1}{(2\pi)^3} \int\!\!\int\!\!\int_0^{2\pi} d\vec{k}\, \frac{1}{y - \hat{J}(\vec{k})}. \quad (2.19)$$

The analytic properties of this integral depend mainly on the small-$\vec{k}$ behavior of $\hat{J}(\vec{k})$ and are well known both for short-[1] and long-range[12] interactions. At this point we remark only that in lattice dimension three and higher, the integral remains bounded as $y \to \hat{J}(0)$. Upon using the relation $\hat{J}(\vec{k}) = -\hat{J}(\vec{\pi} - \vec{k})$, we can alternatively write

$$W_3(y) = \frac{1}{(2\pi)^3} \int\!\!\int\!\!\int_{-\pi/2}^{\pi/2} d\vec{k}\, \frac{2y}{y^2 - \hat{J}^2(\vec{k})}. \quad (2.20)$$

It follows then by inspection that (2.13) and (2.14) take the form

$$1 = \tfrac{1}{2} k_B T (z_B/z_A)^{1/2} W_3(k_B T z_A^{1/2} z_B^{1/2}) + \frac{1}{2N} \frac{z_B}{z_A z_B - \beta^2 \hat{J}^2(0)} + (M+M')^2, \quad (2.21)$$

$$1 = \tfrac{1}{2} k_B T (z_A/z_B)^{1/2} W_3(k_B T z_A^{1/2} z_B^{1/2}) + \frac{1}{2N} \frac{z_A}{z_A z_B - \beta^2 \hat{J}^2(0)} + (M-M')^2. \quad (2.22)$$

All these equations are valid as long as the quadratic form (2.9) remains negative definite. This imposes on z_A and z_B the restriction

$$z_A z_B > \beta^2 \hat{J}^2(0), \quad z_A, z_B > 0. \quad (2.23)$$

In the usual[1] spherical model the transition is brought about by the fact that below the critical temperature the field z, which is a solution to the "saddle-point equation" [the analog of our equations (2.21) and (2.22)], sticks at the limit of its definition range. In much the same manner we will find a region of values of H, H', and T in which the spherical fields z_A and z_B stick on the hyperbola $z_A z_B = \beta^2 \hat{J}^2(0)$. We will refer to this region as the ordered region. This region is separated by a critical surface or phase boundary from the rest of the (H, H', T) space where regular solutions to the spherical-constraint equations are found.

We want to analyze the properties of the model in the ordered region. In that case, the product $z_A z_B$ becomes, in the thermodynamic limit, asymptotically equal to $\beta^2 \hat{J}^2(0)$. In this limit we introduce the parameter $\lambda > 0$ by writing

$$z_A = \lambda \beta \hat{J}(0), \quad z_B = \lambda^{-1} \beta \hat{J}(0). \quad (2.24)$$

In addition, we introduce a parameter which determines the rate at which $z_A z_B - \beta^2 \hat{J}(0)$ approaches zero as the thermodynamic limit is taken. Specifically, we define x^2 by

$$\lim_{N\to\infty} \frac{1}{2N} \frac{z_B}{z_A z_B - \beta^2 \hat{J}^2(0)} = \frac{x^2}{\lambda} \quad (2.25)$$

or, equivalently,

$$\lim_{N\to\infty} \frac{1}{2N} \frac{z_A}{z_A z_B - \beta^2 \hat{J}^2(0)} = x^2 \lambda. \quad (2.26)$$

Inserting these relations in (2.21) and (2.22) shows that the spherical constraint in the ordered region can be written

$$1 = (T/T_0 + x^2)\lambda^{-1} + (M+M')^2, \quad (2.27)$$

$$1 = (T/T_0 + x^2)\lambda + (M-M')^2. \quad (2.28)$$

Here T_0 is defined by

$$W_3(\hat{J}(0)) = 2/kT_0. \quad (2.29)$$

It will become clear that T_0 is the critical temperature in zero field. In the case of nearest-neighbor interactions with coupling J, its value is fixed by $kT_0/J \simeq 3.957$. Finally, on using (2.24), we notice that Eqs. (2.17) and (2.18), determining H and H' in the ordered region, can be written in the form

$$H/\hat{J}(0) = (2 + \lambda + \lambda^{-1}) M + (\lambda - \lambda^{-1}) M', \quad (2.30)$$

$$H'/\hat{J}(0) = (\lambda - \lambda^{-1}) M + (\lambda + \lambda^{-1} - 2) M'. \quad (2.31)$$

