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# Critical Behavior of Magnets with Dipolar Interactions. II. Feynman-Graph Expansion for Ferromagnets near Four Dimensions

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The Feynman-graph-expansion approach of Wilson is used to study the critical behavior for $0 < t = (T/T_c) - 1 \ll G/Ja^d$ and $H = 0$ of an isotropic ferromagnet in $d = 4 - \epsilon (\epsilon > 0)$ dimensions with exchange and dipolar interactions between $d$-component spins. [Here $G = (g\mu_B)^2/2$ measures the dipole-dipole interaction strength, $J$ is the short-range exchange parameter, and $a$ is the lattice spacing.] The susceptibility and the two-spin correlation functions are calculated to first order in $\epsilon$, and agree with previous work, based on the renormalization-group approach. In addition the correlation function for transverse-spin fluctuations at $T_c$ is investigated, yielding the critical exponent $\eta \approx 20\epsilon^2/867$ [whereas for short-range exchange forces one has $\eta \approx 3\epsilon^2/12^2$]. The limiting angular dependence of the four-spin correlation function is obtained.

## I. INTRODUCTION

The magnetic "classical" interaction between the dipoles in a ferromagnet becomes important in determining the critical behavior, if $t = (T - T_c)/T_c \ll G/Ja^d$, where $G = \frac{1}{2}(g\mu_B)^2$ measures the strength of the dipole-dipole interaction, $J$ is the short-range exchange parameter, $a$ is the nearest-neighbor lattice constant, and $d$ is the dimensionality. In Paper I of this series,[1] the exact recursion relations of Wilson's renormalization group[2] were applied to derive the critical exponents $\gamma$, $\nu$ and $\eta$, to first order in $\epsilon = 4 - d$, for a $d$-dimensional ferromagnetic system with dipole-dipole interactions between its $d$-component spins. It is apparent from the formulas in I, that the renormalization-group-recursion-relation approach becomes very cumbersome if one wishes to extend the discussion to order $\epsilon^2$. This method, in addition, did not give the correlation functions themselves very explicitly. (They were estimated by taking a finite number of renormalization steps, until the renormalized correlation length became of order $a$, the lattice spacing.)

The main purpose of the present paper is thus to investigate explicitly the behavior of the correlation functions near the critical point and, in particular, to calculate the exponent $\eta$ to second order in $\epsilon$. Following Wilson[3] we will employ a direct Feynman-graph expansion of the partition function. As we shall see, the critical behavior found in I for the two-spin correlation function and the susceptibility will be confirmed. In addition, we are able to describe the behavior of the four-spin correlation function, and to derive the exponent $\eta$ to order $\epsilon^2$.

The notation will be the same as in I, and we shall work in the framework of the same continuous-spin $s^4$ model. In order to make calculations easier, we replace the $|\vec{q}| = 1$ momentum cutoff of I (see, e.g., the beginning of Sec. IV in I) by an additional $q^4$ term in the Hamiltonian. The partition function thus becomes

$$Z = \int_\sigma e^{\bar{\mathcal{H}}}, \tag{1}$$

with

$$\bar{\mathcal{H}} = -\frac{1}{2}\sum_{\alpha\beta}\int_{\vec{q}} \{[r + q^2 + q^4 + f_0(q^\alpha)^2]\delta_{\alpha\beta} + (g_0 - h_0 q^2)(q^\alpha q^\beta/q^2)\}\sigma^\alpha_{\vec{q}}\sigma^\beta_{-\vec{q}} - u_0 \sum_{\alpha\beta}\int_{\vec{q}_1}\int_{\vec{q}_2}\int_{\vec{q}_3} \sigma^\alpha_{\vec{q}_1}\sigma^\alpha_{\vec{q}_2}\sigma^\beta_{\vec{q}_3}\sigma^\beta_{-\vec{q}_1-\vec{q}_2-\vec{q}_3} - v_0\sum_\alpha \int_{\vec{q}_1}\int_{\vec{q}_2}\int_{\vec{q}_3}\sigma^\alpha_{\vec{q}_1}\sigma^\alpha_{\vec{q}_2}\sigma^\alpha_{\vec{q}_3}\sigma^\alpha_{-\vec{q}_1-\vec{q}_2-\vec{q}_3} - \tfrac{1}{2}(r_0 - r)\sum_\alpha\int_{\vec{q}}\sigma^\alpha_{\vec{q}}\sigma^\alpha_{-\vec{q}}, \tag{2}$$

where all integrations now run over the whole $\vec{q}$ space. All the other definitions are the same as in I, namely,

$$r_0 = k(T - T_0)/\tilde{J}\pi^2, \quad \tilde{J} = (c/2d)J - 2a_4 G a^{-d},$$

$$kT_0 = cJ + 2Ga^{-d}a_3, \quad f_0 = 2Ga^{-d}a_5/\tilde{J},$$

$$g_0 = 2Ga^{-d}a_1/\tilde{J}\pi^2, \quad h_0 = 2Ga^{-d}a_2/J,$$

$$u_0 = u(kT/\tilde{J})^2\pi^{d-4}, \quad v_0 = v(kT/\tilde{J})^2\pi^{d-4}, \tag{3}$$

