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Critical Behavior of Magnets with Dipolar Interactions. IV. Anisotropy

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Renormalization-group recursion formulas for magnets with dipolar interactions, derived in previous papers, are generalized to include the case of anisotropic exchange and dipolar forces. For a ferromagnet with a dominant m isotropy (the original spins having $n = d = 4 - \epsilon > m > 1$ components), the critical exponent ν is $2\nu = 1 + (1/2)(m + 2)^2\epsilon/(m^2 + 10m + 12) + O(\epsilon^2)$ [to be compared with $2\nu = 1 + (1/2)(m + 2)\epsilon/(m + 8) + O(\epsilon^2)$ for the pure-exchange case]. For antiferromagnets, the pure-exchange exponents apply.

I. INTRODUCTION

In recent work,[1-4] the behavior of magnets with both exchange forces and dipole-dipole interactions between d-component spins in $d=4-\epsilon$ dimensions was investigated for small ϵ. In this analysis, only isotropic interactions were considered, both for the exchange and the dipolar terms. In another study, Fisher and Pfeuty[5] and Wegner[6] showed how anisotropic exchange forces can be treated in the framework of renormalization-group recursion relations; the crossover from isotropic behavior to anisotropic behavior was investigated for pure exchange forces. The purpose of the present work is to generalize the former results to interactions which are anisotropic in spin space.

We shall follow closely the notation of I, and emphasize only the changes due to the anisotropy. The Hamiltonian to be discussed is

$$\mathcal{H} = -\frac{1}{2}\sum_{\alpha,\vec{R},\vec{R}'} J_\alpha(\vec{R}-\vec{R}')S^\alpha_{\vec{R}}S^\alpha_{\vec{R}'} - G\sum_{\alpha,\beta,\vec{R},\vec{R}'}\frac{\partial^2}{\partial R^\alpha\,\partial R^\beta}\left(|\vec{R}-\vec{R}'|^{2-d}\right)S^\alpha_{\vec{R}}S^\beta_{\vec{R}'}, \quad (1)$$

where the $J_\alpha(\vec{R}-\vec{R}')$ are the anisotropic exchange coefficients (assumed to be of sufficiently short range) and $G=\frac{1}{2}(g\mu_B)^2$ (g is the gyromagnetic ratio and μ_B is the Bohr magneton). The case of anisotropic g factors, when g^2 is to be replaced by $g_\alpha g_\beta$ inside the sum over components, will be discussed in Sec. V.

After translating to a continuous-spin model, with the weight factor,

$$\exp\left(-\frac{1}{2}\sum_{\vec{R}} S^2_{\vec{R}} - u\sum_{\vec{R}}(S^2_{\vec{R}})^2 - v\sum_{\vec{R},\alpha}(S^\alpha_{\vec{R}})^4\right), \quad (2)$$

Fourier transforming the spin variables and renormalizing them, we obtain the partition function

$$Z=\int_\sigma e^{\overline{\mathcal{H}}}, \quad (3)$$

with the reduced Hamiltonian

$$\overline{\mathcal{H}} = -\frac{1}{2}\int_{\vec{q}} U_2^{0,\alpha\beta}(\vec{q})\sigma^\alpha_{\vec{q}}\sigma^\beta_{-\vec{q}} - \sum_{\alpha\beta}(u^0_{\alpha\beta}+\delta_{\alpha\beta}v^0_\alpha) \times\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}\sigma^\alpha_{\vec{q}}\sigma^\alpha_{\vec{q}'}\sigma^\beta_{\vec{q}''}\sigma^\beta_{-\vec{q}-\vec{q}'-\vec{q}''}. \quad (4)$$