We show now that the parameter x^2 plays the role of an order parameter. To this end we calculate the Fourier transforms of the three relevant spin-spin correlation functions of the model, which are the correlation function for the sublattice A, g_{AA}, for the sublattice B, g_{BB}, and between the sublattices, g_{AB}. We recall from (2.9) that the only off-diagonal interaction terms are of the form $S_{\vec{k}}S^*_{\vec{\pi}-\vec{k}}$. It follows readily that

$$g_{AA}(\vec{k}) = \tfrac{1}{2}\langle (S_{\vec{k}} + S_{\vec{\pi}-\vec{k}})^2\rangle - \tfrac{1}{2}\langle S_{\vec{k}} + S_{\vec{\pi}-\vec{k}}\rangle^2 = \frac{\tfrac{1}{2}z_B}{z_A z_B - \beta^2\hat{J}^2(\vec{k})}, \tag{2.32}$$

$$g_{BB}(\vec{k}) = \tfrac{1}{2}\langle (S_{\vec{k}} - S_{\vec{\pi}-\vec{k}})^2\rangle - \tfrac{1}{2}\langle S_{\vec{k}} - S_{\vec{\pi}-\vec{k}}\rangle^2 = \frac{\tfrac{1}{2}z_A}{z_A z_B - \beta^2\hat{J}^2(\vec{k})}, \tag{2.33}$$

$$g_{AB}(\vec{k}) = \tfrac{1}{2}\langle S^2_{\vec{k}}\rangle - \tfrac{1}{2}\langle S^2_{\vec{\pi}-\vec{k}}\rangle - \tfrac{1}{2}\langle S_{\vec{k}}\rangle^2 + \tfrac{1}{2}\langle S_{\vec{\pi}-\vec{k}}\rangle^2 = \frac{-\tfrac{1}{2}\beta\hat{J}(\vec{k})}{z_A z_B - \beta^2\hat{J}^2(\vec{k})}. \tag{2.34}$$

In the ordered region, where the relations (2.25) or (2.26) defining x^2 apply, we find for $k=0$ the results

$$\lambda g_{AA}(0) = \lambda^{-1} g_{BB}(0) = -g_{AB}(0) \simeq x^2 N. \tag{2.35}$$

In lattice space this means that the correlations, in the limit of large distances, approach a constant which is proportional to x^2. We conclude that all these correlation functions indeed exhibit long-range order in the ordered region and identify x as the corresponding order parameter. We stress, however, that when $H \neq 0$, one cannot obtain the value of x from the value of the staggered magnetization in the limit $H' \to 0$, the latter value being zero as stated already in Sec. I.

Furthermore, we will find that the staggered susceptibility χ' remains finite, when $H \neq 0$, throughout the ordered region and on the phase boundary. This fact, which in view of the long-range order of the correlation functions might at first sight seem puzzling, can be understood when one notices that it is only the susceptibility (or fluctuation) at *constant fields* which is directly connected to the long-range behavior of the correlation functions. In the present case we find from (2.12) that the staggered susceptibility χ'_{MSM} in the MSM, which is the susceptibility calculated at constant spherical fields, is given by

$$\chi'_{\mathrm{MSM}} = -\left(\frac{\partial^2 f}{\partial^2 H'}\right)_{z_A, z_B} = \frac{\beta[2\beta\hat{J}(0) + z_A + z_B]}{4[z_A z_B - \beta^2\hat{J}(0)]}. \tag{2.36}$$

Using the relations (2.32)–(2.34), we find that it is indeed this susceptibility which directly reflects the long-range behavior of the correlation functions through the relation

$$\chi'_{\mathrm{MSM}} = \lim_{\vec{k}\to 0} \beta[\tfrac{1}{2}g_{AA}(\vec{k}) + \tfrac{1}{2}g_{BB}(\vec{k}) + g_{AB}(\vec{k})]. \tag{2.37}$$

We conclude that χ'_{MSM} is infinite throughout the ordered region. The susceptibility in the spherical model, where the sum of the squares of the spins is held fixed, can be obtained by allowing for the H' dependence of z_A and z_B. One has

$$\chi' = \chi'_{\mathrm{MSM}} + \frac{\partial M'}{\partial z_A}\frac{\partial z_A}{\partial H'} + \frac{\partial M'}{\partial z_B}\frac{\partial z_B}{\partial H'}. \tag{2.38}$$

From (2.17) and (2.18) one can show that $\partial M'/\partial z_A$ and $\partial M'/\partial z_B$ also diverge in the ordered region in such a way as to yield a finite value for χ', as will be found later.

Before we go on to calculate M' and χ' in the ordered region, we want to find the phase boundary which demarcates this region. At a given value of λ the value of M and M' on the phase boundary can be found by setting the order parameter x equal to 0 in (2.27) and (2.28). Using the result in (2.30) and (2.31), we find that H and H' on the phase boundary are given by

$$H/\hat{J}(0) = \pm(1+\lambda)(1-t/\lambda)^{1/2} \pm (1+\lambda^{-1})(1-t\lambda)^{1/2}, \tag{2.39}$$

$$H'/\hat{J}(0) = \pm(\lambda-1)(1-t/\lambda)^{1/2} \pm (1-\lambda^{-1})(1-t\lambda)^{1/2}, \tag{2.40}$$

in which the same combination of + and − signs is to be chosen in both equations, while t is the reduced temperature, $t = T/T_0$.

Consider now a fixed value of $t<1$. We can then regard λ as a parameter that runs from t to t^{-1}; Eqs. (2.39) and (2.40) hence describe a smooth curve in the H, H' plane which represents the phase boundary (or critical line) in this plane at a temperature $T = tT_0$. Note that $\lambda = 1$ implies $H' = 0$, which means that the spherical fields are equal in zero staggered field.

