Calculation of the Critical Exponent η via Renormalization-Group Recursion Formulas*

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This paper presents an extension of Wilson's renormalization-group calculation of Ising-model critical exponents to include calculation of the critical exponent η. New recursion formulas are derived using the simplest set of consistent approximations which allow a nonzero η. They are intended to demonstrate, qualitatively, how nonzero values for η are consistent with the renormalization-group approach; they do not represent systematic, quantitative improvements to Wilson's earlier calculation of the exponents ν and γ. The equations are solved both by ϵ expansion about four dimensions and by numerical integration in three dimensions. To order ϵ^2 we obtain $\eta = 0.05\epsilon^2$. Numerical results in three dimension are $\eta = 0.058$, $\nu = 0.588$, and $\gamma = 1.14$. The relation $\gamma = (2-\eta)\nu$ is confirmed.

I. INTRODUCTION

In a recent series of papers[1,2] Wilson has developed the renormalization-group approach as a powerful tool for understanding critical phenomena. In particular he has derived and solved recursion formulas[3] which represent the qualitative behavior of the renormalization-group transformation on a generalization of the Ising-model ferromagnet. The model Hamiltonian used has the Landau-Ginsberg[4] form

$$\mathcal{H} = -\int_{\mathfrak{x}} P(s(\vec{x})) - \tfrac{1}{2}K\int_{\mathfrak{x}} [\vec{\nabla}s(x)]^2 , \qquad (1.1)$$

where $\int_{\mathfrak{x}}$ means $\int d^d x$, K is a constant, and $P(s(\vec{x}))$ is a function of the continuous spin variable $s(\vec{x})$. Using the recursion formulas, Wilson calculates the critical exponents ν and γ in impressive agreement with numerical calculations based on high-temperature series expansions.[5] However, approximations involved in his derivation of the recursion formulas restrict the value of the exponent η to be zero, presenting the interesting question of whether η would be different than zero in a more accurate derivation of the recursion formulas.

In this paper I extend Wilson's renormalization-group calculation of Ising-model critical exponents to include calculation of the critical exponent η. New recursion formulas are derived using the simplest consistent set of approximations which allow for a nonzero η. They are intended to demonstrate, qualitatively, how nonzero values for η are consistent with the renormalization-group approach; they do not represent systematic quantitative improvements to Wilson's earlier calculations of the exponents ν and γ.

The value for η determined by these formulas in three dimensions is $\eta = 0.058$, which is in quite good agreement with the numerical result $\eta = 0.056$[5] determined from conventional high-temperature-expansion methods. The exponent relation $\gamma = (2-\eta)\nu$ is also verified, and the exponents ν and γ are computed, giving in three dimensions $\nu = 0.588$, $\gamma = 1.14$, which disagree with the high-temperature-expansion results[5] ($\nu = 0.643$, $\gamma = 1.25$) but are consistent with the qualitative nature of the equations.

The equations are derived as a correction to Wilson's earlier recursion formula which restricted the value of η to be zero because of approximations made in its derivation. The approximations arose in the course of functionally integrating out from the partition function for Eq. (1.1) the high-momentum components of the spin variable $s(\vec{x})$ so as to define an equivalent effective interaction in terms of the low-momentum components alone. We shall see later[6] that the approximation that restricted the value of η to be zero was the replacement within a phase-space cell of the slowly varying, low-momenta spin variable [$s'(\vec{x}/2)$ in Wilson's scaled notation] by its value at the center of the cell, $s'(\vec{x}_0/2)$. Instead, we make a Taylor-series expansion of $s'(\vec{x}/2)$ about $\vec{x}_0/2$,

$$s'(\vec{x}/2) = s'(\vec{x}_0/2) + \vec{\nabla}s'(\vec{x}_0/2)\cdot(\vec{x}-\vec{x}_0) + \tfrac{1}{2}\nabla_i\nabla_j s'(\vec{x}_0/2)(\vec{x}-\vec{x}_0)_i(\vec{x}-\vec{x}_0)_j + \cdots , \qquad (1.2)$$

and expand $P(s'(\vec{x}/2))$ about $s'(\vec{x}_0/2)$ to order $[\vec{\nabla}s' \times(\vec{x}_0/2)]^2$ or $[\nabla^2 s'(\vec{x}_0/2)]$. This expansion generates new terms in the recursion formulas contributing to the $[\vec{\nabla}s(\vec{x})]^2$ part of the Hamiltonian. It is these terms which produce a nonzero value for η.

A consistent treatment of the new terms requires a larger space of Hamiltonians of the form

$$\mathcal{H} = -\int_{\mathfrak{x}} P(s(\vec{x})) - \tfrac{1}{2}K\int_{\mathfrak{x}} R(s(\vec{x}))[\vec{\nabla}s(\vec{x})]^2 , \qquad (1.3)$$

where $R(s(\vec{x}))$ is an arbitrary function of $s(\vec{x})$. Then the new recursion formulas can be written in the following form. Let

$$I_l(z) = \int_{-\infty}^{\infty} dy \exp[-y^2 W_l(z) - \tfrac{1}{2}Q_l(y+z) - \tfrac{1}{2}Q_l(-y+z)] \qquad (1.4)$$

and define the "expectation value"

$$\langle F_l^n(x)\rangle \equiv [I_l(z)]^{-1}\int_{-\infty}^{\infty} dy[\tfrac{1}{2}F_l(y+z) + \tfrac{1}{2}F_l(-y+z)]^n \exp[\cdots], \quad (1.5)$$

where $[\cdots]$ represents the expression in brackets in the previous equation, and $Q_l(y)$ and $W_l(y)$ are dimensionless forms of the functions $P_l(s(\vec{x}))$ and $R_l(s(\vec{x}))$ written in terms of a scaled spin variable. Then