The momentum variables are now restricted by $0<|\vec{q}|<1$. The pair spin interaction is

$$U_2^{0,\alpha\beta}(\vec{q}) = [r^0_\alpha + q^2 + f_0(q^\alpha)^2]\delta_{\alpha\beta} + (g_0 - h_0 q^2)(q^\alpha q^\beta/q^2), \quad (5)$$

with parameters

$$r^0_\alpha = [kT-(cJ_\alpha+2\tilde{G}a_3)]/\tilde{J}_\alpha\pi^2,$$
$$u^0_{\alpha\beta} = u(kT)^2\pi^{d-4}/\tilde{J}_\alpha\tilde{J}_\beta, \quad (6)$$
$$v^0_\alpha = v(kT/\tilde{J}_\alpha)^2\pi^{d-4},$$
$$\tilde{J}_\alpha = (c/2d)J_\alpha - 2\tilde{G}a_4, \quad \tilde{G}=Ga^{-d}. \quad (7)$$

In Eqs. (6) and (7) we have, for convenience, assumed J_α to be nonzero only for the c nearest neighbors although this restriction is easily lifted. All the other quantities remain as defined in I; in particular, g_0, h_0, and f_0 are proportional to a_1, a_2, and a_5, where the a_i are the coefficients of various terms in the Fourier transform of the dipolar part in the Hamiltonian (1).

In Sec. II, the recursion relations derived in I are generalized to include the anisotropic exchange forces. Section III includes a discussion of the various possible types of critical behavior and of the crossovers between them. The critical exponent ν is evaluated for each case. In Sec. IV the discussion is extended to include anisotropic g factors, while in Sec. V the results are applied to antiferromagnets and to ferromagnets with anisotropic lattices.

II. RECURSION RELATIONS

By a simple generalization of the results of I, the propagator for the ϵ expansion now becomes, for $\vec{q}\neq 0$,

$$G^{\alpha\beta}(\vec{q}) = \frac{1}{r_\alpha^0 + q^2 + f_0(q^\alpha)^2} \times \left(\delta_{\alpha\beta} - \frac{g_0 - h_0 q^2}{q^2 + (g_0 - h_0 q^2)Q^2} \frac{q^\alpha q^\beta}{r_\beta^0 + q^2 + f_0(q^\beta)^2}\right), \tag{8}$$

with

$$Q^2 = \sum_\gamma \frac{(q^\gamma)^2}{r_\gamma^0 + q^2 + f_0(q^\gamma)^2} . \tag{9}$$

As in I, we shall ignore, from now on, the dipolar anisotropic terms involving f_0. Thus, we may also drop v_α^0 in (4). The justification for this has been discussed thoroughly in Sec. IX of I. In short, the parameter v_α^l becomes of order $u_{\alpha\beta}^l$ only for very large l, and therefore its effects are expected to be felt only at temperatures so close to T_c, as to be beyond present experimental resolution.

The recursion formula for $U_2^{l,\alpha\beta}(\vec{q})$ then becomes

$$U_2^{l+1,\alpha\beta}(\vec{q}) = \zeta_l^2 b^{-d}\Big[(r_\alpha^l + b^{-2}q^2)\delta_{\alpha\beta} + (g_l - b^{-2}h_l q^2)\frac{q^\alpha q^\beta}{q^2} + 4\Big(\delta_{\alpha\beta}\sum_\gamma u_{\alpha\gamma}^l \int_{\vec{q}_1}^{>} G^{\gamma\gamma}(\vec{q}_1) + 2u_{\alpha\beta}^l \int_{\vec{q}_1}^{>} G^{\alpha\beta}(\vec{q}_1)\Big)$$
$$- 32\sum_{\gamma\delta} u_{\alpha\delta}^l u_{\beta\gamma}^l \int_{\vec{q}_1}^{>}\int_{\vec{q}_2}^{>} [G^{\alpha\beta}(\vec{q}_1)G^{\gamma\delta}(\vec{q}_2)G^{\gamma\delta}(\vec{q}_1 + \vec{q}_2 + b^{-1}\vec{q}) + 2G^{\alpha\gamma}(\vec{q}_1)G^{\delta\gamma}(\vec{q}_2)G^{\beta\delta}(\vec{q}_1 + \vec{q}_2 + b^{-1}\vec{q})] + \cdots\Big], \tag{10}$$

