Critical Behavior of Magnets with Dipolar Interactions. I. Renormalization Group near Four Dimensions

Amnon Aharony and Michael E. Fisher
Baker Laboratory, Cornell University, Ithaca, New York 14850
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The exact renormalization-group approach of Wilson is used to study the critical behavior for $T > T_c$, $H = 0$, and small $\epsilon > 0$, of an isotropic ferromagnetic system in $d = 4 - \epsilon$ dimensions, with exchange and dipolar interactions between d-component spins. Normal isotropic Heisenberg behavior with $1/\gamma \approx 1/2\nu \approx 1 - \epsilon/4$ (to first order in ϵ) is retained for $t = (T/T_c) - 1 \ll G/Ja^d$, where $G = (g\mu_B)^2/2$ measures the strength of the dipole-dipole interactions, J is the short-range exchange parameter, and a is the lattice spacing. When $t^\phi \approx G/Ja^d$, where $\phi \approx 1 + \epsilon/4$, crossover occurs to a characteristic dipolar behavior described by a new fixed point of the recursion relations. For $t \ll G/Ja^d$ one thus finds $1/\gamma \approx 1/2\nu \approx 1 - 9\epsilon/34$ {and, for spins of nd components, $1/\gamma \approx 1/2\nu \approx 1 - [(6n+3)/2(6n+11)]\epsilon$, which agrees with spherical-model results when $n \to \infty$}. In the dipolar regime the spin-correlation function $\langle s_q^\alpha s_{-q}^\beta \rangle$ has a factor $[\delta_{\alpha\beta} - (q^\alpha q^\beta/q^2)]$, which suppresses longitudinal spin fluctuations; the susceptibilities $\chi^{\alpha\alpha}$ display the expected demagnetization effects. It is found that dipolar anistropies deriving from the lattice structure produce weak instabilities which should be hard to detect although their effects are not fully elucidated. Extensions of the results to nonzero magnetic fields, and to anistropic exchange interactions are indicated; the experimental situation is mentioned briefly.

I. INTRODUCTION

It has been recognized for some time that sufficiently long-ranged interactions between the spins in a magnetic spin system should lead to values for critical exponents differing from those appropriate to short-range interactions. Even for short-range interactions, the range of the interaction is correlated with the size of the critical region outside which the classical Landau or phenomenological theory for phase transitions applies.[1] The addition of a long-range interaction might thus be expected to bring the critical behavior of a system closer to the classical one. In fact, a calculation of the critical exponents of a system with an interaction $r_{ij}^{-d-\sigma}(\vec{S}_i \cdot \vec{S}_j)$, using renormalization-group techniques,[2] shows that the classical behavior is achieved for $\sigma < \frac{1}{2}d$, and gives the deviation of the critical exponents from the classical values as a function of $(\sigma - \frac{1}{2}d)$ for $\sigma < 2$.

Since a dipole-dipole coupling between spins exists in all real magnetic materials, it might be expected that on approach to the critical point, the behavior of such materials should, owing to the long range of the dipole interactions, deviate from that obtained with the short-range interaction. For the case of uniaxial ferroelectrics with dipolar interactions, Larkin and Khmel'nitskii[3] have, indeed, predicted only logarithmic deviations from the classical theory.

Dipole-dipole interactions have been investigated in the past by several approaches. The most commonly used procedure has been based on Van Vleck's moment expansion.[4] Another method used to obtain the ground state of the system is that of Luttinger and Tisza.[5] These authors find that the ground state of a pure dipole-dipole system on an fcc lattice is antiferromagnetic.

One of the difficulties arising with dipole-dipole interactions is the shape-dependence of some of the thermodynamic properties. This can be viewed as a result of the conditional convergence of d-dimensional lattice sums of the type

$$\Phi^{\alpha\beta} = -\sum_j \left(\frac{\delta_{\alpha\beta}}{x_{ij}^d} - d\,\frac{x_{ij}^\alpha x_{ij}^\beta}{x_{ij}^{d+2}} \right). \tag{1}$$

The familiar practical case is, of course, $d = 3$. These shape dependences have been investigated by Levy[6] using diagrammatic techniques. As Levy notes, the technical difficulties occur only when one has expressions that contain the difference between two diverging sums. The shape dependence of the susceptibility, as found by Levy for the Ising case, and by Marquard[7] for the general case, is of the form predicted by macroscopic theory, namely, the observed susceptibility χ is given by

$$\chi^{-1}(H,T) = X^{-1}(H,T) - \Phi, \tag{2}$$

where Φ follows from (1), and $X(H,T)$ is "intrinsic" to the system, i.e., shape independent. (In the general case, χ, X, and Φ are tensors and χ^{-1} is the inverse tensor of χ.) The sum Φ is related to the demagnetization factor for the shape of the system (which must be an ellipsoid if simple results are to apply).

As noted already by Lorentz,[8] the sum Φ vanishes for cubic symmetry in spherical samples. In *zero* external field, no shape dependence of the free energy is to be expected[9] even below T_c, presumably because the system always breaks up into domains. In the present paper we shall consider only the case of zero field above T_c, so that we shall not have to worry about the shape dependence (except when discussing the susceptibility).

Another approach to the study of dipole-dipole interactions was proposed by Lax[10] who used the spherical model approximation. The critical behavior of the system can be derived from a form similar to that obtained for the spherical model with short-range forces. The only way the dipole-dipole interactions enter is through the eigenvalues $\lambda_\alpha(\vec{p})$ of a 3×3 matrix $A^{\alpha\beta}(\vec{p})$, which is the Fourier transform of the dipole-dipole interaction. Lax does not discuss the explicit momentum dependence of these eigenvalues, but he suggests that behavior near the critical point should be similar to that of the short-range case.

In all the work reviewed above, no direct attempt has been made to find the critical exponents for a system which has both isotropic (Heisenberg) ferromagnetic exchange forces and dipole-dipole interactions. The recent work of Wilson[11,12] on the renormalization-group approach to critical-point behavior now makes it possible to attack this important question, and to investigate the deviations from the short-range behavior due to the existence of dipole-dipole forces. In the present paper we consider this problem for cubic lattices in zero field, above T_c, and calculate the critical exponents for $d=4-\epsilon$ dimensions,[12] to lowest order in ϵ. These first-order results already serve to reveal the differences between systems with and without dipole-dipole interactions. We find that for a temperature range close enough to T_c, all the critical exponents change.

In Sec. II we discuss the appropriate Hamiltonian for the system, and show that it is necessary to go beyond the leading low-momentum representation of the dipolar interactions. The Gaussian model for the system is considered in Sec. III; a modified spin-spin correlation function results, which has a complicated angular dependence. In Sec. IV we introduce the continuous-spin s^4 model, and obtain and discuss the exact renormalization-group recursion formulas for the case of a fully isotropic system in which terms of only cubic symmetry, deriving from the lattice structure, are neglected or are absent. Section V contains a derivation of the crossover exponent ϕ, which determines the change over from the scaling behavior characteristic of the d-dimensional short-range Heisenberg fixed point to that of a new fixed point, which is determined by the isotropic dipole-dipole interactions. This fixed point is discussed further in Sec. VI, where the new value of the critical exponent ν for the range of correlation is derived. The susceptibility, its exponent, and the correlation functions are considered in further detail in Sec. VII. Section VIII is devoted to the derivation of the more complicated recursion relations required to deal with the complete anisotropic dipolar interactions. The fixed points of these equations are discussed in Sec. IX, where an unexpected instability, albeit rather weak, is discovered with respect to the anisotropic coupling. Finally, Sec. X includes a discussion of possible extensions of the model and a brief reference to the experimental situation.

Appendix A includes an extension of Ewald's method to the calculation of the d-dimensional Fourier transform of the general dipole-dipole potential; Appendix B includes a list of angular integrals needed in these types of calculations; and Appendix C includes a separate investigation of the nd-component model and the spherical model, which also arises as the limit $n\to\infty$.[13,14] Finally, in Appendix D an initial discussion of the case of uniaxial dipolar interactions is presented.

In following papers the Feynman-graph technique introduced by Wilson, is used to calculate the exponent η to second order in ϵ, and the effects of dipolar interactions on the critical behavior of *anti*ferromagnets and of systems with anisotropic exchange interactions are studied.

II. HAMILTONIAN

The simplest Hamiltonian with isotropic Heisenberg exchange and dipolar interactions may be written

$$\mathcal{H}=-\tfrac{1}{2}J\sum_{\vec{R}}\sum_{\vec{\delta}}\vec{s}_{\vec{R}}\cdot\vec{s}_{\vec{R}+\vec{\delta}} - G\sum_{\vec{R}\neq\vec{R}'}\sum_{\alpha,\beta}\frac{\partial^2}{\partial R^\alpha\,\partial R'^\beta}\left(\frac{1}{|\vec{R}-\vec{R}'|^{d-2}}\right)s^\alpha_{\vec{R}}\,s^\beta_{\vec{R}'}\,. \tag{3}$$

where $\vec{s}_{\vec{R}}=(s^\alpha_{\vec{R}})$ is a classical spin vector of unit length located at the site $\vec{R}$ of a d-dimensional lattice of cubic symmetry and coordination number c. The vectors $\vec{\delta}$ of length a (the lattice spacing) run over the c nearest-neighbor sites of the origin site. It will be convenient in most places to measure all distances in units of a or, equivalently, to set $a=1$. As usual J denotes the exchange energy, while $G=\frac{1}{2}(g\mu_B)^2$ measures the strength of the dipole-dipole interactions (in terms of the Bohr magneton μ_B and the gyromagnetic ratio g; magnetic monopoles are normalized so that their "Coulomb" potential is m^2/r^{d-2}). Note that the ratio G/Ja^d is the basic dimensionless parameter. Since the dipole-dipole interaction, when written out in full, involves scalar products such as $\vec{s}_{\vec{R}}\cdot(\vec{R}-\vec{R}')$, the dimensionality of the spin vectors should equal that of the space, namely, d.

A natural extension of the model to a system in which the order parameter will have more components, may be constructed by associating n

"subcomponents" $s^{\alpha i}$ ($i = 1, 2, \ldots, n$) with each principal spin component s^{α} ($\alpha = 1, \ldots, d$). In the limit $n \to \infty$ such an nd-component model is expected to yield spherical model results.[13,14] This model is discussed in Appendix C.

For the short-range terms in (3) we have taken a fully isotropic d-component (or, more generally, nd-component) ferromagnetic Heisenberg interaction. The extension to anisotropic interactions may be made,[14] but will complicate all the expressions. It is simpler in the first instance to consider any explicit anisotropic terms as perturbations about our present results.

In order to apply the renormalization group approach in a straightforward way we extend the model by allowing continuous spins of unbounded magnitude[11,12,14,15] which, however, are restricted by a single-spin weighting factor of the form $\exp[-Q(s^{\alpha}_{\vec{R}})]$, where $Q(s)$ contains quadratic and quartic powers. Within this framework the partition function is

$$Z = \int_{\vec{s}} e^{\bar{\mathcal{H}}}, \tag{4}$$

where $\int_{\vec{s}}$ denotes integration over all the spin variables $s^{\alpha}_{\vec{R}}$ and where the reduced Hamiltonian is

$$\bar{\mathcal{H}} = -\frac{\mathcal{H}}{k_B T} - \frac{1}{2}\sum_{\vec{R}} \vec{s}^{\,2}_{\vec{R}} - u\sum_{\vec{R}} (\vec{s}^{\,2}_{\vec{R}})^2 - v\sum_{\vec{R}}\sum_{\alpha} (s^{\alpha}_{\vec{R}})^4. \tag{5}$$

The $\vec{s}^{\,2}_{\vec{R}}$ and the $(\vec{s}^{\,2}_{\vec{R}})^2$ terms in (5) are the usual isotropic weight factors used previously in the s^4 model, when rotational invariance exists. The last term $\sum_{\alpha}(s^{\alpha}_{\vec{R}})^4$ is similar to the one used by Wilson and Fisher[12] for two spin components. [In fact, our v is equivalent to their $(u_0 - \frac{1}{2}g_0)$.] This term is added here for two purposes: One is similar to that of Wilson and Fisher, namely, to try to obtain fixed points with different symmetries from the model. The other is to allow for those terms in the Hamiltonian that have only cubic symmetry, instead of full rotational symmetry. Such terms arise directly from dipolar interactions on a lattice and are propagated by the renormalization group recursion relations as will be discussed in some detail below.

It is convenient to transform the spins to new variables,

$$\vec{\sigma}_{\vec{q}} = \sum_{\vec{R}} e^{-i\vec{q}\cdot\vec{x}}\vec{s}_{\vec{R}}\quad \vec{x} = \vec{R}/a\ , \tag{6}$$

where $\vec{q}$ is a dimensionless wave vector with, for a simple cubic lattice, $|q^{\alpha}| \leq \pi$.

For a cubic lattice, the Hamiltonian then becomes approximately

$$\mathcal{H} = -\tfrac{1}{2}cJ\int_{\vec{q}} (1 - \tfrac{1}{2}d^{-1}q^2)\sum_{\alpha} \sigma^{\alpha}_{\vec{q}}\,\sigma^{\alpha}_{-\vec{q}} + Ga^{-d}\int_{\vec{q}}\sum_{\alpha\beta} A^{\alpha\beta}(\vec{q})\sigma^{\alpha}_{\vec{q}}\,\sigma^{\beta}_{-\vec{q}}\ , \tag{7}$$

where $\int_{\vec{q}}$ is an abbreviation for $(2\pi)^{-d}\int d^dq$. The first term in (7) represents the short-range interaction to order q^2. (Retention of the q^4 and higher-order terms should not alter the character of our results.[15]) The dipole-dipole function $A^{\alpha\beta}(\vec{q})$ is defined by

$$A^{\alpha\beta}(\vec{q}) = -\left(\frac{\partial^2}{\partial x^{\alpha}\partial x^{\beta}}\sum_{l}{}' \frac{e^{i\vec{q}\cdot\vec{x}_l}}{|\vec{x}_l - \vec{x}\,|^{d-2}}\right)_{\vec{x}=0}, \tag{8}$$

where the prime denotes omission of the term with $\vec{x}_l = 0$. This function is the analog of Lax's[10] tensor $\lambda(\vec{q})$. In contrast to the numerical methods suggested by Lax, we propose to consider the exact behavior of $A^{\alpha\beta}(\vec{q})$ for small values of $|\vec{q}\,|$. Using Ewald's method[16] for the summation of the first few terms in a Taylor series of (8) in powers of the q^{α}, we find, from Appendix A,

$$A^{\alpha\beta}(\vec{q}) = a_1\frac{q^{\alpha}q^{\beta}}{q^2} - a_2 q^{\alpha}q^{\beta} - [a_3 + a_4 q^2 - a_5(q^{\alpha})^2]\,\delta_{\alpha\beta} + O(q^4, (q^{\alpha})^4, (q^{\alpha})^2(q^{\beta})^2), \tag{9}$$

where the a_i are constants. The terms of order q^2 and $(q^{\alpha})^2$ were ignored in the work of Larkin and Khmel'nitzkii[3] and of Vaks, Larkin, and Pikin,[17] who have considered some aspects of the dipole-dipole interactions. As we shall see, this may, in fact, lead to correct results, but it clearly requires justification. As mentioned above the terms of order q^4, $(q^{\alpha})^4$, etc., will, by virtue of the renormalization group recursion, correspond to irrelevant variables, and hence may be ignored as regards leading order critical behavior.[15]

Note that the cubic (as against full-rotational) symmetry of the system is reflected only through the coefficient a_5 in Eq. (9). For "fully isotropic" systems with complete rotational symmetry, such as a "liquid ferromagnet," and maybe even for randomly oriented systems, such as crystalline powders, we expect that this anisotropic term may be neglected. As we shall see later, however, this term, when it is present, plays an important role in determining the critical behavior of the system.

