de Haas–van Alphen Effect and the Specific Heat of an Electron Gas*

T. Holstein, R. E. Norton, and P. Pincus†
Department of Physics, University of California, Los Angeles, California 90024
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A model electron gas is studied in order to provide a microscopic justification of the Shoenberg conjecture that the harmonic content of the de Haas–van Alpen oscillations of the magnetization $\vec{M}$ can properly be explained by replacing the applied field $\vec{H}$ in the elementary expression for $\vec{M}(\vec{H})$ by the total induction $\vec{B} = \vec{H} + \vec{M}$. The model contains a current-current interaction between the electrons; and it is shown that the unscreened long-range nature of this interaction leads to a term in the low-temperature specific heat proportional to $T \ln T$. For a gas with terrestrial densities the coefficient of this anomalous term is very small and probably beyond detection.

I. INTRODUCTION

In 1962, Shoenberg[1] observed in some noble metals that the de Haas–van Alphen (dHvA) oscillations of the diamagnetic susceptibility were particularly rich in harmonic content. This led him to the conjecture that this structure could be explained if one assumed that the diamagnetic contribution to the magnetization $\vec{M}$ of the sample was periodic in the inverse magnetic induction B^{-1} instead of the usual H^{-1} (where H is the applied field, corrected for demagnetizing effects). This would lead to a nonlinear functional relationship $\vec{M}(\vec{H}+\vec{M})$ whose effects would be accentuated by the sharp oscillatory nature of the dHvA effect. Pippard[2] extended these ideas in a thermodynamic theory of the dHvA effect and indeed found that by taking $\vec{M}(\vec{B})$ instead of $\vec{M}(\vec{H})$ it was possible to account for Shoenberg's observations. More recently, Condon[3] and Halloran and Hsu,[4] on the basis of dHvA and thermomagnetic effect observations in beryllium, have postulated the existence of a domain structure, i.e. a nonuniform diamagnetism. Azbel,[5] Privorotskii[6] and Lee, Greene, and Quinn[7] have discussed such domain formation on the assumption that the electrons indeed move in the field $\vec{H}+\vec{M}(\vec{r})$, where $\vec{r}$ is the position of the electron under consideration. It is the primary purpose of this paper to give a microscopic justification of this replacement of the applied field by local induction.

Our model consists of an electron gas with a positively charged background in the presence of an applied, uniform magnetic field. In addition to other interactions—e.g., the Coulomb potential between the electrons and between the electrons and the background—the electrons interact with each other via a static current-current potential which, to order v^2/c^2, takes into account the force between the electrons arising from the exchange of transverse photons. The details of the model are described in Sec. II.

It is shown in Sec. III that if the current-current interaction is considered in the Hartree approximation, the Shoenberg conjecture of replacing $\vec{M}(\vec{H})$ by $\vec{M}(\vec{H}+\vec{M})$ follows in a straightforward manner from the structure of the thermodynamic potential. In Sec. IV we argue that the non-Hartree effects of the current-current interaction are safely negligible, and we discuss the dHvA oscillations in the light of these conclusions.

In contrast to the Coulomb potential, there is no screening of the current-current interaction at zero frequency (for normal nonsuperconducting systems). This feature is explicit from the calculations of Appendix A, wherein the corrected current-current potential (photon propagator) is obtained by iterating the photon-self-energy part in the absence of an applied magnetic field. In Appendix B we consider the electron-exchange-self-energy part to first order in this corrected interaction. It is shown there that the lack of static screening requires the frequency derivative of the real part of the electron self-energy $\Sigma_p(\xi)$ to diverge as $\ln|\xi-\mu|$ when both p and ξ approach the Fermi surface. Similarly, the imaginary part of $\Sigma_p(\xi)$ vanishes like $|\xi-\mu|$ in this limit, in contrast to the quadratic behavior characteristic° of short-range interactions.

Because this anomalous behavior of $\Sigma_p(\xi)$ might raise doubts concerning our neglect of the exchange contributions of the current-current interaction, and also because the results are of interest themselves, in Sec. V we consider the effect of these anomalies on the low-temperature specific heat. It is shown that they lead to a term in the specific heat proportional to $T\ln T$, which is indeed at variance with the behavior linear in T characteristic[9] of systems with short-range interactions. However, because the current-current potential is essentially a relativistic effect, the coefficient of the $T\ln T$ part of the specific heat is roughly $\sim \alpha v_F/c \sim 10^{-5}$–$10^{-6}$ ($\alpha=\frac{1}{137}$) that of the dominant term proportional to T. We feel that this small relative magnitude also characterizes the long-range exchange effects of the current-current interaction on the dHvA oscillations.

II. THE CURRENT-CURRENT INTERACTION

The Hamiltonian density for a system of nonrelativistic electrons in interaction with a vector potential $\vec{A}$ is

$$\mathcal{H} = \mathcal{H}_0^\gamma + \frac{\hbar^2}{2m}\left(\vec{\nabla} + \frac{ie}{\hbar c}\vec{A}\right)\psi^\dagger \cdot \left(\vec{\nabla} - \frac{ie}{\hbar c}\vec{A}\right)\psi + U, \qquad (2.1)$$

where $\mathcal{H}_0^\gamma$ is the Hamiltonian density of the free radiation field and ψ is the second-quantized electron field normalized so that $\psi^\dagger\psi$ is the electron density. The symbol U in Eq. (2.1) contains all other potential interactions of the electrons, including the Coulomb interaction between the electrons and between the electrons and the background.

