Critical Behavior of Magnets with Dipolar Interactions. V. Uniaxial Magnets in d Dimensions

Amnon Aharony
Baker Laboratory, Cornell University, Ithaca, New York 14850
(Received 11 May 1973)

The exact renormalization-group approach is used to study the critical behavior for $T > T_c$, $H = 0$ of a uniaxial ferromagnetic (or ferroelectric) system in d dimensions, with exchange and dipolar interactions between the (single-component) spins. Normal Ising-like behavior is retained for $t = T/T_c - 1 \gg \hat{g} = (g\mu_B)^2/Ja^d$, where J is the exchange parameter, $g\mu_B$ is the magnetic moment per spin, and a is the lattice spacing. Crossover to a characteristic dipolar behavior occurs when $t^{\phi} \approx \hat{g}$, where $\phi = 1 + \epsilon/6$ (to first order in $\epsilon = 4 - d$). For $t \ll \hat{g}$, the leading temperature singularity in the Fourier transform of the spin-spin correlation function $\Gamma(\vec{q})$ becomes $\xi^2 \times [1 + (\xi q)^2 - h_0(\xi q^z)^2 + g_0(q^z/q)^2]^{-1}$, where h_0 and g_0 are of order $\hat{g}$, and $\xi(t)$ varies as $t^{-1/2}$ for $d > 3$, as $t^{-1/2}|\ln t|^{1/6}$ for $d = 3$, and as $t^{-\nu}$ with $1/2\nu = 1 - (3-d)/6 + O((3-d)^2)$ for $d < 3$. The susceptibility displays the expected demagnetization effects, namely, $(\chi^{-1} - g_0\mathfrak{D}) \propto \xi^{-2}$. The experimental situation is mentioned briefly.

I. INTRODUCTION

In recent work,[1–3] the behavior of magnets with both exchange and dipole-dipole interactions has been investigated using the technique of renormalization-group recursion relations.[4] d-dimensional systems with either isotropic d-component spins or with a dominant m-component isotropy were studied, with $1 < m \le d$. It was found that the dipolar interactions both increase the critical exponents (compared to pure exchange forces) and change the angular dependence of the correlation function.

It was discovered in the previous work,[5] that the case $m = 1$ is special in that the propagator for the Feynman-graph expansion goes to zero in the dipolar limit, so that the nontrivial fixed point, which, in general, yields the above results, now apparently disappears. The problem of $m = 1$, namely, of the uniaxial ferromagnetic (ferroelectric) phase transitions, has been treated in pioneering work by Larkin and Khmel'nitskii,[6] using Feynman-graph expansions for $d = 3$. Their result is, that all the thermodynamic quantities behave classically, except for logarithmic corrections.

The method of renormalization-group recursion relations has several advantages over direct Feynman-graph expansions. The main advantage is the ability to deduce from the recursion relations the regions of temperature and of other parameters in which the system is close to a given fixed point, and thence to estimate crossovers between different types of behavior. As was illustrated in Papers I and in IV, the introduction of the dipolar forces indeed increases the number of fixed points, and leads to a complicated evolution of the system as the renormalization process proceeds. In this paper the problem of the unaxial ferromagnets is discussed using the renormalization-group approach, thereby completing the previous studies.

In Sec. II we recall the definitions of the Hamiltonian and the partition function, and discuss the Gaussian model, which already gives some feeling for the critical behavior of the system. Section III includes a derivation of the recursion relations, while Sec. IV elucidates their fixed points and the crossovers between them. Typical dipolar behavior is found; this is further discussed in Secs. V and VI for $d \ge 3$, and in Sec. VII for $d < 3$. Section VIII discusses the results and refers briefly to the experimental situation.

II. GAUSSIAN MODEL

As discussed in Appendix D of I, we start with the Hamiltonian

$$\mathcal{H} = -\tfrac{1}{2} J \sum_{\vec{R}} \sum_{\vec{\delta}} s_{\vec{R}} s_{\vec{R}+\vec{\delta}} - G \sum_{\vec{R} \ne \vec{R}'} \frac{\partial^2}{\partial z^2} (|\vec{R} - \vec{R}'|^{2-d}) s_{\vec{R}} s_{\vec{R}'}, \quad (1)$$

where $s_{\vec{R}}$ is a one-component classical spin vector of unit length, pointing in the direction of the axis of spatial anisotropy (the z axis), and located at the site $\vec{R}$ of a d-dimensional lattice of cubic symmetry and coordination number c. The vectors $\vec{\delta}$ of length a run over the c nearest-neighbor sites of the origin site. In what follows we shall take $a = 1$. As usual, J denotes the exchange energy, while $G = \frac{1}{2}(g\mu_B)^2$ measures the strength of the dipole-dipole interaction.

The partition function is written

$$Z = \int_s e^{\bar{\mathcal{H}}}, \quad (2)$$

with

$$\bar{\mathcal{H}} = -\frac{\mathcal{H}}{k_B T} - \frac{1}{2} \sum_{\vec{R}} s_{\vec{R}}^2 - u \sum_{\vec{R}} s_{\vec{R}}^4, \quad (3)$$

where the last two terms come from the weighting factor, which is introduced on passing to the continuous-spin s^4 model.[4]

The Fourier transform of the dipolar part of the Hamiltonian has been discussed throughly in I. For the uniaxial ferromagnetic case, with the axis of spatial anisotropy along the z axis, the result near $\vec{q}=0$ is

$$-\frac{\partial^2}{\partial z^2}\left(\sum_l{}' e^{i\vec{q}\cdot\vec{R}_l}|\vec{R}_l-\vec{R}|^{2-d}\right)_{\vec{R}=0} = a_1\left(\frac{q^z}{q}\right)^2 - a_2(aq^z)^2 - a_3 - a_4(aq)^2 + O((aq)^4,\ (aq^z)^4)\ . \qquad (4)$$

Values of the a_i's are given in Appendix A of I. [Here a_2 replaces $a_2 - a_5$ of Eq. (9) in I.]