The fact that $A^{\alpha\beta}(\vec{q})$ is defined by a poorly convergent series is reflected through the nonuniqueness of the limit of $A^{\alpha\beta}(\vec{q})$ as $\vec{q} \to 0$. Precisely at $\vec{q} = 0$, the value of $A^{\alpha\beta}(0)$ should coincide with $-\Phi^{\alpha\beta}$ of Eq. (1), and therefore should be shape dependent. Noting from Eq. (A17) of Appendix A that $a_1 = da_3$, the limit of $A^{\alpha\alpha}(\vec{q})$ as $\vec{q} \to 0$ may assume any value in the range $-a_1 d^{-1}$ to $a_1(1 - d^{-1})$ depending on the direction from which $\vec{q}$ approaches zero. (For the simple cubic lattice $a_1 = 4\pi$, and the usual range of values of the demagnetization factor is recovered.) The limiting behavior of $A^{\alpha\beta}(\vec{q})$ as $\vec{q} \to 0$, is important for the case of finite magnetic fields; this may be seen, e.g., in the expressions given by Lax[10] who extracts the shape dependence by claiming that the combination $[-A^{\alpha\beta}(0) + 4\pi L^{\alpha\beta}]$ is shape independent. Here $L^{\alpha\beta}$ is the shape-dependent depolarization tensor, which is related to $\Phi^{\alpha\beta}$ of Eq. (1). However, it is not really clear how Lax takes the limit $\vec{q}\to 0$ in order to obtain his $A^{\alpha\beta}(0)$.

Except for the special value of $A^{\alpha\beta}(\vec{q})$ at $\vec{q}=0$, there are no shape-dependent terms in the Hamiltonian $\mathcal{H}$, and we may therefore proceed as normally. Neglecting fourth-order terms in q^α, the Hamiltonian in (5) now becomes

$$\bar{\mathcal{H}}=-\frac{1}{2}\int_{\vec{q}}\sum_{\alpha\beta}\tilde{U}_2^{\alpha\beta}(\vec{q})\,\sigma_{\vec{q}}^{\alpha}\sigma_{-\vec{q}}^{\beta}-\sum_{\alpha\beta}(u+\delta_{\alpha\beta}v)\times\int_{\vec{q}}\int_{\vec{q}_1}\int_{\vec{q}_2}\sigma_{\vec{q}}^{\alpha}\sigma_{\vec{q}_1}^{\alpha}\sigma_{\vec{q}_2}^{\beta}\sigma_{-\vec{q}-\vec{q}_1-\vec{q}_2}^{\beta}, \quad (10)$$

where

$$\tilde{U}_2^{\alpha\beta}(\vec{q})=\left[\tilde{r}+\left(\frac{\tilde{J}}{k_BT}\right)q^2+\left(\frac{2\tilde{G}a_5}{k_BT}\right)(q^\alpha)^2\right]\delta_{\alpha\beta}+\left(\frac{2\tilde{G}}{k_BT}\right)(a_1-a_2q^2)\,\frac{q^\alpha q^\beta}{q^2}\ , \quad (11)$$

$$\tilde{r}=1-(cJ+2a_3Ga^{-d})/k_BT\,, \quad (12)$$

$$\tilde{J}=\tfrac{1}{2}cd^{-1}J-2a_4\,,\ \tilde{G}=Ga^{-d}\ . \quad (13)$$

Clearly, the sign of the interaction parameter $\tilde{J}$ can have a crucial effect on the behavior of the system. If the dipole-dipole interaction is much stronger than the exchange interaction, and if a_4 is positive (see Table I) then $\tilde{J}$ will be negative, and we shall have to be careful in using the Gaussian approximation for the system, since the positivity of the matrix $\tilde{U}_2^{\alpha\beta}(\vec{q})$ may be in question. (This might be related to the fact that for a pure dipole-dipole interaction, Luttinger and Tisza[5] find an antiferromagnetic ground state.) If Ga^{-d} is much larger than J, the $(q^\alpha)^2$ term becomes the important one, and as we shall see, this will need special treatment. In the present investigation we shall assume that

$$G/Ja^d\ll 1. \quad (14)$$

As we shall see, this will lead to new critical behavior for

TABLE I. Coefficients in the Taylor expansion of $A^{\alpha\beta}(\vec{q})$ for three-dimensional cubic lattices.

Lattice	sc	bcc	fcc
c	6	8	12
v_a	1	$4.3^{-3/2}$	$2^{-1/2}$
a_1	4π	$3^{3/2}\pi$	$4\pi 2^{1/2}$
a_3	$\frac{4}{3}\pi$	$3^{1/2}\pi$	$(\frac{4}{3}\pi)2^{1/2}$
a_2	1.2755	1.341	2.002
a_4	0.1649	−0.247	−0.237
a_5	1.7700	0.632	1.289

$$t=(T-T_c)/T_c\ll G/Ja^d\,, \quad (15)$$

whereas for $t\gg G/Ja^d$ the normal d-component isotropic-Heisenberg-model behavior will be realized. Finally, for $\tilde{J}>0$, we can rescale the spins $\sigma_{\vec{q}}^{\alpha}$ so that $\bar{\mathcal{H}}$ becomes

$$\bar{\mathcal{H}}_0=-\frac{1}{2}\sum_{\alpha\beta}\int_{\vec{q}}U_2^{0,\alpha\beta}(\vec{q})\,\sigma_{\vec{q}}^{\alpha}\sigma_{-\vec{q}}^{\beta}-\sum_{\alpha\beta}(u_0+\delta_{\alpha\beta}v_0)\times\int_{\vec{q}}\int_{\vec{q}_1}\int_{\vec{q}_2}\sigma_{\vec{q}}^{\alpha}\sigma_{\vec{q}_1}^{\alpha}\sigma_{\vec{q}_2}^{\beta}\sigma_{-\vec{q}-\vec{q}_1-\vec{q}_2}^{\beta}\ , \quad (16)$$

with

$$U_2^{0,\alpha\beta}(\vec{q})=[r_0+q^2+f_0(q^\alpha)^2]\,\delta_{\alpha\beta}+g_0(q^\alpha q^\beta/q^2)-h_0q^\alpha q^\beta\,, \quad (17)$$

where

$$r_0=\frac{\tilde{r}k_BT}{\tilde{J}}=\frac{k_BT-(cJ+2a_3Ga^{-d})}{\frac{1}{2}cd^{-1}J-2a_4Ga^{-d}}=\frac{k_B(T-T_0)}{\tilde{J}}\,, \quad (18)$$

with

$$k_BT_0=cJ+2a_3Ga^{-d} \quad (19)$$

and, with $\tilde{J}$ defined as in (13),

$$f_0=\frac{2a_5G}{\tilde{J}a^d},\quad g_0=\frac{2a_1G}{\tilde{J}a^d}\,,\quad h_0=\frac{2a_2G}{\tilde{J}a^d}\,,\quad u_0=u\left(\frac{k_BT}{\tilde{J}}\right)^2,\quad v_0=v\left(\frac{k_BT}{\tilde{J}}\right)^2. \quad (20)$$

III. GAUSSIAN MODEL

In the Gaussian model one ignores the σ^4 terms in (16), i.e., we set $u_0=v_0=0$. The spin-spin correlation function in this case is easily seen to be

$$\langle\sigma_{\vec{q}}^{\alpha}\sigma_{\vec{q}'}^{\beta}\rangle_0=Z_0^{-1}\int_\sigma\sigma_{\vec{q}}^{\alpha}\sigma_{\vec{q}'}^{\beta}\,e^{\bar{\mathcal{H}}_0^0}=\delta(\vec{q}+\vec{q}')\,G^{\alpha\beta}(\vec{q})\,, \quad (21)$$

where $\int_\sigma$ denotes an integral over all spins, $\bar{\mathcal{H}}_0^0$ is given by (16) with $u_0=v_0=0$, $Z_0=\int_\sigma e^{\bar{\mathcal{H}}_0^0}$, and $G^{\alpha\beta}(\vec{q})$ is the inverse $d\times d$ matrix of $U_2^{0,\alpha\beta}(\vec{q})$. This inverse matrix has a simple form when $f_0=0$, and since, under the assumption (14), we have $f_0\ll 1$, we shall expand $G^{\alpha\beta}$ in a power series in f_0. For $f_0=0$, and $\vec{q}\neq 0$, we find

$$G_0^{\alpha\beta}(\vec{q})=\frac{1}{r_0+q^2}\left(\delta_{\alpha\beta}-\frac{q^\alpha q^\beta}{q^2}\right)+\frac{1}{r_0+g_0+(1-h_0)q^2}\,\frac{q^\alpha q^\beta}{q^2}\ . \quad (22)$$

Note that this expression does not apply for $\vec{q}=0$ because of the undefined value of $U_2^{0,\alpha\beta}(\vec{q})$ at this point (see Sec. VII). Since by (20) the condition $a_5=0$ implies $f_0=0$, the function $G_0^{\alpha\beta}(\vec{q})$ represents the complete classical correlation function for the fully isotropic case exemplified by the "liquid ferromagnet" mentioned after (9). We thus consider the anisotropic deviations from full rotational symmetry as perturbations. We shall see in Sec.

IX, however, that this may not be completely satisfactory when f_0 is not small.

If we use the full expression (17), with $f_0 \neq 0$, we find the propagator can be written explicitly as

$$G^{\alpha\beta}(\vec{q}) = \frac{1}{r_0 + q^2 + f_0(q^\alpha)^2} \times \left(\delta_{\alpha\beta} - \frac{(g_0 - h_0 q^2) q^\alpha q^\beta}{[q^2 + (g_0 - h_0 q^2) Q][r_0 + q^2 + f_0(q^\beta)^2]}\right), \tag{23}$$

$$Q = \sum_\gamma (q^\gamma)^2 / [r_0 + q^2 + f_0(q^\gamma)^2].$$

Clearly, for f_0, h_0, $g_0 \ll r_0$, the correlation function becomes simply $\delta_{\alpha\beta}/(r_0 + q^2)$, which is the usual result for the d-component Heisenberg Gaussian model. Note that the spin fluctuations are quite isotropic. For nonzero g_0, and for temperatures close enough to T_c, we have $r_0 \ll g_0$, and so the second term in (22) can be neglected for small enough values of $|\vec{q}|$, explicitly for $q^2 \ll g_0$. In this range, by expanding (23) in powers of f_0, one finds

$$G^{\alpha\beta}(\vec{q}) \approx \frac{1}{r_0 + q^2}\left[\delta_{\alpha\beta} - \frac{q^\alpha q^\beta}{q^2} - \frac{f_0 q^2}{r_0 + q^2} B^{\alpha\beta}\left(\frac{q^\gamma}{|\vec{q}|}\right) + \frac{f_0^2 q^4}{(r_0 + q^2)^2} C^{\alpha\beta}\left(\frac{q^\gamma}{|\vec{q}|}\right) + O(f_0^3)\right] + O(g_0^{-1}), \tag{24a}$$

with

$$B^{\alpha\beta}(y^\gamma) = (y^\alpha)^2 \delta_{\alpha\beta} - \left((y^\alpha)^2 + (y^\beta)^2 - \sum_\delta (y^\delta)^4\right) y^\alpha y^\beta \tag{24b}$$

and

$$C^{\alpha\beta}(y^\gamma) = (y^\alpha)^4 \delta_{\alpha\beta} - [(y^\alpha)^4 + (y^\alpha)^2 (y^\beta)^2 + (y^\beta)^4] y^\alpha y^\beta + \left([(y^\alpha)^2 + (y^\beta)^2] \sum_\gamma (y^\gamma)^4 + \sum_\gamma (y^\gamma)^6\right) \times y^\alpha y^\beta - \left(\sum_\gamma (y^\gamma)^4\right)^2 y^\alpha y^\beta. \tag{24c}$$

This expansion is valid only for $f_0 \ll 1$. For larger values, the full propagator (23) must be used, although (14) may then not be valid.

Now for a given wave vector $\vec{q}$ one can define the longitudinal spin component as

$$\sigma_{\vec{q}}^{\parallel} = \vec{\sigma}_{\vec{q}} \cdot \vec{q} / |\vec{q}| = \sum_\alpha \sigma_{\vec{q}}^\alpha q^\alpha \Big/ \sum_\alpha (q^\alpha)^2.$$

Correspondingly, the longitudinal spin fluctuations are described by the correlation function

$$\langle \sigma_{\vec{q}}^{\parallel} \sigma_{-\vec{q}}^{\parallel} \rangle = \sum_\alpha \sum_\beta q^\alpha q^\beta \langle \sigma_{\vec{q}}^\alpha \sigma_{-\vec{q}}^\beta \rangle / q^2 = \sum_\alpha \sum_\beta q^\alpha q^\beta G^{\alpha\beta}(\vec{q}) / q^2. \tag{25}$$

It is then straightforward to check that the leading forms in (24) imply the *vanishing* of the longitudinal correlation function [up to terms of $O(g_0^{-1})$]. This is particularly easy to see in the limit $q^2 \ll r_0/f_0$, where the factor in large square brackets reduces to $\delta_{\alpha\beta} - (q_\alpha q_\beta / q^2)$. Thus, we find that for temperatures close to the critical point $T_c = T_0$ ($r_0 = 0$), the longitudinal spin fluctuations of long wavelength are strongly suppressed.[18] There will, of course, remain relatively small contributions from the second term in (22), but these will not diverge at T_c. This in turn may be related to the fact that appropriate susceptibilities do not diverge at T_c for dipolar materials!

It is clear from (24) that the magnitude of the remaining, transverse spin fluctuations are still determined by $1/r_0$. Quite generally, (24) may be written asymptotically in the scaled form

$$G^{\alpha\beta}(\vec{q}) \approx \frac{1}{q^2} D^{\alpha\beta}\left(\frac{|\vec{q}|}{r_0^{1/2}};\ \frac{q^\gamma}{|\vec{q}|}\right) \quad (r_0,\ q^2 \ll g_0), \tag{26}$$

which enables one to recognize $r_0^{-1/2}$ as the correlation length $\xi(T)$. Thus, the critical exponent ν for the Gaussian case is $\frac{1}{2}$, as in the normal classical theory for short-range interactions. At the critical point itself ($T = T_c = T_0$, $r_0 = 0$), the scaling function becomes merely the critical amplitude

$$D^{\alpha\beta}(\infty; y^\gamma) = \delta_{\alpha\beta} - y^\alpha y^\beta - f_0 B^{\alpha\beta}(y^\gamma) + f_0^2 C^{\alpha\beta}(y^\gamma) + O(f_0^3), \tag{27}$$

where $B^{\alpha\beta}(y^\gamma)$ and $C^{\alpha\beta}(y^\gamma)$ were defined in (24). As indicated above, this amplitude can vanish only for longitudinal combinations of the form $\sum_\alpha \sum_\beta y_\alpha y_\beta D^{\alpha\beta}$. Thus, in general, we have $G^{\alpha\beta}(\vec{q}) \sim 1/q^2$ and so may conclude that the exponent η vanishes (as expected for a Gaussian model).