The vector potential $\vec{A}$ is the sum of an external field $\vec{A}^{ex}$, whose source is currents outside the system, and a radiation field $\vec{A}^r$ arising from currents induced in the medium. That is,

$$\vec{A} = \vec{A}^{ex} + \vec{A}^r, \qquad (2.2)$$

$$\vec{\nabla} \times \vec{A}^{ex} = \vec{H}^0, \qquad (2.3)$$

and

$$\Box^2 \vec{A}^r = -\left[\vec{j} - \frac{1}{\nabla^2}(\vec{\nabla}\cdot\vec{j})\right] \qquad (2.4)$$

where $\Box^2 = \nabla^2 - (1/c^2)(\partial^2/\partial t^2)$, and the operator for the electric current $\vec{j}$ given from (2.1) by

$$\vec{j} = -\frac{\partial\mathcal{H}}{\partial\vec{A}} = \frac{e\hbar}{2imc}(\psi^\dagger\vec{\nabla}\psi - \vec{\nabla}\psi^\dagger\psi) - \frac{e^2}{mc^2}\vec{A}\psi^\dagger\psi. \qquad (2.5)$$

In Sec. III, we discuss features of the thermodynamic potential Ω viewed as the sum of all closed linked graphs constructed according to the rules of Luttinger and Ward.[10] The internal potential lines in these graphs come from the potential interactions in U in Eq. (2.1) and from contractions of pairs of A^r's which arise in evaluating the grand partition sum as a power series in the electric charge e. Each contraction generates an internal photon line which connects between vertices of the kind shown in Fig. 1. These vertices, which refer separately to the two parts of the electric current in Eq. (2.5), also serve as terminals for interactions with the external field $\vec{A}^{ex}$. Each distinct way of connecting the ends of internal photon lines and the external field lines to these vertices leads to an additional expression to be included in the sum for Ω.

It is adequate for our purposes to consider the photon propagator at zero temperature. With the definition

$$D_{ij}(k, k_0) \equiv \frac{i}{(2\pi)^4}\int d^3x\, dt\, e^{-i\vec{k}\cdot\vec{x}+ik_0 t} \times \langle T(A_i(x,t)A_j(0))\rangle, \qquad (2.6)$$

it is essentially evident from Eq. (2.4) that we can take the bare noninteracting photon propagator to be

$$D_{ij}^0(k, k_0) = \frac{1}{k^2 - k_0^2 - i\epsilon}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right). \qquad (2.7)$$

The internal photon-propagator lines reflect the effects of a retarded current-current interaction between the electrons arising from the exchange of transverse photons. The retarded nature of this interaction is manifest from the appearance of the frequency k_0 in the denominator of (2.7). We could ignore this retardation by replacing the propagator in Eq. (2.7) by an instantaneous potential

$$\tilde{D}_{ij}^0(k) = \frac{1}{k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right). \qquad (2.7')$$

The remarks and calculations in the remaining sections are based on the model of the current-current interaction obtained by using the expression in Eq. (2.7′) for the photon propagator. Since the current vertices at the ends of the propagator lines are essentially of order v/c, this approximation retains the effects of the current-current interaction in the first order of v^2/c^2. Higher-order relativistic corrections are not included in this study.

III. MAGNETIZATION IN THE HARTREE APPROXIMATION

We define the Hartree approximation with respect to the current-current interaction of Sec. II as the approximation of retaining only those graphs for Ω which are separated into two disconnected pieces when any propagator line associated with the potential in (2.7′) is severed.

If $\Omega_H(\vec{H}_0)$ is the Hartree approximation for the thermodynamic potential in the presence of an external magnetic field $\vec{H}_0$ [see Eq. (2.3)], then the magnetization density $\vec{M}_H(\vec{H}_0)$ defined in terms of the total induction $\vec{B}_H$ by

$$\vec{B}_H = \vec{H}_0 + \vec{M}_H \qquad (3.1)$$

FIG. 1. Elementary vertices for the current-current interaction which refer, respectively, to the two parts of the electric current in Eq. (2.5).

is given by the functional derivative

$$\vec{M}_H(\vec{r}) = -\,\delta\Omega_H/\delta\vec{H}_0(\vec{r}). \tag{3.2}$$

Within the medium, the magnetization M_H differs from the magnetic dipole moment per unit volume to the extent that the magnetic field H differs from the external field H_0; for a sample with demagnetizing factor N, the magnetization M_H is equal to $1-N$ times the dipole moment density.

Suppose we were to completely ignore the current-current interaction but replace the external field $\vec{H}_0$ by $\vec{B}_H$ in Eq. (3.1). We would then obtain a magnetization density

$$\vec{M}_0^B(\vec{r}) = -\,\delta\Omega_0(\vec{B}_H)/\delta\vec{B}_H(\vec{r}), \tag{3.3}$$

where Ω_0 is the sum over only those closed-linked graphs which contain no internal propagator lines associated with the potential in (2.7′).

In the remainder of this section, we wish to show that the expressions in (3.2) and (3.3) are equal. This equality is equivalent to the conjecture of Shoenberg,[1] once the adequacy of the Hartree approximation for the magnetization density is established. This latter point is discussed in Sec. IV.

An elegant form for the thermodynamic potential Ω, derived in Ref. 10 and applied by Luttinger[11] to the theory of the dHvA effect, is

$$\Omega = -\beta^{-1}\sum_l \mathrm{Tr}\{\ln[\epsilon + \Sigma(\xi_l) - \xi_l] + G(\xi_l)\Sigma(\xi_l)\}e^{\xi_l 0_+} + \Omega'\{G\}, \tag{3.4}$$

where $\beta^{-1} = kT$, ϵ is the single electron energy matrix, $\Sigma(\xi_l)$ is the proper electron-self-energy part, $G(\xi_l)$ is the full electron-Green's-function matrix, and where[10,11] Ω' is the sum of all closed-linked skeleton graphs with the internal electron lines associated with the full Green's function G.