In Fig. 1 the phase boundaries are drawn for several values of t. The boundary clearly lies symmetrically in the four quadrants. For $t \neq 0$ a closed region is obtained inside which $x \neq 0$. When $T \to 0$ the boundary tends asymptotically to infinity along the line $H' = H - 2\hat{J}(0)$. It is evident from Fig. 1 that there are transitions even when $H' \neq 0$.

To calculate the staggered magnetization M' in the ordered region, we first eliminate x^2 from (2.27) and (2.28), which yields

$$\lambda^2 - 1 = \lambda^2(M+M')^2 - (M-M')^2. \tag{2.41}$$

Note further that (2.30) and (2.31) imply

$$\lambda = (H+H')/(H-H'). \tag{2.42}$$

When we use this relation in (2.43) and eliminate M' with the help of (2.30), we find the staggered magnetization in the ordered region ($H \neq 0$) to be

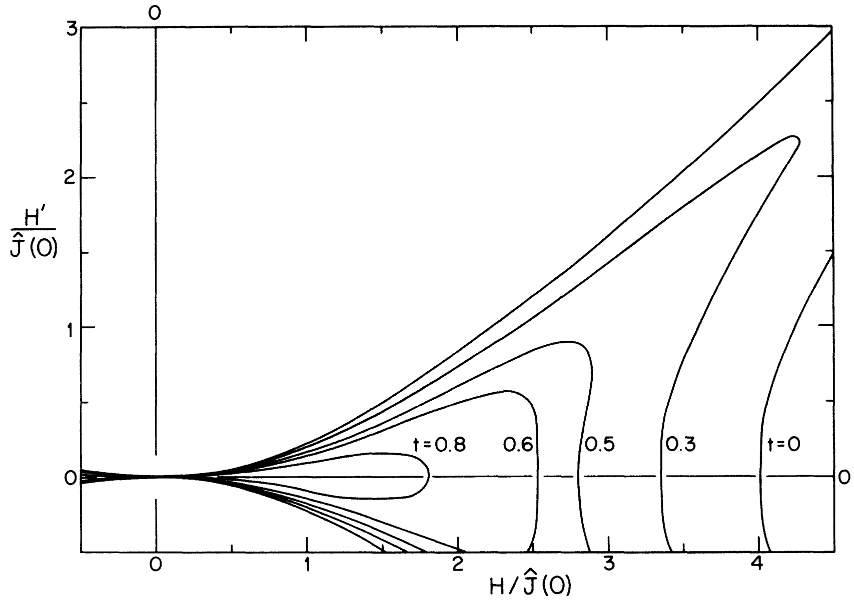

FIG. 1. Phase boundary of the antiferromagnetic SM at fixed temperature $T = tT_0$, where T_0 is the zero-field critical temperature.

$$M' = 4\hat{J}(0)\,\frac{H^2 H'}{(H^2 - H'^2)^2} - \frac{1}{4}\,\frac{H'}{\hat{J}(0)}\ . \tag{2.43}$$

Note that inside the ordered region, M' does not depend on the temperature. It is obvious from this formula that when $H \neq 0$, there is no spontaneous magnetization, i.e., $\lim_{H'\to 0} M' = 0$. Further, it is also clear that the staggered susceptibility $\chi' = \partial M'/\partial H'$ does not diverge inside the ordered region provided $H \neq 0$.

Figure 2 presents a sketch of M' vs H' at $T = \frac{1}{2} T_0$ for several values of H. With the help of the formulas [(2.39) and (2.40)] for the phase boundary, one also finds the critical values of M' at this boundary as a function of H'. The resultant curve is represented by a dotted line. Beyond this line the expression for M' deviates from that found in the ordered region. We will see later, however, that M' is still continuous with, for dimensions $d \leq 4$, a continuous first-order derivative at this point. In addition, the critical value of M', when one approaches the point H, $H' = 0$ along the phase boundary, tends to the value $(1-t)^{1/2}$. This is exactly the value of the order parameter in zero fields. When $H = 0$, one does therefore obtain the order parameter as a spontaneous staggered magnetization in the limit $H' \to 0$, as illustrated in Fig. 2. This confirms what was anticipated in Sec. I; at $H = 0$ there is no "spin-flip" phenomenon in the n-vector models and consequently one does not expect the SM to exhibit an anomalous behavior in this case.

To discuss the behavior of the model near the phase boundary we first ask how the order parameter x vanishes on approach of the phase boundary from inside the ordered region. Suppose the values of H and H' are fixed. At the critical temperature, where $x = 0$, we have from (2.27)

$$1 = [T_c(H, H')/T_0]\lambda^{-1} + (M + M')^2, \tag{2.44}$$

while at lower temperatures

$$1 = (T/T_0 + x^2)\,\lambda^{-1} + (M + M')^2, \tag{2.45}$$

with the same value of M and M' since, in the ordered region, the magnetizations do not depend on the temperature. Subtracting the two equations yield

$$x = \left(\frac{T_c(H, H') - T}{T_0}\right)^{\beta}. \tag{2.46}$$

The critical exponent β has the value $\beta = \frac{1}{2}$, which is the value normally found in all spherical models.