$$Q_{l+1}(y) = -2^d[\ln I_l(2^{-d/2}\alpha_l y) - \ln I_l(0)], \quad (1.6)$$

$$W_{l+1}(y) = \tfrac{1}{8}\alpha_l^2 C_d \left\langle \left(\left\langle \frac{dQ_l(z)}{dz} \right\rangle - \frac{dQ_l(z)}{dz} \right)^2 \right\rangle - \tfrac{1}{4}\alpha_l^2\langle W_l(z)\rangle, \quad (1.7)$$

$$z = 2^{-d/2}\alpha_l y \quad (1.8)$$

to order $[\vec{\nabla}s'(\vec{x}_0/2)]^2$, where C_d is a constant (depending on dimension), and α_l is determined by normalizing $W_{l+1}(0)=1$. Since the fluctuation term $\langle(\langle dQ_l(z)/dz\rangle - dQ_l(z)/dz)^2\rangle$ is zero for $z=0$, [7] the normalization condition determining α_l can be written

$$W_{l+1}(0) = \tfrac{1}{4}\alpha_l^2\langle W_l(0)\rangle = 1.$$

The critical point is then determined by finding the fixed point $Q_c(y)$, $W_c(y)$, and α_c of the recursion formulas. The exponent η is related to the fixed-point value α_c according to the formula $\alpha_c = 2^{1-\eta/2}$. Hence η will be greater than zero if $\langle W_c(0)\rangle > 1$.

These formulas will be derived and discussed in subsequent sections of this paper. However, we remark at this time that the above equations will reduce to Wilson's recursion formulas if one neglects the contribution from fluctuations in the derivative of the probability distribution

$$\left\langle \left(\left\langle \frac{dQ_l(z)}{dz} \right\rangle - \frac{dQ_l(z)}{dz} \right)^2 \right\rangle . \quad (1.9)$$

It is these fluctuations, to order $[\vec{\nabla}s'(\vec{x}_0/2)]^2$, that drive $\eta > 0$.

In Sec. II I present my derivation of the new recursion formulas and discuss how the exponents are determined. I verify the relation $\gamma = (2-\eta)\nu$ in the context of the equations. In Sec. III I study the equations using two approaches. Using analytic ϵ expansion about dimensionality four I show the existence of a nontrivial fixed point at which $\eta = 0.05\,\epsilon^2$, to order ϵ^2. The recursion formulas are also solved by numerical integration in three dimensions, the results being stated at the beginning of this introduction. Section IV is the concluding discussion.

II. ANALYSIS OF THE PARTITION FUNCTION

A. Derivation of the Recursion Formulas

We wish to evaluate the partition function

$$Z = \int_{\{s(\vec{x})\}} \exp\{-\int_{\vec{x}} P(s(\vec{x})) - \tfrac{1}{2}K\int_{\vec{x}} R(s(\vec{x}))[\vec{\nabla}s(\vec{x})]^2, \quad (2.1)$$

where $\int_{\{s(\vec{x})\}}$ stands for functional integration over the functional variable $s(\vec{x})$. Consider the term $\int_{\vec{x}} P(s(\vec{x}))$. Using the phase-space-cell analysis of Wilson[1] we write

$$\int_{\vec{x}} \text{ as } \sum_{\vec{m}} \int_{\vec{x}\in \text{box } \vec{m}} \quad (2.2)$$

and decompose the spin variable in the standard separation of high- and low-frequency components

$$s(\vec{x}) = \sum_{\vec{m}} \psi_{\vec{m}0}(\vec{x}) s_{\vec{m}0} + 2^{-d/2}\alpha s'(\vec{x}/2) . \quad (2.3)$$

Then using the expansion described in Sec. I, we can approximate the P integral as

$$\int_{\vec{x}\in \text{box } \vec{m}} P(s(\vec{x})) \simeq \tfrac{1}{2}w^{-1}[P(w^{1/2}s_{\vec{m}0} + u_{\vec{m}}) + P(-w^{1/2}s_{\vec{m}0} + u_{\vec{m}})] + 2^{(-d-2)/2}\alpha\left[V^+\frac{dP}{ds}(w^{1/2}s_{\vec{m}0} + u_{\vec{m}}) + V^-\frac{dP}{ds}(-w^{1/2}s_{\vec{m}0} + u_{\vec{m}})\right][\nabla^2 s'(\vec{x}_0/2)] + 2^{-d-1}\alpha^2\left[V^+\frac{d^2P}{ds^2}(w^{1/2}s_{\vec{m}0} + u_{\vec{m}}) + V^-\frac{d^2P}{ds^2}(-w^{1/2}s_{\vec{m}0} + u_{\vec{m}})\right][\vec{\nabla}s'(\vec{x}_0/2)]^2, \quad (2.4)$$

where $u_{\vec{m}} = 2^{-d/2}\alpha s'(\vec{x}_0/2)$, [8] and $\psi_{\vec{m}0}(\vec{x})$ is approximated as in Fig. 1, taking the value $w^{1/2}$ over half the box and $-w^{1/2}$ over the other half, the box having volume w^{-1}. The terms $V^{\pm}$ correspond to the volume integrals of the expansion variable $(\vec{x}-\vec{x}_0)_i^2$ taken over the regions where $\psi_{\vec{m}0}(\vec{x}) = \pm w^{1/2}$. They will be evaluated later. The expansion term linear in $\vec{\nabla}s'(\vec{x}_0/2)\cdot(\vec{x}-\vec{x}_0)$ vanishes upon integration over the box, and hence is absent from Eq. (2.4).