where $\int_{\vec{q}_1}^{>}$ denotes integration over $b^{-1} < |\vec{q}_1| < 1$. In addition, ζ_l is the spin rescaling factor, which is chosen to be

$$\zeta_l^2 = b^{d+2-\eta_l}, \tag{11}$$

with $\eta_l = O((u_{\alpha\beta}^l)^2)$, so that the coefficient of q^2 will remain fixed equal to unity as the recursion relation is iterated.

For the fourth-order spin interactions we similarly find, to second order in $u_{\alpha\beta}^l$,

$$\sum_{\alpha\beta} u_{\alpha\beta}^{l+1} \int_{\vec{q}}\int_{\vec{q}_1}\int_{\vec{q}_2} \sigma_{\vec{q}}^\alpha \sigma_{\vec{q}_1}^\alpha \sigma_{\vec{q}_2}^\beta \sigma_{-\vec{q}-\vec{q}_1-\vec{q}_2}^\beta = \zeta_l^4 b^{-3d}\Big[\sum_{\alpha\beta} u_{\alpha\beta}^l \int_{\vec{q}}\int_{\vec{q}_1}\int_{\vec{q}_2} \sigma_{\vec{q}}^\alpha \sigma_{\vec{q}_1}^\alpha \sigma_{\vec{q}_2}^\beta \sigma_{-\vec{q}-\vec{q}_1-\vec{q}_2}^\beta - 4\sum_{\alpha\beta\gamma\delta} u_{\alpha\gamma}^l u_{\beta\delta}^l$$
$$\times\Big(\int_{\vec{q}_1}^{>} G^{\gamma\delta}(\vec{q}_1)G^{\gamma\delta}(\vec{q}_1)\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''} \sigma_{\vec{q}}^\alpha \sigma_{\vec{q}'}^\alpha \sigma_{\vec{q}''}^\beta \sigma_{-\vec{q}-\vec{q}'-\vec{q}''}^\beta + 4\int_{\vec{q}_1}^{>} G^{\gamma\beta}(\vec{q}_1)G^{\gamma\delta}(\vec{q}_1)\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''} \sigma_{\vec{q}}^\alpha \sigma_{\vec{q}}^\alpha \sigma_{\vec{q}'}^\beta \sigma_{-\vec{q}-\vec{q}'-\vec{q}''}^\delta$$
$$+ 4\int_{\vec{q}'}^{>} G^{\alpha\beta}(\vec{q}_1)G^{\gamma\delta}(\vec{q}_1)\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''} \sigma_{\vec{q}}^\alpha \sigma_{\vec{q}'}^\beta \sigma_{\vec{q}''}^\gamma \sigma_{-\vec{q}-\vec{q}'-\vec{q}''}^\delta\Big)\Big] . \tag{12}$$

The recursion relation for the dominant dipolar parameter g_l remains as in the isotropic case, namely,

$$g_{l+1} = b^{2-\eta_l} g_l . \tag{13}$$

Hence, if $g_0 \neq 0$, we see that g_l goes rapidly to infinity. After a few steps of renormalization, g_l will become large enough, so that the propagator (8) may be well approximated by

$$G^{\alpha\beta}(\vec{q}) = \frac{1}{r_\alpha^l + q^2}\left(\delta_{\alpha\beta} - \frac{1}{Q^2}\frac{q^\alpha q^\beta}{r_\beta^2 + q^2}\right) + O\left(\frac{1}{g_l}\right) . \tag{14}$$