Next we calculate the behavior of M and M' on approach of the phase boundary from outside the ordered region. In this case the product $z_A z_B$ will no longer equal $\beta^2\hat{J}(0)$. It is then convenient to introduce the auxiliary parameter ζ by

$$z_A = \zeta\lambda\beta\hat{J}(0), \quad z_B = \zeta\lambda^{-1}\beta\hat{J}(0), \quad \zeta > 1. \tag{2.47}$$

Note that for $\zeta \to 1$ the phase boundary is approached. The original spherical-constraint equations [(2.21) and (2.22)] then read

$$1 = \tfrac{1}{2}k_B T\lambda^{-1} W(\hat{J}(0)\zeta) + (M + M')^2, \tag{2.48}$$

$$1 = \tfrac{1}{2}k_B T\lambda W(\hat{J}(0)\zeta) + (M - M')^2. \tag{2.49}$$

Note that, if $\zeta > 1$, the terms proportional to N^{-1} drop out in the thermodynamic limit. The relations (2.17) and (2.18) determining H and H' are now given by

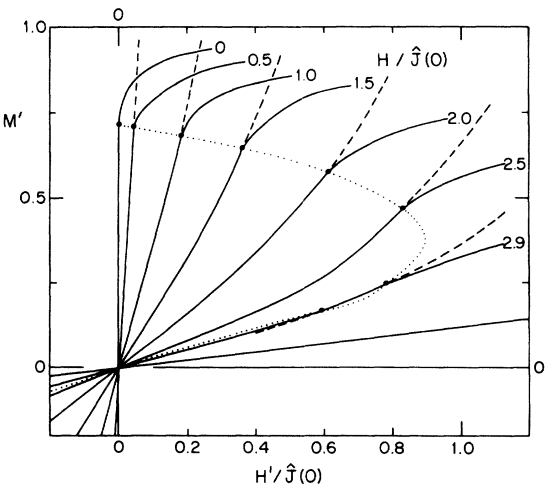

FIG. 2. Staggered magnetization M' as function of the staggered field H' at a temperature $T=\frac{1}{2}T_0$. Curves corresponding to several values of the uniform field H are shown. The dashed lines represent the analytic continuation of M' from the ordered region. The dotted line gives the critical values of M' on the phase boundary, $T=\frac{1}{2}T_0$.

$$H/\hat{J}(0)=(2+\zeta\lambda+\zeta\lambda^{-1})M+(\zeta\lambda-\zeta\lambda^{-1})M' , \quad (2.50)$$

$$H'/\hat{J}(0)=(\zeta\lambda-\zeta\lambda^{-1})M+(\zeta\lambda+\zeta\lambda^{-1}-2)M' . \quad (2.51)$$

Consider again fixed values of H and H' on the phase boundary. Suppose the temperature is raised a small amount ϵ above the critical temperature $T_c(H,H')$ corresponding to H and H'. The variables ζ, λ, M, and M' will then deviate from their values 1, λ_0, M_0, and M'_0 at the critical temperature by amounts that we denote by, respectively, $\Delta\zeta$, $\Delta\lambda$, ΔM, and $\Delta M'$. To calculate $\Delta\lambda$, ΔM, and $\Delta M'$ in terms of $\Delta\zeta$, we eliminate the term with W in (2.50) and (2.51) to obtain

$$\lambda-\lambda^{-1}=\lambda(M+M')^2-\lambda^{-1}(M-M')^2 . \quad (2.52)$$

Equations (2.50)–(2.53) are equally valid at the critical temperature as at higher temperatures. Using this fact and retaining only terms linear in the deviations from the critical values we find, for $H\neq 0$,

$$\Delta M=-\left[\lambda_0(M_0^2-M_0'^2)\,(HM_0+H'M_0')^{-1}\left(\frac{\partial H'}{\partial\lambda}\right)_{\lambda_0}\right]\Delta\zeta , \quad (2.53)$$

$$\Delta M'=\left[\lambda_0(M_0^2-M_0'^2)\,(HM_0+H'M_0')^{-1}\left(\frac{\partial H}{\partial\lambda}\right)_{\lambda_0}\right]\Delta\zeta , \quad (2.54)$$

$$\Delta\lambda=G\,\Delta\zeta . \quad (2.55)$$

Here G is a constant that we need not specify further. The symbols $(\partial H/\partial\lambda)_{\lambda_0}$ and $(\partial H'/\partial\lambda)_{\lambda_0}$ stand for the derivatives of the expressions (2.39) and (2.40), respectively, defining the phase boundary. It follows that there is no first-order contribution in $\Delta\zeta$ to ΔM or $\Delta M'$ when the H or H' axis, respectively, happens to be parallel to the phase boundary. At these special points, M or M' have a singularity which is even weaker than what will be found at the other points of the phase boundary. This phenomenon is expected from a general geometrical study[13] of many-component systems. It might appear from (2.53) and (2.54) that the point at which $M_0=M'_0$ is also special. Calculation of $(\partial H'/\partial\lambda)_{\lambda_0}$ and $(\partial H/\partial\lambda)_{\lambda_0}$ shows, however, that both derivatives contain a factor $(M_0^2-M_0'^2)^{-1}$, so that a finite proportionality factor is found also in this case.