The matrix ϵ in Eq. (3.4) is essentially the Hamiltonian for a single electron in the presence of the external field. In the model leading to (3.3) this external field is $\vec{B}_H$, but the self-energy parts Σ, which occur explicitly in (3.4) as well as implicitly in G, arise only from the interactions contained in the potential U of Eq. (2.1). Let us compare this circumstance with that which obtains for the thermodynamic potential $\Omega_H(\vec{H}_0)$ in (3.2). In this latter instance there is an additional contribution to Σ arising from the effects of the current-current interaction in the Hartree approximation. But this additional part of Σ simply gives the difference between the single-electron energies in the field $\vec{B}_H$ and in the field $\vec{H}_0$, so that the combination $\epsilon + \Sigma(\xi_l)$ occurring in (3.4) is the same for both versions of the theory. Thus, it follows from (3.4) that

$$\Omega_H(\vec{H}_0) = \Omega_0(\vec{B}_H) - \beta^{-1}\sum_l \mathrm{Tr}[G(\xi_l)\Sigma_H(\xi_l)]e^{\xi_l 0_+} + \Omega_H'\{G\}, \tag{3.5}$$

where Σ_H is the Hartree part of the electron-self-energy graph, and where $\Omega_H'\{G\}$ is the closed-linked Hartree contribution to $\Omega'\{G\}$ shown in Fig. 2. It is equal to

$$\Omega_H'\{G\} = (2\beta)^{-1}\sum_l \mathrm{Tr}[G(\xi_l)\Sigma_H(\xi_l)]e^{\xi_l 0_+}, \tag{3.6}$$

so that

$$\Omega_H(\vec{H}_0) = \Omega_0(\vec{B}_H) - \Omega_H'\{G\}. \tag{3.7}$$

The expression for $\Omega_H'\{G\}$ in Eq. (3.6) corresponding to the graph in Fig. 2 can be written

$$\Omega_H' = -\tfrac{1}{2}\int d^3x\, d^3y\, [\vec{\nabla}\times\vec{M}_H(x)]_i \times \tilde{D}_{ij}^0\,(x-y)[\vec{\nabla}\times\vec{M}_H(y)]_j, \tag{3.8}$$

where $\tilde{D}_{ij}^0\,(x)$ is the Fourier transform of the potential in Eq. (2.7′):

$$\tilde{D}_{ij}^0\,(x) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, \frac{1}{k^2}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) . \tag{3.9}$$

It is evident that only the term proportional to δ_{ij} in (3.9) contributes to (3.8). If we also recognize that $\vec{\nabla}\cdot\vec{M}_H$ vanishes, since both $\vec{B}_H$ and $\vec{H}_0$ in (3.1) have zero divergence, we obtain, upon substituting (3.9) into (3.8) and integrating by parts,

$$\Omega_H' = -\tfrac{1}{2}\int d^3y\, M_H^2(y). \tag{3.10}$$

Inserting this result into (3.7) and making use of (3.2) and (3.3), we obtain, by varying both sides of (3.7) with respect to $\vec{H}_0$,

$$\vec{M}_H(x) - \vec{M}_0^B(x) = \int d^3y\; [\vec{M}_0^B(y) - \vec{M}_H(y)]\; \frac{\delta M_H(y)}{\delta\vec{H}_0(x)}\,. \tag{3.11}$$

We can thus conclude that

$$\vec{M}_H = \vec{M}_0^B, \tag{3.12}$$

which is our desired result.

IV. dHvA OSCILLATIONS

It has been shown by Luttinger[11] and somewhat more generally by Bychkov and Gor'kov[12] that with the restriction of short-range interactions, and with the neglect of terms of order $(e\hbar H/2mc\mu)^{1/2}$ compared to unity (μ is the chemical potential),

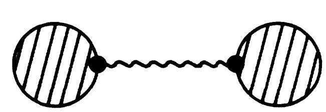


FIG. 2. Hartree contribution to Ω'; the wiggly line is the potential in Eq. (2.7′), and the bubbles contain all closed-linked skeleton graphs constructed according to the rules of Ref. 10 and in the Hartree approximation for the current-current potential.

the calculation of Lifshitz and Kosevich[13] can be applied to a system of interacting electrons to calculate the oscillating part of the magnetization in terms of the quasiparticle parameters and the features of the Fermi surface at its extrema. Each extremum in the cross-sectional area enclosed by the Fermi surface, where the normal to the area is parallel to the applied field, gives rise to its own characteristic oscillations of $\vec{M}$ vs $\vec{H}$.

As discussed further below, the current-current interaction of Sec. II is not sufficiently short range to be included in the arguments of Refs. 11 and 12. However, we have seen in Sec. III that if $\vec{M}(\vec{H}_0)$ can be calculated in the absence of this interaction, then the effect of including it in the Hartree approximation is simply to replace $\vec{M}(\vec{H}_0)$ by $\vec{M}(\vec{B})$. As mentioned in the previous paragraph, the oscillating part of $\vec{M}(\vec{H}_0)$ can essentially be calculated. Further, any nonoscillating part of $\vec{M}(\vec{H}_0)$, although it becomes, in principle, an oscillating part once H_0 is replaced by B, is essentially unaffected by this replacement—and therefore is irrelevant for the dHvA oscillations—since for the metals of interest $M \lesssim 10^{-5}H$, and it is only the very rapid variation of the oscillating part of $\vec{M}$ with $\vec{H}$ that allows an appreciable effect[1] to be realized when $\vec{H}$ is replaced by $\vec{B}$. Thus, it follows from these considerations that to the extent the current-current interaction is adequately taken into account in the Hartree approximation, the results of Refs. 11 and 12 can immediately be generalized to include the current-current interaction by simply replacing $\vec{M}(\vec{H})$ by $\vec{M}(\vec{B})$, as suggested by Shoenberg.

