# Surface-Impedance Theory for Pure Type-I Superconductors in Large Static Magnetic Fields*

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A microscopic nonphenomenological theory for the interaction of electromagnetic radiation with pure Pippard superconductors is presented to explain the changes in surface impedance due to large static magnetic fields. The static magnetic field penetrating the metal's surface creates a momentum-dependent potential. Quasiparticles near the Fermi surface find that such a potential strongly modifies their wave function and energy-excitation spectrum. An algorithm is developed that allows surface impedances to be determined despite the difficulties caused by spatial inhomogeneity, anisotropy, and nonlocal electrodynamics.

## I. INTRODUCTION

Over the past twenty years, experimentalists have been measuring the changes in surface impedance of pure type-I superconductors due to a large static magnetic field at the surface.[1-6] The principle features observed are that the magnetic field increases the superconducting surface resistance $R_s$ for "low temperatures" ($T \lesssim 0.5T_c$) and "high temperatures" ($T \gtrsim 0.9T_c$), but can strongly decreases $R_s$ for the "intermediate $T$" ($0.5T_c \lesssim T \lesssim 0.9T_c$). At "low $T$" the superconducting surface reactance $X_s$ is always increased, while at "intermediate and high $T$," the changes can be either positive or negative, depending on the frequency of the radiation $\omega$.

To calculate surface impedances for this situation, we must cope with two complexities: (i) The coupled second-order differential equations of superconductivity are nonlinear. (ii) The relation between the current density $\vec{j}(\vec{r})$ and the electric field $\vec{E}(\vec{r})$ [or the associated vector potential $\vec{A}(\vec{r})$] is nonlocal; current at one point inside the superconductor depends on the field strength in the surrounding region, which means that finding the field distribution necessitates solving a complicated integrodifferential equation. And the kernel of this integrodifferential equation must be found from (i) above. Techniques employed when there are no static magnetic fields present[7,9] are no longer valid, since the static field introduces spatial inhomogeneity and anisotropy.

Our vehicle for discussing this problem will be the Bogolubov canonical transformation from interacting electron states to noninteracting quasiparticle states.[8] Using this transformation, we derive in Sec. II the general form of the current-versus-field relation for spatially varying situations. Further evaluation of the latter results require the explicit form for the Bogolubov transformation coefficients; this we do in Sec. III. Section IV then develops the algorithm to calculate the surface impedance. Combining all the previously developed machinery, we display some results of the theory and compare to data in Sec. V. Section VI considers the question of whether a large static magnetic field can alter the gap function or order parameter. A discussion of rival theories and some other comments are found in Sec. VII.

## II. $\vec{j}(\vec{A})$ RELATION

Any discussion of electrodynamics inside a metal requires knowledge of a current-versus-field relation. In this section we derive a particular form for the current as a function of a weak vector potential useful for surface-impedance calculations.

Abrikosov, Gorkov, and Khalatnikov (AGK)[9] show that to first order in $\vec{A}(x)$, the physical current $\vec{j}(x)$ is

$$\vec{j}(x) = \frac{ie^2}{4m^2}(\nabla_{\vec{x}} - \nabla_{\vec{x}'})_{\vec{x}'\to\vec{x}} \int_{-\infty}^{t_x} \vec{A}(y)\cdot(\nabla_{\vec{y}} - \nabla_{\vec{y}'})_{\vec{y}'\to\vec{y}} \times \sum_{\alpha,\beta} \langle \psi_\beta^\dagger(y')\psi_\beta(y)\psi_\alpha^\dagger(x')\psi_\alpha(x) - \psi_\alpha^\dagger(x')\psi_\alpha(x)\psi_\beta^\dagger(y')\psi_\beta(y)\rangle_T d^4y - \frac{Ne^2}{m}\vec{A}(x), \quad (2.1)$$

where $x = (\vec{x}, t_x)$, $\alpha$ and $\beta$ are electron-spin indices, $\psi^\dagger$ ($\psi$) is an electron-field creation (annihilation) operator, $N = \sum_\alpha \langle \psi_\alpha^\dagger(x)\psi_\alpha(x)\rangle_T$ is the density of electrons in the system, $\langle\ \rangle_T$ denotes thermal averaging over a grand canonical ensemble, and we set $\hbar = c = 1$ everywhere. All particle field operators are in the interaction representation if they have an explicit time dependence; otherwise they are Schrödinger operators.