Using this form, $\overline{\mathcal{H}}$ may be written

$$\overline{\mathcal{H}} = -\tfrac{1}{2}\int_{\vec{q}} U_2^0(\vec{q})\,\sigma_{\vec{q}}\,\sigma_{-\vec{q}} - u_0\int_{\vec{q}}\int_{\vec{q}_1}\int_{\vec{q}_2}\sigma_{\vec{q}}\,\sigma_{\vec{q}_1}\sigma_{\vec{q}_2}\sigma_{-\vec{q}-\vec{q}_1-\vec{q}_2}\ , \qquad (5)$$

where $\int_{\vec{q}}$ means $(2\pi)^{-d}$ times the integral over $0 < |\vec{q}| < 1$, while

$$U_2^0(\vec{q}) = r_0 + q^2 - h_0(q^z)^2 + g_0(q^z/q)^2\ , \qquad (6)$$

$$r_0 = k(T-T_0)/\tilde{J}\pi^2\ , \qquad (7)$$

$$kT_0 = cJ + 2a_3 G a^{-d}, \qquad (8)$$

$$\tilde{J} = \tfrac{1}{2}cd^{-1}J - 2a_4 G a^{-d}, \qquad (9)$$

$$h_0 = 2a_2 G/\tilde{J}a^d,\quad g_0 = 2a_1 G/\tilde{J}a^d\pi^2, \qquad (10)$$

$$u_0 = u(kT/\tilde{J})^2\pi^{d-4}\ . \qquad (11)$$

Note that $\vec{q}$ in Eq. (6) is measured in units of a^{-1}. We actually take $a=1$, and the factors a^{-d} in Eqs. (8)–(10) have been inserted for clarity. Finally, the $\sigma_{\vec{q}}$ are the Fourier transforms of the spins $s_{\vec{R}}$, rescaled by the factor $(\tilde{J}/kT\pi^{d+2})^{1/2}$.

In the Gaussian model one has $u_0 = 0$, so that the spin-spin correlation function is simply the inverse of $U_2^0(\vec{q})$, namely, for $\vec{q}\neq 0$,

$$G(\vec{q}) = [r_0 + q^2 - h_0(q^z)^2 + g_0(q^z/q)^2]^{-1}. \qquad (12)$$

Thus, $G(\vec{q})$ exhibits the "classical" form of the "dipolar" spin-spin correlation function. For small h_0 and g_0, G clearly reduces to the usual Ising-like (zero-order) propagator $(r_0+q^2)^{-1}$. Remembering that $r_0 \propto (T-T_0)$ [see Eq. (7)] shows that $\gamma=1$, $\nu=\frac{1}{2}$, $\eta=0$. As g_0 increases, the divergence of G as $|\vec{q}|$ approaches zero for $r_0=0$, becomes dependent on the direction from which $\vec{q}=0$ is approached. In the limit $g_0\to\infty$, the correlation function diverges only if the point $\vec{q}=0$ is approached in a direction perpendicular to the z axis, namely, with $q^z\equiv 0$. Thus, the "soft modes" of this model are only those spin waves with the wave vectors perpendicular to the axis of spatial anisotropy, namely, the z axis. This result is similar to those obtained in Papers I and in IV for $m>1$. When $q^z=0$, the remaining parts in the correlation function have the usual Ornstein-Zernike form, and again $\gamma=1$, $\nu=\frac{1}{2}$, $\eta=0$.

For $\vec{q}\equiv 0$, the form (12) does not apply, and one must use the shape-dependent value of $U_2^0(0)$, as discussed in I. This gives

$$G(0) = [r_0 + g_0\mathfrak{D}]^{-1}\ , \qquad (13)$$

with

$$\mathfrak{D} = d^{-1} - (d-2)\Phi/a_1\ , \qquad (14)$$

where Φ is the shape-dependent sum

$$\Phi = -\sum_j R_{ij}^{-d}(1 - d\cos^2\theta_{ij}) \qquad (15)$$

(θ_{ij} is the angle between $\vec{R}_{ij}$ and the z axis), which does not depend on i for an ellipsoidally shaped sample with an axis aligned parallel to the z axis.

III. RECURSION RELATIONS

We are now ready to generalize the Gaussian model by letting $u_0\neq 0$, and expanding (2) in powers of u_0.

Following the usual renormalization procedure,[4] we can thus construct the recursion formulas. For $U_2(\vec{q})$ we find

$$U_2^{l+1}(\vec{q}) = \zeta_l^2 b^{-d}\Big[r_l + b^{-2}q^2 + (g_l - b^{-2}h_l q^2)\times\left(\frac{q^z}{q}\right)^2 + 12u_l\int_{\vec{q}_1}^{>} G(\vec{q}_1) - 96u_l^2\int_{\vec{q}_1}^{>}\int_{\vec{q}_2}^{>} G(\vec{q}_1)G(\vec{q}_2)\times G(\vec{q}_1+\vec{q}_2+b^{-1}\vec{q}) + O(u_l^3)\Big]\ , \qquad (16)$$

where $\int_{\vec{q}_1}^{>}$ denotes integration over $b^{-1}<|\vec{q}|<1$.

As usual, we arrange to keep the coefficient of q^2 constant and equal to unity. (However, see remarks in Sec. VIII.) We therefore choose the spin renormalization factor according to

$$\zeta_l^2 = b^{d+2-\eta_l},\quad \eta_l = O(u_l^2). \qquad (17)$$

The $\vec{q}$ dependence of the graphical integrals [e.g., the second integral in Eq. (16)] enters only through combinations such as $\vec{q}_1+\vec{q}_2+b^{-1}\vec{q}$. Since $|\vec{q}_1| > b^{-1}$, $|\vec{q}_2| > b^{-1}$ and since $|b^{-1}\vec{q}| \ll b^{-1}$, all these integrals are analytic in $\vec{q}$, and therefore they will not regenerate terms of the form $(q^z/q)^2$. Therefore, the recursion relation for g_l is[2]

$$g_{l+1} = b^{2-\eta_l}g_l\ . \qquad (18)$$

Consequently, the initial condition $g_0\neq 0$ will lead to $g_l = b^{(2-\eta)l}g_0\to\infty$ (where η is an appropriate average over η_l).