The current-current interaction of Sec. II is proportional to $\alpha v^2/c^2$ ($\alpha = \frac{1}{137}$) and is therefore inherently very small for the nonrelativistic systems of interest here. Consequently, it should certainly be adequate when considering its possible effects beyond the Hartree approximation to restrict our attention to its first-order contribution in the electron-exchange-self-energy part. In the next paragraph we state why we feel that these exchange contributions to the character of the dHvA oscillations can safely be ignored.

In contrast to the Coulomb interaction, the current-current interaction is unscreened at zero frequency. As discussed in the next Sec., this feature leads to deviations of the properties of a Fermi liquid from those obtained from the quasiparticle picture; in particular, the low-temperature specific heat is shown to have an anomalous term proportional to $T \ln T$ in addition to the dominant term proportional to T. However, because of the inherent smallness of the current-current interaction, this anomaly is very small and probably beyond detection. It is our opinion, although we must admit some mild doubts in this regard, that this same degree of smallness characterizes the effect of the current-current exchange term on the dH-vA oscillations. That is, despite the fact that it is formally of the same order as the Hartree self-energy part, the effect of the long-range interaction in the exchange term is confined to a narrow frequency interval within an integral; and further, we see no mechanism for the enhancement of this exchange part—in contrast to the Hartree self-energy—where the inherent v^2/c^2 is counterbalanced by its very rapid variation with the applied field.

V. ANOMALY IN THE SPECIFIC HEAT

In this section we wish to show that the lack of static screening of the current-current interaction of Sec. II leads to an anomaly in the low-temperature specific heat of the form $T \ln T$. However, as can be seen from Eq. (5.15), the coefficient accompanying this behavior is so small that it will be very difficult, if not impossible, to detect this deviation from the predominant dependence proportional to T. Nevertheless, we find the existence of this anomaly interesting in itself, and discuss it here for this reason, as well as because we feel that its smallness lends support for the view stated in Sec. IV that the effect on the dH-vA oscillations of the current-current exchange interaction can safely be ignored. This conclusion is relevant because the long-range nature of this interaction does not allow its effects to be included in the results of Refs. 11 and 12.

In Appendix A the class of diagrams in Fig. 3 are considered in the absence of an applied magnetic field. They are shown to yield an effective transverse the photon propagator

$$D_{ij}(k,\omega) = \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\frac{1}{k^2 + A(k,\omega)}, \qquad (5.1)$$

where [see (A4)]

$$\mathrm{Re}A(k,\omega) \xrightarrow{k,\,\omega/k\to 0} \sim \left(\frac{\omega}{k}\right)^2 \qquad (5.2a)$$

and

$$\mathrm{Im}A(k,\omega) \xrightarrow{k,\,\omega/k\to 0} \sim \frac{\omega}{k} \ . \qquad (5.2b)$$

It is apparent from this behavior of $A(k,\omega)$ that the current-current interaction is unscreened at zero frequency.

In Appendix B the photon propagator obtained in Appendix A is used to calculate the self-energy part shown in Fig. 3. It is shown there that this lack of screening causes the momentum $p(\xi)$ at which the electron propagator has a pole

$$[\xi - \epsilon_p - \Sigma_p(\xi)]\big|_{p=p(\xi)} = 0 \qquad (5.3)$$

to have an anomalous logarithmic behavior as the frequency ξ approaches the chemical potential

$$p(\xi) \xrightarrow{\xi\to\mu} p_F - \frac{\alpha}{3\pi}\left(\frac{m^*}{m}\right)^2 (\xi-\mu)\ln|\xi-\mu| + O(\xi-\mu), \tag{5.4}$$

where m^* is defined in (B6).

To show that the logarithmic term in (5.4) implies a term behaving as $T\ln T$ in the low-temperature specific heat, we consider the formula of Luttinger[9] [Eq. (46) of Ref. 9] for the temperature-dependent part of the thermodynamic potential at low temperature

$$\Omega = -\sum_p \frac{1}{2\pi i}\int_{-\infty}^{\infty} dx\,\{\ln[\epsilon_p + \Sigma_p(x-i\epsilon) - x + i\epsilon] - \text{c.c.}\}\,\frac{1}{e^{\beta(x-\mu)}+1} . \tag{5.5}$$

If we replace the sum over p by an integral and integrate by parts, we can rewrite (5.5) as

$$\Omega = \frac{V}{12\pi^3 i}\int_0^\infty p^3\,dp\int_{-\infty}^{\infty} dx\left\{\left[\frac{1}{\epsilon_p + \Sigma_p(x-i\epsilon) - x + i\epsilon}\times\left(\frac{\partial\epsilon_p}{\partial p} + \frac{\partial\Sigma_p(x-i\epsilon)}{\partial p}\right) - \text{c.c.}\right]\frac{1}{e^{\beta(x-\mu)}+1}\right\} , \tag{5.6}$$

where V is the volume. It is shown in Ref. 9 [see Eq. (40) of Ref. 9] that the derivative with respect to T of Ω is the negative of the specific heat C at low temperature. Thus, by differentiating the relation in (5.6) with respect to T, we obtain

$$C = -\frac{kV}{12\pi^3 i}\int_{-\infty}^{\infty} dy\int_0^\infty p^3\,dp\left[\frac{1}{\epsilon_p + \Sigma_p(\mu+y/\beta-i\epsilon) - (\mu+y/\beta-i\epsilon)}\left(\frac{\partial\epsilon_p}{\partial p} + \frac{\partial\Sigma_p(\mu+y/\beta-i\epsilon)}{\partial p}\right) - \text{c.c.}\right]\frac{ye^y}{(e^y+1)^2} \tag{5.7}$$

We wish to show that the logarithmic behavior in (5.4) produces a term in (5.7) behaving as $T\ln T$.