Simple symmetry arguments can now be used, to show that $\int_{\vec{q}}^{>} G^{\alpha\beta}(\vec{q})$ vanishes for $\alpha \neq \beta$. The recursion formula for r_α^l therefore simplifies to

$$r_\alpha^{l+1} = b^{2-\eta_l}\Big[r_\alpha^l + 4\Big(\sum_\gamma{}' u_{\alpha\gamma}^l \int_{\vec{q}_1}^{>} G^{\gamma\gamma}(\vec{q}_1) \times 2u_{\alpha\alpha}^l \int_{\vec{q}_1}^{>} G^{\alpha\alpha}(\vec{q}_1)\Big) + \cdots\Big]. \tag{15}$$

Assuming $u_{\alpha\beta}^l$ to be of order ϵ, it is clear, exactly as for the pure exchange forces,[5] that we cannot have all the r_α approach a fixed point. Indeed, unless we choose the temperature T in (6) in such a way that, for some value of α, the r_α^l change slowly with l, *all* the r_α will diverge rapidly [owing to the factor $b^{2-\eta_l}$ in Eq. (15)]. This conclusion is independent of the explicit form of the integrals in (15), since they are regular functions of the r_α, which become of order $1/r_\alpha$ for large r_α.

Accordingly, let us consider[5] a dominant m isotropy, namely, $J_\alpha = J_0$ for $\alpha \leq m < d$, and $J_\alpha < J_0$ for $\alpha > m$. (The assumption that J_0 be the largest J_α is required owing to the fact that the corresponding critical temperature, to be defined shortly, will be the first one to be encountered when lowering the temperature.) We can now choose T so that r_α^0 for $\alpha \leq m$, will obey

$$(b^{-2} - 1) r_\alpha^0 \approx 4u_{\alpha\alpha}^0\Big(2\int_{\vec{q}_1}^{>} G^{\alpha\alpha}(\vec{q}_1) + \sum_\gamma \frac{J_0}{J_\gamma}\int_{\vec{q}_1}^{>} G^{\gamma\gamma}(\vec{q}_1)\Big). \tag{16}$$

With this choice, $r^l = r_\alpha^l$ for $\alpha \leq m$ will change slowly with l, while r_β^l for $\beta > m$, will grow exponentially. After a few iterations, we can thus neglect terms of order $1/r_\beta^l$ with $\beta > m$. The sum Q^2 of (9) then becomes

$$Q^2 = \frac{1}{r^l + q^2} \sum_{\gamma \le m} (q^\gamma)^2 , \tag{17}$$

while the propagator (14) takes the form

$$G_l^{\alpha\beta}(\vec{q}) \approx \frac{1}{r^l + q^2}\left(\delta_{\alpha\beta} - q^\alpha q^\beta \Big/ \sum_{\gamma\le m}(q^\gamma)^2\right) \quad (1 \le \alpha\,,\ \beta \le m),$$

$$\approx 0 \qquad \text{otherwise.} \tag{18}$$

This propagator is very similar to the dipolar propagator discussed in I: After suppressing long-range correlations between spin components which interact via the subdominant exchange couplings, it also suppresses long-range correlations between "longitudinal" components of the remaining effective m-dimensional spins. The angular integrals are thus to be performed in an m-dimensional space, whereas the final q integration is in the original d-dimensional space. Explicitly, for $d = 4 - \epsilon$ and small ϵ, we find, by the previous methods, the results

$$\int_{\vec{q}_1}^{>} G_l^{\alpha\beta}(\vec{q}) = K_4[\tfrac{1}{2}(1 - b^{-2}) - r_l \ln b + O(r_l^2, \epsilon)]\left(1 - \frac{1}{m}\right)\delta_{\alpha\beta}, \tag{19}$$

$$\int_{\vec{q}}^{>} G_l^{\alpha\beta}(\vec{q}) G_l^{\gamma\delta}(\vec{q}) = K_4[-\ln b + O(r_l, \epsilon)]$$
$$\times\left[\left(1 - \frac{2}{m}\right)\delta_{\alpha\beta}\delta_{\gamma\delta} + \frac{1}{m(m+2)}(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma} + \delta_{\alpha\gamma}\delta_{\beta\delta})\right], \tag{20}$$

with

$$K_d^{-1} = 2^{d-1}\pi^{d/2}\Gamma(\tfrac{1}{2}d) . \tag{21}$$

The factor $(1 - m^{-1})$ in Eq. (19) comes from the angular integral of the propagator (18). It reflects the suppression of the "longitudinal" components of the effective m-dimensional spins. Note that for $m = 1$ the propagator (18) exactly vanishes, and one has to go to higher-order terms.