To find the dependence of ΔM and $\Delta M'$ on the distance ϵ from the critical temperature, we return to (2.49). The function W is known to behave for $\zeta\simeq 1$ as

$$W(\hat{J}(0)\zeta)=(2/k_B T_0)(1-D\Delta\zeta^{1/(1-\alpha)}+\cdots) , \quad (2.56)$$

in which, for short-range interactions, $\alpha=-1$ when the lattice dimension $d=3$, $\alpha=0$ (logarithmatic) when $d=4$, and $\alpha=0$ when $d\geq 5$. Further, D denotes a constant. When we apply this, together with the results (2.53)–(2.55), to Eq. (2.52) we find

$$1=\lambda_0(T_c+\epsilon)T_0^{-1}(1-D'\Delta\zeta^{1/(1-\alpha)})+(M_0-M_0')^2, \quad (2.57)$$

in which the terms linear in $\Delta\zeta$, arising from λ, M, and M', are incorporated in the constant D' when $d\geq 5$. At the critical temperature we have from (2.28), with $x=0$,

$$1=\lambda_0 T_c/T_0+(M_0-M_0')^2 . \quad (2.58)$$

Subtracting the two equations we find

$$\epsilon=T_c D'\Delta\zeta^{1/(1-\alpha)}+\cdots . \quad (2.59)$$

We conclude, then, that when $H\neq 0$ and the phase boundary does not happen to be parallel to either the H or H' axis, ΔM and $\Delta M'$ are, approaching the phase boundary from outside the ordered region, in leading order given by

$$\Delta M\simeq\Delta M'\simeq\epsilon^{1-\alpha} . \quad (2.60)$$

Here $\epsilon=T-T_c(H,H')$ and the values of α are given above. An analogous result was found by Mazo[5] for ΔM at $H'=0$ and $d=3$. When we compare the present result with the corresponding behavior of the energy[2] in the spherical model, we see that both M and M' behave as energylike variables on approach of the phase boundary ($H\neq 0$). This agrees again with the general predictions of Griffiths and Wheeler,[13] since both H and H' are fields which have their axes in the plane of the ordered region (which would normally have been the coexistence region). In particular, it follows for a three-dimensional lattice that the susceptibilities χ and χ' (like C_v) remain finite and are continuous across the phase boundary ($H \neq 0$).

In closing this section we want to come back to the significance of the special choice (2.2) that was made for the interaction. A more general interaction, that includes also *intra*sublattice couplings will in no way alter the main conclusions. The formulas will be somewhat more complicated. The equation for the phase boundary which presently depends only on the interaction through the constants $\hat{J}(0)$ and T_0, will depend on further details of the interaction. In addition, the staggered magnetization will depend also on the temperature in the ordered region.

III. ISOTROPIC n-VECTOR MODELS

In this section we want to compare the shape of the phase boundary previously calculated with that of an antiferromagnetic isotropic-spin model, with spin dimensionality $n \geq 2$. The similarity that we want to demonstrate is already clear in the mean-field (MF) approximation, to which we confine ourselves.

The MF approximation does not alter the main qualitative results of Sec. II. To see this note first that the MF approximation can be obtained by rescaling the potential so that it becomes infinitely weak and infinitely long ranged.[14] Then recall that the details of the potential entered only in (2.31) and (2.32) through the constants $\hat{J}(0)$, which is not altered by the rescaling, and through T_0. In the MF approximation we have therefore only to replace T_0 by its mean-field value

$$T_0 = 2\hat{J}(0)/k. \tag{3.1}$$

Consider now an isotropic-spin model with energy

$$E = \sum_{\vec{i},\vec{j}} nJ_{\vec{i},\vec{j}}\vec{\sigma}_{\vec{i}}\cdot\vec{\sigma}_{\vec{j}} - nH\sum_{\vec{j}}\sigma^z_{\vec{j}} - nH'\left(\sum_{\vec{j}\in A}\sigma^z_{\vec{j}} - \sum_{\vec{j}\in B}\sigma^z_{\vec{j}}\right). \tag{3.2}$$

Here $\vec{\sigma}_{\vec{j}}$ is an n-dimensional vector subject to the restraint $\sigma^2_{\vec{j}} = 1$. All other symbols have the same meaning as in (2.1). The couplings are "renormalized" with a factor n to facilitate the comparison between different n values.

The results of the MF approximation applied to this model are as follows. (The detailed derivations can be found in the Appendix.) There is a region where two phases with different signs for the spontaneous staggered magnetization coexist. The spontaneous magnetization is in a direction orthogonal to the applied fields H and H'. The boundary of the coexistence region, the critical surface, is given by

$$H/J(0) = \pm S_A(\lambda)(1+\lambda) \pm S_B(\lambda)(1+\lambda^{-1}), \tag{3.3}$$

$$H'/J(0) = \pm S_A(\lambda)(\lambda - 1) \pm S_B(\lambda)(1-\lambda^{-1}), \tag{3.4}$$

in which $S_A(\lambda)$ and $S_B(\lambda)$ are the sublattice magnetization in the z direction. At the critical surface they satisfy

$$S_A = \frac{I_{n/2}(t^{-1}n\lambda S_A)}{I_{n/2-1}(t^{-1}n\lambda S_A)}, \tag{3.5}$$

$$S_B = \frac{I_{n/2}(t^{-1}n\lambda^{-1}S_B)}{I_{n/2-1}(t^{-1}n\lambda^{-1}S_B)}. \tag{3.6}$$

Here I_ν denotes the Bessel function of imaginary argument and t is the reduced temperature, $t = \frac{1}{2}kT/\hat{J}(0) = T/T_0$. Positive solutions exist if $t < 1$ and $t < \lambda < t^{-1}$.