The integral over p in (5.7) is

$$\int_0^\infty p^3\,dp(\cdots) = \int \bar{p}^3(z)\,\frac{dz}{z} - \text{c.c.}, \tag{5.8}$$

where z is the energy denominator in (5.7), and where $\bar{p}(z)$ is the root of

$$[\epsilon_p + \Sigma_p(\mu+y/\beta-i\epsilon) - (\mu+y/\beta)]_{p=\bar{p}(z)} = z. \tag{5.9}$$

The integration contour in (5.8) traverses a path from the left to the right above the real z axis along the contour for which $\bar{p}(z)$ is real. Thus, Eq. (5.8) can be rewritten

$$\int \bar{p}^3(z)\,\frac{dz}{z} - \text{c.c.} = \int \bar{p}^3(z)\left(\frac{dz}{z} - \frac{dz^*}{z^*}\right) . \tag{5.10}$$

The imaginary part of z along the integration contour is the imaginary part of $\Sigma_{\bar{p}(z)}(\mu+y/\beta-i\epsilon)$. For short-range interactions this[8] is of order β^{-2}. However, as shown in Appendix B, the absence of static screening of the current-current interaction allows this imaginary part to be as large as $\sim\beta^{-1}$ when the z contour in (5.8) and (5.10) is in the neighborhood of $z=0$. Thus, with the neglect of terms that are evidently $\lesssim O(\beta^{-2})$, we can replace $\bar{p}^3(z)$ in the second term on the right-hand side of (5.10) by

$$\bar{p}^3(z) \simeq \bar{p}^3(z^*) + 3\bar{p}^2(z)(z-z^*)\,\frac{dp}{dz} \tag{5.11}$$

to rewrite (5.10) as

$$\int \bar{p}^3(z)\,\frac{dz}{z} - \text{c.c.} = i\,\text{Im}\int\left[\bar{p}^3(z)\,\frac{dz}{z} - \bar{p}^3(z^*)\,\frac{dz^*}{z^*}\right] + 3i\,\text{Re}\int\frac{p^2\,\text{Im}z(p)}{z(p)}\,dp. \tag{5.12}$$

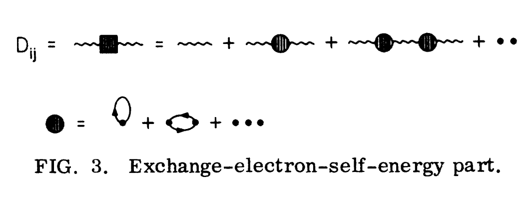
It is shown in Appendix C that the contribution to (5.7) from the second term on the right-hand side of (5.12) is $< O(T^{1+\alpha}\ln T)$, where $\alpha>0$. Thus, since our purpose is to show that the leading part of (5.7) is $\sim T\ln T$, we need keep only the first integral on the right-hand side of (5.12). This integral proceeds to the right along a contour slightly above the real z axis and then back to the left along the complex conjugate contour. However, it is evident that at the extremities of the contours $\text{Im}z \sim O(\beta^{-2})$, so that the two contours can be joined to give a single closed contour enclosing the origin in a clockwise sense. Thus, the relation in (5.12) can be taken to be

$$\int \bar{p}^3(z)\,\frac{dz}{z} - \text{c.c.} = -2\pi i\,\text{Re}\,\bar{p}^3(0). \tag{5.13}$$

But we see by comparing (5.3) and (5.9) that $\bar{p}(0) = p(\mu+y/\beta)$. Thus, from (5.13) and (5.8) we obtain for the specific heat in (5.7),

$$C = \frac{kV}{6\pi^2}\,\text{Re}\int_{-\infty}^{\infty} dy\,p^3\,\frac{\mu+y}{\beta}\left(u+\frac{y}{\beta}\right)\frac{ye^y}{(e^y+1)^2} , \tag{5.14}$$

which with (5.4) leads to



(a)

(b)

FIG. 3. Exchange-electron-self-energy part.

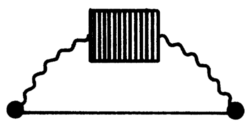


FIG. 4. Corrected current-current potential.

$$C=-\frac{\alpha k^2 V}{36\pi}\,p_F^2\left(\frac{m^*}{m}\right)^2 T\ln T+O(T). \tag{5.15}$$

The specific heat in (5.15) has an anomalous $T\ln T$ behavior arising from the lack of screening of the current-current interaction. However, it is easy to verify that the coefficient of the $T\ln T$ term is $\sim\alpha(v_F/c)(m^*/m)^2\sim 10^{-5}$ that of the dominant term proportional to T.

We expect that an analogous anomalous contribution of the current-current interaction to the dHvA oscillations is correspondingly small.