We evaluate (2.1) using the Bogolubov canonical transformation from interacting electron states to noninteracting quasiparticle states. The transformation on particle field operators is written compactly as

$$\psi_\alpha(\vec{x}) = \sum_j \left[\gamma_{j\alpha} u_j(\vec{x}) + S_{\alpha'}\gamma^\dagger_{j\alpha'} v^*_j(\vec{x})\right], \quad \alpha' \neq \alpha \tag{2.2}$$

where $S_{\alpha'}$ is 1 for spin up and $-1$ for spin down, $\gamma^\dagger$ ($\gamma$) create (annihilate) noninteracting Fermion excitations or quasiparticles. In order that this transformation exist, $u$ and $v$ must obey some conditions known as Bogolubov's equations (see Sec. III). $u$ and $v$ have a physical interpretation: $u_j(\vec{x})$ is the amplitude to find a particlelike quasiparticle in state $j$ at position $\vec{x}$, while $v_j(\vec{x})$ is the amplitude to find an antiparticlelike (or holelike) quasiparticle in state $j$ at position $\vec{x}$.

For operators evolving in the interaction representation

$$\psi_\alpha(x) = \sum_j \left[\gamma_{j\alpha}(t) u_j(x) + S_{\alpha'}\gamma^\dagger_{j\alpha'}(t) v^*_j(x)\right], \tag{2.3}$$

where

$$\gamma_{j\alpha}(t) = e^{iH_m t}\gamma_{j\alpha} e^{-iH_m t} \tag{2.4}$$

and

$$H_m = \sum_{n,\alpha} \epsilon_n \gamma^\dagger_{n\alpha}\gamma_{n\alpha}, \tag{2.5}$$

so that

$$H_m |m\rangle = \epsilon_m |m\rangle; \tag{2.6}$$

$H_m$ is the Hamiltonian of the quasiparticles, and $\epsilon_n$ is the excitation energy of a quasiparticle in state $|n\rangle$.

Writing $\vec{j}(x)$ in terms of $\gamma$'s yields sums over many terms, each of which is a product of four $\gamma$'s. But only some of these terms are nonvanishing upon taking thermal averages. Since the thermal average is a weighted sum of diagonal matrix elements, only terms of the form $\gamma^\dagger_i\gamma_i\gamma^\dagger_j\gamma_j$ (or permutations in ordering) can contribute. A typical term for $\langle\ \rangle_T$ in Eq. (2.1) is

$$u^*_l(\vec{y}')u_m(\vec{y})u^*_n(\vec{x}')u_p(\vec{x})\langle\gamma^\dagger_{l\beta}(t_y)\gamma_{m\beta}(t_y)\gamma^\dagger_{n\alpha}(t_x)\gamma_{p\alpha}(t_x)$$
$$-\gamma^\dagger_{n\alpha}(t_x)\gamma_{p\alpha}(t_x)\gamma^\dagger_{l\beta}(t_y)\gamma_{m\beta}(t_y)\rangle_T$$
$$= u\cdots u[\delta_{\alpha\beta}\delta_{lp}\delta_{mn}[f_T(\epsilon_l) - f_T(\epsilon_m)]e^{i(\epsilon_l-\epsilon_m)(t_y-t_x)}],$$

where $f_T(\epsilon_l) = \langle\gamma^\dagger_l\gamma_l\rangle_T = 1/(e^{\epsilon_l/T}+1)$ is just the Fermi distribution function for noninteracting Fermions, and the exponential in time results from

$$\langle\cdots\gamma_j(t)\rangle_T = \langle\cdots e^{iH_m t}\gamma_j e^{-iH_m t}\rangle_T = e^{-i\epsilon_j t}\langle\cdots\gamma_j\rangle_T.$$

Using a sinusoidal $\vec{A}$ field which vanishes as $t_y \to -\infty$,

$$\vec{A}(y) = \vec{A}(\vec{y})e^{i(\omega - i\delta)t_y}, \quad \delta \to 0^+ \tag{2.7}$$

allows us to perform, then, the time integration in Eq. (2.1). As one expects intuitively, all terms for the current vary sinusoidally with frequency $\omega$, too. We have $\vec{j}(x) \sim \vec{j}(\vec{x})e^{i\omega t_x}$. Henceforth, we consider only the $\omega$ Fourier component.