In the following paper[19] we discuss the detailed form of the correlations for the full s^4 model and show that in leading orders in ϵ, the form of the angular dependence is similar, and that only the values of ν and η have to be modified.

IV. RENORMALIZATION-GROUP RECURSION FORMULAS

We now proceed to derive the renormalization-group recursion formulas. Since, by (17), the interaction $U_2^{\alpha\beta}(\vec{q})$ includes the terms $q^\alpha q^\beta / q^2$, $q^\alpha q^\beta$, etc., we cannot use Wilson's approximate recursion relation;[11,12] accordingly we have to work with the exact approach. As explained in Ref. 15, we take the $\vec{q}$ integrals in (16) to be over the range $|\vec{q}| < 1$ (instead of the appropriate cell in the reciprocal lattice $|q^\alpha| < \pi$), with the justification that only small values of $|\vec{q}|$ will play an important role. (This results in an effective rescaling of all spins by a factor $\pi^{1+(d/2)}$, of r_0 and g_0 by π^{-2}, and of u_0 and v_0 by a factor π^{d-4}.)

At each step of renormalization we integrate all momenta in the range $b^{-1} < |\vec{q}| < 1$, change the variables of integration to $\vec{q}' = b\vec{q}$, and rescale the spin variables to $\vec{\sigma}_{\vec{q}} \to \zeta \vec{\sigma}'_{b\vec{q}}$. Starting from the expression (16) for $\mathcal{H}_0$, we thus obtain a sequence of re-

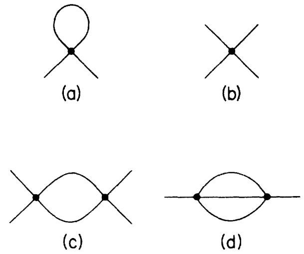

FIG. 1. Set of graphs needed for calculation to first and second order in u_l and v_l.

duced Hamiltonians $\bar{\mathcal{H}}_l$ of the same form, with r_l, f_l, h_l, g_l, u_l, and v_l replacing r_0, f_0, h_0, g_0, u_0, and v_0. The terms with u_l and v_l will be considered as small perturbations, their contributions being evaluated by a Feynman-graph expansion using the propagator (23). We thus find

$$\bar{\mathcal{H}}_{l+1} = -\tfrac{1}{2}\zeta_l^2 b^{-d} \times \Bigg[\sum_{\alpha\beta} \int_{|\vec{q}|<1} \Bigg([r_l + b^{-2}q^2 + b^{-2} f_l (q^\alpha)^2]\, \delta_{\alpha\beta} + (g_l - b^{-2} h_l q^2) \frac{q^\alpha q^\beta}{q^2}\Bigg)\, \sigma_{\vec{q}}^\alpha \sigma_{-\vec{q}}^\beta + \text{graphs}\Bigg], \qquad (28)$$

where, to second order in u_l and v_l we have to consider the graphs shown in Fig. 1.

It is evident from (24) that the terms involving f_l will make all the expressions rather complicated; however, we shall see that they may have a significant effect on the results. Accordingly, we shall divide our treatment of the recursion relations into two parts, considering first the case of $f_0 = 0$, i.e., full isotropy. As noted in the discussion of expression (5), the parameter v in the four-spin part of the effective Hamiltonian, also breaks the rotational symmetry; but, as we shall demonstrate later, it only enters for $f_0 \neq 0$. At this point, therefore, we set $f_0 = v_0 = 0$ and restrict attention to the parameters r_l, h_l, g_l, and u_l; this means that we can take $G^{\alpha\beta}$ equal to $G_0^{\alpha\beta}$ as defined in (22). We shall return to f_l and v_l in Sec. VIII.

We first investigate the form of the pair interaction potential $U_2^{l+1,\alpha\beta}(\vec{q})$. From the graphs (a) and (d) of Fig. 1, we find

$$U_2^{l+1,\alpha\beta}(\vec{q}) = \zeta_l^2 b^{-d}\Bigg[(r_l + b^{-2}q^2)\,\delta_{\alpha\beta} + (g_l - b^{-2}h_l q^2)\,\frac{q^\alpha q^\beta}{q^2} + 4u_l\Bigg(\sum_\gamma \int_{\vec{q}_1}^{>} G^{\gamma\gamma}(\vec{q}_1) + 2\int_{\vec{q}_1}^{>} G^{\alpha\beta}(\vec{q}_1)\Bigg) - 32u_l^2 \sum_{\gamma,\delta} \int_{\vec{q}_1}^{>}\int_{\vec{q}_2}^{>} [G^{\alpha\beta}(\vec{q}_1) G^{\gamma\delta}(\vec{q}_2) G^{\gamma\delta}(\vec{q}_1 + \vec{q}_2 + b^{-1}\vec{q}) + 2G^{\alpha\gamma}(\vec{q}_1) G^{\delta\gamma}(\vec{q}_2) G^{\beta\delta}(\vec{q}_1 + \vec{q}_2 + b^{-1}\vec{q})]\Bigg], \qquad (29)$$

where $\int_{\vec{q}}^{>}$ denotes the integration $(2\pi)^{-d}\int d^d q$ over the range $1 \geq |\vec{q}| > b^{-1}$. Clearly, the first-order terms, coming from graph (a) of Fig. 1 will not affect the $\vec{q}$ dependence of $U_2^{l+1,\alpha\beta}(\vec{q})$ and, hence, will contribute only to r_{l+1}. The second-order term, arising from graph (*d*) of Fig. 1 contributes to h_{l+1} and also produces a term proportional to q^2, which will modify the coefficient of q^2 in U_2^{l+1}; this, in turn, determines the scale factor ζ_l which is always chosen so that this coefficient remains fixed equal to unity.

The contributions of the (a) terms of Fig. 1 are readily computed, using the formulas for the angular integrals discussed in Appendix B. (See also Ref. 20.) The result for r_{l+1} is

$$r_{l+1} = \zeta_l^2 b^{-d}\{r_l + 4u_l K_d d^{-1}(d+2) \times [(d-1)A_{10}^d + A_{01}^d] + O(u_l^2)\}, \qquad (30)$$

where the basic integrals $A_{nm}^d = A_{nm}^d(r_l, g_l, h_l)$ are defined by

$$A_{nm}^d(r,g,h) = \int_{b^{-1}}^{1} \frac{q^{d-1}dq}{(r+q^2)^n[r+g+(1-h)q^2]^m}, \qquad (31)$$

while the numerical coefficient K_d is defined in (B4).

We now consider the contributions of graph (d) of Fig. 1. A typical term is

$$\bar{X}^{\alpha\beta\gamma\delta}(\vec{q}) = -\int_{\vec{q}_1}^{>}\int_{\vec{q}_2}^{>} G^{\alpha\gamma}(\vec{q}_1) G^{\gamma\delta}(\vec{q}_2) \times G^{\delta\beta}(\vec{q}_1 + \vec{q}_2 + b^{-1}\vec{q}). \qquad (32)$$

For $\vec{q} = 0$, this will give a contribution of order u_l^2 to (30). We are most interested in the contribution to the $\vec{q}$-dependent parts of $U_2^{l,\alpha\beta}(\vec{q})$, and therefore it is sufficient to look at

$$X^{\alpha\beta\gamma\delta}(\vec{q}) = \bar{X}^{\alpha\beta\gamma\delta}(\vec{q}) - \bar{X}^{\alpha\beta\gamma\delta}(0) = -\int_{\vec{q}_1}^{>}\int_{\vec{q}_2}^{>} G^{\alpha\gamma}(\vec{q}_1) G^{\gamma\delta}(\vec{q}_2) \times [G^{\delta\beta}(\vec{q}_1 + \vec{q}_2 + b^{-1}\vec{q}) - G^{\delta\beta}(\vec{q}_1 + \vec{q}_2)]. \qquad (33)$$

Since we are interested only in the terms involving $(q^\alpha q^\beta / q^2)$ and q^2, $(q^\alpha)^2$, or $q^\alpha q^\beta$, it is sufficient to study (33) for very small values of $|\vec{q}|$. Now the ranges of integration are $0 < b^{-1} < |\vec{q}_1|, |\vec{q}_2| < 1$; hence, $G^{\delta\beta}(\vec{q}_1 + \vec{q}_2 + b^{-1}\vec{q})$ is analytic in $\vec{q}$ for small $|\vec{q}|$. Thus, we get no contribution to g_{l+1}. It seems clear that this argument extends to higher-order diagrams. Accordingly, we obtain simply

$$g_{l+1} = \zeta_l^2 b^{-d} g_l. \qquad (34)$$

Next we examine a few typical terms in (33). Since the contribution $X^{\alpha\beta\gamma\delta}$ is to be multiplied by terms of order u_l^2, and since we will assume u_l to be small, namely, of order $\epsilon = 4 - d$, we shall take

$r=0$ and compute the integrals at $d=4$. The simplest term in (33) is then

$$X_1^{\alpha\beta\gamma\delta}=-\delta_{\alpha\beta}\delta_{\alpha\gamma}\delta_{\alpha\delta}\int_{\vec{q}_1}^{>}\frac{1}{q_1^2}\int_{\vec{q}_2}^{>}\frac{1}{q_2^2}\times\left(\frac{1}{(\vec{q}_1+\vec{q}_2+b^{-1}\vec{q})^2}-\frac{1}{(\vec{q}_1+\vec{q}_2)^2}\right), \quad (35)$$

which occurs also in the standard short-range problem.

Assuming $|\vec{q}|$ to be small, so that in effect $|\vec{q}_1+b^{-1}\vec{q}|>b^{-1}$, the q_2 integration gives

$$\int_{q_2}^{>}\left(\frac{1}{(\vec{q}_1+\vec{q}_2+b^{-1}\vec{q})^2}-\frac{1}{(\vec{q}_1+\vec{q}_2)^2}\right)=\tfrac{1}{2}K_4\left[\frac{1}{b^2}\left(\frac{1}{q_1^2}-\frac{1}{(\vec{q}_1+b^{-1}\vec{q})^2}\right)+\ln q_1^2-\ln(\vec{q}_1+b^{-1}\vec{q})^2\right], \quad (36)$$

and hence to order q^2 we find

$$X_1^{\alpha\beta\gamma\delta}=\tfrac{1}{4}\delta_{\alpha\beta}\delta_{\alpha\gamma}\delta_{\alpha\delta}K_4^2q^2b^{-2}\ln b. \quad (37)$$

Similar contributions proportional to q^2 come from other parts of (35). Finally, after performing the summations on γ and δ the q^2 term in (29) becomes

$$\zeta_l^2 b^{-d-2}[1+PK_4^2u_l^2\ln b]q^2\approx\zeta_l^2b^{-d-2+\eta_l}q^2, \quad (38)$$

where the exponent η_l is defined to second order in u_l by

$$\eta_l=P(g_l,\,h_l)K_4^2u_l^2, \quad (39)$$

and where the coefficient P is a function of g_l and h_l which enter through the propagator [see (22) and (23)] via integrals as $A_{nm}^d(r,g,h)$ defined in (31) (but recall that r has been set equal to zero in these terms). In order that the coefficient of q^2 in $\bar{\mathcal{H}}_l$ remains equal to unity we choose the scale factor ζ_l to satisfy

$$\zeta_l^2=b^{d+2-\eta_l}. \quad (40)$$

A fuller calculation of the exponent η (to be presented elsewhere[19]) shows that $P(g_l,h_l)$ is positive in the regions of interest and indeed, approximately equal to $8(d+2)\simeq 48$ for $g=0$, and to $26\frac{2}{3}$ for $g\to\infty$.

Another typical term in (33) is

$$X_2^{\alpha\beta\gamma\delta}=\delta_{\gamma\delta}\int_{\vec{q}_1}^{>}\frac{q_1^\alpha q_1^\beta}{q_1^4}\int_{\vec{q}_2}^{>}\frac{1}{q_2^2}\left(\frac{1}{(\vec{q}_1+\vec{q}_2+b^{-1}\vec{q})^2}-\frac{1}{(\vec{q}_1+\vec{q}_2)^2}\right)\approx-\tfrac{1}{12}K_4^2\delta_{\gamma\delta}(\delta_{\alpha\beta}q^2-q^\alpha q^\beta)b^{-2}\ln b, \quad (41)$$

where the calculation was performed with similar approximations, and the additional assumption $b\gg 1$, which enables one to ignore b^{-2} as compared with $b^{-2}\ln b$. The expression in (41) will contribute both to η_l through the q^2 term, and to h_{l+1} through the $q^\alpha q^\beta$ part. Finally, after summing over all the contributions of this kind, we obtain, for the $q^\alpha q^\beta$ terms in (29), the result

$$-\zeta_l^2b^{-d-2}q^\alpha q^\beta[h_l-P_1K_4^2u_l^2\ln b],$$

where P_1 is again a function of g_l and h_l. When (40) is used, we obtain the recursion relation

$$h_{l+1}=b^{-\eta_l}[h_l-P_1K_4^2u_l^2\ln b]. \quad (42)$$

As regards the g_l and h_l dependence of P_1 we note that the full expression for a term as (41) involves, in place of one factor $1/q_1^2$, the form

$$\left(\frac{1}{q_1^2}-\frac{1}{g+(1-h)q_1^2}\right)=\frac{g-hq_1^2}{g_1^2[g+(1-h)q_1^2]}$$

as may be seen from (22). It follows from these observations (a) that $P_1(g,h)$ *vanishes* when $g=h=0$, and (b) that P_1 is *linear* in g and h for small g and h. For large g, which will be relevant later, P_1 becomes independent of both g and h, and hence reduces to a number P_1 of order unity. Similar remarks apply to $P(g,h)$.