The recursion relations for r_l and h_l now become

$$r_{l+1} = b^{2-\eta_l}[r_l + 12u_l \int_{\vec{q}_1}^{>} G(\vec{q}_1) + O(u_l^2)] \quad (19)$$

and

$$h_{l+1} = b^{-\eta_l}[h_l - P_1(g_l, h_l)u_l^2 \ln b], \quad (20)$$

where $P_1(g_l, h_l)$ is a complicated function, resulting from the second integral in Eq. (16).

In a similar manner we can obtain the recursion formula for u_l, which reads

$$u_{l+1} = b^{4-d}[u_l - 36u_l^2 \int_{\vec{q}_1}^{>} [G(\vec{q}_1)]^2 + O(u_l^3)]. \quad (21)$$

In what follows, we shall see that it is much more convenient to consider the differential form of the recursion relations.[7] Letting $b = e^{\delta l}$, with $\delta l \ll 1$, we find, for small r_l,

$$\frac{dr_l}{dl} = (2-\eta_l)r_l + 12u_l A(g_l - h_l) - 12u_l B(g_l - h_l)r + O(u_l^2,\ u_l r_l^2) \quad (22)$$

and

$$\frac{du_l}{dl} = (4-d)u_l - 36u_l^2 B(g_l - h_l) + O(u_l^3, u_l^2 r_l), \quad (23)$$

where, from Eqs. (19) and (21),

$$A(x) = \frac{K_{d-1}}{2\pi} \int_0^{\pi} \frac{\sin^{d-2}\theta\, d\theta}{1 + x\cos^2\theta}, \quad (24)$$

$$B(x) = \frac{K_{d-1}}{2\pi} \int_0^{\pi} \frac{\sin^{d-2}\theta\, d\theta}{(1 + x\cos^2\theta)^2}, \quad (25)$$

$$K_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(\tfrac{1}{2}d). \quad (26)$$

The difference $g-h$ appears in Eqs. (22) and (23) since we are integrating over $(r + q^2 - hq^2\cos^2\theta + g\cos^2\theta)^{-1}$ over $e^{-\delta l} \leq |\vec{q}| \leq 1$, and then letting $\delta l \to 1$. Note that in our units $a = 1$, and therefore h and g have the same dimensions.

A simple calculation gives

$$A(x) = K_d[1 - x/d + O(x^2)],$$
$$B(x) = K_d[1 - 2x/d + O(x^2)] \quad (27)$$

for small x. For $g,\ h \to 0$, Eqs. (22) and (23) thus reduce to the usual Ising-model recursion formulas.[4] If $g_0,\ h_0 \neq 0$, it is clear from Eqs. (18) and (20) that after a few steps $g_l \gg h_l$, and therefore we need not study the detailed form of $P_1(g_l,\ h_l)$ in (20), since the only combination which enters in (22) and in (23) is $g-h$. However, h may become important when $g_0 = 0$, e.g., for an antiferromagnet or for a spatially anisotropic system.[8]

For $x \gg 1$ one easily finds

$$A(x) = 2b_d K_d x^{-1/2} + O(x^{-3/2}),$$
$$B(x) = b_d K_d x^{-1/2} + O(x^{-3/2}), \quad (28)$$

with

$$b_2 = \tfrac{1}{2}, \quad b_3 = \tfrac{1}{4}\pi, \quad b_4 = 1. \quad (29)$$

The x dependence of A and B in this limit is independent of d, since only the values of θ near $\frac{1}{2}\pi$ are important in the integrals (24) and (25).

From (28), (22), and (23) one sees that if g is large, the parameter u will always be multiplied by $g^{-1/2}$, so that the natural expansion parameter now becomes $ug^{-1/2}$, instead of u. This is the crucial point, which enables one to study the critical behavior even for large values of u and of g.

IV. FIXED POINTS AND CROSSOVER EXPONENTS

As mentioned above, when $g_0 = h_0 = 0$ we recover the usual Ising-model recursion relations, which yield two fixed points, namely, for $d > 4$ the Gaussian fixed point,

$$u^* = r^* = g^* = h^* = 0, \quad (30)$$

with classical exponents, and for $d < 4$ the nontrivial Ising-like fixed point,

$$g^* = h^* = 0, \quad r^* = -\tfrac{1}{6}\epsilon, \quad K_d u^* = \tfrac{1}{36}\epsilon, \quad (31)$$

where $\epsilon = 4 - d$. Clearly, for our expansion in u to be justified, u^* must be small, and therefore ϵ must be small and the expressions in (31) for r^* and for u^* must be considered as the first terms in an expansion in ϵ. However, experience with recursion relations and with ϵ expansions suggests, that even for $\epsilon = 2$, namely, for $d = 2$, the results to order ϵ^2 are reasonable approximations (note in this connection the factor of $\frac{1}{36}$ in the expression for $K_d u^*$, which is the natural expansion parameter).