The $R(s(\vec{x}))$ integral can be expanded in the same way to generate higher-order corrections. However, inasmuch as we assume departure of $R(s(\vec{x}))$ from 1 will be small [already of order $[\vec{\nabla}s'(\vec{x}_0/2)]^2$ relative to $P(s(\vec{x}))$ terms], we choose to ignore these corrections. [9] We may then write the gradient term as

$$\int_{\vec{x}} R(s(\vec{x}))[\vec{\nabla}s(\vec{x})]^2 \approx \sum_{\vec{m}}\{\rho_d R(u_{\vec{m}})s_{\vec{m}0}^2 + 2^{-d-1}\alpha^2 w^{-1}$$

$$\times [R(w^{1/2}s_{\vec{m}0}+u_{\vec{m}})+R(-w^{1/2}s_{\vec{m}0}+u_{\vec{m}})]$$
$$\times [\vec{\nabla}s'(\vec{x}_0/2)]^2\}\,, \quad (2.5)$$

where

$$\rho_d = \int_{\vec{x}} [\vec{\nabla}\psi_{\vec{m}0}(\vec{x})]^2\,,$$

independent of $\vec{m}$.

Following Wilson's analysis, coupling terms from different boxes (proportional to $s_{\vec{m}0}s_{\vec{n}0}$ with $\vec{m}\neq\vec{n}$) have been neglected. We have also assumed, as in Fig. 1, that contributions of $\vec{\nabla}\psi_{\vec{m}0}(\vec{x})$ to various integrands is appreciably nonzero only when $\psi_{\vec{m}0}(\vec{x})=0$.[10] Thus we can approximate $[\vec{\nabla}\psi_{\vec{m}0}(\vec{x})]^2$ by the δ function $\rho_d\delta^{(d)}(\vec{x}-\vec{x}')$, where $\psi_{\vec{m}0}(\vec{x}')=0$; hence the term $\rho_d R(u_{\vec{m}})$. There are no cross terms proportional to $\vec{\nabla}s'(\vec{x}_0/2)\cdot\vec{\nabla}\psi_{\vec{m}0}(\vec{x})s_{\vec{m}0}$ as these have vanished through the $\vec{x}$ integration.

We can now do the functional integration $s_{\vec{m}0}$. Making the substitutions

$$y_{\vec{m}} = (K\rho_d/2)^{1/2}s_{\vec{m}0}\,, \quad z_{\vec{m}} = (K\rho_d/2w)^{1/2}u_{\vec{m}}\,, \quad (2.6)$$

$$Q(y) = w^{-1}P((2w/K\rho_d)^{1/2}y)\,, \quad (2.7)$$

$$W(y) = R((2w/K\rho_d)^{1/2}y)\,, \quad (2.8)$$

our integration becomes

$$M(z)=\int_{-\infty}^{\infty} dy \exp\left\{-y^2W(z)-\tfrac{1}{2}Q(y+z)-\tfrac{1}{2}Q(-y+z)-2^{(-d-3)/2}\alpha(K\rho_d w)^{1/2}\left[V^+\frac{d}{dy}Q(y+z)+V^-\frac{d}{dy}Q(-y+z)\right][\nabla^2 s'(\vec{x}_0/2)]\right.$$
$$-2^{-d-2}\alpha^2K\rho_d\left[V^+\frac{d^2}{dy^2}Q(y+z)+V^-\frac{d^2}{dy^2}Q(-y+z)\right][\vec{\nabla}s'(\vec{x}_0/2)]^2$$
$$\left.-2^{-d-2}\alpha^2Kw^{-1}[W(y+z)+W(-y+z)][\vec{\nabla}s'(\vec{x}_0/2)]^2\right\}\,. \quad (2.9)$$

Then the partition function is

$$Z=\prod_{\vec{m}}\prod_{l=1}^{\infty}\int_{-\infty}^{\infty} ds_{\vec{m}l}\prod_{\vec{m}}\left\{(2/K\rho_d)^{1/2}M(z_{\vec{m}})\right\}\,. \quad (2.10)$$

We consider all terms in $[\nabla^2 s'(\vec{x}_0/2)]$ or $[\vec{\nabla}s'(\vec{x}_0/2)]^2$ small and expand the exponential to that order in those terms. Then using Eqs. (1.4) and (1.5) we can write

$$M(z)\simeq I(z)\left\{1-2^{(-d-3)/2}\alpha(K\rho_d w)^{1/2}V\right.$$
$$\times\left\langle\frac{dQ(z)}{dz}\right\rangle[\nabla^2 s'(\vec{x}_0/2)]$$
$$-2^{-d-1}\alpha^2K\left[\frac{\rho_d V}{2}\left\langle\frac{d^2Q(z)}{dz^2}\right\rangle\right.$$
$$\left.\left.+w^{-1}\langle W(z)\rangle\right][\vec{\nabla}s'(\vec{x}_0/2)]^2\right\}\,, \quad (2.11)$$

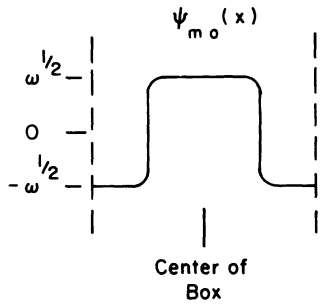

FIG. 1. Approximation of the position-space wave packet $\psi_{\vec{m}0}(\vec{x})$ within the position-space box $\vec{m}$.

where $V=V^++V^-$, and we have used the fact that $Q(y)$ and $W(y)$ are even functions.