At this stage we have derived all the recursion formulas which are related to $U_2^{l,\alpha\beta}(\vec{q})$ for the fully isotropic case. Next we must obtain the recursion formula for u_l. Graph (b) of Fig. 1 is trivial, and we go on to consider graph (c) of that figure: its contribution to (28) is

$$4\zeta_l^4b^{-3d}u_l^2\sum_{\alpha,\beta,\gamma,\delta}\left(\int_{\vec{q}_1}^{>}G^{\gamma\delta}(\vec{q}_1)G^{\gamma\delta}(\vec{q}_1)\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}\sigma_{\vec{q}}^\alpha\sigma_{\vec{q}'}^\alpha\sigma_{\vec{q}''}^\beta\sigma_{-\vec{q}-\vec{q}'-\vec{q}''}^\beta+4\int_{\vec{q}_1}^{>}G^{\gamma\beta}(\vec{q}_1)G^{\gamma\delta}(\vec{q}_1)\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}\sigma_{\vec{q}}^\alpha\sigma_{\vec{q}'}^\alpha\sigma_{\vec{q}''}^\beta\sigma_{-\vec{q}-\vec{q}'-\vec{q}''}^\delta+4\int_{\vec{q}_1}^{>}G^{\alpha\beta}(\vec{q}_1)G^{\gamma\delta}(\vec{q}_1)\int_{\vec{q}}\int_{\vec{q}'}\int_{\vec{q}''}\sigma_{\vec{q}}^\alpha\sigma_{\vec{q}'}^\beta\sigma_{\vec{q}''}^\gamma\sigma_{-\vec{q}-\vec{q}'-\vec{q}''}^\delta\right). \quad (43)$$

Note that while the $\vec{q}_1$ integrals are restricted by $1\geq|\vec{q}_1|>b^{-1}$, the integrals on $\vec{q}$, $\vec{q}'$, $\vec{q}''$ run over the full range $|\vec{q}|<1$ (since $\sigma_{\vec{q}}^\alpha$ is already renormalized by ζ_l).

A typical integral in (43) is (see Appendix B)

$$K_d^{-1}\int_{\vec{q}_1}^{>}G^{\alpha\gamma}(\vec{q}_1)G^{\beta\delta}(\vec{q}_1)=\delta_{\alpha\gamma}\delta_{\beta\delta}\,d^{-1}\times[(d-2)A_{20}^d+2A_{11}^d]+[d(d+2)]^{-1}\times(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\delta}\delta_{\gamma\beta})\times(A_{20}^d-2A_{11}^d+A_{02}^d), \quad (44)$$

where A_{nm}^d was defined in (31). On collecting up terms to leading order in ϵ (so that $d\simeq 4$), the recursion formula for u_l is found to be

$$u_{l+1}=b^{\epsilon}[u_l-2K_4u_l^2(17A_{20}^4+2A_{11}^4+5A_{02}^4)+O(u_l^3)], \quad (45)$$

where we have used (40) for ζ_l.

The formulas (30), (34), (42), and (45) constitute the complete set of recursion relations for the coefficients r_l, g_l, h_l, and u_l for $\epsilon=4-d$ small (and for the fully isotropic case $f_l=v_l=0$).

V. FIXED POINTS AND CROSSOVER EXPONENT

We are now in a position to study the fixed points of the dipolar Hamiltonian for the fully isotropic case $f_l=v_l=0$. We start with (34), the recursion relation for g_l. With the choice of (40) for ζ_l, this recursion relation becomes

$$g_{l+1}=b^{2-\eta_l}g_l. \quad (46)$$

By (39), η_l is of order u_l^2. Since u_l will turn out to approach a fixed-point value of order ϵ, the exponent η_l is of order ϵ^2. Accordingly, (46) has only the two fixed points, $g^*=0$ and $g^*=\infty$. Even a small value of g_0 will thus result in an exponentially fast growth of g_l as l increases. For $g_0=0$, which by (20) also implies $h_0=f_0=0$, the propagator (23) becomes simply $\delta_{\alpha\beta}/(r_0+q^2)$, and we obtain the known recursion relations for the isotropic d-component Heisenberg model.[14] These relations have two fixed points, namely, the Gaussian fixed point

$$u^*=v^*=r^*=g^*=h^*=f^*=0, \quad (47)$$

with classical exponents, and the nontrivial isotropic fixed point

$$g^*=h^*=f^*=v^*=0,\quad u^*=\epsilon/48K_4,\quad r^*=-\tfrac{1}{4}\epsilon, \quad (48)$$

up to correction of order ϵ^2. To obtain the nontrivial fixed point, we have taken $d=4$ in (30) and used the results

$$A_{20}^4=A_{11}^4=A_{02}^4=\ln b,\quad A_{10}^4=A_{01}^4=\tfrac{1}{2}(1-b^{-2}) \quad \text{for } r=g=h=0, \quad (49)$$

which follow easily from the definition (31). The corresponding critical exponents are, of course, the same as may be obtained from Refs. 14 and 15 by putting $n=d=4-\epsilon$, namely,

$$2\nu=1+\frac{d+2}{2(d+8)}\epsilon+O(\epsilon^2)=1+\tfrac{1}{4}\epsilon+O(\epsilon^2), \quad (50)$$

$$\eta=\frac{d+2}{2(d+8)^2}\epsilon^2+O(\epsilon^3)=\tfrac{1}{48}\epsilon^2+O(\epsilon^3). \quad (51)$$

In what follows we shall refer to this fixed point as the "Heisenberg fixed point." It may be noted parenthetically, that if one uses $n=d=4-\epsilon$ for the three-dimensional Heisenberg model, and calculates to *second* order in $\epsilon(=1)$, the results are in better numerical agreement with those found by series expansions, than are the results obtained by fixing $n=3$. This suggests that $n=d(=4-\epsilon)$ might generally be a better description for the Heisenberg model; indeed, physically, the number of spin components bears a direct relation to the spatial dimensionality.

For small deviations of the parameters from the fixed-point values given above, one may linearize the recursion relations. Now the parameter g_l is coupled only to r_l and u_l through the $A_{nm}^d(r,g,h)$ terms in (30) and (45). Therefore, to study the instability associated with g_l, we may ignore the equation for h_l and simply put $h_l=h^*=0$. Near either of the fixed points mentioned, we expect the correlation length to behave as

$$\xi(T,g)\approx t^{-\nu}X(g/t^{\phi})=g^{-\nu/\phi}\tilde{X}(g/t^{\phi}), \quad (52)$$

where ϕ is the crossover exponent[14,21] and, as in (15), $t=(T-T_c)/T_c$.

Since the exponent ν is known [by (50) for the nontrivial fixed point, while $\nu=\frac{1}{2}$ for the Gaussian point], it is sufficient for calculating ϕ to obtain the exponent $\nu_G=\nu/\phi$, which describes the instability associated directly[14] with the dipolar term g_l. We may therefore also replace η_l by its fixed-point value, namely, the true exponent η. In addition we observe that

$$-\left(\frac{\partial A_{10}^4}{\partial r}\right)=-\left(\frac{\partial A_{01}^4}{\partial g}\right)=A_{20}^4=A_{02}^4=\ln b+O(\epsilon) \quad \text{at } r=r^*=O(\epsilon) \text{ and } g=h=0. \quad (53)$$

Finally then, the necessary linearized recursion relations are, to leading order,

$$r_{l+1}=b^{2-\eta}[r_l-4K_4(d+2)u^*\ln b(r_l+d^{-1}g_l)],\qquad g_{l+1}=b^{2-\eta}g_l. \quad (54)$$

(We could, of course, set $d=4$ here since we are working only to order ϵ.) If we search for solutions of the forms,

$$\Delta r_l=r_l-r^*=A\Lambda^l,\quad \Delta g_l=g_l-g^*=g_l=B\Lambda^l, \quad (55)$$

we immediately find two eigenvalues Λ. The first of these is

$$\Lambda_1=b^2[1-4K_4u^*(d+2)\ln b]+O(\epsilon^2) \quad (56)$$

and has the eigenvector $A=1$, $B=0$. It thus corresponds to the temperature instability (associated with r_0) and leads to the previously quoted values of the exponent ν. The second eigenvalue is simply $\Lambda_2=b^{2-\eta}$, and corresponds to the eigenvector $A=-d^{-1}+O(\epsilon)$, $B=1$, namely to the perturbation Hamiltonian

$$\mathcal{H}^1=\int_{\vec{q}}\sum_{\alpha\beta}\left(\frac{q^\alpha q^\beta}{q^2}-\frac{\delta_{\alpha\beta}}{d}\right)\sigma_{\vec{q}}^{\alpha}\sigma_{-\vec{q}}^{\beta}. \quad (57)$$

This is recognized just as the long-wave expression for the dipole-dipole interactions [see the formulas (7), (9), and (A17)]. From Λ_2 we immediately have

$$\nu_G = \ln b/\ln\Lambda_2 = \tfrac{1}{2}[1+\tfrac{1}{2}\eta + O(\eta^2)] \quad (58)$$

and hence for the nontrivial Heisenberg fixed point, we find

$$\phi = \nu/\nu_G = 1 + \tfrac{1}{4}\epsilon + O(\epsilon^2). \quad (59)$$

For the Gaussian fixed point $\nu_G = \nu = \frac{1}{2}$, and so $\phi = 1$.

From this it is clear that when

$$t = (T - T_c)/T_c \lesssim g_0^{1/\phi}, \quad (60)$$

the effect of the dipole-dipole interactions cannot be ignored, and one has to investigate the full nonlinear set of recursion relations.

VI. FULLY ISOTROPIC DIPOLAR BEHAVIOR

As noted above it follows from (34) that any nonzero initial value of the dipolar amplitude parameter $g_0(>0)$ implies that g_l diverges to infinity as $l\to\infty$ so that, close to T_c, a new form of critical behavior takes over. To study this, consider all the remaining recursion relations; the parameter g_l enters the equations only through the integrals $A^d_{nm}(r_l, g_l, h_l)$ defined in (31) and through the similar coefficients $P(g_l, h_l)$, $P_1(g_l, h_l)$. Clearly as $g_l\to\infty$ all the integrals A^d_{nm} involving g_l (i.e., for $m>0$) will approach zero and will hence drop out of the equations. Similarly $P(g_l, h_l)$, and $P_1(g_l, h_l)$, approach numbers P and P_1. Equivalently, we may then ignore the second term of order $1/g_0$ in the expression (22) for the propagator. The consequence is that we obtain a new set of reduced recursion relations corresponding to $g_l = g^* = \infty$. To determine the characteristic dipolar exponents we examine the fixed point (r^*, h^*, u^*) of these reduced equations.

Consider first the relation (42) for h_l; assuming u^* to be of order ϵ yields

$$\eta h^* = -P_1 K_4^2 u^{*2} + O(\epsilon^3), \quad (61)$$

where $\eta = \eta^*$ is determined by (39). Since η is thence of order ϵ^2 this yields

$$h^* = -(P_1/P) + O(\epsilon), \quad (62)$$

so that h^* is of order unity. However, h_l appears only in the same propagator factors $[g_l + r_l + (1 - h_l)q^2]^{-1}$ as does g_l; since these expressions vanish in the dipolar limit $g_l\to\infty$ we may also ignore h_l in the sequel.

Taking the limit $g\to\infty$ in (45) and in (30), and using (49), leads to a new "dipolar fixed point"

$$u^* = \frac{\epsilon \ln b}{34K_4 A^4_{20}(r=0)} = \frac{\epsilon}{34K_4} + O(\epsilon^2) \quad (63)$$

and

$$r^* = -\frac{18K_4 u^* A^4_{10}(r=0)}{1-b^{-2}} + O(\epsilon^2) = -\tfrac{9}{34}\epsilon + O(\epsilon^2). \quad (64)$$

To obtain the corresponding dipolar exponents we linearize (30) about this set of fixed point values [using (53) once more]. To order ϵ it is sufficient to consider the r_l equation alone, with the other parameters at their fixed point values. This yields

$$\Delta r_{l+1} = r_{l+1} - r^* = b^2[1 - \tfrac{9}{17}\epsilon\ln b + O(\epsilon^2)]\,\Delta r_l. \quad (65)$$

From the corresponding eigenvalue Λ, we finally conclude

$$2\nu = 2\ln b/\ln\Lambda = 1 + \tfrac{9}{34}\epsilon + O(\epsilon^2). \quad (66)$$

The significance of this value will be discussed further below, but it is worth remarking that it is very close to the isotropic result (50) despite the significant difference in the corresponding values of u^*. This is principally because the integrals for graphs (a) and (c) both change by nearly the same factor when g_l goes from 0 to ∞.

VII. DIPOLAR BEHAVIOR OF CORRELATION FUNCTION AND OF SUSCEPTIBILITY

As in (21), the two-spin correlation function $\Gamma^{\alpha\beta}(\vec{q})$ is defined generally by

$$\langle\sigma^\alpha_{\vec{q}}\sigma^\beta_{\vec{q}'}\rangle = \frac{1}{Z}\int_\sigma \sigma^\alpha_{\vec{q}}\sigma^\beta_{\vec{q}'} e^{\bar{\mathcal{H}}_0} = \delta(\vec{q}+\vec{q}')\,\Gamma^{\alpha\beta}(\vec{q}), \quad (67)$$

where $\bar{\mathcal{H}}$ is now the full expression in (16). If $|\vec{q}|$ and $|\vec{q}'|$ are smaller than b^{-l}, we can replace $\bar{\mathcal{H}}_0$ in (67) by $\bar{\mathcal{H}}_l$ and appropriately renormalize the spins, to show

$$\delta(\vec{q}+\vec{q}')\,\Gamma^{\alpha\beta}(\vec{q}) \equiv \frac{\zeta^{2l}}{Z_l}\int_\sigma \sigma^\alpha_{b^l\vec{q}}\sigma^\beta_{b^l\vec{q}'} e^{\bar{\mathcal{H}}_l} = b^{l(d+2-\eta)}\Gamma^{\alpha\beta}_l(b^l\vec{q})\,b^{-ld}\delta(\vec{q}+\vec{q}'),$$

where ζ is the geometric mean of the ζ_j for $j=1, 2, \ldots, l$. Hence we find

$$\Gamma^{\alpha\beta}(\vec{q}) = b^{l(2-\eta)}\Gamma^{\alpha\beta}_l(b^l\vec{q}), \quad (68)$$

where $\Gamma^{\alpha\beta}_l(\vec{q})$ is the correlation function calculated with $\bar{\mathcal{H}}_l$. In the present section we will still consider the fully isotropic case, namely, $f_l = v_l = 0$. In the definition corresponding to (67) we can now develop in a Taylor series in u_l, thus finding

$$\Gamma^{\alpha\beta}(\vec{q}) = b^{l(2-\eta)}\Big[G^{\alpha\beta}_l(b^l\vec{q}) - 4u_l\sum_\gamma\Big(G^{\alpha\gamma}_l(b^l\vec{q})G^{\beta\gamma}_l(-b^l\vec{q})\sum_\delta\int_{\vec{q}'}G^{\delta\delta}_l(\vec{q}') + 2\sum_\delta G^{\alpha\gamma}_l(b^l\vec{q})G^{\delta\beta}_l(-b^l\vec{q})\int_{\vec{q}'}G^{\gamma\delta}(\vec{q}')\Big) + O(u_l^2)\Big], \quad (69)$$

where $G^{\alpha\beta}_l$ is now given by (22) with r_l, g_l, and h_l replacing r_0, g_0, and h_0. Now if l is large enough, so that $g_l \gg 1$, and if $\vec{q}\neq 0$, we may ignore the terms in $G^{\alpha\beta}_l(b^l\vec{q})$ containing g_l and take

$$G^{\alpha\beta}_l(\vec{q}) \simeq \frac{1}{r_l+q^2}\left(\delta_{\alpha\beta} - \frac{q^\alpha q^\beta}{q^2}\right), \quad \vec{q}\neq 0. \quad (70)$$