As discussed in I, we expect that for small deviations of all the parameters from their fixed-point values (30) or (31), the correlation length will behave as

$$\xi(T, g) = t^{-\nu} X(g/t^{\phi}) = g^{-\nu/\phi} \tilde{X}(g/t^{\phi}), \quad (32)$$

where ν is the exponent related to the temperature instability of the fixed point, and $\nu_G = \nu/\phi$ is the exponent for the dipolar instability of the fixed point. (ϕ is the relevant crossover exponent.[9])

Linearizing (22) and (18) with $u = u^*$ and $h = h^* = 0$ yields

$$\frac{d}{dl}(r_l - r^*) = (2 - \eta - 12u^* K_d)(r_l - r^*) - \frac{12u^* K_d(g_l - g^*)}{d}, \quad (33)$$

$$\frac{d}{dl}(g_l - g^*) = (2 - \eta)(g_l - g^*).$$

The trial forms

$$r_l - r^* = Ae^{\lambda l}, \quad g_l - g^* = Be^{\lambda l}$$

yield two solutions: the first is related to the temperature instability, having $A = 1$, $B = 0$ and

$$\nu = 1/\lambda_1 = \tfrac{1}{2}[1 + \tfrac{1}{6}\epsilon + O(\epsilon^2)]. \quad (34)$$

The second is associated with the dipolar instability, having $A=-1/d$, $B=1$ [note that $a_3=a_1/d$ in Eq. (4) (see Ref. 2)], with

$$\nu_G=1/\lambda_2=1/(2-\eta). \tag{35}$$

Thus the crossover exponent from the Ising-like fixed point is

$$\phi=\nu/\nu_G=\nu(2-\eta)=\gamma=1+\tfrac{1}{6}\epsilon+O(\epsilon^2). \tag{36}$$

Similarly, $\phi=1$ for the Gaussian fixed point.

Thus, when

$$t=T/T_c-1\lesssim g_0^{1/\phi}, \tag{37}$$

the effects of dipole-dipole interactions become important.

V. DIPOLAR BEHAVIOR FOR $d\geq 3$

When g_0 does not vanish, g_l will become large after a few interations, and one must solve the full recursion formulas (18), (22), and (23). The solution of (18) is simply

$$g_l\approx g_0e^{2l}, \tag{38}$$

where we have taken $\eta\simeq 0$, assuming $u_lg_l^{-1/2}$ to be small [see Eq. (17)]. As noted after (29), $u_lg_l^{-1/2}$ is the actual expansion parameter (see below). Substituting (38) in (23), and neglecting h, gives

$$\frac{du_l}{dl}=(4-d)u_l-36u_l^2B(g_0e^{2l}) \tag{39}$$

for small u and r. This equation can be integrated to yield

$$u_l=e^{\epsilon l}\bigg/\left(\frac{1}{u_0}+36\int_0^l B(g_0e^{2l'})e^{\epsilon l'}dl'\right). \tag{40}$$

As long as $g=g_0e^{2l}$ is very small, so that [see Eq. (27)]

$$B(g_0e^{2l})\approx B(0)=K_d, \tag{41}$$

solution (40) reduces to

$$u_l\approx\epsilon/[36K_d+(\epsilon/u_0-36K_d)e^{-\epsilon l}]. \tag{42}$$

Hence u_l approaches the nontrivial Ising-like fixed point (31) for $\epsilon>0$, but the Gaussian fixed point (30) for $\epsilon<0$. At $\epsilon=0$ ($d=4$) solution (42) must be replaced by

$$u_l=u_0/(1+36K_du_0l), \tag{43}$$

which leads to logarithmic corrections.[7]

As l grows larger, the approximation (41) becomes unjustified. Assuming that u_0 is close to $\epsilon/36K_d$, u_l will first remain close to this fixed point, but will gradually deviate from u^*, as $B(g_0e^{2l})$ becomes smaller. [Note that $B(g)$ is a monotonic decreasing function of g.] Linearizing (39) near the fixed point (31) leads to

$$\frac{d}{dl}(u_l-u^*)=\epsilon\left(1-\frac{2}{K_d}B(g_0e^{2l})\right)(u_l-u^*), \tag{44}$$

which, for $\epsilon>0$, yields

$$u_l-u^*=(u_0-u^*)\exp\left(\epsilon l-\frac{2\epsilon}{K_d}\int_0^l B(g_0e^{2l'})dl'\right). \tag{45}$$

For small g, this shows that (u_l-u^*) varies as $e^{-\epsilon l}$, but as B becomes smaller, u_l gradually crosses over to the divergent behavior $e^{\epsilon l}$, and leaves the linear region.

If g_0 is large, so that we can use (28) for B, we find

$$u_l=e^{\epsilon l}\bigg/\left(\frac{1}{u_0}+\frac{36b_dK_dg_0^{-1/2}(e^{(\epsilon-1)l}-1)}{\epsilon-1}\right)\quad\text{for }\epsilon\neq 1 \tag{46}$$

or

$$u_l=\frac{g_0^{1/2}e^l}{36b_3K_3(l+\bar{l})},\quad \bar{l}=\frac{g_0^{1/2}}{36b_3K_3u_0}\quad\text{for }\epsilon=1. \tag{47}$$

Thus, for $\epsilon>0$ u_l will eventually leave the neighborhood of the fixed points (30) and (31), and start growing. The case $\epsilon=0$ ($d=4$) is a borderline, where u_l approaches the value $u_0/(1+36b_4K_4g_0^{-1/2}u_0)$. For $d=3$, u_l has the special form (47), which leads to logarithmic terms (see below).

Since u_l is increasing with l, the validity of our expansions in u_l [e.g., in Eq. (16)] has to be carefully examined. Of course, we can always expand Z_l in powers of u_l, but *a priori* it is not clear that we are justified in truncating the series after u_l or u_l^2. To justify this truncation, we follow Larkin and Khmel'nitskii[6] in noting that the true small parameter for the expansion is not u_l, but rather $u_lg_l^{-1/2}$. This follows directly from our remark at the end of Sec. III. Thus, if $u_lg_l^{-1/2}$ is small, we may neglect higher orders in $u_lg_l^{-1/2}$ (even though u_l itself may be large, so that an expansion of the partition function Z may not be truncated). As we shall see in Sec. VI, a similar argument allows us to truncate the series for the correlation function and for the susceptibility.[10]

We can now proceed with the recursion relation for r [Eq. (22)]. If we choose

$$r_0=\frac{-6u_0A(g_0-h_0)}{1-6u_0B(g_0-h_0)}\approx-6u_0A(g_0-h_0), \tag{48}$$

then, initially, r_l will change very slowly with l. If $g_0e^{2l}\ll 1$, and if $u_0=u^*$ [as given by Eq. (31)], then r_l will also approach its fixed-point value (31). As l increases, u_l starts to increase, according to (46) or (47), and A and B start to decrease. In the limit of large g, (22) becomes

$$\frac{dr_l}{dl}=2r_l+12b_dK_du_lg_l^{-1/2}e^{-l}(2-r_l)+O(u_l^2g_l^{-1}). \tag{49}$$