Following Wilson we define a new interaction $\mathcal{H}_1$, such that

$$Z\propto\prod_{\vec{m}}\prod_{l=0}^{\infty}\int_{-\infty}^{\infty} ds'_{\vec{m}l}\,e^{\mathcal{H}_1}\,. \quad (2.12)$$

Then

$$\mathcal{H}_1=\sum_{\vec{m}}\ln[M(z_{\vec{m}})/I(0)] \quad (2.13)$$

making the standard separation of $I(0)$. We want $\mathcal{H}_1$ to be of the same form as $\mathcal{H}$ so we expand the logarithm to order $[\nabla^2 s'(\vec{x}_0/2)]$ or $[\vec{\nabla}s'(\vec{x}_0/2)]^2$ in those terms, and convert the $\sum_{\vec{m}}$ back to $\int_{\vec{x}}$. We show in Appendix A that, to order $\nabla^2 s(\vec{x})$, we may replace

$$\sum_{\vec{m}} f(s(\vec{x}_{\vec{m}}/2)) - 2^d w\int_{\vec{x}} f(s(\vec{x}))\,. \quad (2.14)$$

Thus, substituting $\frac{1}{2}\vec{\nabla}_{\vec{x}/2}\,s'(\vec{x}_0/2)$ for $\vec{\nabla}s'(\vec{x}_0/2)$ we can write

$$\mathcal{H}_1\simeq 2^d w\int_{\vec{x}}[\ln I(z)-\ln I(0)]$$
$$-\frac{2^d w}{4}\int_{\vec{x}}\left\{2^{(-d-3)/2}\alpha(K\rho_d w)^{1/2}V\left\langle\frac{dQ(z)}{dz}\right\rangle[\nabla^2 s'(\vec{x})]\right.$$
$$\left.+2^{d-1}\alpha^2K\left[\frac{\rho_d V}{2}\left\langle\frac{d^2Q(z)}{dz^2}\right\rangle+w^{-1}\langle W(z)\rangle\right][\vec{\nabla}s'(\vec{x})]^2\right\}\,, \quad (2.15)$$

$$z=(K\rho_d/2w)^{1/2}2^{-d/2}\alpha s'(\vec{x})\,. \quad (2.16)$$

Integrating the $[\nabla^2 s'(\vec{x})]$ term by parts [see Appendix A, Eq. (A8)], we see that we can now write

$$\mathcal{H}_1 = -\int_{\vec{x}} P_1[s'(\vec{x})] - \tfrac{1}{2}K\int_{\vec{x}} R_1(s'(\vec{x}))[\vec{\nabla}s'(\vec{x})]^2 , \tag{2.17}$$

where

$$P_1(s'(\vec{x})) = -2^d w[\ln I(z) - \ln I(0)] , \tag{2.18}$$

$$R_1(s'(\vec{x})) = \tfrac{1}{8}\alpha^2 C_d \left\langle \left(\left\langle \frac{dQ(z)}{dz} \right\rangle - \frac{dQ(z)}{dz} \right)^2 \right\rangle - \tfrac{1}{4}\alpha^2 \langle W(z) \rangle , \tag{2.19}$$

$$C_d = w\rho_d V . \tag{2.20}$$

We now have $\mathcal{H}_1$ in the same form as $\mathcal{H}$ and can set up recursion formulas to carry out the integrations for each momentum shell. Letting $\mathcal{H} = \mathcal{H}_0$ we define

$$Q_l(y) = w^{-1} P_l((2w/K\rho_d)^{1/2} y) , \tag{2.21}$$

$$W_l(y) = R_l((2w/K\rho_d)^{1/2} y) \tag{2.22}$$

and allow independent scale factors α_l for each l, to be fixed by the requirement $W_{l+1}(0) = 1$. Then we obtain the recursion formulas, Eqs. (1.4)–(1.8), stated in the Introduction. The quantity C_d, which is dimensionless, is evaluated in Appendix B.

B. Exponent Relations

To determine the critical exponents we look at the spin-spin correlation function. Its Fourier transform $g(\vec{k})$ has the approximate form[11]

$$g(\vec{k}) \simeq \alpha_0^2 \cdots \alpha_{l-1}^2 (K\rho_d/2)^{-1} G_l(0) \tag{2.23}$$

for $\vec{k}$ lying in the lth momentum shell, i.e., $|\vec{k}| \sim 2^{-l}$. $G_l(z)$ is defined as

$$G_l(z) = [M_l(z)]^{-1} \int_{-\infty}^{\infty} dy\, y^2 \exp\{\cdots\} , \tag{2.24}$$

where the argument of the exponential can be read from Eq. (2.9) defining $M_l(z)$.

By appropriate choice of $Q_0(y)$ and $W_0(y)$ we expect to see critical behavior manifest in a fixed point of the recursion formulas:

$$Q_l(y) \to Q_c(y) , \quad W_l(y) \to W_c(y) , \quad \alpha_l \to \alpha_c$$

after decay of the initial transients corresponding to the "irrelevant variables."[12] Therefore, at the critical temperature $g(\vec{k})$ will have the form

$$g(\vec{k}) \simeq \alpha_0^2 \cdots \alpha_t^2 (\alpha_c^2)^{l-t-1} (K\rho_d/2)^{-1} G_c(0) , \tag{2.25}$$

where $\alpha_0^2 \cdots \alpha_t^2$ represents the initial transient decay to the fixed point. In the limit of large l, small k, we can neglect the departure of the transient terms from α_c and write

$$\lim_{\vec{k}\to 0} g(\vec{k}) \propto \alpha_c^{2l} (K\rho_d/2)^{-1} G_c(0) . \tag{2.26}$$

The critical exponent η is defined by the relation

$$\lim_{\vec{k}\to 0} g(\vec{k}) \propto \frac{1}{k^{2-\eta}} . \tag{2.27}$$

This in turn implies $\alpha_c = 2^{1-\eta/2}$.