At criticality, i.e., when $T=T_c$, the parameter

r_l approaches r^* as $l\to\infty$, and $G_l^{\alpha\beta}(\vec{q})$ is always large for small $\vec{q}$. On the other hand, if there is an initial departure, however small, from criticality, as in the case $T>T_c$, then r_l will eventually start to deviate rapidly from the fixed point. In the linear region, where (65) applies, r_l will vary according to

$$\Delta r_l = r_l - r^* \sim (T-T_c)\Lambda^l. \quad (71)$$

Finally, Δr_l grows too large and leaves the linear region. However, the full recursion formulas may be integrated to yield the functional dependence of r_l on l, on $(T-T_c)$, and on the other variables. The eigenvalue of the linearized recursion relation for u_l is $b^{-\epsilon}=1+O(\epsilon)$ so that if $u_0\simeq u^*$, u_l will deviate from the fixed point much less rapidly than r_l. Similar arguments show that we may assume u_l, h_l, and g_l^{-1} to be close to their fixed point values. With the replacements $u_l=u^*=O(\epsilon)$, $h_l=h^*$, and $g_l\simeq\infty$, the recursion relation for r_l becomes

$$\Delta r_{l+1} = b^{2-\eta}[1-18K_4u^*\ln b]\,\Delta r_l + 9K_4 b^{2-\eta}u^* r_l \ln[(1+r_l)/(1+b^2r_l)] + O(\epsilon^2). \quad (72)$$

From this we see that r_l deviates from the exponential form $(T-T_c)\Lambda^l$, of (71), only by terms of order ϵ. Thus, we write

$$\Delta r_l = A(T-T_c)\Lambda^l + \epsilon r_l^1 = r_l^0 + \epsilon r_l^1. \quad (73)$$

One may proceed with the iteration of (72) until $r_l^0\simeq 1$. At this stage the reduced correlation length $\xi_l=\xi(r_l, u_l)$ will also be of order unity.[11,12] Since by construction of the renormalization group one has $\xi_l=b^{-l}\xi(r_0, u_0)$ and $\xi_0=\xi(r_0,u_0)\sim(T-T_c)^{-\nu}$, the exponent ν is determined by the value of Λ, as in (66). When $r_l^0=1$ the propagator $G_l^{\alpha\beta}$ becomes of order $(1+\epsilon r_l^1+q^2)^{-1}$, so that the term proportional to u_l in (69) indeed gives contributions only of order ϵ. In fact, if this expression for $G_l^{\alpha\beta}$ is expanded in powers of ϵr_l^1 one finds that the first-order term exactly cancels the u_l terms in (69). This will be shown more generally in Paper II.[19] In order to demonstrate its plausibility, suppose $b\gg 1$. In this case we may assume that r_{l-1} is still in the linear region, namely, $r_{l-1}^1\simeq 0$, and hence,

$$\Delta r_{l-1} = r_{l-1}^0 = A(T-T_c)\Lambda^{l-1} = \Lambda^{-1}r_l^0 = \Lambda^{-1}.$$

The recursion relation (72) then yields

$$1+\epsilon r_l^1 = \Delta r_l = b^{2-\eta}\Lambda^{-1}\left[1-\frac{9\epsilon}{34}\ln\left(\frac{1+\Lambda^{-1}}{b^{-2}+\Lambda^{-1}}\right)+O(\epsilon^2)\right],$$

and hence finally

$$r_l = 1-\tfrac{9}{34}\epsilon(1-\ln 2) \quad (74)$$

and

$$G_l^{\alpha\beta}(\vec{q}) = \left(\frac{\delta_{\alpha\beta}-(q^\alpha q^\beta/q^2)}{1+q^2}\right) \times[1+\tfrac{9}{34}\epsilon(1-\ln 2)(1+q^2)^{-1}+O(\epsilon^2)]. \quad (75)$$

Returning to (69) we may replace b^{-l} by $\xi_l/\xi_0=\xi^{-1}$ as explained, and assume $(q\xi)^2\ll 1$, to find

$$\Gamma^{\alpha\beta}(\vec{q}) \simeq b^{l(2-\eta)}\left[G_l^{\alpha\beta}(b^l\vec{q}) - \frac{9K_4u_l}{r_l+q^2}\left(\delta_{\alpha\beta}-\frac{q^\alpha q^\beta}{q^2}\right)\times[1-r_l\ln(1+r_l^{-1})]+O(\epsilon^2)\right]$$

$$\simeq \frac{C}{t^\gamma}\,\frac{1}{1+\xi^2q^2}\left(\delta_{\alpha\beta}-\frac{q^\alpha q^\beta}{q^2}\right)[1+O(\epsilon^2)]. \quad (76)$$

where, as before, $t=(T/T_c)-1$ and

$$\xi(T)=\xi_0/t^\nu, \quad (77)$$

while

$$\gamma=(2-\eta)\nu. \quad (78)$$

The constant C depends on all the irrelevant variables. The correction terms become important when $(\xi q)^2$ increases to order unity. Indeed when $T\to T_c$ and $\xi\to\infty$ these terms must modify the $(1+\xi^2q^2)^{-1}$ behavior to $(\xi^2q^2)^{-1+\eta/2}$ so that at the critical point $\Gamma^{\alpha\beta}(\vec{q})$ varies as $1/q^{2-\eta}$.

Note that in (76) we have used only the first term in the expression (22) for the propagator $G_l^{\alpha\beta}$. It is instructive to examine the other part of G_l which involves g_l and h_l. On assuming $u_l=u^*$, $\eta_l=\eta^*=\eta$, etc., and using (46) for g_l, the second term in (22) gives the additional contribution

$$\frac{b^{l(2-\eta)}}{r_l+g_l+(1-h_l)b^{2l}q^2}\,\frac{q^\alpha q^\beta}{q^2} = \frac{Ct^{-\gamma}}{1+Ct^{-\gamma}g_0+(1-h_l)\xi^2q^2}\,\frac{q^\alpha q^\beta}{q^2}. \quad (79)$$

Now h_l changes slowly with l, and has a finite fixed-point value. Hence we may assume $h_l\simeq h^*$ [see Eq. (62)]. Clearly, for $0<q^2\xi^2\ll 1$, the expression in (79) does not diverge at T_c. It is important, however, for the calculation of correlations between the longitudinal spin components. As a result of these considerations we conclude that (76) remains valid for the transverse fluctuations when $0<\xi^2q^2\ll 1$ even when the effects of the g and h corrections are taken into account.

The relation (78) confirms the standard scaling relation[1,21] and, in combination with (66), gives

$$\gamma = 1+\tfrac{9}{34}\epsilon+O(\epsilon^2). \quad (80)$$

Up to this point we have considered only the case $\vec{q}\neq 0$. For $\vec{q}=0$ most of the previous discussion still applies, but (70) is no longer valid [see the note after Eq. (22)]. Instead, we must now write

$$G_l^{\alpha\beta}(0) = [U_2^l(0)^{-1}]^{\alpha\beta}, \quad (81)$$

where $U_2^{l,\alpha\beta}(0)$ is the correct shape-dependent value of $U_2^{l,\alpha\beta}(\vec{q})$ at $\vec{q}=0$, namely [by (17), (9), (A17), (8), and (1)],

$$U_2^{l,\alpha\beta}(0)=r_l\delta_{\alpha\beta}+g_l\psi^{\alpha\beta}, \qquad (82)$$

where

$$\psi^{\alpha\beta}=\frac{1}{d}\,\delta_{\alpha\beta}+\frac{1}{a_1}\,A^{\alpha\beta}(0)=\frac{1}{d}\,\delta_{\alpha\beta}-\frac{(d-2)}{a_1}\,\Phi^{\alpha\beta}. \qquad (83)$$

We thus return to the question of the shape dependence of the susceptibility. As usual,[6] we shall assume that the sample has ellipsoidal boundaries. With this assumption, $\Phi^{\alpha\beta}$, defined in (1), is independent of i and is diagonal. Therefore, we have

$$\psi^{\alpha\beta}=\mathfrak{D}^\alpha\delta_{\alpha\beta}, \qquad (84)$$

where $\mathfrak{D}^\alpha$ depends on d, $\Phi^{\alpha\alpha}$, and a_1, and hence

$$G_l^{\alpha\beta}(0)=\delta_{\alpha\beta}[r_l+g_l\mathfrak{D}^\alpha]^{-1}. \qquad (85)$$

When Eq. (46) is used for g_l, (74) for r_l, and repeating the previous arguments, we now find that the susceptibility tensor has the form

$$\chi^{\alpha\beta}\propto\Gamma^{\alpha\beta}(0)=\delta_{\alpha\beta}b^{l(2-\eta)}\Gamma_l^{\alpha\alpha}(0)=\delta_{\alpha\beta}b^{l(2-\eta)}\left(G_l^{\alpha\alpha}(0)-\frac{9K_4u_l}{(r_l+g_l\mathfrak{D}^\alpha)^2}[1-r_l\ln(1+r_l^{-1})]+O(\epsilon^2)\right)$$

$$=\delta_{\alpha\beta}b^{l(2-\eta)}\left[\frac{1}{1+b^{l(2-\eta)}g_0\mathfrak{D}^\alpha}\left(1+\frac{9\epsilon(1-\ln 2)}{34(1+b^{l(2-\eta)}g_0\mathfrak{D}^\alpha)}\right)-\frac{9\epsilon(1-\ln 2)}{34(1+b^{l(2-\eta)}g_0\mathfrak{D}^\alpha)^2}+O(\epsilon^2)\right], \qquad (86)$$

so that finally

$$1/\chi^{\alpha\alpha}\propto C^{-1}t^\gamma+g_0\mathfrak{D}^\alpha+O(\epsilon^2). \qquad (87)$$

The omitted constant of the proportionality arises through the spin rescaling introduced in (16) and is equal to

$$k_BT/(\tfrac{1}{2}cd^{-1}J-2a_4Ga^{-d})\,\pi^{1+d/2}.$$

The amplitude C in (87) is the same as in (76). We shall show in Paper II (Ref. 19) that all the higher-order terms in ϵ in (87) are in fact canceled by a mechanism similar to the one discussed above.

Thus, $\chi^{\alpha\alpha}$ will diverge only if $\mathfrak{D}^\alpha=0$. (This occurs in three dimensions, for example, for a cylindrical sample with α parallel to the cylinder axis.) In all other cases, the exponent γ describes only the approach of $(\chi^{\alpha\alpha})^{-1}$ to $g_0\mathfrak{D}^\alpha$ as $T\to T_c$. However, if one corrects the experimental data by subtracting the demagnetizing field from the external field in the usual way, one obtains only the $(T-T_c)^\gamma$ part on the right-hand side of (87); one can thus measure γ directly, as in the absence of dipolar interaction. But even so, $\chi^{\alpha\alpha}$ is not just a limit of $\Gamma^{\alpha\beta}(\vec{q})$ as $\vec{q}\to 0$, because the angular dependence of $\Gamma^{\alpha\beta}(\vec{q})$ remains singular. The usual methods of observing $\Gamma^{\alpha\beta}(\vec{q})$ are through neutron scattering experiments which, for small momentum transfer, measure[22] only spin flucutations which are transverse to $\vec{q}$. In addition, one has to be careful in interpreting these experiments, since the relaxation time of the spin fluctuations are also governed by the dipole-dipole interactions in the region we are considering.[23] The difficulties in making a comparison of $\chi^{\alpha\alpha}$ with the limit of $\Gamma^{\alpha\alpha}(\vec{q})$ in materials for which dipole-dipole forces are important, has been noticed recently by Arrott, Heinrich, and Noakes.[24] We feel that the present analysis clarifies some of these difficulties.

The validity of the macroscopic expectation[6,7] (2), has been checked here to order ϵ. A discussion of the ϵ^2 terms will be presented in Paper II.[19]

VIII. RECURSION RELATIONS FOR ANISOTROPIC DIPOLAR INTERACTIONS

As seen in Appendix A, the value of the coefficient a_5 of the term $\delta_{\alpha\beta}(q^\alpha)^2$ in the dipolar interaction matrix (9) is nonzero for the standard cubic lattices in three dimensions. However, we are treating the problem near four dimensions and, without further calculation, it is not clear how a_5 depends on the dimensionality. It might, indeed, be small compared to ϵ. Barring such a possibility, however, our treatment up to this point applies only to a fully isotropic or "liquid" ferromagnet. For the general case of a cubic lattice we must consider both the parameters f_0 and v_0 arising in Eqs. (16) and (17). Consequently, the full propagator equation (23) must be used and, since the terms proportional to f_0 will induce an anisotropic four-spin coupling, we must replace u_l at every graphical vertex by $u_l+\delta_{\alpha\beta}v_l$, where α and β are the indices of the two pairs of lines entering into that vertex. When the steps of Sec. IV are repeated, the recursion relation (30) for r_l now becomes

$$r_{l+1}=\zeta_l^2b^{-d}(r_l+4K_dd^{-1}[(d+2)u_l+3v_l]$$
$$\times\{(d-1)A_{10}^d+A_{01}^d-(d+2)^{-1}f_l$$
$$\times[(d-1)A_{20}^{d+2}+3A_{02}^{d+2}]+O(f_l^2)\}+O(u_l^2,\,u_lv_l,\,v_l^2)). \qquad (89)$$

The recursion relation (34) for g_l is unaffected by the inclusion of f_l and v_l. However, relations (38) and (42) for ζ_l and h_l do have to be modified, since u_l^2 will be replaced everywhere by a quadratic form in u_l and v_l. Thus, (39) becomes

$$\eta_l=PK_4^2(u_l^2+P'u_lv_l+P''v_l^2), \qquad (90)$$

while (42) is replaced by

$$h_{l+1}=b^{-\eta_l}[h_l-P_1K_4^2(u_l^2+P_1'u_lv_l+P_1''v_l^2)\ln b], \qquad (91)$$

where P', P'', P_1', and P_1'' are also functions of g and of h (we take $r=0$ here), which will attain definite limits of order unity when g and h approach zero or $g\to\infty$. Owing to the complexity of the calculations, however, we have not attempted to obtain complete expressions for these coefficients.