For $\epsilon < 1$ ($d > 3$), one finds, from (46) and (38),

$$u_l g_l^{-1/2} = e^{(\epsilon-1)l} \Big/ \left(\frac{g_0^{1/2}}{u_0} + \frac{36 b_d K_d (e^{(\epsilon-1)l} - 1)}{\epsilon - 1} \right) \to 0, \tag{50}$$

and hence the recursion relation (49) reduces to the "classical" form

$$\frac{dr_l}{dl} = 2r_l, \tag{51}$$

leading to the usual classical behavior. Thus, the borderline between classical and nonclassical behavior is moved from $d = 4$ (for pure exchange Ising models) to $d = 3$ (for exchange plus dipolar interactions).

For $\epsilon = 1$ ($d = 3$), the second term on the right-hand side of (49) is slowly decreasing, and, using (47), we have

$$\frac{dr_l}{dl} = 2r_l + \frac{\frac{1}{3}(2 - r_l)}{l + \bar{l}} + O(r_l^2). \tag{52}$$

In order to find an approach to a fixed point, we return to condition (48). For large values of g_0 this reads

$$r_0 = -12 u_0 b_3 K_3 g_0^{-1/2} = -1/3\bar{l} \tag{53}$$

[see Eq. (47)]. For small $u_0 g_0^{-1/2}$, $\bar{l}$ is very large and $|r_0|$ is very small. The resulting evolution is now

$$r_l \approx -1/3(l + \bar{l}), \tag{54}$$

so that r_l approaches the "Gaussian"-fixed-point value $r^* = 0$. However, this is not to be identified with the "true" Gaussian fixed point (30), since both u^* and g^* now are infinite, and only the ratio $(ug^{-1/2})^*$ approaches zero.

If (53) is not satisfied, then we are not at the critical point, and the solution to (52) becomes

$$r_l = r_0 e^{2l} (l/\bar{l} + 1)^{-1/3} \tag{55}$$

up to corrections of order $\ln(l + \bar{l})$.

VI. CRITICAL BEHAVIOR OF SUSCEPTIBILITY AND OF CORRELATION FUNCTION FOR $d \geq 3$

At each step of renormalization the effective correlation length decreases by a factor $b^{-1} = e^{-\delta l}$, so that

$$\xi(r_l, u_l, g_l) = e^{-l} \xi(t), \tag{56}$$

where $t = T/T_c - 1$. As shown in Sec. VII of I, the correlation function $\Gamma(\vec{q})$, defined by

$$\langle \sigma_{\vec{q}} \sigma_{\vec{q}'} \rangle = \frac{1}{Z} \int_\sigma \sigma_{\vec{q}} \sigma_{\vec{q}'} e^{\bar{\mathcal{H}}_0} = \delta(\vec{q} + \vec{q}') \Gamma(\vec{q}), \tag{57}$$

may be calculated using $\bar{\mathcal{H}}_l$ instead of $\bar{\mathcal{H}}_0$, according to

$$\Gamma(\vec{q}) = e^{l(2-\eta)} \Gamma_l(e^l \vec{q}), \tag{58}$$

provided $|\vec{q}| < e^{-l}$. If $\xi q < 1$, we may thus repeat the renormalization iterations until $\xi(r_l, u_l, g_l)$ is of order unity (equal to the lattice spacing a), and at the same time r_l becomes of order one.

For $d > 3$, r_l varies as $r_0 e^{2l}$, and therefore a simultaneous solution of $r_l = 1$ and $\xi(r_l, u_l, g_l) = 1$ gives

$$\xi^2 = e^{2l} = r_0^{-1} \propto (T - T_0)^{-1}. \tag{59}$$

Hence the critical exponent ν is equal to $\frac{1}{2}$, and the correlation function is

$$\Gamma(\vec{q}) = \xi^{2-\eta} \Gamma_l(\xi\vec{q}) = \xi^{2-\eta} \left(G_l(\xi\vec{q}) - 12 G_l^2(\xi\vec{q}) u_l \int_{\vec{q}_1} G_l(\vec{q}_1) + 32 G_l^2(\xi\vec{q}) u_l^2 \int_{\vec{q}_1} \int_{\vec{q}_2} G_l(\vec{q}_1) G_l(\vec{q}_2) G_l(\vec{q}_1 + \vec{q}_2 + \vec{q}) + O(u_l^3) \right)$$

$$= C(T - T_0)^{-1} \left\{ \left[1 + (\xi q)^2 - h_0(\xi q^*)^2 + g_0\left(\frac{q^*}{q}\right)^2 \right]^{1} + O\left((T - T_0)^{1-\epsilon} [1 + (\xi q)^2 - h_0(\xi q^*)^2 + g_0(q^*/q)^2]^{-2}\right) \right\}, \tag{60}$$

for $\epsilon < 1$ and $q\xi < 1$, where C is a constant. As in I, the order $(u_l g_l^{-1/2})$ terms exactly cancel terms which come from the deviation of r_l from unity in $G_l(\xi\vec{q})$. The order u_l^2 term in (60) becomes of order $u_l^2 g_l^{-1}$ after performing the integral. Using Eqs. (50) and (59),

$$u_l^2 g_l^{-1} \propto e^{2(\epsilon-1)l} \propto \xi^{2(\epsilon-1)} \propto (T - T_0)^{1-\epsilon};$$

this shows the origin of the last term in (60). As ϵ approaches 1 this correction clearly becomes more and more important. However, the leading singularity is represented by the first term in (60), and exhibits all the properties we described following (12), in relation to the Gaussian model.