Proceeding in the above manner for all terms in Eq. (2.1), calculating the trivial sums over $\delta$ functions then puts the current density $\vec{j}(\vec{x})$ into the following form:

$$\vec{j}(\vec{x}) = \vec{j}_1(\vec{x}) - (Ne^2/m)\vec{A}(\vec{x}), \tag{2.8a}$$

where

$$j_1(x) = \frac{e^2}{2m^2}(\nabla_{\vec{x}} - \nabla_{\vec{x}'})_{\vec{x}'\to\vec{x}} \int d\vec{y}\,\vec{A}(\vec{y})\cdot(\nabla_{\vec{y}} - \nabla_{\vec{y}'})_{\vec{y}'\to\vec{y}} \sum_{l,m}\Bigg[[f_T(\epsilon_l) - f_T(\epsilon_m)]$$
$$\times\left(\frac{u_l(\vec{x})u^*_m(\vec{x}')u_m(\vec{y})u^*_l(\vec{y}') - v^*_m(\vec{x})v_l(\vec{x}')u_m(\vec{y})u^*_l(\vec{y}')}{\epsilon_l - \epsilon_m + \omega - i\delta} + \frac{u_m(\vec{x})u^*_l(\vec{x}')v^*_m(\vec{y})v_l(\vec{y}') - v^*_l(\vec{x})v_m(\vec{x}')v^*_m(\vec{y})v_l(\vec{x}')}{-(\epsilon_l - \epsilon_m) + \omega - i\delta}\right)$$
$$+[1 - f_T(\epsilon_l) - f_T(\epsilon_m)]\left(-\frac{u_m(\vec{x})v_l(\vec{x}')v^*_m(\vec{y})u^*_l(\vec{y}') + u_l(\vec{x})v_m(\vec{x}')v^*_m(\vec{y})u^*_l(\vec{y}')}{\epsilon_l + \epsilon_m + \omega - i\delta}\right.$$
$$\left.+\frac{v^*_m(\vec{x})u^*_l(\vec{x}')u_m(\vec{y})v_l(\vec{y}') + v^*_l(\vec{x})u^*_m(\vec{x}')u_m(\vec{y})v_l(\vec{y}')}{-(\epsilon_l + \epsilon_m) + \omega - i\delta}\right)\Bigg]. \tag{2.8b}$$

We now specialize all further developments to a particular geometry. The superconducting metal's surface lies in the $y$-$z$ plane. It is infinite in $y$ and $z$, and semi-infinite in the $+x$ direction. Now note that there are two sets of fields to consider—those from the static magnetic field and those due to the high-frequency weak radiation impinging upon the surface. The vector potential $\vec{A}_{dc}$, due to the $\vec{H}_{dc}$, defines the $y$ axis. The vector potential $\vec{A}_{ac}$ due to the radiation is linearly polarized in the $y$-$z$ plane and makes an angle $\theta$ with respect to the $y$ axis. Each $\vec{A}$ field has associated with it a current $\vec{j}$ in the metal, and this "screening" current is parallel to its respective $\vec{A}$ field. It is to be noted here that we are treating $\vec{A}_{ac}$ and $\vec{A}_{dc}$ separately. The present section considers $\vec{A}_{ac}$ as a small perturbation on the system and the current-versus-field relations are $\vec{j}_{ac}$ as a function of $\vec{A}_{ac}$. $\vec{A}_{dc}$ is not small, and its effects are to be included in the $u$, $v$ functions.