We must now also derive the recursion relation for f_l. A simple symmetry argument shows that if $G_0^{\alpha\beta}$, as defined in Eq. (22), is used everywhere in the calculation of the graph (d) of Fig. 1, instead of the full propagator $G^{\alpha\beta}$, no $(q^\alpha)^2\delta_{\alpha\beta}$ terms will be generated except for those already included in the $hq^\alpha q^\beta$ term. To first order in f, one therefore has to consider integrals such as

$$X_3^{\alpha\beta\gamma\delta}=\delta_{\gamma\delta}f\int_{\vec{q}_1}^{>}\frac{1}{q_1^2}\left\{\left(\frac{q_1^\alpha}{q_1}\right)^2\delta_{\alpha\beta}+\left[\sum_\epsilon\left(\frac{q_1^\epsilon}{q_1}\right)^4-\left(\frac{q_1^\alpha}{q_1}\right)^2-\left(\frac{q_1^\beta}{q_1}\right)^2\right]\frac{q_1^\alpha q_1^\beta}{q_1^2}\right\}$$
$$\times\int_{\vec{q}_2}^{>}\frac{1}{q_2^2}\left(\frac{1}{(\vec{q}_1+\vec{q}_2+b^{-1}\vec{q})^2}-\frac{1}{(\vec{q}_1+\vec{q}_2)^2}\right)=-\delta_{\gamma\delta}\frac{fK_4^2}{30b^2}\ln b\left\{\delta_{\alpha\beta}[q^2-(q^\alpha)^2]+\tfrac{3}{4}q^\alpha q^\beta\right\}. \qquad (92)$$

(Note that all the numerical coefficients follow simply from the angular integration formulas discussed in Appendix B for $d=4$.) Summing all the contributions of order fu^2, we finally find that the $(q^\alpha)^2\delta_{\alpha\beta}$ term in (29) yields, to leading order,

$$f_{l+1}=\zeta_l^2b^{-d-2}f_l[1+P_2K_4^2(u_l^2+P_2'u_lv_l+P_2''v_l^2)\ln b]$$
$$=f_l\{1-[\eta_l-P_2K_4^2(u_l^2+P_2'u_lv_l+P_2''v_l^2)]\ln b\}, \qquad (93)$$

where we have used (38) and (90) to eliminate ζ_l. The coefficients P_2, P_2', and P_2'' again depend on g_l and h_l but have not been calculated explicitly. Now if the factor in curly brackets is less than unity, f_l will decay with increasing l so that f_0 represents an irrelevant variable; conversely if the factor exceeds unity, f_l would be unstable under the recursion and would correspond to a relevant variable. Clearly, this point will be determined by the relative signs and magnitudes of the coefficients P, P', and P'' in (90) and P_2, P_2', and P_2'' in (93). For the particular terms we have studied the former are positive and dominate. We believe this will remain true when all the terms are included. In support we first note that the contributions not calculated explicitly involve products of a larger number of components $q^\alpha q^\beta q^\delta\cdots$; as can be seen from Appendix B the values of the integrals decrease with increasing numbers of components. Secondly, a full Feynman-graph calculation[19,20] of the exponent η, which is of a similar character and includes contributions from the same graphs, indicates that the sign of P is the same as the sign of the partial contribution (37), namely, positive. Although P_2 may well remain positive it is reasonable to expect $P>P_2$ (and similarly for P_2', P_2'', etc.), since the integrals for P_2 involve products of more components than those for P and hence should be smaller (as explained). Indeed, a comparison of (37) and (92) shows that the magnitude of the contribution to P_2 is $\frac{2}{15}$ of the corresponding contribution to P. In the vicinity of the Heisenberg fixed point (48) we have in fact been able to calculate[25] P and P_2 (by expanding all expressions to first order in h and in f and taking $g=0$). We find that for this region $P(g=0,\ h=0)=48$, $P_2(g=0,\ h=0)=\frac{80}{3}$, so that P does indeed exceed P_2. In summary, the recursion relation (93) may be written

$$f_{l+1}=b^{-\tilde{\eta}_l}f_l+O(u_l^2f_l^2,\ u_l^3f_l,\ u_lv_lf_l^2), \qquad (94)$$

where $\tilde{\eta}_l$ is a quadratic form in u_l and in v_l in which we expect the coefficient of u_l^2 to be positive. As a matter of fact, if this presumption failed we could, alternatively, choose to normalize ζ_l through the $(q^\alpha)^2$ term in $U_2^{l,\alpha\beta}$, instead of through the q^2 term. This would interchange the roles of η and $(\eta-\tilde{\eta})$, and we could then proceed in the same way. However, this possibility seems unlikely and we shall not pursue it further. It should also be noted that since η and $\tilde{\eta}$ are of order ϵ^2, our first-order results will remain valid as far as the equation for f_l is concerned.

We consider next the graph (c) of Fig. 1. The typical integral (44) now has an additional term, of order f, and the sums over components in (43) now contain varying factors $(u_l+v_l\delta_{\alpha\beta})(u_l+v_l\delta_{\delta\beta})$ instead of the constant u_l^2. Finally, we find

$$u_{l+1}=b^\epsilon[u_l-E_{20}u_l^2-E_{11}u_lv_l-E_{02}v_l^2+E_{20}'f_lu_l^2+E_{11}'f_lu_lv_l+E_{02}'f_lv_l^2]+O(f_l^2u_l^2,\ u_l^3,\ u_l^2v_l,\ \dots), \qquad (95)$$

$$v_{l+1}=b^\epsilon[v_l-F_{11}u_lv_l-F_{02}v_l^2+F_{20}'f_lu_l^2+F_{11}'f_lu_lv_l+F_{02}'f_lv_l^2]+O(f_l^2u_l^2,\ u_l^3,\ u_l^2v_l,\ \dots), \qquad (96)$$

where the coefficients E_{ij}, E_{ij}', F_{ij}, and F_{ij}' are linear combinations of the integrals A_{nm}^d for $d=4$ and 6 and n, $m=0, 1, 2, 3$, and so depend on r_l, g_l, and h_l. The explicit formulas for these coefficients are presented in Table II. The expression for $E_{20}(r_l, g_l, h_l)$ has already appeared in (45). Note

TABLE II. Coefficients for the u_l and v_l recursion relations.

$$E_{20}=2K_4(17A_{20}^4+2A_{11}^4+5A_{02}^4)$$
$$E_{11}=4K_4(5A_{20}^4-A_{11}^4+2A_{02}^4)$$
$$E_{02}=3K_4(\tfrac{1}{2}A_{20}^4-A_{11}^4+\tfrac{1}{2}A_{02}^4)$$
$$E'_{20}=\tfrac{1}{5}K_4(56A_{30}^6-17A_{21}^6+39A_{12}^6+36A_{03}^6)$$
$$E'_{11}=\tfrac{1}{10}K_4(67A_{30}^6-37A_{21}^6+24A_{12}^6+66A_{03}^6)$$
$$E'_{02}=\tfrac{1}{20}K_4(18A_{30}^6-9A_{21}^6-27A_{12}^6+18A_{03}^6)$$
$$F_{11}=4K_4(7A_{20}^4+4A_{11}^4+A_{02}^4)$$
$$F_{02}=K_4(21A_{20}^4+12A_{11}^4+3A_{02}^4)$$
$$F'_{20}=\tfrac{1}{5}K_4(-7A_{30}^6-11A_{21}^6+12A_{12}^6+6A_{03}^6)$$
$$F'_{11}=\tfrac{1}{5}K_4(31A_{30}^6-19A_{21}^6+78A_{12}^6+30A_{03}^6)$$
$$F'_{02}=\tfrac{1}{10}K_4(57A_{30}^6-12A_{21}^6+99A_{12}^6+36A_{03}^6)$$

the presence in (96) of a term proportional to $f_l u_l^2$; this serves to generate the anisotropic four-spin coupling with parameter v_l, even if $v_0=0$.

IX. BEHAVIOR WITH ANISOTROPIC INTERACTIONS

The complete set or recursion relations for anisotropic dipolar interactions, involving the parameters r_l, g_l, h_l, f_l, u_l, and v_l are (89) (with $d=4$), (34), (91), (93), (95), and (96). We will discuss, as far as we are able, the various fixed points and their characters. The results are summarized in Table III.

First note that the Gaussian and Heisenberg fixed points, (47) and (48), are still solutions of the equations; both are clearly unstable with respect to the interaction $q^\alpha q^\beta/q^2$ with parameter g_l. Even for $g_l=0$, the Gaussian fixed point is unstable with respect to both u_l and v_l (with eigenvalues $\Lambda\simeq b^\epsilon$). It turns out,[25] however, that the Heisenberg fixed point is also unstable with respect to v_l. Our present interest is in ferromagnets with dipolar interactions, $g_0\neq 0$. Accordingly, the crossover from these fixed points, discussed in Sec. V, will still occur; but we have to determine the behavior to which the system actually crosses over as t becomes much less than $g_0^{1/\phi}$. To do so, we search for additional fixed points of the recursion relations. For $g=f=h=0$, the addition of the parameter v adds one new fixed point, namely,

$$g^*=h^*=f^*=u^*=0,\quad v^*=\epsilon/36K_4,\quad r^*=-\tfrac{1}{6}\epsilon\,, \tag{97}$$

which corresponds simply to $n=d=4-\epsilon$ *uncoupled* Ising models with appropriate exponents.[12] Conversely, for $g\neq 0$, the fixed-point condition $g^*=\infty$ applies, and by Eq. (31) we can ignore all integrals A_{nm}^d with $m>0$. The isotropic dipolar fixed point, discussed in Sec. VI, is then still a fixed point of the new recursion formulas, with

$$g^*=\infty\,,\quad f^*=v^*=0\,,$$
$$h^*=-P_1/P\,,\quad u^*=\epsilon/34K_4,\quad r^*=-\tfrac{9}{34}\epsilon\,. \tag{98}$$

An additional anisotropic dipolar fixed point can arise from a simultaneous solution of (93), (95), and (96). To first order in f, this means that u^* and v^* must satisfy

$$\tilde{\eta}(u^*,\, v^*)=0\,; \tag{99}$$

note by (90), (93), and (94) that $\tilde{\eta}_l$ is a quadratic form in u_l and in v_l although the numerical values of the coefficients are not explicitly known. If (99) has a solution, with $u^*\neq 0$ and with the ratio $x=v^*/u^*$ real, then we can divide (95) by (96) to obtain

$$x=\frac{56x+42x^2+(f/5)(14-62x-57x^2)}{68+40x+3x^2-(f/5)(112+67x+9x^2)}\,, \tag{100}$$

where we have used Table II, set $A_{nm}^d=0$ for $m>0$, and noted that $A_{30}^6=A_{20}^4$ when $r\to 0$. This relation

TABLE III. Fixed points of the anisotropic-dipolar model.

Fixed point	Operators and corresponding parameters					
	Two-spin $\sigma_{\vec q}^\alpha\sigma_{-\vec q}^\beta$				Four-spin $\sigma_{\vec q}^\alpha\sigma_{\vec q'}^\alpha\sigma_{\vec q''}^\beta\sigma_{-\vec q-\vec q'-\vec q''}^\beta$	
	$\delta_{\alpha\beta}$	$q^\alpha q^\beta/q^2$	$-q^\alpha q^\beta$	$(q^\alpha)^2\delta_{\alpha\beta}$	1	$\delta_{\alpha\beta}$
Note $n=d=4-\epsilon$	r	g	h	f	K_4u	K_4v
Gaussian	0	0	0	0	0	0
Heisenberg	$-\frac{1}{4}\epsilon$	0	0	0	$\frac{1}{48}\epsilon$	0
Ising	$-\frac{1}{6}\epsilon$	0	0	0	0	$\frac{1}{36}\epsilon$
Isotropic dipolar	$-\frac{9}{34}\epsilon$	∞	$-P_1/P$	0	$\frac{1}{34}\epsilon$	0
Anisotropic dipolar	$-\frac{9}{34}\epsilon+O(f^*)$	∞	$-P_1/P+O(f^*)$	$\lvert f^*\rvert\ll 1$	$\frac{1}{34}\epsilon+O(f^*)$	$\frac{7}{1020}\epsilon f^*$
(two possibilities)	?	∞	?	$O(1)$?	?	?

may be solved for f^*, and then one can substitute $v^* = xu^*$ into (95) and solve for u^*. Of course this procedure is subject to our original perturbation assumption that $f \ll 1$; thus it will be justified only if it actually yields a solution $f^* \ll 1$. If f is indeed much less than unity the only value of x allowed by (100) is

$$x = \tfrac{7}{30} f^* + O(f^{*2}) . \qquad (101)$$

Hence we also have $|x| \ll 1$ or $|v^*| \ll u^*$. This effectively brings us back close to the isotropic dipolar fixed point (98); the previous values thus apply with corrections of order f^*, specifically,

$$g^* = \infty , \quad v^* = \tfrac{7}{30} u^* f^* , \quad h^* = -\frac{P_1}{P} + O(f^*),$$
$$u^* = \left(\frac{\epsilon}{34K_4}\right) + O(f^*), \quad r^* = -\tfrac{9}{34}\epsilon + O(f^*) . \qquad (102)$$

The critical exponents derived in Secs. VI and VII will also have corrections of order f^*; but if f^* is sufficiently small, these may be neglected.

The existence of the fixed point (102) is, of course, still subject to the existence of an appropriate small real root x of the quadratic equation (99). If, in actuality, there is no such solution with $f^* \ll 1$, there still may be a fixed point with a relatively large value of f^*. However, our basic expansion (24) for the propagator would then be no longer valid, and the full propagator (23) would have to be used.

With the information summarized in Table III before us, we may now discuss the evolution of the various parameters as l increases. Clearly, g_l will always tend rapidly to infinity. The initial values of u_0 and of v_0 are characteristic of the continuous-spin model, and we are effectively free to choose them. If both u_0 and v_0 are chosen of order ϵ, it is clear from the recursion relations that u_l and v_l will change relatively slowly with l, specifically at rates proportional to ϵ. By the same token the rates of changes of f_l and of h_l will be only of order ϵ^2.

The behavior of r_l, on the other hand, depends strongly on the initial value r_0. For most initial values, r_l will grow rapidly and indefinitely with an eigenvalue of order b^2. The value of r_0 is, of course, fixed by the temperature T via (18). From (89) we see that only for r_0 in the vicinity of

$$\begin{aligned} r_0^c = & -3K_4(1-b^{-2})^{-1}(v_0+2u_0) \\ & \times[3A_{10}^4 + A_{01}^4 - \tfrac{1}{2} f_0 (A_{20}^6 + A_{02}^6)] \\ \simeq & -3K_4(u_0 + \tfrac{1}{2}v_0)[3 - \tfrac{1}{2}f_0 + (1+g_0-h_0)^{-1} \\ & - \tfrac{1}{2} f_0 (1+g_0-h_0)^{-2}] , \end{aligned} \qquad (103)$$

will r_l change slowly with l, the changes then being of order ϵ. Relation (103) may thus be taken as defining the critical temperature T_c. The initial values g_0, f_0, and h_0 are fixed by the dipolar interactions in the original Hamiltonian through (20).

As mentioned, we may choose u_0 to be of order ϵ. It is also natural to take $v_0 = 0$; but when $f_0 \neq 0$ the recursion relation (96) leads to $v_l = O(f_0 u_0^2) = O(\epsilon^2)$ for small l. Furthermore, all the fixed points discussed above are found to be *unstable* with respect to v. Specifically, at the Gaussian fixed point the eigenvalue is simply $\Lambda_v = b^\epsilon > 1$. At the Heisenberg fixed point an explicit second-order calculation[25] shows that

$$\Lambda_v \approx b^{\epsilon^2/12} \quad (\text{Heisenberg } n = d = 4 - \epsilon) . \qquad (104)$$

For the isotropic dipolar fixed point the recursion relation (96) yields

$$\Lambda_v \approx b^\epsilon (1 - F_{11} u^*) \approx b^{3\epsilon/17} \quad (\text{isotropic dipolar}) , \qquad (105)$$

where we have used (49) and the values listed in Tables II and III. Finally, if there is an anisotropic dipolar fixed point with $|f^*| \ll 1$, this last value applies up to corrections of order f^*.