The shape of the critical surface can easily be obtained, by graphical methods, from these equations. Figure 3 shows the phase boundary in the (H, H') plane at $t = 0.5$ for $n = 2, 3,$ and ∞. We see that the shape of the critical line is quite similar to that of the transition line of Sec. II. The coexistence region shrinks with increasing n and converges toward the $n=\infty$ result. This $n=\infty$ result is, in fact, identical to the phase boundary found in Sec. II. In order to show this we use the relation[4]

$$\lim_{n\to\infty} \frac{I_{n/2}(\alpha n)}{I_{n/2-1}(\alpha n)} = \frac{2\alpha}{1+[1+(2\alpha)^2]^{1/2}}. \tag{3.7}$$

On substitution in (3.5) and (3.6) one obtains two equations which can easily be solved with the result

$$S_A = [1 - t/\lambda]^{1/2}, \quad S_B = (1 - t\lambda)^{1/2}. \tag{3.8}$$

Inserting these results into (3.3) and (3.4) shows that the critical fields at $n=\infty$ satisfy equations that are identical to (2.31) and (2.32).

Finally we note that for all $n \geq 2$ the critical fields in the ground state may be found by setting S_A, S_B equal to unity in (3.3) and (3.4). This result must hold independently of the MF approximation. Note that the critical value of H at $H' = 0$ ($\lambda = 1$) is $4J(0)$, which is twice as large as what would be expected for the corresponding Ising ($n = 1$) model.

IV. PLANAR SPHERICAL MODEL

In Sec. III, we have seen that the antiferromagnetic SM, despite its scalar character, is closely similar to a n-vector model ($n \geq 2$). The important distinction remaining is that the SM does not allow a spontaneous magnetization in an orthogonal direction. This suggests the introduction of a planar SM, where the spins are vectors in a plane.[8] Consider the energy expression

$$E = \sum_{\vec{i},\vec{j}} 2J_{\vec{i},\vec{j}}(\sigma^z_{\vec{i}}\sigma^z_{\vec{j}} + \sigma^x_{\vec{i}}\sigma^x_{\vec{j}}) - 2H\sum_{\vec{j}}\sigma^z_{\vec{j}} - 2H'_{\parallel}\left(\sum_{\vec{j}\in A}\sigma^z_{\vec{j}} - \sum_{\vec{j}\in B}\sigma^z_{\vec{j}}\right) - 2H'_{\perp}\left(\sum_{\vec{j}\in A}\sigma^x_{\vec{j}} - \sum_{\vec{j}\in B}\sigma^x_{\vec{j}}\right), \tag{4.1}$$

where the spins have to satisfy the spherical constraints

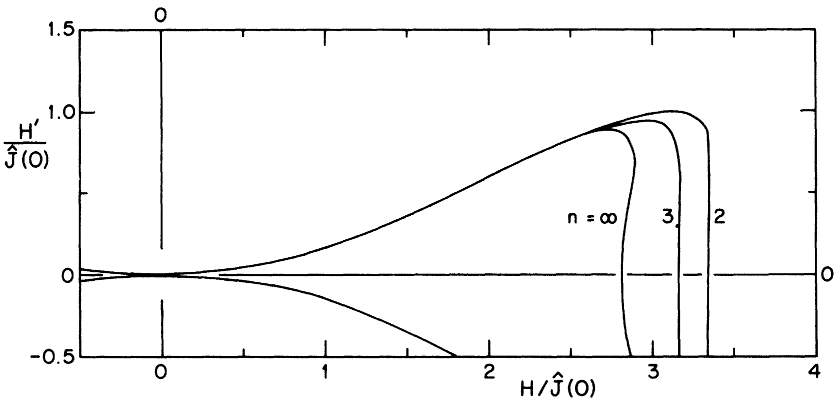

FIG. 3. Phase boundary of an isotropic n-vector model for $n=2$, 3, and ∞ at a temperature half the zero-field critical temperature according to the mean-field approximation.

$$\sum_{\vec{l}\in A}(\sigma_{\vec{l}}^x)^2+(\sigma_{\vec{l}}^z)^2=\tfrac{1}{2}N,\quad \sum_{\vec{l}\in B}(\sigma_{\vec{l}}^x)^2+(\sigma_{\vec{l}}^z)^2=\tfrac{1}{2}N. \tag{4.2}$$

For convenience we have also introduced a staggered field $H'_\perp$ in the direction orthogonal to the uniform field.