Getting back to further processing of the $\vec{j}$-vs-$\vec{A}$ relation, we write

$$\vec{A}_{ac}(\vec{x}) = (\vec{e}_y \cos\theta + \vec{e}_z \sin\theta) A_{ac}(x) \quad , \tag{2.9}$$

where $\vec{e}_{y,z}$ are unit vectors in the $y$ or $z$ direction, and $x$ refers to the coordinate on the $x$ axis. [There will no longer be any reference to $x = (\vec{x}, t)$ as a four-vector.] Note that $A_{ac}(x)$ varies only with $x$; we have an infinite plane wave impinging upon the surface. Due to translational invariance in $y$ and $z$ directions, $u$, $v \sim u(x)$, $v(x)e^{i(K_y y + K_z z)}$. After using this and Eq. (2.9) in Eq. (2.8), performing the $A \cdot \nabla$ differentiations, taking limits, doing the integrations over the pure exponentials (which give $\delta$ functions), and making some simple rearrangements which depend on $l$, $m$ being dummy indices, we can obtain the following result:

$$j_1(x) = \int_0^\infty Q_1(x, x') A_{ac}(x')\, dx \quad , \tag{2.10a}$$

$$Q_1(x, x') = \frac{2e^2}{m^2} (2\pi)^2 \sum_{l,m} (K_y^l \cos\theta + K_z^l \sin\theta)^2 \Big[ [f_T(\epsilon_l) - f_T(\epsilon_m)] \delta(K_y^l - K_y^m) \delta(K_z^l - K_z^m)$$
$$\times \left( \frac{[u_l(x) u_m^*(x) + v_l(x) v_m^*(x)][u_m(x') u_l^*(x') + v_m(x') v_l^*(x')]}{\epsilon_m - \epsilon_l - \omega + i\delta} \right) + \tfrac{1}{2} [1 - f_T(\epsilon_l) - f_T(\epsilon_m)] \delta(K_y^l + K_y^m) \delta(K_z^l + K_z^m)$$
$$\times \left( \frac{[u_l(x) v_m(x) - u_m(x) v_l(x)][u_l^*(x') v_m^*(x') - u_m^*(x') v_l^*(x')]}{\epsilon_m + \epsilon_l + \omega - i\delta} + \frac{[u_m^*(x) v_l^*(x) - u_l^*(x) v_m^*(x)][u_m(x') v_l(x') - u_l(x') v_m(x')]}{\epsilon_m + \epsilon_l - \omega + i\delta} \right) \Big] \quad , \tag{2.10b}$$

$$\vec{j}_1(x) = (\vec{e}_y \cos\theta + \vec{e}_z \sin\theta) j_1(x) \quad . \tag{2.10c}$$

The last equation says that the current flows in the same direction as the $A$ field's linear polarization.

Equation (2.10) for $\vec{j}(\vec{A})$ is now in a form amenable to physical interpretation. Two distinct physical processes are represented: The $\vec{A}$ field causes thermally excited quasiparticles to make transitions from one state to another state. The $\vec{A}$ field causes the creation or destruction of a pair of excitations; this pair is made up of a quasiparticle and an antiquasiparticle. (These processes are directly analogous to electron-positron theory in quantum electrodynamics; the rest-mass energy $mc^2$ plays a role very similar to the superconducting energy gap $\Delta$.)