where $J$ is the exchange parameter, $c$ is the coordination number of the (cubic) lattice, $G = \frac{1}{2}(g\mu_B)^2$ ($g$ is the gyromagnetic ratio per spin), $a$ is the nearest-neighbor distance (with units normally chosen so that $a=1$), and $u$ and $v$ are the coefficients of the $s^4$ and $s_\alpha^4$ terms in the weighting factor. Finally, $a_1$, $a_2$, $a_3=a_1/d$, $a_4$, and $a_5$ are the coefficients in a Taylor expansion of the Fourier transform of the dipole-dipole interaction

$$-\left(\frac{\partial^2}{\partial x^\alpha\,\partial x^\beta}\sum_l{}'\,|\vec{x}_l-\vec{x}|^{2-d}e^{i\vec{q}\cdot\vec{x}_l}\right)_{\vec{x}=0}$$
$$=A^{\alpha\beta}(\vec{q})=a_1\left(\frac{q^\alpha q^\beta}{q^2}\right)-a_2q^\alpha q^\beta-[a_3+a_4q^2-a_5(q^\alpha)^2]\times\delta_{\alpha\beta}+O((q^\alpha)^4,(q^\alpha q^\beta)^2,q^4). \quad (4)$$

In the language of the renormalization group, the coefficient of the additional $q^4$ term in (2) is irrelevant. (This is why we ignored all terms of order $q^4$ in I.) Its only function is to suppress all contributions to integrals from the high-momentum ($|\vec{q}|\gg 1$) region. We choose here a different cutoff form than the one used by Wilson in Ref. 3. As explained by Nickel,[4] this choice may change the numerical values of various integrals [and, indeed, allow some to diverge—see Eqs. (20)–(22) below], but it does not change the values of the critical exponents. It also should not change the forms of the asymptotic scaling functions.

The last (mass) term in (2) is introduced so that $r$ can be related to the inverse susceptibility which vanishes as $T\to T_c$. As discussed in Sec. VII of I, the susceptibility is expected (for an ellipsoidal sample) to have the form

$$\chi^{\alpha\beta}=\delta_{\alpha\beta}/(r+g_0D^{\alpha\alpha}), \quad (5)$$

where

$$D^{\alpha\alpha}=d^{-1}+a_1^{-1}A^{\alpha\alpha}(0), \quad (6)$$

is a "demagnetization" factor. We therefore define $r$ by (5), and use the last term in (2) to relate $r$ to $r_0$ and, hence, to the temperature.

The last three terms in (2) will be considered as perturbations. Thus $e^{\overline{\mathcal{H}}}$ will be expanded in powers of $u_0$, $v_0$, and $r_0-r$, and used to calculate the correlation functions

$$\Gamma^{\alpha\beta}(\vec{q},r)=\langle\sigma_{\vec{q}}^\alpha\sigma_{\vec{q}'}^\beta\rangle/(2\pi)^d\delta^d(\vec{q}+\vec{q}')$$
$$=Z^{-1}\int_\sigma\sigma_{\vec{q}}^\alpha\sigma_{\vec{q}'}^\beta\exp\overline{\mathcal{H}}/[(2\pi)^d\delta^d(\vec{q}+\vec{q}')] \quad (7)$$

and

$$\Gamma^{\alpha\beta\gamma\delta}(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4,r)=\frac{\langle\sigma_{\vec{q}_1}^\alpha\sigma_{\vec{q}_2}^\beta\sigma_{\vec{q}_3}^\gamma\sigma_{\vec{q}_4}^\delta\rangle_c}{(2\pi)^d\delta^d(\vec{q}_1+\vec{q}_2+\vec{q}_3+\vec{q}_4)}, \quad (8)$$

where the subscript $c$ denotes the cumulant or "connected part" of the expectation value.

As discussed by Wilson,[3] these correlation functions can be expected to exhibit scaling behavior only if the parameters $u_0$, $v_0$, etc., are chosen equal (or very close) to the renormalization group fixed point values, $u^*$, $v^*$, etc. From now on we shall restrict our discussion to the isotropic dipolar fixed point discussed in I.

As shown in Sec. VII of I, we can calculate $\langle\sigma_{\vec{q}}^\alpha\sigma_{-\vec{q}}^\beta\rangle$ in (7) for $|\vec{q}|<b^{-l}$, by using $\overline{\mathcal{H}}_l$ instead of $\overline{\mathcal{H}}$ and $\zeta^l\sigma_{\vec{q}}^\alpha$ instead of $\sigma_{\vec{q}}^\alpha$. (We assume $u_0=u^*$, $v_0=v^*=0$, and hence $u_l\approx u^*$, $v_l\approx 0$, $\eta_l\approx\eta^*=\eta$, $\zeta_l\approx\zeta^*=\zeta$.) The same holds for $\langle\sigma_{\vec{q}_1}^\alpha\sigma_{\vec{q}_2}^\beta\sigma_{\vec{q}_3}^\gamma\sigma_{\vec{q}_4}^\delta\rangle$ in (8). Only when one of the momenta $|\vec{q}|$ is of order $b^{-l}$, do we have to return to (7) and (8) and perform the integration over the whole range of $\vec{q}$ values (instead of the partial ranges $b^{-1}<|\vec{q}|<1$ used in I). If this value of $l$ is large enough, and if we are close enough to $T_c$, we may neglect all irrelevant parameters (assuming they are initially small enough). This allows us to ignore $h_0$ and $1/g_0$ in the following calculations. In any case, for small enough $q^2$ (to be defined later), terms involving $1/g_0$ and $h_0$ may be considered as perturbations to our results, provided we assume

$$t=(T-T_c)/T_c\ll G/Ja^d\ll 1. \quad (9)$$

We shall restrict our discussion to this range.