The recursion relations for r^l and for $u^l = u^l_{\alpha\beta}$ for α, $\beta \le m$, now become

$$r^{l+1} = b^{2-\eta_l}\{r^l + 4(m+2)(1 - m^{-1})K_4 u^l$$
$$\times[\tfrac{1}{2}(1 - b^{-2}) - r^l \ln b + O(r^2)] + O((u^l)^2)]\} \tag{22}$$

and

$$u^{l+1} = b^\epsilon\left[u^l - 4K_4(u^l)^2 \ln b\left(m + 7 - \frac{12}{m} + \frac{12}{m(m+2)}\right)\right]. \tag{23}$$

For $\beta > m$ we find $r_\beta^{l+1} \approx b^{2-\eta_l} r_\beta^l \to \infty$ as $l \to \infty$. Similarly, when either α or β exceeds m we have $u_{\alpha\beta}^{l+1} \approx b^\epsilon u^l_{\alpha\beta} \to \infty$. (Note that the latter fact need not worry us, since $u_{\alpha\beta}$ always appears together with an integral involving $G^{\alpha\beta}$, which behaves as does $\delta_{\alpha\beta}/r_\alpha$ or $1/r_\alpha r_\beta$, and hence goes to zero as $b^{(\epsilon-2)l}$.)

III. FIXED POINTS AND EXPONENTS

First we note that all the fixed points which were discussed in I are still fixed points of the present problem. However, they all are unstable with respect to the r_β^0 for $\beta > m$. In particular, for $g_0 = 0$ we retain the Heisenberg-like fixed point and the Fisher-Pfeuty anisotropic fixed points.[5] For $g_0 \ne 0$ and anisotropic exchange, the Heisenberg fixed point becomes unstable with respect to both g_0 and r_β^0 for $\beta > m$. If the anisotropy is very small, namely,

$$r_\beta^0 - r^0 \ll g_0 , \tag{24}$$

then we can expect that the system will first crossover from the Heisenberg fixed point to the dipolar fixed point described in I, and then—much closer to T_c—crossover again, owing to the instability with respect to r_β^0 to a new fixed point. If condition (24) is reversed, the system will first crossover from the Heisenberg isotropic behavior to the Fisher-Pfeuty anisotropic behavior, and only then crossover to the new dipolar fixed point.

In any case we can write the correlation length near the Heisenberg fixed point in the form

$$\xi(t) = t^{-\nu} X\left(\frac{g_0}{t^{\phi_d}}, \frac{r^0_{m+1} - r^0}{t^\phi}, \frac{r^0_{m+2} - r^0}{t^\phi}, \ldots\right), \quad t = \frac{T}{T_c} - 1 , \tag{25}$$

where ϕ_d is the dipolar crossover exponent found in I, and where ϕ is the anisotropy crossover exponent found by Fisher and Pfeuty[5] and by Wegner.[6]

To find the new fixed point, we solve Eqs. (22) and (23). The solutions are

$$r^* = -\tfrac{1}{2}(m+2)(1 - m^{-1})\epsilon \Big/ \left(m + 7 - \frac{12}{m} + \frac{12}{m(m+2)}\right) + O(\epsilon^2)$$
$$= \frac{-\frac{1}{2}(m+2)^2\epsilon}{m^2 + 10m + 12} + O(\epsilon^2) , \tag{26}$$

$$u^* = \epsilon \Big/ 4K_4\left(m + 7 - \frac{12}{m} + \frac{12}{m(m+2)}\right) + O(\epsilon^2)$$
$$= \frac{m(m+2)\epsilon}{4K_4(m-1)(m^2 + 10m + 12)} + O(\epsilon^2) . \tag{27}$$

We note that $h^* = O(1)$, but we do not have to consider h^l, since $g^* = \infty$.