If we are above the critical temperature, $Q_l(y)$ will diverge away from $Q_c(y)$ as we integrate out momenta smaller than the inverse correlation length ξ^{-1}. Then $g(\vec{k})$ will have the form

$$g(\vec{k}) \simeq \alpha_0^2 \cdots \alpha_t^2 (\alpha_c^2)^{l_M - t - 1} \alpha_{l_M+1}^2 \cdots \alpha_{l-1}^2 (K\rho_d/2)^{-1} G_l(0) \tag{2.28}$$

for $|\vec{k}| < 2^{-l_M} = \xi^{-1}$. The α_l for $l > l_M$ reflect the divergence of $Q_l(y)$ away from the fixed point. Since $Q_l(y)$ becomes Gaussian for large l, $T > T_c$, α_l approaches its Gaussian value 2 required by the normalization condition.[12] Consequently, for large l, $G_l(0)$ behaves like $2^{2(l_M - l)}$. Thus, for small $|\vec{k}|$,

$$g(\vec{k}) \propto \alpha_c^{2l_M} = \xi^{2-\eta} . \tag{2.29}$$

As we approach the critical temperature ξ diverges as $(K - K_c)^{-\nu}$, so

$$g(k) \propto (K - K_c)^{-(2-\eta)\nu} , \quad |\vec{k}| < \xi^{-1} . \tag{2.30}$$

Since the magnetic susceptibility χ is the correlation function $g(\vec{k})$ evaluated at $\vec{k} = 0$, we have

$$\chi \sim (K - K_c)^{-(2-\eta)\nu} . \tag{2.31}$$

The divergence of χ at the critical temperature is usually written $\chi \sim (K - K_c)^{-\gamma}$, so we have the exponent relation $\gamma = (2 - \eta)\nu$.[13]

The exponent ν is determined from the linearized recursion formulas as discussed by Wilson.[1] It will not be discussed here.

III. SOLUTION OF THE RECURSION FORMULAS

A. ϵ Expansion

We now look at solutions to the recursion formulas. For the Gaussian model, $Q_0(y) = r_0 y^2$, $W_0(y) = 1$, we see immediately that the fluctuation term

$$\left\langle \left(\left\langle \frac{dQ_0(z)}{dz} \right\rangle - \frac{dQ_0(z)}{dz} \right)^2 \right\rangle$$

is zero, and hence the relation

$$W_{l+1}(y) = \tfrac{1}{4}\alpha_l^2 \langle W_l(2^{-d/2}\alpha_l y) \rangle = \tfrac{1}{4}\alpha_l^2 \tag{3.1}$$

determines the trivial fixed point $W_c(y) = 1$, $\alpha_c = 2$. The recursion formula for $Q_l(y)$ now has Wilson's form and determines the trivial fixed point $Q_c(y) = 0$, as discussed in detail in Wilson's paper.

More interesting are small departures from the Gaussian model. It can be shown that functions of the form

$$Q_l(y) = r_l y^2 + \lambda_l y^4 , \quad \lambda_l \ll r_l \tag{3.2}$$

$$W_l(y) = 1 + s_l y^2 , \quad s_l \sim O(\lambda_l^2) \tag{3.3}$$

are preserved by the recursion formulas, provided one calculates only to order λ_l^2. Using the method of ϵ expansion about dimension four,[2] $d = 4 - \epsilon$, we can write the recursion formulas, to order ϵ^2, as

$$r_{l+1} = 4[r_l + 3\lambda_l - 3\lambda_l r_l + \tfrac{1}{2}s_l - 9\lambda_l^2] , \tag{3.4}$$

$$\lambda_{l+1} = (1+\epsilon \ln 2)\lambda_l - 9\lambda_l^2 , \tag{3.5}$$

$$s_{l+1} = \tfrac{1}{4}s_l + 9C_4\lambda_l^2 , \tag{3.6}$$

$$1 = \tfrac{1}{4}\alpha_l^2 + \tfrac{1}{2}s_l , \tag{3.7}$$

where the last equation represents the normalization requirement $W_{l+1}(0)=1$.

From Eqs. (3.4)–(3.7) we see that, to order ϵ^2, the behavior of s_l is controlled by the behavior of λ_l. Starting from the arbitrary (small) s_0, s_l will approach the fixed point s_c determined by the asymptotic behavior $\lambda_l \to \lambda_c$. We see from Eq. (3.6) that in the absence of the driving term the variable s_l will decrease to zero with successive iterations of the recursion formulas. Hence s_0 is an "irrelevant variable" with respect to the Gaussian fixed point in the sense of Kadanoff.[14,15] Therefore once $\lambda_0 > 0$ and s_0 are arbitrarily specified, we need only fix r_0 at its critical value $r_0 = r_0(\lambda_0, s_0)$ in order to reach the nontrivial fixed point as $l \to \infty$. So, in zero magnetic field, we still need specify only one external condition (the temperature) in order to see critical behavior.

The normalization condition implies η acquires a nonzero value first in order ϵ^2. Writing $\alpha_l^2 = 2^{2-\eta_l} \simeq 4(1-\eta_l \ln 2)$ we can then solve the recursion formulas to determine the nontrivial fixed point:

$$r_c = -\tfrac{4}{9}\epsilon \ln 2 + O(\epsilon^2) , \tag{3.8}$$

$$\lambda_c = \tfrac{1}{9}\epsilon \ln 2 + O(\epsilon^2) , \tag{3.9}$$

$$s_c = \tfrac{4}{27} C_4 \epsilon^2 \ln^2 2 + O(\epsilon^3) , \tag{3.10}$$

$$\eta = \tfrac{2}{27} C_4 \epsilon^2 \ln 2 + O(\epsilon^3) . \tag{3.11}$$

We also have the trivial Gaussian fixed point:

$$r_c = \lambda_c = s_c = \eta = 0 . \tag{3.12}$$

Substituting for numerical factors we find, to order ϵ^2,

$$\eta = 0.050\ \epsilon^2 . \tag{3.13}$$

Here we have *qualitative* agreement with *exact* results to order ϵ^2 which give $\eta = \frac{1}{54}\epsilon^2$.[2,16] More than qualitative agreement cannot be expected. The other exponents, ν and γ, are unchanged, to order ϵ, from the results of Wilson and Fisher's analysis[2] of the $\eta = 0$ recursion formulas. They will not be discussed further here.