However, the coefficient $f_0$, which introduces cubic anisotropy, does, in fact, lead to new relevant operators. Nevertheless, we demonstrated in Sec. VIII of I that the effects remain small for a very large range of $l$ provided $f_0$ is small enough. Moreover, if the system is "fully isotropic," i.e. rotationally invariant, then $f_0=0$ and may certainly be ignored; this case was exemplified in I by a "liquid ferromagnet." In what follows we want to investigate the isotropic dipolar fixed point found in I, and hence we shall set $f_0=0$. Likewise we shall not have to retain the anisotropic four-spin terms of the form

$$v_0\delta_{\alpha\beta}\sigma_{\vec{q}}^\alpha\sigma_{\vec{q}'}^\alpha\sigma_{\vec{q}''}^\beta\sigma_{-\vec{q}-\vec{q}'-\vec{q}''}^\beta\ . \quad (10)$$

These entered in I, even with $v_0=0$, since under renormalization they are generated by the $f$ terms. Our starting point is thus the expression (2), with $f_0=v_0=0$. The range of validity of the results was discussed at length in Sec. IX of I.

With this introduction we are ready to use the Feynman-graph expansions, and to discuss the behavior of the magnetostatic susceptibility (Sec. II), the two-spin, and the four-spin correlation functions (Secs. III and IV). A comparison of their scaling properties enables us to choose the appropriate value of $u_0$, and thus obtain the exponents $\gamma$ and $\eta$ to lowest order in $\epsilon$ (Sec. V). The Appendix includes the details of the calculations involved in deriving the two-spin correlation function, and hence the exponent $\eta$, to order $\epsilon^2$.

## II. SUSCEPTIBILITY

As discussed in I, the propagator for the Feynman-graph expansion is given by the two-spin correlation function of the unperturbed Hamiltonian. For

Hamiltonian (2), with $u_0=v_0=f_0=0$ and $r_0=r$, this becomes

$$G^{\alpha\beta}(\vec{q})=\frac{\delta_{\alpha\beta}-(q^\alpha q^\beta/q^2)}{r+q^2+q^4}+\frac{q^\alpha q^\beta/q^2}{r+g_0+(1-h_0)q^2+q^4}, \tag{11}$$

provided $\vec{q}\neq 0$. For $0<q^2\ll r\ll g_0$, we may approximate (11) by

$$G^{\alpha\beta}(\vec{q},r)=\frac{\delta_{\alpha\beta}-(q^\alpha q^\beta/q^2)}{r+q^2+q^4}. \tag{12}$$

It is easy to see that the corrections to this form are of order $1/g_0$. In the calculation of the critical exponents $\eta$ and $\gamma$, we shall expand various correlation functions to obtain terms proportional to $\ln q$ and to $\ln r$, respectively. One may check easily that none of the correcting terms will contribute such logarithms. For example, the last term in (11) will generate terms such as $|\ln(r+g_0)| \approx |\ln g_0| \ll |\ln r|$ for $1>g_0\gg r$. The corrections may thus be ignored.

As discussed at length in Sec. VII of I, the expressions (11) or (12) are valid only for $\vec{q}\neq 0$. For $\vec{q}=0$, $G^{\alpha\beta}(\vec{q})$ is the inverse matrix of

$$U^{\alpha\beta}=r\delta_{\alpha\beta}+g_0\psi^{\alpha\beta}, \tag{13}$$

where

$$\psi^{\alpha\beta}=d^{-1}\delta_{\alpha\beta}+a_1^{-1}A^{\alpha\beta}(0)=D^{\alpha\alpha}\delta_{\alpha\beta} \tag{14}$$

(the last step is justified for a sample with ellipsoidal boundaries) and hence

$$G^{\alpha\beta}(0,r)=\delta_{\alpha\beta}/(r+g_0D^{\alpha\alpha}). \tag{15}$$

We are now ready to calculate the two-spin correlation function using (7). From the same Feynman graphs employed in I, Fig. 1, we find

$$\begin{aligned}\Gamma^{\alpha\beta}(\vec{q},r)=G^{\alpha\beta}(\vec{q},r)&-4u_0\sum_\gamma\Big(G^{\alpha\gamma}(\vec{q},r)G^{\beta\gamma}(-\vec{q},r)\sum_\delta\int_{\vec{q}_1}G^{\delta\delta}(\vec{q}_1,r)\\&+2\sum_\delta G^{\alpha\gamma}(\vec{q},r)G^{\delta\beta}(-\vec{q},r)\int_{\vec{q}_1}G^{\gamma\delta}(\vec{q}_1,r)\Big)-(r_0-r)\sum_\gamma G^{\alpha\gamma}(\vec{q},r)G^{\beta\gamma}(-\vec{q},r)\\&+32u_0^2\sum_{\gamma\delta\epsilon\eta}\Big(G^{\delta\alpha}(\vec{q},r)G^{\eta\beta}(-\vec{q},r)\int_{\vec{q}_1}\int_{\vec{q}_2}G^{\gamma\epsilon}(\vec{q}_1,r)G^{\delta\eta}(\vec{q}_2,r)G^{\gamma\epsilon}(\vec{q}_1+\vec{q}_2+\vec{q},r)\\&+2G^{\delta\alpha}(\vec{q},r)G^{\epsilon\beta}(-\vec{q},r)\int_{\vec{q}_1}\int_{\vec{q}_2}G^{\gamma\epsilon}(\vec{q}_1,r)G^{\gamma\eta}(\vec{q}_2,r)G^{\delta\eta}(\vec{q}_1+\vec{q}_2+\vec{q},r)\Big)+O(u_0^3).\end{aligned} \tag{16}$$