Consider first the scattering process. Since the incident photon has momentum only normal to the surface and none in the plane of the surface, only the $x$ component of a quasiparticle's momentum can change in a transition. So the $y$ and $z$ component remain the same. This is expressed through the $\delta$ functions $\delta(K_y^l - K_y^m)$ and $\delta(K_z^l - K_z^m)$. If the transition is from a state of lower energy to one of higher energy, there must be some nonzero probability for the lower state to contain a quasiparticle, and some nonzero probability for the upper state to have an empty slot available. The net current, though, is proportional to the number of upward transitions minus the number of downward transitions. This fact is expressed through the Fermi factor difference $f_T(\epsilon_l) - f_T(\epsilon_m)$. [Recall that if $f_T(\epsilon)$ is the probability for $\epsilon$ to be occupied, then $1 - f_T(\epsilon)$ is the probability for $\epsilon$ to be empty.] Now suppose the incident photon annihilates at $x'$. There is an amplitude for it to scatter a quasiparticle from state $l$ to state $m$. But since a state has amplitude $u$ to be particlelike and amplitude $v$ to be antiparticlelike, we must consider the quantum-mechanical interference and add the amplitudes of the processes: particle-to-particle $u_m(x') u_l^*(x')$ plus antiparticle-to-antiparticle $v_m(x') v_l^*(x')$ transitions. Similarly, a current found at $x$ generated by the transition from state $l$ to state $m$ could either be due to a particle-particle transition or to an antiparticle-antiparticle transition; hence the factor $u_l(x) u_m^*(x) + v_l(x) v_m^*(x)$. The factor $1/(\epsilon_m - \epsilon_l - \omega + i\delta)$ splits into a real part $\mathcal{P}[1/(\epsilon_m - \epsilon_l - \omega)]$ and an imaginary part $-i\pi\delta(\epsilon_m - \epsilon_l - \omega)$ via a familiar identity. It will be seen later that the imaginary part is related to power absorbed by the metal in which case energy must be conserved in the scattering process. This is expressed via the $\delta$-function part. The real part, which does not conserve energy, thus considers only virtual processes, and these are related to the diamagnetic or screening properties of a material. We discuss this later after developing more formalism.

Next, consider the pair processes. Once again transverse momentum must be conserved. If initially there is no pair, transverse momentum is zero; after a pair is created, there is total $y$ momentum $K_y^l + K_y^m$ for the excitations put in states $l$, $m$. Thus there are the factors $\delta(K_y^l + K_y^m)$ and $\delta(K_z^l + K_z^m)$. In order to send a pair of excitations into states $l$, $m$, these states must be empty, or if a pair is to be annihilated from $l$, $m$, there must be a pair located in $l$, $m$; hence the factor $1 - f_T(\epsilon_l) - f_T(\epsilon_m)$. This includes up-minus-down transitions, as before. The energy denominators express conservation of energy for absorption

processes and yield virtual processes for non-energy-conserving processes. Note that since $\epsilon_m$, $\epsilon_l$, and $\omega$ are always positive, the term involving $\delta(\epsilon_m+\epsilon_l+\omega)$ never contributes to the absorption. In fact, this term more nearly corresponds to the destruction of a pair creating a photon and, hence, depleting current in the material. A photon annihilating at $x'$ can put a particle in $m$ and an antiparticle in $l$, or a particle in $l$ and an antiparticle in $m$. The amplitude for this is $u_m(x')v_l(x') - u_l(x')v_m(x')$, and the explanation of the other matrix elements is obvious.

The factor $(K_y^l\cos\theta + K_z^l\sin\theta)^2$ is the coupling between the photon's polarization and an absorbing quasiparticle's transverse motion. A quasiparticle has the strongest interaction when it is moving exactly parallel or antiparallel to the direction of linear polarization.

## III. SOLUTIONS OF BOGOLUBOV'S EQUATIONS IN A LARGE STATIC MAGNETIC FIELD

Further evaluation of Eq. (2.10) requires explicit forms for $u$, $v$ when there are large static magnetic fields at the metal's surface. Bogolubov's equations for this case are

$$\epsilon_n u_n(\vec{r}) = \{(1/2m)[-i\nabla - e\vec{A}_{dc}(\vec{r})]^2 - E_F\}u_n(\vec{r}) + \Delta(\vec{r})v_n(\vec{r}), \quad (3.1a)$$

$$\epsilon_n v_n(\vec{r}) = -\{(1/2m)[i\nabla - e\vec{A}_{dc}(\vec{r})]^2 - E_F\}v_n(\vec{r}) + \Delta^*(\vec{r})u_n(\vec{r}), \quad (3.1b)$$

$$\Delta(\vec{r}) = V_{BCS}\sum_n v_n^*(\vec{r})u_n(\vec{r})[1-2f_T(\epsilon_n)] \ . \quad (3.2)$$

$\Delta(\vec{r})$ is the order parameter, $V_{BCS}$ is the electron-electron interaction potential, and $E_F$ is the Fermi energy of the electron ensemble. In addition, there is a normalization condition to be satisfied:

$$\int d\vec{r}\,[|u_n(\vec{r})|^2 + |v_n(\vec{r})|^2] = 1 \ . \quad (3.3)$$

We shall find some approximate solutions of these equations following closely a method proposed by Pincus.[10]