The eigenvalue of the linearized r equation is easily found to be

$$\Lambda = b^\lambda = b^2\left(1 - \frac{(m+2)^2\epsilon \ln b}{m^2 + 10m + 12} + O(\epsilon^2)\right), \tag{28}$$

so that the critical exponent for the correlation length is

$$\nu = \frac{\ln b}{\ln \Lambda} = \frac{1}{\lambda} = \frac{1}{2}\left(1 + \frac{(m+2)^2\epsilon}{2(m^2 + 10m + 12)} + O(\epsilon^2)\right). \tag{29}$$

As discussed in I, we feel that the equivalent of the Heisenberg model with dipolar interactions is obtained with $m=n=d=4-\epsilon$. (In fact this gives better agreement with the pure-exchange high-temperature-series exponents too!) For this case, (29) correctly reproduces the previous[1-3] isotropic dipolar result

$$\nu=\tfrac{1}{2}[1+\tfrac{9}{34}\epsilon+O(\epsilon^2)] \quad (m=4-\epsilon). \tag{30}$$

There are now two ways to obtain the case of *planar isotropy*: We could either put $m=d-1=3-\epsilon$, or put $m=2$. The two should coincide for $d=3$. Since m enters only in the order ϵ term in (29), the former is equivalent to $m=3$, unless we know the ϵ^2 terms. This reflects on the precision we can get after truncating the ϵ expansion. Of course, we expect that the two approaches will give the same limit if the ϵ-expansion converges and if we shall have enough terms. To be consistent with I, we choose to describe the planar-isotropic case by $m=d-1=3-\epsilon$, and similarly we shall describe the "biplanar-anisotropic" case by $m=d-2=2-\epsilon$. In each case, we have to compare the resulting exponent with the dominant m-isotropic pure-exchange results, which are[5]

$$\nu_{ex}=\tfrac{1}{2}\{1+[(m+2)/2(m+8)]\epsilon+O(\epsilon^2)\}. \tag{31}$$

We thus find

$$\left.\begin{aligned}\nu&=\tfrac{1}{2}[1+\tfrac{25}{102}\epsilon+O(\epsilon^2)]\\ \nu_{ex}&=\tfrac{1}{2}[1+\tfrac{25}{110}\epsilon+O(\epsilon^2)]\end{aligned}\right\}(m=d-1) \tag{32}$$

and

$$\left.\begin{aligned}\nu&=\tfrac{1}{2}[1+\tfrac{2}{9}\epsilon+O(\epsilon^2)]\\ \nu_{ex}&=\tfrac{1}{2}[1+\tfrac{2}{10}\epsilon+O(\epsilon^2)]\end{aligned}\right\}(m=d-2). \tag{33}$$

We see that the dipolar exponents are always larger than the pure-exchange ones, the difference being

$$\nu-\nu_{ex}=\frac{(m+2)\epsilon}{(m+8)(m^2+10m+22)}+O(\epsilon^2). \tag{34}$$

The case $m=1$ deserves special treatment. As we noted after (21), the propagator (18) vanishes identically for $m=1$, so that we no longer have the nontrivial dipolar fixed point. Therefore, we do not expect (29) to give a physically correct exponent for $m=1$, although mathematically one could analytically continue (29) to approach $m=1$, and find $\nu=1+\frac{9}{46}\epsilon$, or use (33) for $d=3$. The function $\nu(m)$ has therefore different limits on approaching $m=1$ from different directions. The point therefore clearly deserves special treatment. A preliminary treatment of this case was given in Appendix D of I, and further treatment will be given elsewhere.[7]

For $m=2$ or $m=d-1$ we get, for $\epsilon=1$, $\nu=0.611$ or $\nu=0.622$, so we may state that $\nu=0.617\pm0.005$, and the ambiguity mentioned above is not crucial.