The discussion must now be divided into two parts; (i) $\vec{q}=0$, and (ii) $\vec{q}\neq 0$. For $\vec{q}=0$, we want $\chi^{\alpha\beta}$ to be given by (5). Together with (15) this means that the sum of all terms in (16) after the first, should vanish; this leads to

$$\begin{aligned}(r_0-r)(r+g_0D^{\alpha\alpha})^{-2}\delta_{\alpha\beta}=&-4u_0(r+g_0D^{\alpha\alpha})^{-2}\\&\times\Big(\delta_{\alpha\beta}\sum_\delta\int_{\vec{q}_1}G^{\delta\delta}(\vec{q}_1,r)\\&+2\int_{\vec{q}_1}G^{\alpha\beta}(\vec{q}_1,r)\Big)+O(u_0^2).\end{aligned} \tag{17}$$

From the angular integrals listed in Appendix B of I, we find

$$\int_{\vec{q}_1}G^{\alpha\beta}(\vec{q}_1,r)=\delta_{\alpha\beta}K_d(1-d^{-1})I(r), \tag{18}$$

where

$$K_d^{-1}=2^{d-1}\pi^{d/2}\Gamma(\tfrac{1}{2}d),\quad I(r)=\int_0^\infty\frac{q^{d-1}dq}{r+q^2+q^4}. \tag{19}$$

Thus, (17) implies

$$r_0-r=-4u_0K_d(d+2)(1-d^{-1})I(r)+O(u_0^2). \tag{20}$$

At $T_c$ we want $r$ to vanish and so

$$r_{0c}=-4u_0K_d(d+2)(1-d^{-1})I(0)+O(u_0^2), \tag{21}$$

which by (3) may be used to calculate the corrections to $T_c$. Clearly, $I(r)$ and hence $T_c$, depends on the form of the cutoff. For $d\leq 4$ our choice does not actually suffice to make $I(0)$ finite in (21). However, only the combination $[I(r)-I(0)]$ enters into the calculations of the critical exponent $\gamma$ (see below) and this is well defined. To obtain a more realistic estimate of the shift in $T_c$ we may replace our cutoff by the original cutoff $|\vec{q}|<1$ to obtain

$$I(0)\Rightarrow\tilde{I}(0)=\int_0^1\frac{q^{d-1}dq}{q^2}=\frac{1}{d-2}, \tag{22}$$

and hence

$$\frac{k(T_c-T_0)}{J\pi^2}=-4u_0K_d\left(\frac{d+2}{d-2}\right)\left(1-\frac{1}{d}\right)+O(u_0^2). \tag{23}$$

Returning to (20), we now find

$$r_0-r_{0c}=r[1+4u_0K_d(d+2)(1-d^{-1})I_1(r)], \tag{24}$$

where

$$I_1(r)=\frac{I(0)-I(r)}{r}=\int_0^\infty \frac{q^{d-1}\,dq}{(q^2+q^4)(r+q^2+q^4)}$$

$$=-\frac{1}{\epsilon}(1-r^{-\epsilon/2})+O(1), \qquad (25)$$

where $\epsilon=4-d\to 0$. The left-hand side of (24) is, by Eq. (3), equal to $k(T-T_c)/\tilde{J}\pi^2$. Expanding (24) in powers of $\epsilon$, assuming $u_0=O(\epsilon)$, and neglecting terms of order $\epsilon^2$ now gives

$$(T-T_c)\propto r[1-9K_4u_0\ln r+O(\epsilon^2)]=r^{1/\gamma}, \qquad (26)$$

with

$$1/\gamma=1-9K_4u_0+O(\epsilon^2). \qquad (27)$$

We thus confirm, to order $\epsilon$, form (5) of $\chi^{\alpha\beta}$, with $r\propto(T-T_c)^\gamma$. The actual value of $\gamma$ still depends on $u_0$; the determination of $u_0$ will be discussed in Sec. IV.

## III. TWO-SPIN CORRELATIONS

We turn now to the case of $\vec{q}\neq 0$. Since the integral (18) appears in Eq. (16) for any value of $\vec{q}$, it is easy to see that our choice of $r$ in Eq. (20), ensures that the $u_0$ and the $r_0-r$ terms in Eq. (16) cancel for any value of $\vec{q}$. Therefore, to order $\epsilon$, we also have

$$\Gamma^{\alpha\beta}(\vec{q},r)=(r+q^2+q^4)^{-1}[\delta_{\alpha\beta}-(q^\alpha q^\beta/q^2)]+O(\epsilon^2), \qquad (28)$$

with $r\propto(T-T_c)^\gamma$. This confirms the conclusions of I. The $u_0^2$ terms in Eq. (16) will give corrections of order $\epsilon^2$ both to $r$ (namely, to $T_c$ and to $\gamma$) and to the $q$ dependence of $\Gamma^{\alpha\beta}$. We shall restrict our attention to the latter, by considering only the critical-point $\vec{q}$ dependence of $\Gamma^{\alpha\beta}(\vec{q},0)$. Since $u_0=O(\epsilon)$, we may calculate all the integrals in the last term of (16) at $d=4$ and with $r=0$. Furthermore, the terms with $\vec{q}=0$ in this expression will contribute only to the susceptibility (i.e., to $r_0-r$). It is sufficient, therefore, to consider the expression