The problem may now be handled by exactly the same steps as in Sec. II. One obtains the saddle-point equations

$$1=\frac{1}{(2\pi)^3}\int d\vec{k}\,\frac{z_B}{z_Az_B-\beta^2\hat{J}^2(\vec{k})}+(S_A^\perp)^2+(M+S)^2, \tag{4.3}$$

$$1=\frac{1}{(2\pi)^3}\int d\vec{k}\,\frac{z_A}{z_Az_B-\beta^2\hat{J}^2(\vec{k})}+(S_B^\perp)^2+(M-S)^2, \tag{4.4}$$

where $S_A^\perp$ and $S_B^\perp$ are the sublattice magnetization of the A and B sublattice and are given by

$$S_A^\perp=\frac{\beta H'_\perp[\beta\hat{J}(0)+z_B]}{2[z_Az_B-\beta^2\hat{J}^2(0)]}, \tag{4.5}$$

$$S_B^\perp=\frac{\beta H'_\perp[\beta\hat{J}(0)+z_A]}{2[z_Az_B-\beta^2\hat{J}^2(0)]}. \tag{4.6}$$

All other symbols have the same meaning as in Sec. II. One need not include a term representing the lowest-eigenvalue term of the original sum because the $S_A^\perp$ and $S_B^\perp$ terms will prevent the lowest eigenvalue from vanishing as long as $H'_\perp\neq 0$.

The coexistence region is now found when z_Az_B becomes asymptotically equal to $\beta^2J^2(0)$ in the limit $H'_\perp\to 0$. Suppose that in this limit we again have $z_A=\beta J\lambda$ and $z_B=\beta J/\lambda$. On defining further

$$\lim_{H'_\perp\to 0}\frac{\beta^2JH'_\perp(\lambda^{-1/2}+\lambda^{1/2})}{2[z_Az_B-\beta^2\hat{J}^2(0)]}=x, \tag{4.7}$$

it follows that

$$\lim_{H'_\perp\to 0}(S_A^\perp)^2=\frac{x^2}{\lambda},\quad \lim_{H'_\perp\to 0}(S_B^\perp)^2=x^2\lambda. \tag{4.8}$$

The other two terms in (4.3) and (4.4) reduce in the same manner as before and we arrive at equations which are formally identical to (2.29) and (2.30). However, the significant difference is that the quantity x^2 can now through (4.8) be related to the *transverse* spontaneous order in contrast to the scalar model where x^2 could not be interpreted as a spontaneous magnetization. This provides an explicit demonstration of the "vector character" of the spherical model and of the spin-flip transition expected on heuristic grounds.

V. CONCLUSION

We have seen that the antiferromagnetic SM reveals its hidden high-spin-dimensionality character much more clearly than the corresponding ferromagnetic model. Despite the scalar character, which does not allow a transverse order, the phase boundary of the model mimics closely the phase diagram of an isotropic n-vector model in which there is a spontaneous sublattice magnetization orthogonal to the magnetic field.

This result has a bearing on the question whether the antiferromagnetic SM can exhibit a tricritical point. Tricritical points are believed[15] to occur in antiferromagnetic Ising models with an additional ferromagnetic coupling between the sites of each sublattice. Such an additional interaction, however, would *not* alter our results in any essential way. One might thus well ask whether an isotropic n-vector model will exhibit a tricritical point. In a metamagnet the tricritical point is that point of the critical line in the (H, T) plane at which the transition changes from continuous to first order. Some reflection shows that increasing H in the ground state of an isotropic (antiferromagnetic) n-vector model will gradually turn the spins over toward the field direction, thereby leading to a continuous transition. It is then hard to imagine that a higher temperature a first-order transition could occur. This heuristic argument excludes the possibility of a tricritical point. In fact its absence in some spherical models has been checked explicitly.

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APPENDIX: MEAN-FIELD TREATMENT

A good way to obtain the mean-field (MF) approximation is to introduce the normalized distribution function

$$\rho=[z(a)z(b)]^{-N/2}\exp\left(\vec{a}\cdot\sum_{\vec{j}\in A}\vec{\sigma}_{\vec{j}}+\vec{b}\cdot\sum_{\vec{j}\in B}\vec{\sigma}_{\vec{j}}\right), \quad \text{(A1)}$$

$$z(a)=\int_{|\vec{\sigma}|=1}d\vec{\sigma}\, e^{\vec{a}\cdot\vec{\sigma}}. \quad \text{(A2)}$$

The free energy corresponding to ρ is given by

$$F=\langle E\rangle_\rho+kT\,\langle\ln\rho\rangle_\rho, \quad \text{(A3)}$$

where E is the energy expression given in (3. 2). The minimum of this expression over all possible distribution functions would give the true free energy; when we minimize with respect to the class of distribution functions defined by (A1) we find the MF approximation to the free energy.

When we differentiate (A3) with respect to $\vec{a}$ and $\vec{b}$ and equate the result to zero in order to obtain the extremes of F, we find the familiar self-consistency equations

$$n^{-1}\vec{a}=-2\beta\hat{J}(0)\vec{S}_B+(\beta H+\beta H')\hat{z}, \quad \text{(A4)}$$

$$n^{-1}\vec{b}=-2\beta\hat{J}(0)\vec{S}_A+(\beta H-\beta H')\hat{z}, \quad \text{(A5)}$$

in which $\hat{z}$ is a unit vector along the z axis, and $\vec{S}_A$, $\vec{S}_B$ are the sublattice magnetization of the A and B lattice, namely,

$$\vec{S}_A=z(a)^{-1}\int_{|\vec{\sigma}|=1}d\vec{\sigma}\, e^{\vec{a}\cdot\vec{\sigma}}\vec{\sigma}, \quad \text{(A6)}$$

$$\vec{S}_B=z(b)^{-1}\int_{|\vec{\sigma}|=1}d\vec{\sigma}\, e^{\vec{b}\cdot\vec{\sigma}}\vec{\sigma}. \quad \text{(A7)}$$