Consider once again the semi-infinite geometry of Sec. II. Inside the metal $H_{dc}(\vec{r}) \cong \vec{e}_z He^{-x/\lambda}$, where $\lambda$ is the static-magnetic-field penetration depth. Working in the $\nabla\cdot\vec{A}=0$ gauge, then,

$$\vec{A}_{dc}(\vec{r}) = \vec{e}_y H\lambda e^{-x/\lambda} \ , \quad (3.4)$$

since $\vec{H} = \nabla\times\vec{A}$. Using Eq. (3.4) and $u(\vec{r})$, $v(\vec{r}) = u(x)$, $v(x)e^{i(K_y y + K_z z)}$ simplifies Eqs. (3.1) and (3.2) to

$$\left(\epsilon_n - \xi_t + \frac{1}{2m}\frac{d^2}{dx^2} - V(x)\right)u_n(x) - \Delta(x)v_n(x) = 0 \ , \quad (3.5a)$$

$$\left(\epsilon_n + \xi_t - \frac{1}{2m}\frac{d^2}{dx^2} - V(x)\right)v_n(x) - \Delta(x)u_n(x) = 0 \ , \quad (3.5b)$$

$$V(x) = (eH\lambda/m)K_y e^{-x/\lambda} \ , \quad (3.5c)$$

$$\xi_t = K_y^2/2m + K_z^2/2m - K_F^2/2m \ , \quad (3.5d)$$

$$\Delta(x) = V_{BCS}\sum_n v_n^*(x)u_n(x)[1-2f_T(\epsilon_n)] \ . \quad (3.6)$$

$K_F$ is the Fermi momentum and $E_F = K_F^2/2m$ is the Fermi energy; our model assumes a spherical Fermi surface. In what follows, it will be convenient to use also a set of momentum variables defined by $K_y = K_\rho\cos\theta$, $K_z = K_\rho\sin\theta$; $K_\rho$ is called the transverse momentum and, hence, $\xi_t$ will be a transverse energy relative to the Fermi energy. The quantity designated $V(x)$ is so labeled because it acts somewhat like a potential barrier or wall, depending on the polarity of $K_y$. Since $\lambda$ is purely real for static fields, $V(x)$ is always purely real. We have ignored the $A_{dc}^2$ term in Eqs. (3.1a) and (3.1b), since $eA_{dc} \ll K_y$ for all $K_y \gtrsim 10^{-4}K_F$, and the $K_y$ of interest in all subsequent work is $K_y \tilde{=} K_F$.

It would now be very helpful if we somehow knew in advance the spatial dependence of the gap function $\Delta(x)$. For an infinite superconductor considered by BCS,[11] $\Delta(x)$ was a constant, making solution of Bogolubov's equations trivial. But in the presence of a large static magnetic field, we can no longer be so sure that $\Delta(x)$ is still constant.

Bogolubov's equations are sufficiently complicated that we cannot solve for $u$ and $v$ while simultaneously satisfying the self-consistency relation on $\Delta$, in the presence of large static magnetic fields. Therefore we shall make a guess at what the final self-consistent $\Delta$ might be, solve Bogolubov's equations for the $u$, $v$ functions, and check via the self-consistency relation just how reasonable was our guess. (This matter is considered further in Sec. VI.) We shall try $\Delta$ independent of $H_{dc}$ and spatially constant.

Even with the assumption of a constant gap, Bogolubov's equations are still unmanageable. The trouble lies with $V(x)$. Any step-function approximation to the exponential in $V(x)$ will lead to a set of coupled second-order differential equations with constant coefficients and that can be handled reasonably. We shall content ourselves, first, with a single step and thus make the replacement

$$V(x) \to V_e(x) = \begin{cases} V_e, & x \le \lambda_e \\ 0, & x > \lambda_e \end{cases} \ . \quad (3.7)$$

$V_e$, the strength of the effective potential, and $\lambda_e$, the spatial extent of the effective potential, are the only parameters in this theory. We shall pick $\lambda_e = 2\lambda$, $0 \le \Delta$, $V_e \sim HK_y$; this is not unique, but seems reasonable to us. If the results are suspects, a many-step approximation to the expo-

nential can be tried; it will not be so sensitive to parameter choices.