$$\Gamma^{\alpha\beta}(\vec{q},0)=G^{\alpha\beta}(\vec{q},0)+32u_0^2\sum_{\gamma\delta\epsilon\eta}\Big(G^{\delta\alpha}(\vec{q},0)G^{\eta\beta}(-\vec{q},0)\int_{\vec{q}_1}\int_{\vec{q}_2}G^{\gamma\epsilon}(\vec{q}_1,0)G^{\delta\eta}(\vec{q}_2,0)$$
$$\times[G^{\gamma\epsilon}(\vec{q}_1+\vec{q}_2+\vec{q},0)-G^{\gamma\epsilon}(\vec{q}_1+\vec{q}_2,0)]+2G^{\delta\alpha}(\vec{q},0)G^{\epsilon\beta}(-\vec{q},0)\int_{\vec{q}_1}\int_{\vec{q}_2}G^{\gamma\epsilon}(\vec{q}_1,0)G^{\gamma\eta}(\vec{q}_2,0)$$
$$\times[G^{\delta\eta}(\vec{q}_1+\vec{q}_2+\vec{q},0)-G^{\delta\eta}(\vec{q}_1+\vec{q}_2,0)]\Big)+O(u_0^3). \qquad (29)$$

The integrals which appear here are calculated in detail in the Appendix. The result to order $q^2\ln q$ is given in Eqs. (A9) and (A10). Using this result in Eq. (29), for $0<q^2\ll 1$ [so that we can ignore $q^4$ in the denominator of $G^{\alpha\beta}(\vec{q},0)$, and so that $q^2|\ln q|\gg q^2$], we finally find

$$\Gamma^{\alpha\beta}(\vec{q},0)\approx\frac{1}{q^2}\left(\delta_{\alpha\beta}-\frac{q^\alpha q^\beta}{q^2}\right)[1+\tfrac{80}{3}K_4^2u_0^2\ln q+O(u_0^3\ln q,u_0^2)]\approx\frac{\delta_{\alpha\beta}-(q^\alpha q^\beta/q^2)}{q^{2-\eta}}, \qquad (30)$$

with the exponent $\eta$ given by

$$\eta=\tfrac{80}{3}K_4^2u_0^2+O(u_0^3). \qquad (31)$$

Thus, we obtain the standard $q$ dependence for the transverse correlation function, the only change from the usual Heisenberg model results coming from the angular-dependent factor $\delta_{\alpha\beta}-(q^\alpha q^\beta/q^2)$ and from the value of $\eta$.

## IV. FOUR-SPIN CORRELATIONS

To complete our calculation of the critical indices we still need the value of $u_0$ for which the scaling forms are valid. Following Wilson,[3] we obtain $u_0$ by calculating $u_R$, which, however, now needs to be defined in a somewhat different way. Consider first the four-spin correlation function [Eq. (8)]. Using the same graphs as in I we now find, for small enough values of $|\vec{q}_1|$, $|\vec{q}_2|$, $|\vec{q}_3|$, and $|\vec{q}_4|$,

$$\Gamma^{\alpha\beta\gamma\delta}(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4,r)=-u_0(r+q_1^2)^{-1}(r+q_2^2)^{-1}(r+q_3^2)^{-1}(r+q_4^2)^{-1}\left\{F^{\alpha\beta\gamma\delta}\left(\frac{\vec{q}_1}{q_1},\frac{\vec{q}_2}{q_2},\frac{\vec{q}_3}{q_3},\frac{\vec{q}_4}{q_4}\right)\right.$$
$$-4u_0\left[F^{\alpha\beta\gamma\delta}\left(\frac{\vec{q}_1}{q_1},\frac{\vec{q}_2}{q_2},\frac{\vec{q}_3}{q_3},\frac{\vec{q}_4}{q_4}\right)\sum_{\mu\xi}\int_{\vec{q}}G^{\mu\xi}(\vec{q},r)G^{\mu\xi}(-\vec{q},r)\right.$$

$$+2\sum_{\zeta\eta} F_1^{\alpha\beta\gamma\delta,\zeta\eta}\left(\frac{\vec{q}_1}{q_1},\frac{\vec{q}_2}{q_2},\frac{\vec{q}_3}{q_3},\frac{\vec{q}_4}{q_4}\right)\sum_{\epsilon}\int_{\vec{q}} G^{\epsilon\zeta}(\vec{q}_1,r)G^{\epsilon\eta}(-\vec{q},r)$$

$$+\sum_{\epsilon\mu\zeta\eta} F_2^{\alpha\beta\lambda\delta,\epsilon\mu\zeta\eta}\left(\frac{\vec{q}_1}{q_1},\frac{\vec{q}_2}{q_2},\frac{\vec{q}_3}{q_3},\frac{\vec{q}_4}{q_4}\right)\int_{\vec{q}} G^{\epsilon\mu}(\vec{q},r)G^{\zeta\eta}(-\vec{q},r)\Bigg]+O(u_0^2)\Bigg\} \qquad (32)$$