B. Numerical Results

In three dimensions the non-Gaussian critical behavior was also calculated numerically from the recursion formulas. The functions $Q_l(z)$ and $W_l(z)$ were calculated on a uniformly spaced mesh of points with spacing 0.1 from $z=0$ to $z=5$. Outside of this range the functions were approximated with the simple power behavior derived by analytic investigation of the asymptotic behavior of the recursion formulas

$$Q_l(z) \sim z^{p_l}, \quad p_l = \frac{d \ln 2}{(\frac{1}{2}d \ln 2 - \ln \alpha_l)} , \tag{3.14}$$

$$W_l(z) \sim z^{m_l}, \quad m_l = \frac{-2(\ln 2 - \ln \alpha_l)}{(\frac{1}{2}d \ln 2 - \ln \alpha_l)} , \tag{3.15}$$

though the extrapolated values were of little importance as they typically involved values $Q_l(z) > 100$. The y integration was calculated neglecting the region $|y| > 5.0$. The integrals were calculated by means of Simpson's rule[17] and the new $Q_{l+1}(y)$ and $W_{l+1}(y)$ were determined using quadratic interpolation between the mesh points. All calculations were performed using double-precision arithmetic.

The initial functions were chosen

$$Q_0(y) = r_0 y^2 + \lambda_0 y^4 , \tag{3.16}$$

$$W_0(y) = 1 , \tag{3.17}$$

with λ_0 fixed at 0.1, and r_0 varied to locate the critical point. The critical value r_c was easily located by observing different critical behavior of $Q_l(y)$ for large l whether r_0 was above or below r_c. This behavior is illustrated in Fig. 2 where the two types of noncritical behavior are compared with $Q_c(y)$. The value r_c was determined by specifying r_0 with increasing precision as the number of iterations increased so as to keep $Q_l(y)$ from diverging from $Q_c(y)$ via the noncritical behavior previously mentioned. The value for r_c was found to be

$$r_c = -0.45321\cdots . \tag{3.18}$$

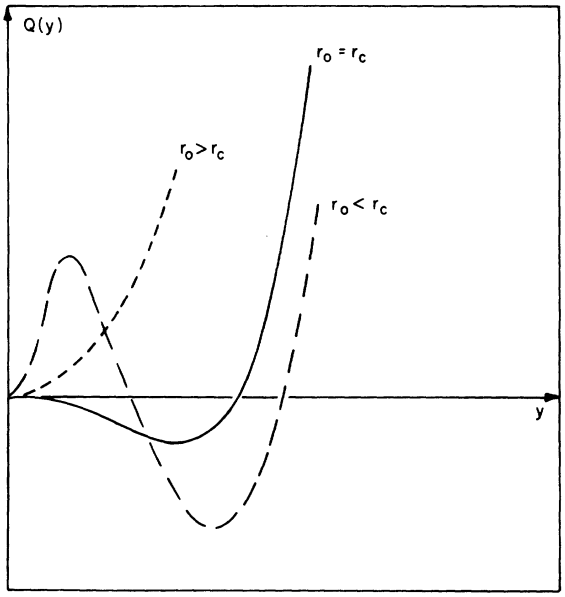

FIG. 2. Qualitative large-l behavior of the function $Q_l(y)$ vs y for values of $r_0 > r_c$, $r_0 = r_c$, and $r_0 < r_c$.

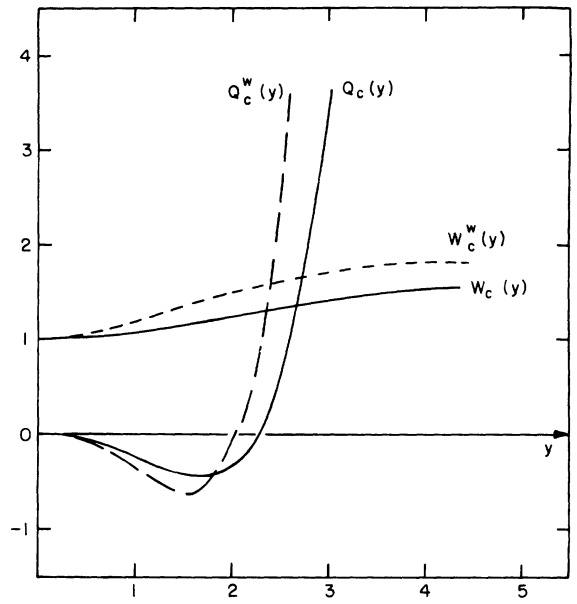

FIG. 3. Critical functions $Q_c(y)$ and $W_c(y)$ vs y compared with the functions $Q_c^w(y)$ and $W_c^w(y)$.

The critical functions $Q_c(y)$ and $W_c(y)$ are plotted in Fig. 3. The critical value of $\alpha_l = \alpha_c$ was found to be

$$\alpha_c = 1.960\,, \tag{3.19}$$

which gives a value of $\eta = 0.058$. Numerical calculations based on high-temperature expansions of the three-dimensional Ising model predict $\eta \simeq 0.056$,[5] so we have impressive agreement with the calculations of this paper.