In order to solve ordinary constant-coefficient differential equations, we must form solutions in each region and match values and appropriate derivatives across the boundaries; at the $x=0$ surface, and at the $x=L_x \gg \lambda$ surface. Take first the problem at the $x=0$ surface.

At the surface of the metal, the current normal to the surface must vanish. However, $j_x \sim u\partial u/\partial x + v\partial v/\partial x$, so either $u$ and $v$ both vanish at $x=0$, or $\partial u/\partial x$ and $\partial v/\partial x$ vanish there, or both do. To pick the proper set, consider the electron density at the surface $N$: $N \sim |u|^2 + |v|^2$. Since the electrons are principally confined to within an angstrom or so of the surface, we imagine our system confined by a large potential barrier at $x=0$, whence $N \to 0 \Rightarrow u,\ v \to 0$ at $x=0$. We thus consider the boundary conditions $u,\ v=0$, while $\partial u/\partial x,\ \partial v/\partial x \neq 0$.

Denote the solutions of Eqs. (3.5a) and (3.5b) in the surface region $x \leq \lambda_e$ where the effective potential is nonzero generally by the subscript 1, and the solutions in the metal's interior $x > \lambda_e$ by the subscript 2. Solutions in each region are of the form $e^{iqx}$. Then straightforward substitution in Eqs. (3.5a) and (3.5b) yields the following general solutions:

$$u_1(x) = A_1(e^{ir_+x} - e^{-ir_+x}) + C_1(e^{ir_-x} - e^{-ir_-x}) \quad , \tag{3.8a}$$

$$v_1(x) = A_1(\epsilon - \xi_r - V_e)(e^{ir_+x} - e^{-ir_+x}) + C_1(\epsilon + \xi_r - V_e)(e^{ir_-x} - e^{-ir_-x}) \quad , \tag{3.8b}$$

$$u_2(x) = A_2 e^{ip_+x} + B_2 e^{-ip_+x} + C_2 e^{ip_-x} + D_2 e^{-ip_-x} \quad , \tag{3.8c}$$

$$v_2(x) = A_2(\epsilon - \xi_p)e^{ip_+x} + B_2(\epsilon - \xi_p)e^{-ip_+x} + C_2(\epsilon + \xi_p)e^{ip_-x} + D_2(\epsilon + \xi_p)e^{-ip_-x} \quad , \tag{3.8d}$$

where

$$r_\pm/K_F = [(-\xi_t \pm \xi_r)/E_F]^{1/2} \quad , \tag{3.9}$$

$$p_\pm/K_F = [(-\xi_t + \xi_p)/E_F]^{1/2} , \tag{3.10}$$

$$\xi_r = [(\epsilon - V_e)^2 - \Delta^2]^{1/2} \quad , \tag{3.11}$$

$$\xi_p = (\epsilon^2 - \Delta^2)^{1/2} \quad , \tag{3.12}$$

and $A_1$, $C_1$, $A_2$, $B_2$, $C_2$, and $D_2$ are constants to be fixed later by imposing boundary conditions on $u$, $v$, and normalization

$$\int_0^{L_x \gg \lambda} [|u(x)|^2 + |v(x)|^2]dx = 1 \quad . \tag{3.13}$$

We have picked the solutions so that the boundary condition at $x=0$ is automatically satisfied. Matching the $u$, $v$ solutions and their derivatives at $x=\lambda_e$ yields four constraints on the coefficients $A_1$, $C_1$, $A_2$, $B_2$, $C_2$, $D_2$:

$$A_1(e^{ir_+\lambda_e} - e^{-ir_+\lambda_e}) + C_1(e^{ir_-\lambda_e} - e^{-ir_-\lambda_e}) = A_2 e^{ip_+\lambda_e} + B_2 e^{-ip_+\lambda_e} + C_2 e^{ip_-\lambda_e} + D_2 e^{-ip_-\lambda_e} \quad , \tag{3.14a}$$