This instability with respect to v is somewhat unexpected. If the connection between n, the number of spin components, and the dimensionality d is relaxed and short-range forces are considered, it is found that to order ϵ there is no instability when $n < 4$.[14] Since the physically relevant case is $n = 3$ (and $d = 3$), one might suspect that there is no such instability in real systems. However, the only route presently open to decide this point seems to be the calculation of ϵ^2 and ϵ^3 corrections in (105) and (104). Until that is done, we are restricted to small ϵ and must face the consequences of the instability.

Thus, even though v_l changes relatively slowly with l it will eventually grow large. This in turn may lead either to a new anisotropic dipolar fixed point with, presumably, $f^* = O(1)$, or to a divergence of v_l to ∞. If the limit is $+\infty$, the recursion relations could yield another fixed point; conversely, if $v_l \to -\infty$ the system probably exhibits a first-order transition in place of a true critical point. This is the situation in the analogous two-component "Baxter-like" model of Wilson and Fisher.[12]

Nevertheless, if we start with $v_0 = 0$, the effects of $v_l \approx \Lambda_v^l f_0 u_0^2 \sim \Lambda_v^l f_0 \epsilon^2$ will remain unfelt in the other recursion relations provided $v_l \ll \epsilon$ or

$$\Lambda_v^l f_0 u_0 \ll 1 , \quad u_0 = O(\epsilon) . \qquad (106)$$

Conversely, g_l will grow much more rapidly, namely, as $\Lambda_g^l \approx b^{(2-\eta)l}$. Since $\Lambda_v = 1 + O(\epsilon)$, it follows that for $r_0 \simeq r_0^c$ and $f_0 \ll 1$ the renormalized Hamiltonian will stay close to the isotropic dipolar fixed point for many iterations before the v instability becomes effective. As explained in Sec. VII, when $T \neq T_c$, the renormalization procedure is to be repeated until the (reduced) correlation length $\xi_l = \xi(r_l, g_l, h_l, f_l, u_l, v_l)$ becomes of order unity (i.e., of order the lattice spacing a); at this stage, see (73), we also have

$$\Delta r_l = r_l - r^* \approx A(T - T_c)\Lambda^l = O(1)\,. \qquad (107)$$

As before, we conclude that if $t = \Delta T/T_c \gtrsim g_0^{1/\phi}$, with $\phi \simeq 1 + \frac{1}{4}\epsilon$ given by (59), the observed behavior will be of the normal Heisenberg character; conversely, for $t \lesssim g_0^{1/\phi}$ crossover to dipolar behavior takes place. Now, as observed, until v_l becomes of order ϵ this behavior will remain of the simple isotropic dipolar character discussed in Secs. VI and VII. By eliminating l between (106) and (107) we hence conclude that isotropic dipolar behavior will be observed provided

$$A(T - T_c) \sim t \gg (f_0 u_0)^{1/\phi_v}\,, \qquad (108)$$

where

$$\phi_v = \ln\Lambda_v / \ln\Lambda \approx \tfrac{3}{34}\epsilon\,. \qquad (109)$$

In evaluating this exponent we have used Eqs. (105), (65), and (66). For $\epsilon \le 1$, we evidently have $1/\phi_v \gtrsim 11$; since, by supposition, g_0 is small, while, by Table I, $f_0 \lesssim \frac{1}{10} g_0$, the lower bound on $(T - T_c)/T_c$ imposed by (108) is very small indeed. In other words, we may ultimately expect some crossover of the form

$$\xi(T, v) \approx t^{-\nu} \overline{X}(f_0 u_0 / t^{\phi_v})\,, \qquad (110)$$

where the exponent ν takes the isotropic dipolar value (66); but, owing to the small value of ϕ_v, it should be hard to detect the implied change over to nonisotropic behavior. What deviations do finally occur, by way of further changes of exponents or to a first-order transition remain, however, to be investigated.

X. DISCUSSION

We have calculated the exponents η, ν, and γ and checked the scaling relation between them [see Eq. (78)] for fully isotropic dipolar interactions. The specific-heat exponent α can be evaluated by considering the energy-energy correlation function but that requires further work. To obtain the exponents β and δ for the magnetization it is necessary to repeat the calculations with the presence of a magnetic field $\vec{B} = [B^\alpha]$, following the lines developed by Brézin, Wallace, and Wilson.[26] This amounts to the addition of a term $\sum_\alpha B^\alpha \sigma_0^\alpha$ to the initial Hamiltonian $\mathcal{H}_0$ in (16), followed by a shifting of the spin variables to $\rho_{\vec{q}}^\alpha = \sigma_{\vec{q}}^\alpha - M^\alpha \delta(\vec{q})$, where the magnetization components M^α are determined so that $\langle \rho_{\vec{q}}^\alpha \rangle \equiv 0$. This procedure yields a new set of interaction parameters, namely, the coefficients of

$$\rho_{\vec{q}}^\alpha,\quad \rho_{\vec{q}}^\alpha \rho_{-\vec{q}}^\beta,\quad \int_{\vec{q}_1}\int_{\vec{q}_2} \rho_{\vec{q}_1}^\alpha \rho_{\vec{q}_2}^\beta \rho_{-\vec{q}_1-\vec{q}_2}^\beta\,,$$

and

$$\int_{\vec{q}_1}\int_{\vec{q}_2}\int_{\vec{q}_3} \rho_{\vec{q}_1}^\alpha \rho_{\vec{q}_2}^\alpha \rho_{\vec{q}_3}^\beta \rho_{-\vec{q}_1-\vec{q}_2-\vec{q}_3}^\beta\,,$$

which are then to be renormalized. The main difficulty then, as in the case of the anisotropic exchange interaction,[14] is that r_0 in the equivalent of (17), now becomes dependent on the component label α, and hence the simple expression (22) for the propagator has to be replaced by a considerably more complicated one [similar to (23) if the parameters r_α are close to one another].

The low-field low-temperature situation ($T \ll T_c$) has been discussed by Holstein and Primakoff,[27] using spin-wave theory. They find a complicated dependence of the magnetization in this region on the magnetic field, which seems not to have the expected demagnetization form. Such complications could still play a role when T is close to but below T_c; the question is still open.

Brézin, Wallace, and Wilson[26] found that all the two-exponent or hyperscaling-exponent relations,[1,21] such as $2 - \eta = d(\delta - 1)/(\delta + 1)$, held at least to order ϵ^2 for short-range interactions. In order to estimate the effects of the dipole-dipole interactions on the other, uncalculated exponents it is reasonable to assume that the hyperscaling relations still apply here. (Nevertheless, it would certainly be

TABLE IV. Critical exponents for small ϵ.

Exponent	Classical	Isotropic short range (d-component spins)	Isotropic short range and dipolar interactions
η	0	$\frac{1}{48}\epsilon^2 + O(\epsilon^3)$	$O(\epsilon^2)$
$1/2\nu$	1	$1 - \frac{1}{4}\epsilon + O(\epsilon^2)$	$1 - \frac{9}{34}\epsilon + O(\epsilon^2)$
$\gamma = (2-\eta)\nu$	0	$1 + \frac{1}{4}\epsilon + O(\epsilon^2)$	$1 + \frac{9}{34}\epsilon + O(\epsilon^2)$
$\alpha = 2 - d\nu$	0	$-\frac{1}{8}\epsilon^2 + O(\epsilon^3)$	$-\frac{1}{34}\epsilon + O(\epsilon^2)$
$\delta = \dfrac{d+2-\eta}{d-2+\eta}$	3	$3 + \epsilon + \frac{11}{24}\epsilon^2 + O(\epsilon^3)$	$3 + \epsilon + \frac{1}{2}\epsilon^2 - 2\eta + O(\epsilon^3)$
$\beta = \frac{1}{2}(d-2+\eta)\nu$	$\frac{1}{2}$	$\frac{1}{2} - \frac{1}{8}\epsilon + O(\epsilon^2)$	$\frac{1}{2} - \frac{2}{17}\epsilon + O(\epsilon^2)$
ϕ_v	$\frac{1}{2}\epsilon$	$\frac{1}{24}\epsilon^2 + O(\epsilon^3)$	$\frac{3}{34}\epsilon + O(\epsilon^2)$

valuable to check this explicitly.) The results obtained from our values for η and ν, using the usual exponent relations, are summarized in Table IV, together with the previously known short-range values. Evidently there are first-order changes in all the exponents (except for η). The differences are, however, surprisingly small amounting, for example, only to $+\frac{1}{68}\epsilon$ in the susceptibility exponent γ. Contrary to what might have been expected, in view of the long-range character of the dipolar forces, on the basis of the calculations by Fisher, Ma, and Nickel[2] for isotropic interaction potentials of the form $1/r^{d+\sigma}$ $(0<\sigma<2)$, the values of γ and ν are increased *away* from the classical values $\gamma=1$ and $\nu=\frac{1}{2}$. However (accepting the hyperscaling relations), the value of β does change towards the classical value $\beta=\frac{1}{2}$, but only by $\frac{1}{136}\epsilon$. Of course, it is by no means clear how large these differences should be numerically for dipolar interactions in three-dimensional systems $(\epsilon=1)$. Even so, our results probably give a valid indication of the qualitative changes to be expected.

Also included in Table IV are the values of the crossover exponent ϕ_v which describes the weak instability of the isotropic behavior, dipolar or otherwise, to four-spin cubically anisotropic interactions. This instability was examined in Sec. IX but it remains to elucidate its ultimate effects although we anticipate that these will be very hard to detect in practice owing to the small value of ϕ_v. For completeness we mention again that the crossover from Heisenberg gehavior to isotropic-dipolar behavior is described by an exponent $\phi=1+\frac{1}{4}\epsilon+O(\epsilon^2)$.

In contrast to the conclusions reached by Larkin and Khmel'nitskii[3] for their uniaxial model, we do not find only logarithmic deviations from classical behavior. It is conceivable, however, that this might be the case if the original dipole-dipole forces are stronger than the isotropic exchange interactions. The uniaxial situation is discussed briefly along the present lines in Appendix D. It is shown that, with a spatially isotropic renormalization, the dipolar fixed point becomes Gaussian in character for small $\epsilon>0$ (i.e., $d \lesssim 4$); logarithmic factors may then enter but they are not studied. (Note that Larkin and Khmel'nitskii concluded there would be logarithmic deviations from classical behavior for $d=3$.) This case is further discussed in Paper V.[32]

It is clear from the form of the propagator graphs involved, that the calculation of the ϵ^2 terms for the exponents, along the present lines using the recursion formulas, would be very complicated. However, following Wilson,[20] a direct Feynman graph expansion may be used to calculate η to order ϵ^2. This will be presented in Paper II.[19]

The experimental situation has been discussed in a preceding paper[28] (in which the results of the present work were briefly summarized). It seems that the theory should be relevant to a fairly large group of ferromagnets such as the europium chalcogenides which have low transition temperatures. The observed values of the exponent γ range between 1.3 and 1.4, and hence cannot yet help in judging the effects of dipole-dipole interactions. [Our predictions from $1/\gamma$ to order ϵ are $\gamma \simeq 1.36$ with the dipole-dipole interaction and $\gamma \simeq 1.33$ for the isotropic $(4-\epsilon)$-component Heisenberg model.]

As already noted at the end of Sec. VII, the existing experimental measurements of the spin fluctuations near T_c are insufficient to check the predicted angular dependence of $\Gamma^{\alpha\beta}(\vec{q})$. It would certainly be interesting to have observational data which could check this feature.

The formulas derived in the present paper are useful for dealing with several other physical situations. The closest problem to that discussed here, concerns the effects of dipolar interactions on *anti*ferromagnets, and the Van Vleck[29] "dipolar" model for explaining ferromagnetic anisotropy. The principal difference between these cases and the present one lies in the absence of a $q^\alpha q^\beta/q^2$ term; as a result there are no changes in the standard critical exponents (see also Ref. 23). These problems will be discussed in future publications.[25]

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APPENDIX A: EWALD'S METHOD IN d DIMENSIONS

We want to calculate the Fourier transform

$$A^{\alpha\beta}(\vec{q},\vec{x}) = -\frac{\partial^2}{\partial x^\alpha\,\partial x^\beta}\sum_l{}' |\vec{x}_l-\vec{x}|^{2-d} e^{i\vec{q}\cdot\vec{x}_l}, \quad \text{(A1)}$$

where the sum runs over all lattice sites except, as indicated by the prime, the origin site $\vec{x}_l=0$. Using the identity[16]

$$|\vec{x}_l-\vec{x}|^{2-d} = \frac{2}{\sqrt{\pi}}\int_0^\infty d\rho \exp(-\rho^2|\vec{x}_l-\vec{x}|^{2(d-2)}), \quad \text{(A2)}$$

we can write

$$A^{\alpha\beta}(\vec{q},\vec{x}) = -\frac{\partial^2}{\partial x^\alpha\,\partial x^\beta}\int_0^\infty d\rho\frac{2}{\sqrt{\pi}}\sum_l \exp[-\rho^2|\vec{x}_l-\vec{x}|^{2(d-2)}+i\vec{q}\cdot(\vec{x}_l-\vec{x})]\, e^{i\vec{q}\cdot\vec{x}} + \frac{\partial^2}{\partial x^\alpha\,\partial x^\beta}(|\vec{x}|^{2-d}). \quad \text{(A3)}$$

Now the expression in braces in the first term is a periodic function of $\vec{x}$ on the lattice, and hence can be replaced by a Fourier sum over the reciprocal lattice, with the coefficients

$$g(h)=v_a^{-1}\int_{\Xi} d^dx\sum_l \exp[-\rho^2|\vec{x}_l-\vec{x}|^{2(d-2)} + i\vec{q}\cdot(\vec{x}_l-\vec{x})]\,e^{-i\vec{q}_h\cdot\vec{x}}, \quad (A4)$$

where Ξ denotes a unit cell of the lattice containing the origin, while v_a is the volume per lattice site and $\vec{q}_h$ is a vector of the reciprocal lattice.