where

$$F^{\alpha\beta\gamma\delta}\left(\frac{\vec{q}_1}{q_1},\frac{\vec{q}_2}{q_2},\frac{\vec{q}_3}{q_3},\frac{\vec{q}_4}{q_4}\right)=\sum_{\epsilon\eta}\left[\left(\delta_{\alpha\epsilon}-\frac{q_1^\alpha q_1^\epsilon}{q_1^2}\right)\left(\delta_{\beta\epsilon}-\frac{q_2^\beta q_2^\epsilon}{q_2^2}\right)\left(\delta_{\gamma\eta}-\frac{q_3^\gamma q_3^\eta}{q_3^2}\right)\left(\delta_{\delta\eta}-\frac{q_4^\delta q_4^\eta}{q_4^2}\right)\right.$$
$$\left.+[2 \text{ permutations}, (\alpha\beta\gamma\delta)\to(\alpha\gamma\beta\delta), (\alpha\beta\gamma\delta)\to(\alpha\delta\beta\gamma)]\right], \qquad (33)$$

$$F_1^{\alpha\beta\gamma\delta,\zeta\eta}=\sum_{\mu}\left[\left(\delta_{\alpha\mu}-\frac{q_1^\alpha q_1^\mu}{q_1^2}\right)\left(\delta_{\beta\mu}-\frac{q_2^\beta q_2^\mu}{q_2^2}\right)\left(\delta_{\gamma\zeta}-\frac{q_3^\gamma q_3^\zeta}{q_3^2}\right)\left(\delta_{\delta\eta}-\frac{q_4^\delta q_4^\eta}{q_4^2}\right)\right.$$
$$\left.+[5 \text{ permutations}, (\alpha\beta\gamma\delta)\to(\alpha\gamma\beta\delta), (\alpha\delta\beta\gamma), (\beta\gamma\alpha\delta), (\beta\delta\alpha\gamma), (\gamma\delta\alpha\beta)]\right], \qquad (34)$$

$$F_2^{\alpha\beta\gamma\delta,\epsilon\mu\zeta\eta}\left(\frac{\vec{q}_1}{q_1},\frac{\vec{q}_2}{q_2},\frac{\vec{q}_3}{q_3},\frac{\vec{q}_4}{q_4}\right)=\left(\delta_{\alpha\epsilon}-\frac{q_1^\alpha q_1^\epsilon}{q_1^2}\right)\left(\delta_{\beta\mu}-\frac{q_2^\beta q_2^\mu}{q_2^2}\right)\left(\delta_{\gamma\zeta}-\frac{q_3^\gamma q_3^\zeta}{q_3^2}\right)\left(\delta_{\delta\eta}-\frac{q_4^\delta q_4^\eta}{q_4^2}\right)$$
$$+[11 \text{ permutations of } (\alpha\beta\gamma\delta)]. \qquad (35)$$

The limiting behavior of $\Gamma^{\alpha\beta\gamma\delta}$ for small values of $\vec{q}_1$ to $\vec{q}_4$ is evidently governed by the angular functions $F_i$, defined in Eqs. (33)–(35).

A detailed calculation of the integrals which appear in Eq. (32) for $d=4$ (see the formulas of Appendix B of I) gives

$$\int_{\vec{q}} G^{\alpha\beta}(\vec{q},r)G^{\gamma\delta}(-\vec{q},r)=[\tfrac{1}{2}\,\delta_{\alpha\beta}\delta_{\gamma\delta}+\tfrac{1}{24}\,(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma})]K_4\int_0^\infty\frac{q^3\,dq}{(r+q^2+q^4)^2}$$
$$=-\tfrac{1}{4}K_4\,[\delta_{\alpha\beta}\delta_{\gamma\delta}+\tfrac{1}{12}\,(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}+\delta_{\alpha\gamma}\delta_{\beta\delta})]\ln r+O(\epsilon\ln r,1), \qquad (36)$$

where we have neglected terms of order $1/g$, etc. Using this expression in (32) gives, after simple algebra,

$$\Gamma^{\alpha\beta\gamma\delta}(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4,r)$$
$$\propto -u_0 r^{-4}F^{\alpha\beta\gamma\delta}\left(\frac{\vec{q}_1}{q_1},\frac{\vec{q}_2}{q_2},\frac{\vec{q}_3}{q_3},\frac{\vec{q}_4}{q_4}\right)$$
$$\times[1+17K_4u_0\ln r+O(u_0^2\ln r,u_0)] \qquad (37)$$

for $q_i^2 \ll r$.

We can therefore define the analog to Wilson's $u_R$ by writing

$$\Gamma^{\alpha\beta\gamma\delta}(\vec{q}_1,\vec{q}_2,\vec{q}_3,\vec{q}_4)\propto -u_R r^{-4}F^{\alpha\beta\gamma\delta}, \qquad (38)$$

where we then must have

$$u_R=u_0[1+17K_4u_0\ln r+O(u_0^2\ln r,u_0)]. \qquad (39)$$

We can now reproduce Wilson's scaling arguments for $u_R$.[3] Since all the angular-dependent functions depend only on $\vec{q}/q$, they remain invariant. Thus, as is to be expected, the previous result still holds, that is

$$u_R\propto r^{(\epsilon-2\eta)/(2-\eta)}=1+\tfrac{1}{2}\,\epsilon\ln r+O(\epsilon^2). \qquad (40)$$

Comparing these last two expression immediately yields a unique value of $u_0$ for which the scaling relation (40) holds, namely,

$$K_4u_0=\tfrac{1}{34}\,\epsilon+O(\epsilon^2). \qquad (41)$$

As anticipated, this is in precise accord with the fixed-point value $u^*$ found in I.