The critical exponent ν was determined, as in Wilson's paper, by observing the exponential growth of the functions $Q_l(y)$ and $W_l(y)$ away from their respective fixed points. The growth of both $Q_l(y)$ and $W_l(y)$ could be fit to the same exponent giving

$$\nu = 0.588\,, \tag{3.20}$$

$$\gamma = (2-\eta)\nu = 1.14\,. \tag{3.21}$$

Numerical calculations based again on high-temperature series expansions of the three-dimensional Ising model predict values $\nu = 0.643$, $\gamma = 1.25$.[5] Wilson's previous calculation using the $\eta = 0$ recursion formulas gives values $\nu = 0.609$, $\gamma = 1.218$.[1] We see that our results do not give as good agreement with numerical calculations as Wilson's. However, they are fully consistent with expectations based on the qualitative nature of this derivation of the recursion formulas.

IV. CONCLUDING DISCUSSION

In this section we examine the qualitative features of the recursion formulas that act to produce $\eta > 0$. As the fluctuation term $\langle(\langle dQ_l(z)/dz\rangle - dQ_l(z)/dz)^2\rangle$ is manifestly greater than zero, for $z \neq 0$, for non-Gaussian interactions its contribution to $W_l(y)$ must be compensated by a decrease in α_l from its value of 2 in Ref. 1 in order that the equations have a fixed point. This can be seen by considering the normalization condition at the fixed point: $W_c(0) = \frac{1}{4}\alpha_c^2 \langle W_c(0)\rangle$. The contributions of the fluctuation term serve to increase $W_c(z)$ relative to its value of 1 at the origin. Thus the expectation value $\langle W_c(0)\rangle > 1$, since it is essentially a weighted average of $W_c(z)$ about its value at the origin. Hence $\alpha_c = 2\langle W_c(0)\rangle^{-1/2}$ will be less than 2.

The size of the fluctuation term determines the size of η. Its contribution is regulated by the presence of $W_l(z)$ in the weight function used to determine the "expectation value," Eq. (1.5), and by α_l, which helps set the scale of variation of $Q_l(y)$ in the recursion formulas. The effect of this regulation is shown in Fig. 3, where we compare the fixed point $Q_c^w(y)$ of Wilson's $\eta = 0$ recursion formulas with $Q_c(y)$. We see that the presence of $W_c(z) > 1$, $\eta > 0$, has acted to smooth the function $Q_c(y)$ making the fluctuation contribution of its derivative smaller.

An approximate upper bound for η from $Q_c^w(y)$ can be determined by computing

$$W_c^w(y) = 1 + \tfrac{1}{2} C_d \left\langle \left(\left\langle \frac{dQ_c^w(z)}{dz} \right\rangle - \frac{dQ_c^w(z)}{dz} \right)^2 \right\rangle, \tag{4.1}$$

$$z = 2^{-d/2+1} y\,, \tag{4.2}$$

which is plotted in Fig. 3, to compare with $W_c(y)$. This provides the approximate bound $\eta \lesssim 0.2$ which is reduced through the self-regulation of $W_l(y)$ to the value obtained in the numerical calculation of Sec. III. We see thus that $\eta > 0$ is a natural feature of the non-Gaussian fixed point, and that the renormalization-group approach illustrates a simple mechanism for its occurrence.

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APPENDIX A

We wish to change from $\sum_{\vec{m}}$ back to $\int_{\vec{x}}$. Consider

$$2^d \int_{\vec{x}} f(s(\vec{x})) = \int_{\vec{x}} f(s(\vec{x}/2))$$

$$= \sum_{\vec{m}} \int_{\vec{x}\in\vec{m}} f[s(\vec{x}_{\vec{m}}/2) + \vec{\nabla}_s(\vec{x}_{\vec{m}}/2)\cdot(\vec{x} - \vec{x}_{\vec{m}})$$

$$+\tfrac{1}{2}\nabla_i\nabla_j s(\vec{x}_{\vec{m}}/2)\,(\vec{x}-\vec{x}_{\vec{m}})_i\,(\vec{x}-\vec{x}_{\vec{m}})_j+\cdots\Big]\,, \quad \text{(A1)}$$

where $x_{\vec{m}}$ is the center of the box $\vec{m}$. As in the expansion of $\int_{\vec{x}} P(s(\vec{x}))$ this becomes

$$\sum_{\vec{m}}\Big\{w^{-1}f(s(\vec{x}_{\vec{m}}/2))+\frac{V}{2}\frac{df}{ds}\,(s(\vec{x}_{\vec{m}}/2))\,[\nabla^2 s(\vec{x}_{\vec{m}}/2)]$$

$$+\frac{V}{2}\frac{d^2f}{ds^2}\,(s(\vec{x}_{\vec{m}}/2))\,[\vec{\nabla}s(\vec{x}_{\vec{m}}/2)]^2 \quad . \quad \text{(A2)}$$

We see to zeroth order

$$\sum_{\vec{m}} f(s(\vec{x}_{\vec{m}}/2))=2^d w\int_{\vec{x}} f(s(\vec{x}))\,. \quad \text{(A3)}$$