IV. ANISOTROPIC g FACTORS

Physically, anisotropy normally enters through the gyromagnetic ratios $\{g_\alpha\}$. For many materials, the gyromagnetic ratios for the various spin components are different; this, in turn, is the main reason for the differences in the exchange coefficients J_α. We can thus bring the coefficient G in Eq. (1) inside the summation and generalize it to the form

$$G_{\alpha\beta}=\tfrac{1}{2}g_\alpha g_\beta\mu_B^2. \tag{35}$$

[This is not, of course, to be confused with the propagator $G^{\alpha\beta}(\vec{q})$.] Then, ignoring the f_0 terms, Eq. (5) becomes

$$U_2^{0,\alpha\beta}(\vec{q})=(r_\alpha^0+q^2)\delta_{\alpha\beta}+(g_{\alpha\beta}^0-h_{\alpha\beta}^0q^2)(q^\alpha q^\beta/q^2), \tag{36}$$

while r_α^0 in Eq. (6) must be replaced by

$$r_\alpha^0=[kT-(cJ_\alpha+\mu_B^2g_\alpha^2a^{-d}a_3)]/\tilde{J}_\alpha\pi^2, \tag{37}$$

with

$$\tilde{J}_\alpha=(c/2d)J_\alpha-\mu_B^2g_\alpha^2a^{-d}a_4. \tag{38}$$

Similarly, the propagator (8) should be replaced by

$$G^{\alpha\beta}(\vec{q})=\frac{1}{r_\alpha^0+q^2}\left(\delta_{\alpha\beta}-B^{\alpha\beta}(\vec{q})\frac{q^\alpha q^\beta}{r_\beta^0+q^2}\right), \tag{39}$$

where the $\{B^{\alpha\beta}(\vec{q})\}$ are the solutions of the linear equations

$$g_{\alpha\beta}^0-h_{\alpha\beta}^0q^2=(r_\alpha+q^2)B^{\alpha\beta}(\vec{q})+\sum_\gamma(g_{\alpha\gamma}^0-h_{\alpha\gamma}^0q^2)B^{\gamma\beta}(\vec{q})\left(\frac{q^\gamma}{q}\right)^2. \tag{40}$$

Formally, we can now repeat all the previous steps. The recursion formula for $g_{\alpha\beta}^l$ remains as in (13); hence we have $g_{\alpha\beta}^l\to\infty$. For large values of l, we may therefore consider only the dominant terms in (40). Suppose, again, that $g_{\alpha\beta}^0=g^0$ for α, $\beta\le m$, and $g_{\alpha\beta}^0<g^0$ for α and/or $\beta>m$. After several iteration steps, this difference will grow larger, and finally we shall be able to approximate (40) by

$$\sum_{\gamma\le m}B^{\beta\gamma}\frac{(q^\gamma)^2}{r^l+q^2}=1 \quad \text{for } \beta\le m$$
$$=O(1/g^l) \quad \text{for } \beta>m. \tag{41}$$

Thus, a possible solution is, for large l, $B^{\beta\gamma}=0$ for β or $\gamma>m$, and

$$B^{\beta\gamma}(\vec{q})\equiv(r^l+q^2)\Big/\sum_{\delta\le m}(q^\delta)^2=\frac{1}{Q^2} \quad \text{for } \beta,\gamma\le m, \tag{42}$$

with Q^2 given by (17). Hence the formulation reduces to the former case, and we may proceed as we did following Eq. (18).

It therefore seems, that even if both the exchange and the dipolar coefficients are anisotropic, there is a nontrivial fixed point for small ϵ. However, the case $m=1$ still needs special treatment.