## V. DISCUSSION

With the value of $u_0$ found in (41) we can now explicitly write the results, (27), (31) and (21), for the exponents $\gamma$ and $\eta$, and for the critical temperature. We find

$$1/\gamma=1-\tfrac{9}{34}\,\epsilon+O(\epsilon^2), \qquad (42)$$

$$\eta=(20/3\cdot 17^2)\epsilon^2+O(\epsilon^3)=0.023\epsilon^2+O(\epsilon^3), \qquad (43)$$

$$k(T_c-T_0)/\tilde{J}\pi^2=-\tfrac{9}{34}\,\epsilon+O(\epsilon^2). \qquad (44)$$

Again, the value of $\gamma$ confirms the calculations in I. Assuming the usual scaling relations we can repeat the calculations of Sec. X of I, to find the other standard exponents; these are not changed much by introducing $\eta$ explicitly, since the value is rather small. The present value of $\eta$ is to be compared with that for the corresponding $(4-\epsilon)$-component Heisenberg model, with short-range interactions, which is[3]

$$\eta_H=(3/12^2)\epsilon^2+O(\epsilon^3)=0.021\epsilon^2+O(\epsilon^3). \qquad (45)$$

We see that the introduction of dipole-dipole forces increases the value of $\eta$, away from the result of phenomenological theory, namely, $\eta=0$.

As already noted in I, the available experiments are far from having the precision necessary to compare values of $\eta$. Nonetheless, it would be well worthwhile to try and improve experimental techniques to a point where measurements of the correlation function at $T_c$ could verify the present predictions.

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## APPENDIX: CALCULATION OF SECOND-ORDER TERMS

Consider first the basic integral combination

$$I^{\alpha\beta}(\vec{q}) = \sum_{\delta}\int_{\vec{q}_1} G^{\alpha\delta}(\vec{q}_1, 0)G^{\beta\delta}(\vec{q}_1+\vec{q}, 0)$$

$$= \delta_{\alpha\beta}\int_{\vec{q}_1} \frac{1}{q_1^2(1+q_1^2)(\vec{q}_1+\vec{q})^2[1+(\vec{q}_1+\vec{q})^2]}$$

$$-2\int_{\vec{q}_1} \frac{q_1^\alpha q_1^\beta}{q_1^4(1+q_1^2)(\vec{q}_1+\vec{q})^2[1+(\vec{q}_1+\vec{q})^2]}$$

$$+\int_{\vec{q}_1} \frac{(q_1^2+\vec{q}_1\cdot\vec{q}_2)q_1^\alpha(\vec{q}_1+\vec{q})^\beta}{q_1^4(1+q_1^2)(\vec{q}_1+\vec{q})^4[1+(\vec{q}_1+\vec{q})^2]}\,. \qquad \text{(A1)}$$

In the calculation of any of the above integrals it is efficacious to use the standard identity

$$(AB)^{-1} = \int_0^1 d\alpha[(1-\alpha)A+\alpha B]^{-2} \qquad \text{(A2)}$$

and to change the variables of integration in order to eliminate the angular dependence in the denominators. For example, the first integral is

$$I_1(\vec{q}) = \int_{\vec{q}_1} \frac{1}{q_1^2(1+q_1^2)(\vec{q}_1+\vec{q})^2[1+(\vec{q}_1+\vec{q})^2]}$$

$$= \int_{\vec{q}_1}\left(\frac{1}{q_1^2}-\frac{1}{1+q_1^2}\right)\left(\frac{1}{(\vec{q}_1+\vec{q})^2}-\frac{1}{1+(\vec{q}_1+\vec{q})^2}\right)$$

$$= \int_0^1 d\alpha \int_{\vec{q}_1}\{[(\vec{q}_1+\alpha\vec{q})^2+\alpha(1-\alpha)q^2]^{-2}$$

$$-[1-\alpha+(\vec{q}_1+\alpha\vec{q})^2+\alpha(1-\alpha)q^2]^{-2}$$

$$-[\alpha+(\vec{q}_1+\alpha\vec{q})^2+\alpha(1-\alpha)q^2]^{-2}$$

$$+[1+(\vec{q}_1+\alpha\vec{q})^2+\alpha(1-\alpha)q^2]^{-2}\}\,. \qquad \text{(A3)}$$

For $d=4$, after changing variables to $\vec{q}\,'=\vec{q}_1+\alpha\vec{q}$, this becomes

$$I_1 = K_4\int_0^1 d\alpha\left[-\ln[\alpha(1-\alpha)]+\ln\left(\frac{1-\alpha}{q^2}+\alpha(1-\alpha)\right)\right.$$

$$\left.+\ln\left(\frac{\alpha}{q^2}+\alpha(1-\alpha)\right)-\ln\left(\frac{1}{q^2}+\alpha(1-\alpha)\right)\right]$$

$$= -K_4\ln q + O(1). \qquad \text{(A4)}$$

We shall see below that we need $I^{\alpha\beta}(\vec{q})$ only to order $\ln q$, and hence the approximation in the last step of (A4) is sufficiently accurate.