It is clear that $\vec{S}_A$ is a vector $\parallel\vec{a}$ with length given by

$$S_A=\frac{d}{da}\ln z(a). \quad \text{(A8)}$$

With the introduction of polar coordinates $z(a)$ can be written as

$$z(a)=C\int_0^\pi e^{a\cos\theta}(\sin\theta)^{n-2}d\theta. \quad \text{(A9)}$$

This integral is directly related to an integral representation of the Bessel function I_ν, so that

$$z(a)=C'a^{1-n/2}I_{n/2-1}(an). \quad \text{(A10)}$$

Using (A8) and a recursion formula for the Bessel functions, one finds

$$S_A=I_{n/2}(a)/I_{n/2-1}(a). \quad \text{(A11)}$$

Similar formulas arise in Stanley's paper[4] on a chain of n-dimensional spins and further details of this function can be found there.

We want to show that the self-consistency relations allow solutions in which $\vec{S}_A$ has a component orthogonal to $\hat{z}$, say in the x direction. Note first that in case $S_{A,x}\neq 0$ the conditions $\vec{S}_A\parallel\vec{a}$ and $\vec{S}_B\parallel\vec{b}$ combined with (A4) and (A5) lead to the relation

$$\vec{S}_B=(H+H')[2J(0)]^{-1}\hat{z}-\lambda\vec{S}_A, \quad \text{(A12)}$$

in which the parameter λ is defined by

$$\lambda=(H+H')/(H-H'). \quad \text{(A 13)}$$

When we insert this in (A4) and (A5) we obtain

$$a=2\beta Jn\lambda S_A,\quad b=2\beta Jn\lambda^{-1}S_B. \quad \text{(A14)}$$

Substituting this in the expression (A11) for the sublattice magnetization yields

$$S_A=\frac{I_{n/2}(2\beta Jn\lambda S_A)}{I_{n/2-1}(2\beta Jn\lambda S_A)}, \quad \text{(A15)}$$

$$S_B=\frac{I_{n/2}(2\beta Jn\lambda^{-1}S_B)}{I_{n/2-1}(2\beta Jn\lambda^{-1}S_B)}. \quad \text{(A16)}$$

When $S_{A,x}\neq 0$ Eqs. (A12), (A15), and (A16) are equivalent to the original self-consistency equations in vector form. We note that (A15) and (A16) are exactly the mean-field equations that would describe a ferromagnetic n-vector model with coupling resp. $J\lambda$ and J/λ. There will therefore exist nontrivial solutions when $T<T_0=2\hat{J}(0)/k$ and $t<\lambda<1/t$, where again $t=T/T_0$.

A convenient way to describe the solutions is to consider at fixed $t<1$ the parameter λ and the value of $S_{A,x}$ as independent variables. From (A15) and (A16) we find the lengths of $\vec{S}_A$ and $\vec{S}_B$ as functions of λ. From (A12) we recall $S_{B,x}=-\lambda S_{A,x}$. Hence we find

$$S_{A,z}^2=S_A^2(\lambda)-S_{A,x}^2, \quad \text{(A17)}$$

$$S_{B,z}^2=S_B^2(\lambda)-\lambda^2S_{A,x}^2. \quad \text{(A18)}$$

Given $S_{A,z}$ and $S_{B,z}$ we can now find H and H' from (A12) and (A13) which can be transformed to

$$H/J=S_{A,z}(1+\lambda)+S_{B,z}(1+\lambda^{-1}), \quad \text{(A19)}$$

$$H'/J=S_{A,z}(\lambda-1)+S_{B,z}(1-\lambda^{-1}). \quad \text{(A20)}$$

It is evident from these equations that at the same value of H, H', and t both $S_{A,x}$ and $-S_{A,x}$ are solutions. This means that two phases with opposite sign of the transverse sublattice magnetization are in coexistence. The critical surface which is the boundary of this coexistence region is found by setting $S_{A,x}=0$, and its equation is consequently

$$H/J=\pm S_A(\lambda)(1+\lambda)\pm S_B(\lambda)(1+1/\lambda), \quad \text{(A 21)}$$

$$H'/J=\pm S_A(\lambda)(\lambda-1)\pm S_B(\lambda)(1-1/\lambda). \quad \text{(A22)}$$

To complete this analysis one should still prove that the solutions that we just obtained are indeed stable solutions, i.e., present a true minimum in the free energy. The best way to do this is to consider the other type of solution which is possible, namely, the one where all vectors are along the z axis. Outside the coexistence region, where there are no other solutions, one of the latter type should certainly be stable. One can now calculate for these solutions the second-order contribution to F from small deviations of $\vec{S}_A$ and $\vec{S}_B$ orthogonal to the z axis. It turns out that the determinant of the quadratic form that one so obtains, passes through zero on the surface defined by (A21) and (A22). This means that the solutions with all vectors parallel to the z axis become unstable with respect to deviations of this axis exactly in the region where we found solutions with transverse components, so that the latter should be stable. We will omit further details.

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