We therefore replace

$$\sum_{\vec{m}}\frac{df}{ds}\,(s(\vec{x}_{\vec{m}}/2))\,[\nabla^2_{\vec{x}/2}s(\vec{x}_{\vec{m}}/2)]$$

$$\to 2^d w\int_{\vec{x}}\frac{df}{ds}\,(s(\vec{x}))\,[\nabla^2 s(\vec{x})]\,, \quad \text{(A4)}$$

$$\sum_{\vec{m}}\frac{d^2f}{ds^2}\,(s(\vec{x}_{\vec{m}}/2))\,[\vec{\nabla}_{\vec{x}/2}s(\vec{x}_{\vec{m}}/2)]^2$$

$$\to 2^d w\int_{\vec{x}}\frac{d^2f}{ds^2}\,(s(\vec{x}))\,[\vec{\nabla}s(\vec{x})]^2 \quad \text{(A5)}$$

and have, to order $\nabla^2 s(\vec{x})$ or $[\vec{\nabla}s(\vec{x})]^2$,

$$\sum_{\vec{m}} f(s(\vec{x}_{\vec{m}}/2))=2^d w\int_{\vec{x}} f(s(\vec{x}))$$

$$-2^{d-3}w^2V\int_{\vec{x}}\frac{df}{ds}\,(s(\vec{x}))\,[\nabla^2 s(\vec{x})]$$

$$-2^{d-3}w^2V\int_{\vec{x}}\frac{d^2f}{ds^2}\,(s(\vec{x}))\,[\vec{\nabla}s(\vec{x})]^2\,. \quad \text{(A6)}$$

Consider

$$\int_{\vec{x}}\frac{df}{ds}(s(\vec{x}))\,[\nabla^2 s(\vec{x})]=-\int_x \vec{\nabla}\left(\frac{df}{ds}\,(s(\vec{x}))\right)\cdot\vec{\nabla}s(\vec{x})$$

$$+\int_{\text{surface}}\frac{df}{ds}\,(s(\vec{x}))\vec{\nabla}s(\vec{x}). \quad \text{(A7)}$$

Periodic boundary conditions make the surface term vanish, and we have

$$\int_{\vec{x}}\frac{df}{ds}\,(s(\vec{x}))\,[\nabla^2 s(\vec{x})]=-\int_x\frac{d^2f}{ds^2}\,(s(\vec{x}))\,[\vec{\nabla}_s(\vec{x})]^2 \quad \text{(A8)}$$

Therefore, to orders $[\nabla^2 s(\vec{x})]$ or $[\vec{\nabla}s(\vec{x})]^2$, we have

$$\sum_{\vec{m}} f(s(\vec{x}_{\vec{m}}/2))=2^d w\int_{\vec{x}} f(s(\vec{x}))\,. \quad \text{(A9)}$$

APPENDIX B

The constant $C_d=w\rho_d V$ is determined from the following:

$$w\equiv\frac{1}{(2\pi)^d}\int_{1\le|\vec{k}|\le 2}d^dk=\frac{\Omega_d}{(2\pi)^d}\,\frac{(2^d-1)}{d}\,, \quad \text{(B1)}$$

where Ω_d is the surface area of a unit sphere in d dimensions:

$$\rho_d\equiv\int_{\vec{x}}[\vec{\nabla}\psi_{\vec{m}0}(\vec{x})]^2=\int k^2\phi^2(k)\,\frac{d^dk}{(2\pi)^d}\,, \quad \text{(B2)}$$

where $\phi(k)=\int_{\vec{x}} e^{i\vec{k}\cdot\vec{x}}\psi(\vec{x})$ and $\int_{\vec{k}}\phi^2(\vec{k})=1$. So ρ_d is approximately equal to the average of k^2 over the momentum shell $1\le|\vec{k}|\le 2$,

$$\rho_d=\left(\frac{2^{d+2}-1}{2^d-1}\right)\frac{d}{d+2}\,, \quad \text{(B3)}$$

$$V=\int_{\vec{x}\in\vec{m}}(\vec{x}-\vec{x}_0)_i^2\,. \quad \text{(B4)}$$

Changing to spherical coordinates $|\vec{x}-\vec{x}_0|=r$, we approximate the position-space region by a sphere of volume w^{-1}. Then

$$V=\int_{\vec{x}\in\vec{m}}\cos^2\theta\; r^{d+1}\,dr\,d\Omega=\frac{\Omega_d}{d}\left(\frac{1}{d+2}\right)\left(\frac{dw^{-1}}{\Omega_d}\right)^{(d+2)/d}\,. \quad \text{(B5)}$$

So

$$C_d=\frac{d}{(d+2)^2}\left(\frac{2^{d+2}-1}{2^d-1}\right)\left(\frac{d^2(2\pi)^d}{\Omega_d^2(2^d-1)}\right)^{2/d}\,. \quad \text{(B6)}$$

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[1]K. G. Wilson, (a) Phys. Rev. B **4**, 3174 (1971); (b) Phys. Rev. B **4**, 3184 (1971).

[2]K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. **28**, 240 (1972).

[3]See also, G. A. Baker, Jr., Phys. Rev. B **5**, 2622 (1972).

[4]V. L. Ginsberg and L. D. Landau, Zh. Eksp. Teor. Fiz. **20**, 1964 (1950).

[5]See, e.g., M. E. Fisher, Rep. Prog. Phys. **30**, 731 (1967).

[6]See K. G. Wilson, Phys. Rev. B **4**, 3184 (1971), Sec. V, p. 3202.

[7]This follows because $Q_l(z)$ is an even function of z.

[8]We understand $\vec{x}_0$ to refer to the center of box $\vec{m}$.

[9]The author has done calculations when these corrections were included and found essentially the same values for the critical exponents.

[10]Again corrections are of higher order and are ignored.

[11]This is the natural extension of K. G. Wilson, in Ref. 1(b), p. 3197.

[12]See Sec. III A below.

[13]By using a different analysis of his own recursion formula, Baker has essentially derived the same result in Ref. 3.

[14]L. P. Kadanoff, Varenna Lecture Notes, 1970 (unpublished).

[15]See also Ref. 1; and K. Wilson and J. Kogut, Princeton Lecture Notes (unpublished).

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[17]See, e.g., F. Hildebrand, *Introduction to Numerical Analysis* (McGraw-Hill, New York, 1956).