This procedure for calculating integrals is even more valuable for integrals such as the second term in (A1), where there is also an angular dependence in the numerator. Such integrals are very difficult to calculate without the change of variables; in fact, it was the impossibility of making a change of variables with a limited range of integrations that prevented us from obtaining $\eta$ to second order by the methods of I. The second integral in (A1) is

$$I_2^{\alpha\beta}(\vec{q}) = \int_{\vec{q}_1}\frac{q_1^\alpha q_1^\beta}{q_1^4(1+q_1^2)(\vec{q}_1+\vec{q})^2[1+(\vec{q}_1+\vec{q})^2]} = -\frac{\partial}{\partial a}\left[(1-a)^{-1}\int_{\vec{q}_1} q_1^\alpha q_1^\beta\left(\frac{1}{a+q_1^2}-\frac{1}{1+q_1^2}\right)\left(\frac{1}{(\vec{q}_1+\vec{q})^2}-\frac{1}{1+(\vec{q}_1+\vec{q})^2}\right)\right]_{a=0}$$

$$= -\frac{\partial}{\partial a}\left[(1-a)^{-1}\int_0^1 d\alpha\, K_d\int_0^\infty q_1^{d-1}dq_1\left(\frac{q_1^2\delta_{\alpha\beta}}{d}+\alpha^2 q^\alpha q^\beta\right)\{[(1-\alpha)a+q_1^2+(1-\alpha)\alpha q^2]^{-2}\right.$$

$$\left.-[(1-\alpha)a+\alpha+q_1^2+(1-\alpha)\alpha q^2]^{-2}-[1-\alpha+q_1^2+(1-\alpha)\alpha q^2]^{-2}+[1+q_1^2+(1-\alpha)\alpha q^2]^{-2}\}\right]_{a=0}$$

$$= -\tfrac{1}{4}K_4\left(\delta_{\alpha\beta}[\ln q+O(1)]-\frac{q^\alpha q^\beta}{q^2}\right)+O(q^2) \qquad \text{(A5)}$$

(in the last step, $d=4$ was explicitly used). Similarly,

$$I_3^{\alpha\beta}(\vec{q}) = \int\frac{(q_1^2+\vec{q}_1\cdot\vec{q})q_1^\alpha(\vec{q}_1+\vec{q})^\beta}{q_1^4(1+q_1^2)(\vec{q}_1+\vec{q})^4[1+(\vec{q}_1+\vec{q})^2]}$$

$$= -\tfrac{1}{4}K_4\delta_{\alpha\beta}\ln q + O(1) \qquad \text{(A6)}$$

and hence

$$I^{\alpha\beta}(\vec{q}) = -\tfrac{3}{4}K_4\delta_{\alpha\beta}[\ln q+O(1)]-\frac{\frac{1}{2}K_4 q^\alpha q^\beta}{q^2}+O(q^2). \qquad \text{(A7)}$$

We now turn to the calculation of the integral

$$J^{\delta\epsilon}(\vec{q}) = \sum_{\gamma\eta} \int_{\vec{q}_1} \int_{\vec{q}_2} G^{\gamma\epsilon}(\vec{q}_1, 0) G^{\gamma\eta}(\vec{q}_2, 0) \times [G^{\delta\eta}(\vec{q}_1 + \vec{q}_2 + \vec{q}, 0) - G^{\delta\eta}(\vec{q}_1 + \vec{q}_2, 0)]$$

$$= \sum_{\gamma} \int_{\vec{q}_1} G^{\gamma\epsilon}(\vec{q}_1, 0)[I^{\gamma\delta}(\vec{q}_1 + \vec{q}) - I^{\gamma\delta}(\vec{q}_1)]. \quad \text{(A8)}$$

This integral may be evaluated by expanding $\ln(\vec{q}_1 + \vec{q})^2$ and $(q_1^\gamma + q^\gamma)(q_1^\delta + q^\delta)/(\vec{q}_1 + \vec{q})^2$ in Taylor series in $2(\vec{q}_1 \cdot \vec{q})(q_1^2 + q^2)^{-1}$, and keeping only terms which are of order $q^2/q_1^2$ (where $q^\alpha q^\beta$ is counted as $q^2$, etc.). These are the only terms which will finally yield results of order $q^2 \ln q$. This is also the reason for our approximations in Eqs. (A4)–(A6). Finally, for $d = 4$,

$$J^{\delta\epsilon}(\vec{q}) = K_4^2 q^2 \ln q\left(\tfrac{1}{6}\,\delta_{\delta\epsilon} + \tfrac{13}{48}\, q^\delta q^\epsilon / q^2\right) + O(q^2). \quad \text{(A9)}$$

The other necessary combination in (29) is similarly obtained:

$$J_1^{\delta\epsilon}(\vec{q}) = \sum_{\gamma\eta} \int_{\vec{q}_1} \int_{\vec{q}_2} G^{\gamma\epsilon}(\vec{q}_1, 0) G^{\delta\eta}(\vec{q}_2, 0) \times [G^{\gamma\eta}(\vec{q}_1 + \vec{q}_2 + \vec{q}, 0) - G^{\gamma\eta}(\vec{q}_1 + \vec{q}_2, 0)]$$

$$= K_4^2 q^2 \ln q\left(\tfrac{1}{2}\,\delta_{\delta\epsilon} + \tfrac{1}{4}\, q^\delta q^\epsilon / q^2\right) + O(q^2). \quad \text{(A10)}$$

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