Changing variables to $\vec{x}\,'=\vec{x}-\vec{x}_l$, we obtain a sum of integrals over the unit cells near $\vec{x}\,'=\vec{x}_l$. This sum is equivalent to an integral over the whole space, namely,

$$g(h)=v_a^{-1}\int d^dx\exp[-\rho^2|\vec{x}|^{2(d-2)}+i(\vec{q}+\vec{q}_h)\cdot\vec{x}] = v_a^{-1}|\vec{q}+\vec{q}_h|^{-d}F_d(\rho|\vec{q}+\vec{q}_h|^{2-d}), \quad (A5)$$

where

$$F_d(z)=\int_0^\infty y^{d-1}\,dy\,f_d(y)\exp(-z^2y^{2(d-2)}), \quad (A6)$$

in which

$$f_d(y)=(2\pi)^{d/2+1}y^{1-d/2}J_{d/2-1}(y) = 4\pi\sin y/y \text{ for } d=3 = 8\pi^3J_1(y)/y \text{ for } d=4\,. \quad (A7)$$

Here $J_\nu(z)$ is a Bessel function, resulting from the angular integral in (A5). Therefore, we obtain

$$\sum_l \exp[-\rho^2|\vec{x}_l-\vec{x}|^{2(d-2)}+i\vec{q}\cdot(\vec{x}_l-\vec{x})] = v_a^{-1}\sum_h|\vec{q}+\vec{q}_h|^{-d}F_d(\rho|\vec{q}+\vec{q}_h|^{2-d})\,. \quad (A8)$$

The integrals over ρ in (A3) may now be divided into two regions: from 0 to R and from R to ∞ Using (A8) only for the first range, we find

$$A^{\alpha\beta}(\vec{q},\vec{x})=v_a^{-1}\sum_h(q^\alpha+q_h^\alpha)(q^\beta+q_h^\beta)|\vec{q}+\vec{q}_h|^{-2}\times G_d(R|\vec{q}+\vec{q}_h|^{2-d})\,e^{i(\vec{q}+\vec{q}_h)\cdot\vec{x}} -\sum_l e^{i\mathbf{q}\cdot\mathbf{x}_l}H_{\alpha\beta}^{d,R}(\vec{x}_l-\vec{x})+\frac{\partial^2}{\partial x^\alpha\,\partial x^\beta}(|\vec{x}|^{2-d}), \quad (A9)$$

where

$$G_d(z)=\frac{2}{\sqrt{\pi}}\int_0^z dxF_d(x) \quad (A10)$$

and

$$H_{\alpha\beta}^{d,R}(\vec{x})=R\frac{\partial^2}{\partial x^\alpha\,\sigma x^\beta}H(R|\vec{x}|^{d-2}), \quad (A11)$$

with

$$H(z)=\frac{2}{x\sqrt{\pi}}\int_x^\infty dy\,e^{-y^2}. \quad (A12)$$

On letting $\vec{x}\to 0$ [see Eq. (8)], we finally find

$$A^{\alpha\beta}(\vec{q})=v_a^{-1}|q|^{-2}q^\alpha q^\beta G_d(R|\vec{q}|^{2-d}) +v_a^{-1}\sum_h{}'(q^\alpha+q_h^\alpha)(q^\beta+q_h^\beta)|\vec{q}+\vec{q}_h|^{-2} \times G_d(R|\vec{q}+\vec{q}_h|^{2-d})-\sum_l{}'H_{\alpha,\beta}^{d,R}(\vec{x}_l)\,e^{i\vec{q}\cdot\vec{x}_l} +\lim_{\vec{x}\to 0}\left(\frac{\partial^2}{\partial x^\alpha\,\partial x^\beta}(|\vec{x}|^{2-d})-H_{\alpha\beta}^{d,R}(\vec{x})\right). \quad (A13)$$

The primes on the summation signs indicate that the terms with $\vec{q}_h=0$ and with $\vec{x}_l=0$ are to be omitted, respectively. The expression (A13) gives the same result for all values of R, but for applications a value of R is chosen which ensures that both series converge rapidly.

It is easy to see that only the first term in (A13) is nonanalytic in $\vec{q}$ near the origin. All the other terms are analytic, and may be expanded in a Taylor series about $\vec{q}=0$. For cubic lattices, this series may be written in the form

$$A^{\alpha\beta}(\mathbf{q})=a_1q^\alpha q^\beta/q^2-a_2q^\alpha q^\beta -[a_3+a_4q^2-a_5(q^\alpha)^2]\,\delta_{\alpha\beta} +O((q^\alpha)^4,\ (q^\alpha)^2(q^\beta)^2,\ q^4,\ \ldots)\,. \quad (A14)$$

As already noted by Cohen and Keffer,[30] who present a detailed investigation of $A^{\alpha\beta}(\vec{q})$ for $d=3$, the coefficients in (A14) may easily be expressed as Ewald series; they clearly depend both on dimentionality and on the lattice structure. The leading coefficient is given simply by

$$a_1=v_a^{-1}G_d(\infty)=(2/v_a\sqrt{\pi})\int_0^\infty dx\,F_d(x); \quad (A15)$$

and its only dependence on the lattice structure is through v_a. Now by definition, the value of the original dipole-dipole interaction vanishes at the origin. This implies the relation

$$\int A^{\alpha\beta}(\vec{q})\,d^dq=0, \quad (A16)$$

which immediately leads to the coefficient relations

$$a_5=a_2+da_4,\quad a_1=da_3\,. \quad (A17)$$

Hence, the dependence of a_3 on the lattice structure is also only through v_0. Table I gives the values of these coefficients for the three-dimensional cubic lattices. (The results for fcc and bcc were extracted from Ref. 30.) In our units, the distance between near neighbors is chosen as $a=1$.

APPENDIX B: SOME ANGULAR INTEGRALS

The integration over $\vec{q}$ space in d dimensions involves expressions of the form

$$I=\int_{\vec{q}}f(\vec{q})=\frac{1}{(2\pi)^d}\int d^dq\,f(\vec{q})\,. \quad (B1)$$

In spherical coordinates, f may involve several angles, so that

$$f(\vec{q})=f(q,\,\theta,\,\varphi,\,\psi,\,\ldots),\tag{B2}$$

and the integral becomes[20]

$$I=\left(K_d\int q^{d-1}dq\int_0^\pi \sin^{d-2}\theta\,d\theta\int_0^\pi \sin^{d-3}\varphi\,d\varphi\cdots f(\mathrm{q})\right.$$

$$\times\left(\int_0^\pi \sin^{d-2}\theta\,d\theta\int_0^\pi \sin^{d-3}\varphi\,d\varphi\cdots\right)^{-1},\tag{B3}$$

where

$$K_d^{-1}=2^{d-1}\pi^{d/2}\Gamma(\tfrac{1}{2}d)\,.\tag{B4}$$

We specifically want to consider functions which involve products of Cartesian components of $\vec{q}$, such as

$$f(\vec{q})=g(q)(q^\alpha/q)^m(q^\beta/q)^n\cdots\,.\tag{B5}$$

From symmetry arguments it is clear that I will vanish unless all exponents in (B5) are even. In all such cases, the angular integrals may be performed with the aid of the identity

$$\int_0^\pi \sin^m\theta\cos^n\theta\,d\theta=B\left(\frac{m+1}{2},\,\frac{n+1}{2}\right)$$

$$=\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)\Big/\Gamma\left(\frac{m+n+2}{2}\right)\,.\tag{B6}$$

Hence we find

$$\int_0^\pi \sin^{d-2}\theta\,d\theta=\Gamma(\tfrac{1}{2}(d-1))\Gamma(\tfrac{1}{2})/\Gamma(\tfrac{1}{2}d)\,.\tag{B7}$$

Denoting the angular part of the integral I by

$$\left\langle\left(\frac{q^\alpha}{q}\right)^m\left(\frac{q^\beta}{q}\right)^n\cdots\right\rangle=\left[\int_0^\pi \sin^{d-2}\theta\,d\theta\right.$$

$$\left.\times\int_0^\pi \sin^{d-3}\varphi\,d\varphi\cdots\left(\frac{q^\alpha}{q}\right)^m\left(\frac{q^\beta}{q}\right)^n\cdots\right]$$

$$\times\left(\int_0^\pi \sin^{d-2}\theta\,d\theta\int_0^\pi \sin^{d-3}\varphi\,d\varphi\cdots\right)^{-1},\tag{B8}$$

we then find

$$\left\langle\frac{q^\alpha q^\beta}{q^2}\right\rangle=\delta_{\alpha\beta}\langle\cos^2\theta\rangle=\left(\int_0^\pi \sin^{d-2}\theta\cos^2\theta\,d\theta\right)$$

$$\times\left(\int_0^\pi \sin^{d-2}\theta\,d\theta\right)^{-1}\delta_{\alpha\beta}=\frac{1}{d}\,\delta_{\alpha\beta},\tag{B9}$$

$$\left\langle\frac{q^\alpha q^\beta q^\gamma q^\delta}{q^4}\right\rangle=\langle\cos^4\theta\rangle=\frac{3}{d(d+2)}\quad(\alpha=\beta=\gamma=\delta)$$

$$=\langle\cos^2\theta\sin^2\theta\cos^2\varphi\rangle=\frac{1}{d(d+2)}\quad(\alpha=\beta\neq\gamma=\delta,\quad\alpha=\gamma\neq\beta=\delta,\quad\text{or }\alpha=\delta\neq\gamma=\beta)$$

$$=0\qquad\text{otherwise},\tag{B10}$$

which can be rewritten

$$\left\langle\frac{q^\alpha q^\beta q^\gamma q^\delta}{q^4}\right\rangle=\frac{1}{d(d+2)}\,(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\gamma\beta})\,.\tag{B11}$$

Similarly, we obtain

$$\left\langle\frac{(q^\alpha)^3q^\beta q^\gamma q^\delta}{q^6}\right\rangle=\frac{1}{d(d+2)(d+4)}$$

$$\times[3(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma})+6\delta_{\alpha\beta}\delta_{\alpha\gamma}\delta_{\alpha\delta}]\tag{B12}$$

and, lastly

$$\left\langle\frac{(q^\alpha)^4(q^\beta)^2(q^\gamma)^2}{q^8}\right\rangle=\frac{3c^{\alpha\beta\gamma}}{d(d+2)(d+4)(d+6)}\,,\tag{B13}$$

where

$$c^{\alpha\alpha\alpha}=35\,,\qquad c^{\alpha\alpha\beta}=c^{\alpha\beta\alpha}=5\,(\alpha\neq\beta)$$

$$c^{\alpha\beta\beta}=3(\alpha\neq\beta),\quad c^{\alpha\beta\gamma}=1\,(\alpha\neq\beta\neq\gamma)\,.\tag{B14}$$

APPENDIX C: *nd*-COMPONENT MODEL

As noted in Sec. II, we may suppose the spin vector has nd components $s^{\alpha,i}$, where the indices α couple to the spatial coordinates as in (3) while i denotes another "internal" degree of freedom. We then have a modified propagator, of the form

$$\overline{G}^{\alpha i\beta j}(\vec{q})=\delta_{ij}G^{\alpha\beta}(\vec{q})\,,\tag{C1}$$

where $G^{\alpha\beta}(\vec{q})$ is still given by (23), and all the sums of the form $\sum_\alpha$ should be replaced by the sums $\sum_{\alpha=1}^d\sum_{i=1}^n$. As in the text, we consider only the isotropic dipolar fixed point, and therefore we take $f_0=v_0=0$. The only change in the recursion relations then results from a different counting of the diagrams. The relations for r and for u, generalizing (30) and (45), are then

$$r_{l+1}=b^2[r_l+6K_4u_l(2n+1)A_{10}^4(r_l)+O(\epsilon^2)]\,,\tag{C2}$$

$$u_{l+1}=b^\epsilon[u_l-4K_4u_l^2A_{20}^4(3n+\tfrac{11}{2})+O(\epsilon^3)]\,.\tag{C3}$$

Following the previous analysis we hence find

$$4K_4u^*=2\epsilon/(6n+11)\,,\tag{C4}$$

and thence the r eigenvalue

$$\Lambda=b^2\{1-[(6n+3)/(6n+11)]\,\epsilon\ln b\,\},\tag{C5}$$

which leads to

$$2\nu=1+[(6n+3)/2(6n+11)]\,\epsilon+O(\epsilon^2)\,.\tag{C6}$$

In the spherical model limit,[13] $n\to\infty$, and we approach the usual result

$$1/2\nu=1-\tfrac{1}{2}\,\epsilon\,,\tag{C7}$$

which is, in fact, exact for $0<\epsilon<2$. This confirms Lax's assertion[10] that the critical behavior in the spherical model is unchanged by the addition of dipole-dipole interactions.

However, it must be remarked, that the instability of the fixed point with respect to the parameter v_l, discussed in Sec. VIII, still exists in the limit $n\to\infty$, so that it is not clear from the present discussion what happens close enough to T_c.

APPENDIX D: UNIAXIAL CASE

The Hamiltonian for the uniaxial case is equivalent to (10), but with the summations over α and β replaced by the single term with $\alpha=\beta=z$. With the appropriate normalizations we find

$$\bar{\mathcal{H}}_0=-\frac{1}{2}\int_{\vec{q}}\left[r_0+q^2+f_0(q^z)^2+g_0\left(\frac{q^z}{q}\right)^2\right]\sigma_{\vec{q}}\sigma_{-\vec{q}} - u_0\int_{\vec{q}}\int_{\vec{q}_1}\int_{\vec{q}_2}\sigma_{\vec{q}}\sigma_{\vec{q}_1}\sigma_{\vec{q}_2}\sigma_{-\vec{q}-\vec{q}_1-\vec{q}_2}, \quad \text{(D1)}$$

where r_0, f_0, g_0, and u_0 are defined in analogy to (18) to (20). [Note that f_0 here replaces the former combination (f_0-h_0).]

For $f_0=g_0=0$ this Hamiltonian yields the usual Ising model results (see, e.g., Ref. 12). For $g_0\neq 0$ we can easily see that (34) still holds, and with (40) this again implies the recursion relation (46). Hence, the only fixed point values of g are 0 or ∞. (However, we must note that this result is based on a spatially *isotropic* renormalization procedure. Use of a different scaling rate in the z direction could well change this result and lead to another fixed point of physical relevance.)

The recursion relations for r and for u are now

$$r'=b^{2-\eta}[r+12u\int^{>}G(\vec{q})\,d\vec{q}\,], \quad \text{(D2)}$$

$$u'=b^{\epsilon}[u-36u^2\int^{>}G(\vec{q})G(-\vec{q})\,d\vec{q}\,], \quad \text{(D3)}$$

with the propagator

$$G(\vec{q})=[r+q^2+f(q^z)^2+g(q^z/q)^2]^{-1}. \quad \text{(D4)}$$

Assuming the arguments in Sec. VI to be valid here also, we see that $G(\vec{q})$ goes to zero for $g\to\infty$ for all $\vec{q}$ with $q^z\neq 0$, hence all the integrals which involve G will vanish in this limit, leading to the result

$$r_{l+1}=b^{2-\eta}r_l,\quad u_{l+1}=b^{\epsilon}u_l\quad(g\to\infty), \quad \text{(D5)}$$

that is, to a Gaussian fixed point.

The exact behavior near this fixed point can, however, be affected by the rate of change of r_l and u_l as g_l approaches infinity. This may lead to logarithmic corrections to the Gaussian-point exponents, such as found by Larkin and Khmel'nitskii.[3] Logarithmic corrections of this sort have been discussed by Wilson,[11] by Wegner,[31] and by Fisher, Ma, and Nickel.[2] We will return to this question in a later work.[32]

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