In a similar way, the susceptibility will have the form

$$\chi \propto e^{(2-\eta)l} \Gamma_l(0) \propto [C^{-1}(T - T_0) + g_0 \mathfrak{D}]^{-1} + O((T - T_0)^{1-\epsilon} [C^{-1}(T - T_0) + g_0 \mathfrak{D}]^{-2}), \tag{61}$$

for $\epsilon < 1$, with $\mathfrak{D}$ defined in (14). The constant of proportionality is related to the spin rescaling, introduced in (5).

To complete the discussion of the case $d > 3$, consider the critical correlations, when $\xi q > 1$. In this case we can continue the renormalization steps only until $|\vec{q}|$ equals e^{-l}, and therefore, using Eq. (58),

$$\Gamma_C(\vec{q}) = |\vec{q}|^{-2+\eta} \Gamma_l(\hat{q})$$

$$= |\vec{q}|^{-2} \{[1+(g_0-h_0)(\hat{q}^z)^2]^{-1}+O(|\vec{q}|^{1-\epsilon})\}, \quad (62)$$

where we have neglected r_l compared to 1, and where $\hat{q}=\vec{q}/|\vec{q}|$. Note that g_0 and h_0 are dimensionless in our units, with $a=1$.

We now turn to the $\epsilon=1$, or $d=3$, case. For $T\neq T_c$, r_l is now given by (55), and therefore a solution of $r_l=1$ and $\xi(r_l,u_l,g_l)=1$ yields

$$\xi^2=e^{2l}\propto r_0^{-1}|\ln r_0|^{1/3}\propto(T-T_0)^{-1}|\ln(T-T_0)|^{1/3}. \quad (63)$$

When $\xi q \ll 1$ this in turn gives

$$\Gamma(\vec{q})\propto \frac{|\ln(T-T_0)|^{1/3}}{(T-T_0)[1+(\xi q)^2-h_0(\xi q^z)^2+g_0(q^z/q)^2]} +O\left(\frac{|\ln(T-T_0)|^{-4/3}}{(T-T_0)[1+(\xi q)^2-h_0(\xi q^z)^2+g_0(q^z/q)^2]^2}\right), \quad (64)$$

where again the order $u_l g_l^{-1/2}$ terms cancel and the last term comes from the contribution of order $u_l^2 g_l^{-1}\propto(l+\bar{l})^{-2}\propto(\ln\xi)^{-2}$.

For $\epsilon=1$, the susceptibility becomes

$$\chi\propto[\xi^{-2}+g_0\mathfrak{D}]^{-1}+O(\xi^{-2}(\ln\xi)^{-2}(\xi^{-2}+g_0\mathfrak{D})^{-2}), \quad (65)$$

while the critical correlation function is

$$\Gamma_C(\vec{q})=|\vec{q}|^{-2}\{[1+(g_0-h_0)(\hat{q}^z)^2]^{-1}+O(|\ln q|^{-1})\}. \quad (66)$$

VII. DIPOLAR BEHAVIOR FOR 2.99 DIMENSIONS

For $d<3$, or $\epsilon>1$, the relation (50) no longer holds. Instead, we have, from Eqs. (47) and (38),

$$u_l g_l^{-1/2}=e^{(\epsilon-1)l}\bigg/\left(\frac{g_0^{1/2}}{u_0}+\frac{36b_dK_d(e^{(\epsilon-1)l}-1)}{\epsilon-1}\right) \to \frac{\epsilon-1}{36b_dK_d}, \quad (67)$$

and therefore our expansion in powers of $u_l g_l^{-1/2}$ may be truncated only for small values of $\epsilon-1$, e.g., $d=2.99$.

For large values of l, we may approximate $u_l g_l^{-1/2}$ using (67), and (49) thus becomes

$$\frac{dr_l}{dl}=2r_l+\tfrac{1}{3}(\epsilon-1)(2-r_l)+O(r_l^2,\ (\epsilon-1)^2), \quad (68)$$

which leads to a new fixed point

$$r^*=-\tfrac{1}{3}(\epsilon-1)+O((\epsilon-1)^2). \quad (69)$$

If we choose r_0 according to (48), r_l will approach (69), and we shall be on the critical locus. If (48) is not satisfied, then eventually r_l will increase, according to

$$(r_l-r^*)\propto(T-T_c)e^{\lambda l}, \quad (70)$$

with

$$\lambda=2-\tfrac{1}{3}(\epsilon-1)+O((\epsilon-1)^2). \quad (71)$$

As usual, the critical exponent ν will therefore be given by

$$1/2\nu=\tfrac{1}{2}\lambda=1-\tfrac{1}{6}(\epsilon-1)+O((\epsilon-1)^2),\quad \epsilon>1$$
$$=1, \quad \epsilon<1. \quad (72)$$

This is to be contrasted with the pure short-range exchange result[4]

$$1/2\nu=1-\tfrac{1}{6}\epsilon+\tfrac{19}{324}\epsilon^2+O(\epsilon^3)\quad(\epsilon>0). \quad (73)$$

Even though $\epsilon-1$ (and ϵ) should both be small, it is instructive to put $d=2$. Then, (73) gives $\nu\simeq 1.16$, in close agreement with the exact result $\nu=1$, whereas (72) gives $\nu\simeq 0.6$, close to the classical value $\nu=0.5$.