V. ANTIFERROMAGNETS AND LATTICE ANISOTROPY

The extension of the present analysis to the antiferromagnetic case, treated in Ref. 4, is straightforward. The propagator (8) is now replaced by

$$G^{\alpha\beta}(\vec{q})=\frac{1}{r_\alpha^0+q^2}\left(\delta_{\alpha\beta}-\frac{h_0q^\alpha q^\beta}{(1+h_0Q^2/q^2)(r_\beta^0+q^2)}\right). \quad (43)$$

For dominant m isotropy, this reduces to

$$G^{\alpha\beta}(\vec{q})\approx\frac{1}{r^I+q^2}\left(\delta_{\alpha\beta}-\frac{h_0q^\alpha q^\beta}{r^I+q^2}+O(h_0^2)\right) \quad \text{for } \alpha,\ \beta\le m$$
$$\approx 0 \quad \text{for } \alpha \text{ or } \beta>m, \quad (44)$$

and we can proceed as in Ref. 4, replacing d by m in all angular integrals and in all α summations. The final result is that the anisotropic antiferromagnet still exhibits only the standard pure short-range exchange fixed point.[5] However, there are also small corrections to the correlation functions, of a form similar to Eq. (44).

So far, nothing specific has been said about lattice anisotropy. This is represented by the parameters f_0 in (5) and v_α^0 in (4). Such anisotropy will be felt, if at all, only very close indeed to T_c. It is thus felt that its effects are beyond present experimental reach. However, we may note that since the actual summations over diagrams are now made for m components, with $m<d$, the v-instability catastrophe discussed in I and in Ref. 4 may actually be avoided. The eigenvalue of v near the nonisotropic nondipolar exchange fixed point, for $m<d=4-\epsilon$, is now

$$\lambda_v=\epsilon\frac{m-4}{m+8}+\epsilon^2\frac{5m^2+14m+152}{(m+8)^3}+O(\epsilon^3). \quad (45)$$

(This follows from the Appendix of Ref. 4.) Hence λ_v is of order ϵ^3 for $m=2$, (at $\epsilon=1$), and very small (though positive!) for $m=3$. Thus, one should still worry about this instability for $m=3$, very close to T_c.

VI. SUMMARY

In conclusion, we have been able to derive the critical exponent ν and the form of the two-spin correlation function [which will be given by the usual pure exchange form, with the new exponents and with the angular part of Eq. (18)] for anisotropically exchange coupled magnets with dipolar interactions. Even in the anisotropic case the effect of the dipolar interactions is evidently to change the critical exponents (truncated at order ϵ) *away* from their classical values. However, the present analysis gives more insight into the meaning of this truncation, and suggests that the higher-order terms in ϵ might reverse this direction, and bring the dipolar critical exponents closer to the classical values.

The susceptibility $\chi^{\alpha\alpha}$ for $\alpha\le m$ will also have the usual form, with demagnetization corrections, and with an exponent $\gamma=2\nu$ (to order ϵ, $\eta=0$), but will be much smaller for $\alpha>m$.

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[1] M. E. Fisher and A. Aharony, Phys. Rev. Lett. **30**, 559 (1973).
[2] A. Aharony and M. E. Fisher, third preceding paper, Phys. Rev. B **8**, 3323 (1973). This paper will be referred to as I throughout the present paper.
[3] A. Aharony, second preceding paper, Phys. Rev. B **8**, 3342 (1973).
[4] A. Aharony, preceding paper, Phys. Rev. B **8**, 3349 (1973).
[5] M. E. Fisher and P. Pfeuty, Phys. Rev. B **6**, 1889 (1972).
[6] F. J. Wegner, Phys. Rev. B **6**, 1891 (1972).
[7] A. Aharony, following paper, Phys. Rev. B **8**, 3363 (1973). See also A. I. Larkin and D. E. Khmel'nitzkii, Zh. Eksp. Teor. Fiz. **56**, 2087 (1969) [Sov. Phys.-JETP **29**, 1123 (1969)].