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APPENDIX A

In this Appendix we outline the calculation of the photon propagator in Fig. (3) in the approximation of retaining only the two-photon self-energy diagrams shown explicitly in Fig. 3(b). As discussed in Appendix B, the aromalous term in the specific heat depends only upon features of the photon propagator which are completely contained in this approximation.

In conformity with our discussion of Eq. (2.7′) in Sec. II, we will drop the frequency dependence of the zero order photon propagator. The graphs of Fig. 3 give

$$\tilde{D}_{ij}(k,\omega)=\left(\delta_{ij}-\frac{k_i k_j}{k^2}\right)\frac{1}{k^2+A(k,\omega)}, \tag{A1}$$

where ($\hbar=c=1$)

$$A(k,\omega)=\frac{4\pi\alpha}{m}\left(n+\frac{1}{m}\int\frac{d^3p}{(2\pi)^3}\,[p^2-(\vec{p}\cdot\hat{k})^2]\times\frac{f(\epsilon_{\vec{p}-\vec{k}/2})-f(\epsilon_{\vec{p}+\vec{k}/2})}{\omega-\epsilon_{\vec{p}+\vec{k}/2}+\epsilon_{\vec{p}-\vec{k}/2}}\right). \tag{A2}$$

Here $\alpha=e^2/4\pi\hbar c=\frac{1}{137}$, n is the electron density, m is the electron mass, f is the Fermi function, and $\epsilon(p)=p^2/2m$. The two terms on the right-hand side of (A2) arise, respectively, from the two parts of the photon self-energy shown explicitly in Fig. 3(b).

We will be particularly interested in the behavior of $A(k,\omega)$ for small k and zero temperature. By using the identity

$$n=2\int\frac{d^3p}{(2\pi)^3}\,f(\epsilon_p), \tag{A3}$$

and setting $f(\epsilon)=\theta(\mu-\epsilon)$, we can perform the integral in (A2) to obtain

$$A(k,\omega)\big|_{T=0}\xrightarrow{k\to 0}\frac{3\omega_p^2}{4}\,x\left[2x+(1-x^2)\ln\frac{x+1}{x-1}\right] \tag{A4}$$

where ω_p is the plasma frequency

$$\omega_p^2=4\pi\alpha n/m \tag{A5}$$

and

$$x=m\omega/kp_F. \tag{A6}$$

APPENDIX B

In this Appendix we derive the zero-temperature features of the exchange-electron-self-energy graph of Fig. 4 which are used in Sec. V.

It is a straightforward application of the diagrammatic techniques[14] to obtain ($\hbar=c=1$)

$$\mathrm{Im}\Sigma_p^{\mathrm{ex}}(\xi+i\epsilon)=\frac{2\alpha}{m^2}\int\frac{d^3k}{(2\pi)^3}\,[p^2-(\vec{p}\cdot\hat{\vec{k}})^2]\times\int_\xi^\mu d\omega\,\mathrm{Im}G^R_{\vec{p}-\vec{k}}(\omega)\,\mathrm{Im}D^R_k(\omega-\xi), \tag{B1}$$

where we have employed the conventional notation of denoting the upper boundary value of the electron Green's function by G^R, as well as for the coefficient of the projection operator in (A1):

$$\tilde{D}^R_k(\omega)\equiv\frac{1}{k^2+A(k,\omega+i\epsilon)}. \tag{B2}$$

We wish to show that $\mathrm{Im}\Sigma_p^{\mathrm{ex}}(\xi+i\epsilon)$ has a part which is essentially of order $|\xi-\mu|$ when both ξ and p approach the Fermi surface simultaneously; if $|\xi-\mu|\to 0$ with p fixed ($p\neq p_F$), then $\mathrm{Im}\Sigma_p^{\mathrm{ex}}\sim(\xi-\mu)^2$ similar to the behavior[8] for all p when only short-range potentials are present. We can therefore use

$$\mathrm{Im}G^R_{\vec{p}-\vec{k}}(\omega)=-\pi\delta(\omega-\epsilon_{\vec{p}-\vec{k}}-\mathrm{Re}\Sigma_{\vec{p}-\vec{k}}(\omega)), \tag{B3}$$

which is generally valid for $\omega\to\mu$. Employing this δ function to integrate over the direction of $\vec{k}$ in (B1), we obtain eventually

$$\mathrm{Im}\Sigma_p^{\mathrm{ex}}(\xi+i\epsilon)\big|_{\xi\simeq\mu}=-\frac{\alpha}{\pi m^2 p}\int_\xi^\mu d\omega\,p(\omega)\times\left(\frac{p(\omega)}{m}+\frac{\partial}{\partial p}\mathrm{Re}\Sigma_p(\omega)\right)^{-1}\times\int_{|p(\omega)-p|}^{p(\omega)+p}k\,dk\left[p^2-\frac{1}{4}\times\left(\frac{k^2-p^2(\omega)+p^2}{k}\right)^2\right]\mathrm{Im}D^R_k(\omega-\xi), \tag{B4}$$

where $p(\omega)$ is defined in Eq. (5.3).