We can now proceed as in Sec. VII, and write expressions for the correlation function and for the susceptibility. For $\xi q<1$, we now find

$$\Gamma(\vec{q})=C(T-T_c)^{-\gamma}\{[1+(\xi q)^2-h_0(\xi q^z)^2 +g_0(q^z/q)^2]^{-1}+O((\epsilon-1)^2)\},\quad \epsilon>1, \quad (74)$$

with

$$\gamma=(2-\eta)\nu=1+\tfrac{1}{6}(\epsilon-1)+O((\epsilon-1)^2). \quad (75)$$

Similarly, the susceptibility becomes

$$\chi\propto[C^{-1}(T-T_c)^{\gamma}+g_0\mathfrak{D}]^{-1}+O((\epsilon-1)^2), \quad (76)$$

while the critical correlation function is

$$\Gamma_C(\vec{q})=|\vec{q}|^{-2+\eta}\{[1+(g_0-h_0)(\hat{q}^z)^2]^{-1}+O((\epsilon-1)^2)\}, \quad (77)$$

where

$$\eta=O((\epsilon-1)^2). \quad (78)$$

VIII. DISCUSSION AND SUMMARY

In a recent paper, Lines[11] investigated the problem of ferroelectric phase transitions using a "correlated-effective-field theory" in three dimensions. He obtained a susceptibility which behave as $\ln(T-T_0)/(T-T_0)$. This is clearly different than Eq. (65), the difference being due to the approximate nature of Lines's approach. Lines also presents an interesting discussion of the meaning of the correlation length ξ in the presence of the dipolar interactions. If one takes a correlation function of the form (64), and Fourier transforms it to real space, one finds

$$\Gamma(\vec{R})\propto R^{-3} \quad (79)$$

as $R\to\infty$ in a general direction (with $T\neq T_c$). The absence of the exponential decay factor $e^{-R/\xi}$, which characterizes correlations in systems with short-range interactions, reflects, of course, the long-range character of the dipolar interactions. However, one need not, in general, *define* the correlation length ξ in terms of the exponential decay; various moments of the correlation function will suffice as well. Thus, from the momentum space correlation function $\Gamma(\vec{q})=\Gamma(q^x,q^y,q^z)$ one can define the correlation length by, say,

$$\xi^2 = \frac{1}{\Gamma(0)} \left(\frac{\partial \Gamma(q^x, q^y, 0)}{\partial (q^x)^2} \right)_{q^x = q^y = 0} . \tag{80}$$

Since scattering experiments observe $\Gamma(\vec{q})$, this definition is, in fact, more directly applicable than one in terms of R-space behavior.[12]

A question that should be raised concerns the fixed-point Hamiltonian. Since both u_l and g_l diverge to infinity at the fixed point, it looks as if $(-\overline{\mathcal{H}}^*)$ is infinite for all nonzero spin values; this is certainly unsatisfactory. To discuss the true situation we may start by choosing

$$\zeta_l^2 = b^d , \tag{81}$$

in place of (17), and then keeping the coefficient of $(q^z/q)^2$ constant, instead of the coefficient of q^2.[13] If we write

$$\overline{\mathcal{H}}_l = -\frac{1}{2} \int_{\vec{q}} \left[r_l + j_l q^2 - h_l (q^z)^2 + g_l \left(\frac{q^z}{q} \right)^2 \right] \sigma_{\vec{q}} \sigma_{-\vec{q}} - u_l \int_{\vec{q}} \int_{\vec{q}_1} \int_{\vec{q}_2} \sigma_{\vec{q}} \sigma_{\vec{q}_1} \sigma_{\vec{q}_2} \sigma_{-\vec{q}-\vec{q}_1-\vec{q}_2} , \tag{82}$$

the choice (81) leads to the recursion relations

$$j_{l+1} = b^{-2-\eta_l} j_l , \quad \eta_l = O(u_l^2), \tag{83}$$

$$r_{l+1} = r_l + 12 u_l j_l^{-1} A[(g_l - h_l)/j_l] - 12 u_l j_l^{-2} r_l B[(g_l - h_l)/j_l] + O(u_l^2, u_l r_l^2), \tag{84}$$

$$u_{l+1} = b^{-d} \{ u_l - 36 u_l^2 j_l^{-2} B[(g_l - h_l)/j_l] + O(u_l^3, u_l^2 r_l) \}, \tag{85}$$

$$h_{l+1} = b^{-2} h_l + O(u_l^2), \tag{86}$$

and

$$g_{l+1} = g_l . \tag{87}$$

Clearly, h_l is an irrelevant parameter, as before. On the other hand, although it follows from (83) that j_l is decreasing with l, it may not be ignored since the reciprocal j_l^{-1} appears in (84) and in (85). For large values of j_l^{-1}, the solution to (85) becomes

$$u_l = g_0^{1/2} e^{-dl} \Big/ \left(\frac{g_0^{1/2}}{u_0} + \frac{36 b_d K_d (e^{(\epsilon-1)l} - 1)}{\epsilon - 1} \right) . \tag{88}$$

Now u_l decreases with l, approaching zero; but although it is thus irrelevant, it should not be ignored! As may be seen, for example, from the last term in (84), the combination that enters into physcial quantities is now

$$g_0^{-1/2} u_l j_l^{-3/2} = e^{(\epsilon-1)l} \times \left(\frac{g_0^{1/2}}{u_0} + \frac{36 b_d K_d (e^{(\epsilon-1)l} - 1)}{\epsilon - 1} \right)^{-1} , \tag{89}$$

which is precisely the analog of $u_l g_l^{-1/2}$ in (50). We can thus proceed as before, and all the results will be the same, except that now we have no infinite parameters. The fixed point Hamiltonian will thus be

$$\overline{\mathcal{H}}^* = -\tfrac{1}{2} g_0 \int_{\vec{q}} \left(\frac{q^z}{q} \right)^2 \sigma_{\vec{q}} \sigma_{-\vec{q}} , \tag{90}$$

and the irrelevant variables j and u will have to be taken into account in studying the critical behavior as $\overline{\mathcal{H}}$ approaches $\overline{\mathcal{H}}^*$.[14]

Unfortunately, there are as yet no long high-temperature series expansions which include dipolar interactions, and therefore we are unable to compare our results with such data. Unfortunately, even if such series existed, our predictions at $d = 3$ would be difficult to test, due to the logarithmic dependence and to the effectively long-range correlations. However, the predictions for $d = 2$ are clear, and series expansions for $d = 2$ may not be too hard to calculate. [For $d = 2$, $|\vec{R} - \vec{R}'|^{2-d}$ in Eq. (1) must be replaced by $\ln|\vec{R} - \vec{R}'|$.] We hope that the present paper will encourage such series calculations to be performed.