For $p \simeq p_F$ it is easy to check that

$$\left[p^2 - \frac{1}{4}\left(\frac{k^2 - p^2(\omega) + p^2}{k}\right)^2\right] \simeq -\tfrac{1}{4}k^2 + p_F^2\left(1 - \frac{[p - p(\omega)]^2}{k^2}\right) . \quad \text{(B5)}$$

The anomalous behavior $\sim |\xi - \mu|$ as p and ξ approach the Fermi surface arises from the fact that the magnetic interaction is unscreened at zero frequency (for normal, not superconducting, systems), and that correspondingly $\mathrm{Im}D_k^R(\omega - \xi)$ is singular when k, $\omega - \xi$, and $\omega - \xi/k$ all approach zero. The first term on the right-hand side of (B5) can therefore be ignored. Further, the leading behavior of $\mathrm{Im}D_k^R(\omega - \xi)$ in this region is contained entirely in the part of D_k^R calculated in Appendix A. We can therefore use the form of $A(k, \omega)$ given in (A4). Defining an effective mass m^* by

$$\frac{p_F}{m^*} \equiv \frac{\partial}{\partial_{p_F}}\left[\epsilon_{p_F} + \mathrm{Re}\Sigma_{p_F}(\mu)\right], \quad \text{(B6)}$$

we can apply these observations to (B4) to obtain

$$\mathrm{Im}\Sigma_p^{\mathrm{ex}}(\xi + i\epsilon)\big|_{\mathrm{FS}} = -\frac{\alpha p_F}{\pi m}\left(\frac{m^*}{m}\right)\int_\xi^\mu d\omega \int_{|p(\omega)-p|}^{2p_F} dk\, k\left(1 - \frac{|p(\omega) - p|^2}{k^2}\right)\mathrm{Im}\,\frac{1}{k^2 + A(k, \omega - \xi + i\epsilon)} , \quad \text{(B7)}$$

which completely contains the anomalous behavior of $\mathrm{Im}\Sigma_p^{\mathrm{ex}}$ on the Fermi surface.

After some effort it is possible to conclude that we need only consider the part of $A(k, \omega - \xi + i\epsilon)$ which dominates for small k and small $\omega - \xi/k$; that is we can approximate $A(k, \omega - \xi + i\epsilon)$ by

$$A(x + i\epsilon) \simeq -\tfrac{3}{4} i\pi\omega_p^2(\omega - \xi)/k, \quad \text{(B8)}$$

which comes from the imaginary part of the logarithm in (A4). Also, it can be verified that we can ignore the term $\sim |p(\omega) - p|^2 k^{-2}$ in the integrand in (B7). With these simplifications, the inner integral in (B7) can be performed with the result that

$$\mathrm{Im}\Sigma_p(\xi + i\epsilon) \underset{p \to p_F}{\xrightarrow{\xi \to \mu}} -\frac{\alpha p_F}{3\pi m}\left(\frac{m^*}{m}\right) \times \int_\xi^\mu d\omega \tan^{-1}\frac{3\pi m\omega_p^2(\omega - \xi)}{4p_F|p(\omega) - p|^3} + O(|\xi - \mu|^{1+\epsilon}). \quad \text{(B9)}$$

If, for example, $p = p(\xi)$ as defined in Eq. (5.3), it is evident that the first term on the right-hand side of (B9) is $\sim |\xi - \mu|$.

We next want to argue that the behavior of $\mathrm{Im}\Sigma_p(\xi)$ in (B9) requires $(\partial/\partial\xi)\mathrm{Re}\Sigma_p(\xi)$ to diverge logarithmically as both p and ξ approach the Fermi surface. Although this could be demonstrated by computing $\mathrm{Re}\Sigma_p(\xi)$ as we have done for $\mathrm{Im}\Sigma_p(\xi)$ above, it is simpler to use the Kramers-Kronig relation

$$\mathrm{Re}\Sigma_p(\xi) = \frac{\mathcal{P}}{\pi}\int \frac{d\xi'}{\xi' - \xi}\,\mathrm{Im}\Sigma_p(\xi'), \quad \text{(B10)}$$

with $\mathrm{Im}\Sigma_p$ given by (B9). Noting that $\mathrm{Im}\Sigma_p(\xi)$ in (B9) is symmetric about $\xi = \mu$, we can differentiate (B10) with respect to ξ and integrate by parts to obtain

$$\frac{\partial\,\mathrm{Re}\Sigma_p^{\mathrm{ex}}(\xi)}{\partial\xi} = \frac{\mathcal{P}}{\pi}\int_\mu^\infty d\xi'\left(\frac{1}{\xi' - \xi} + \frac{1}{\xi' + \xi}\right) \times \frac{\partial}{\partial\xi'}\,\mathrm{Im}\Sigma_p^{\mathrm{ex}}(\xi'). \quad \text{(B11)}$$

Putting $p = p_F$, one can verify from (B9) that

$$\frac{\partial}{\partial\xi'}\,\mathrm{Im}\Sigma_{p_F}(\xi') \xrightarrow{\xi' \to \mu} -\frac{\alpha p_F}{3\pi m}\left(\frac{m^*}{m}\right)\tan^{-1}\frac{3\pi m\omega_p^2(\xi' - \mu)}{|p(\xi') - p_F|^3} \quad \text{(B12)}$$

plus terms which vanish as $\xi' \to \mu$. As $\xi' \to \mu$, the arctangent in (B12) becomes equal to $\frac{1}{2}\pi$. From (B11) we thus obtain

$$\frac{\partial}{\partial\xi}\,\mathrm{Re}\Sigma_p^{\mathrm{ex}}(\xi)\big|_{p=p(\xi)} \xrightarrow{\xi \to \mu} \frac{\alpha p_F}{3\pi m}\left(\frac{m^*}{m}\right)\ln|\xi - \mu| . \quad \text{(B13)}$$

From (B13) we can easily obtain Eq. (5.4). That is, since

$$\frac{dp(\xi)}{d\xi} = \left(1 - \frac{\partial\Sigma_p(\xi)}{\partial\xi}\right)\frac{\partial}{\partial p}\left[\epsilon_p + \Sigma_p(\xi)\right]\big|_{p=p(\xi)} , \quad \text{(B14)}$$

it follows from (B6) and (B13) that near $\xi \to \mu$, the anomalous part of $p(\xi)$ satisfies

$$\frac{dp(\xi)}{d\xi} \xrightarrow{\xi \to \mu} -\frac{\alpha}{3\pi}\left(\frac{m^*}{m}\right)\ln|\xi - \mu| , \quad \text{(B15)}$$

from which Eq. (5.4) is obtained by integration.

APPENDIX C

In this Appendix we show that the second integral on the right-hand side of Eq. (5.12) does not contribute to the specific heat in (5.7) in an order which is larger than $\sim T$ as $T \to 0$. In particular, we wish to show that

$$\mathrm{Re}\int \frac{p^2\,dp\,\mathrm{Im}\Sigma_p(\mu + Ty)}{\mu + Ty - \epsilon_p - \Sigma_p(\mu + Ty)} \xrightarrow{T \to 0} < O(T) . \quad \text{(C1)}$$

From the discussion in Appendix B, we know that $\mathrm{Im}\Sigma_p(\mu + Ty)$ can only be as large as $O(T)$ as $T \to 0$ when p is in the neighborhood of $p_F \approx p(\mu + Ty)$. Hence, the p^2 inside the integral in (C1) can be replaced by p_F^2, and the denominator can be expanded about $p(\mu+Ty)$ [see Eq. (5.3)],

$$\mu + Ty - \epsilon_p - \Sigma_p(\mu+Ty) = -[p - p(\mu+Ty)] \times \frac{\partial}{\partial p}[\epsilon_p + \Sigma_p(\mu+Ty)]_{p=p(\mu+Ty)} . \quad \text{(C2)}$$

The derivative on the right-hand side of (C2) is a finite nonzero constant which also can be taken outside the integral in (C1). Thus, by appealing to (B9) to obtain the form of $\mathrm{Im}\Sigma_p$ near the Fermi surface, it is evident that the inequality in (C1) is satisfied if

$$\mathrm{Re}\int_{-\infty}^{\infty}\frac{dp}{p-p(\mu+Ty)}\int_{\mu}^{\mu+Ty} d\omega \tan^{-1}\times\frac{a(\mu+Ty+\omega)}{|p-p(\omega)|^3} \xrightarrow{T\to 0} < O(T), \quad \text{(C3)}$$

where a is a constant.

If the order of integration in (C3) is changed, and if the integration variable p is replaced by x, where $p - p(\mu+Ty) = x(p(\mu+Ty) - p(\omega))$, then the left-hand side of (C3) becomes

$$\int_{\mu}^{\mu+Ty} d\omega\,\mathcal{P}\int_0^{\infty}\frac{dx}{x}\left[\tan^{-1}\frac{a(\mu+Ty-\omega)}{|p(\mu+Ty)-p(\omega)|^3(1+x)^3} - \tan^{-1}\frac{a(\mu+Ty-\omega)}{|p(\mu+Ty)-p(\omega)|\,|1-x|^3}\right]. \quad \text{(C4)}$$

Consider two regions of x separately:

region 1: $x \lesssim K(Ty)^{-2/3+\alpha}$,

region 2: $x \gtrsim K(Ty)^{-2/3+\alpha}$,

where $0<\alpha<\frac{2}{3}$ and K is a constant. Since [see Eq. (5.4)] $|p(\mu+Ty)-p(\omega)|$ is of order $|\mu+Ty-\omega| \times \ln|\mu+Ty-\omega|$, the argument of the arctangents in (C4) in region 1 are $\gtrsim (Ty)^{-3\alpha}\ln^{-3}|Ty|$. Thus, for sufficiently small T, we can use the relation

$$\tan^{-1}y \simeq \tfrac{1}{2}\pi - y$$

valid for small y. It is then evident that the integrand for the ω integration coming from region 1 is $\lesssim O(|Ty|^{3\alpha}\ln^3|Ty|)$ and the integral in (C4) from this region is $\lesssim O(|Ty|^{1+3\alpha}\ln^3|Ty|)$.

In region 2, $x \gg 1$ for small T, and we can use $[A \equiv a(\mu+Ty-\omega)|p(\mu+Ty)-p(\omega)|^{-3}]$

$$\tan^{-1}\frac{A}{|1\pm x|^3} \simeq \tan^{-1}\left(\frac{A}{x^3}\mp\frac{3A}{x^4}\right) \simeq \tan^{-1}\frac{A}{x^3}\mp\frac{1}{1+A^2/x^6}\,\frac{3A}{x^4} .$$

The difference of the arctangents in (C4) then satisfies the inequality

$$\left|\tan^{-1}\frac{A}{(1+x)^3} - \tan^{-1}\frac{A}{|1-x|^3}\right| < \left|\frac{6A}{x^4}\right| \lesssim (Ty)^{-2-4\alpha+8/3}\ln^{-3}|Ty| \sim (Ty)^{2/3-4\alpha}\ln^{-3}|Ty| .$$

The integral in (C4) coming from region 2 is then less than $\sim |Ty|^{5/3-4\alpha}\ln^{-3}|Ty|$.

Comparing this result with the conclusion above for the contribution to (C4) from region 1, we see that for $0<\alpha<\frac{1}{6}$ both contributions are less than $\sim |Ty|$. The second term on the right-hand side of (5.12) thus gives no contribution to the anomalous term in the specific heat $\sim T\ln T$, as shown in Eq. (5.15), nor for that matter does it contribute to the dominant term proportional to T.

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