The experimental situation is not much better. As discussed in Ref. 1, one needs magnetic materials with a low transition temperature for the dipolar effects to dominate and for crossover (37) to occur. An Ising ferromagnet with a low transition temperature is[15] Tb $(OH)_3$, which has $T_C = (3.72 \pm 0.01)$ °K. In fact, the susceptibility of this material does exhibit behavior similar to the classical Curie law. Specifically, a plot of χ^{-1} vs T appears to approach a straight line only a short distance above T_C, with a slope almost equal to that of the high-temperature Curie-law asymptote. Logarithmic corrections, such as predicted by (65), could be hidden in the experimental errors.

Another group of experiments deals with uniaxial ferroelectrics. The dynamical measurements[12] on $BaTiO_3$ and on triglycine sulfate[16] are consistent with the form (64), without the logarithmic correactions. It would certainly be interesting to test if such corrections could yield a better description of the experimental data.

The author is not aware of any two-dimensional uniaxial ferromagnetic or ferroelectric materials. Note that "two dimensional" here means that both the lattice geometry and the dipole-dipole interaction are two dimensional. Although the former may exist, the latter is certainly difficult to visualize (a system of parallel infinitely long magnetic rods?). If one considers a two-dimensional lattice with a three-dimensional dipole-dipole interaction, our results may change. For example, if the axis of spatial anisotropy is perpendicular to the plane of the lattice, the dipolar interaction is the same as an exchange interaction with $J \propto R^{-3}$, for which Fisher, Ma, and Nickel[7] predict logarithmic corrections to "classical" behavior; for $\sigma = \frac{1}{2} d = 1$ they find that both ξ and χ vary at $t^{-1} |\ln t|^{1/3}$. On the other hand, if the axis of spatial anisotropy is in the lattice plane, then we must repeat the calculation, with a modified form of the dipolar terms [which will now include $|\vec{q}|$, $(q^{z})^2/|\vec{q}|$, etc.].

In conclusion, we have been able to reproduce Larkin and Khmel'nitskii's results for $d=3$, and to extend them to $d>3$ and $d<3$, finding different behavior in these two regions. For the experimentally interesting case, $d=3$, a crossover is predicted from the usual Ising short-range behavior to a typical dipolar behavior, which is much closer to classical behavior, although it still differs from it in several important details. The case $d=2$ may be of theoretical interest, through comparisons with series calculations. The behavior of the Ising-like model is significantly different from the Heisenberg-like model or of systems with m-isotropic interactions with $m>1$.

ACKNOWLEDGMENTS

The author is indebted to Professor Michael E. Fisher for many helpful discussions. Informative conversations with Professors Kenneth G. Wilson and Robert B. Griffiths have also been helpful. The support of the National Science Foundation, partly through the Materials Science Center at Cornell University, and a scholarship received from the Fulbright-Hays Committee, are acknowledged.

---

[1]M. E. Fisher and A. Aharony, Phys. Rev. Lett. **30**, 559 (1973).

[2]A. Aharony and M. E. Fisher, fourth preceding paper, Phys. Rev. B **8**, 3323 (1973). This paper is referred to as I through the present work.

[3]A. Aharony, third preceding paper, Phys. Rev. B **8**, 3342 (1973); second preceding paper, Phys. Rev. B **8**, 3349 (1973); preceding paper, Phys. Rev. B **8**, 3358 (1973).

[4]K. G. Wilson, Phys. Rev. B **4**, 3174 (1971); Phys. Rev. B **4**, 3184 (1971); K. G. Wilson and M. E. Fisher, Phys. Rev. Lett. **28**, 240 (1972); K. G. Wilson and J. Kogut, Phys. Reports (to be published).

[5]See Appendix D in Ref. 2; and also Ref. 3, Paper IV.

[6]A. I. Larkin and D. E. Khmel'nitskii, Zh. Eksp. Teor. Fiz. **56**, 2087 (1969) [Sov. Phys.-JETP **29**, 1123 (1969)].

[7]M. E. Fisher, S.-K. Ma, and B. G. Nickel, Phys. Rev. Lett. **29**, 917 (1972); F. J. Wegner and E. K. Riedel, Phys. Rev. B **7**, 248 (1973).

[8]See Ref. 3, Paper III. The case $g=0$ also describes a spatially anisotropic exchange parameter; see e.g., L. L. Liu and H. E. Stanley, Phys. Rev. Lett. **29**, 927 (1972); T. S. Chang and H. E. Stanley, M.I.T., report of work prior to publication (1973).

[9]M. E. Fisher and D. Jasnow, *Theory of Correlations in the Critical Region* (Academic, New York, to be published).

[10]A similar situation occurs in the case of anisotropic exchange forces, when the parameters $u_{\alpha\beta}$ and r_α associated with the smaller exchange constants J_α both go to infinity, but only combinations such as $u_{\alpha\beta}/r_\alpha$ are relevant for calculations: M. E. Fisher and P. Pfeuty, Phys. Rev. B **6**, 1889 (1972); A. Aharony, Ref. 3, Paper IV.

[11]M. E. Lines, Phys. Rev. B **5**, 3690 (1972).

[12]Y. Yamada, G. Shirane, and A. Linz, Phys. Rev. **177**, 848 (1969).

[13]A different choice of ζ is used also for the long-range exchange interactions: Fisher, Ma, and Nickel, Ref. 7; J. Sak, Phys. Rev. B **8**, 281 (1973).

[14]The need to consider irrelevant operators when the fixed-point Hamiltonian has no lower bound has been discussed by Wegner and Riedel, Ref. 7.

[15]W. P. Wolf, H. E. Meissner, and C. A. Catanese, J. Appl. Phys. **39**, 1134 (1968); C. A. Catanese, A. T. Skjeltorp, H. E. Meissner, and W. P. Wolf, Yale, report of work prior to publication (1972). I thank Professor Wolf for sending me his results prior to publication.

[16]K. Deguchi and E. Nakamura, Phys. Rev. B **5**, 1072 (1972).