$$A_1 r_+(e^{ir_+\lambda_e} + e^{-ir_+\lambda_e}) + C_1 r_-(e^{ir_-\lambda_e} + e^{-ir_-\lambda_e}) = A_2 p_+ e^{ip_+\lambda_e} - B_2 p_+ e^{-ip_+\lambda_e} + C_2 p_- e^{ip_-\lambda_e} - D_2 p_- e^{-ip_-\lambda_e} , \tag{3.14b}$$

$$A_1(\epsilon - \xi_r - V_e)(e^{ir_+\lambda_e} - e^{-ir_+\lambda_e}) + C_1(\epsilon + \xi_r - V_e) \times (e^{ir_-\lambda_e} - e^{-ir_-\lambda_e}) = A_2(\epsilon - \xi_p)e^{ip_+\lambda_e} + B_2(\epsilon - \xi_p)e^{-ip_+\lambda_e} + C_2(\epsilon + \xi_p)e^{ip_-\lambda_e} + D_2(\epsilon + \xi_p)e^{-ip_-\lambda_e} \quad , \tag{3.14c}$$

$$A_1(\epsilon - \xi_r - V_e)r_+(e^{ir_+\lambda_e} + e^{-ir_+\lambda_e}) + C_1(\epsilon + \xi_r - V_e)r_- \times (e^{ir_-\lambda_e} + e^{-ir_-\lambda_e}) = A_2(\epsilon - \xi_p)p_+ e^{ip_+\lambda_e} - B_2(\epsilon - \xi_p)p_+ e^{-ip_+\lambda_e} + C_2(\epsilon + \xi_p)p_- e^{ip_-\lambda_e} - D_2(\epsilon + \xi_p)p_- e^{-ip_-\lambda_e} \quad . \tag{3.14d}$$

Using the states given by (3.8) in the normalization equation (3.13) yields the following:

$$\int_0^{\lambda_e} [|u_1(x)|^2 + |v_1(x)|^2]dx + \int_{\lambda_e}^{\infty} [|u_2(x)|^2 + |v_2(x)|^2]dx = 1$$

implies

$$i\Delta^2 = |A_1|^2[\Delta^2 + (\epsilon - \xi_r - V_e)(\epsilon - \xi_r^* - V_e)]\left(\frac{e^{i(r_+ - r_+^*)\lambda_e} - e^{-i(r_+ - r_+^*)\lambda_e}}{r_+ - r_+^*} - \frac{e^{i(r_+ + r_+^*)\lambda_e} - e^{-i(r_+ + r_+^*)\lambda_e}}{r_+ + r_+^*}\right)$$

$$+ |C_1|^2[\Delta^2 + (\epsilon + \xi_r - V_e)(\epsilon + \xi_r^* - V_e)]\left(\frac{e^{i(r_- - r_-^*)\lambda_e} - e^{-i(r_- - r_-^*)\lambda_e}}{r_- - r_-^*} - \frac{e^{i(r_- + r_-^*)\lambda_e} - e^{-i(r_- + r_-^*)\lambda_e}}{r_- + r_-^*}\right)$$

$$+ A_1 C_1^*[\Delta^2 + (\epsilon - \xi_r - V_e)(\epsilon + \xi_r^* - V_e)]\left(\frac{e^{i(r_+ - r_-^*)\lambda_e} - e^{-i(r_+ - r_-^*)\lambda_e}}{r_+ - r_-^*} - \frac{e^{i(r_+ + r_-^*)\lambda_e} - e^{-i(r_+ + r_-^*)\lambda_e}}{r_+ + r_-^*}\right)$$

$$+ A_1^* C_1[\Delta^2 + (\epsilon - \xi_r^* - V_e)(\epsilon + \xi_r - V_e)]\left(\frac{e^{i(r_+^* - r_-)\lambda_e} - e^{-i(r_+^* - r_-)\lambda_e}}{r_+^* - r_-} - \frac{e^{i(r_+^* + r_-)\lambda_e} - e^{-i(r_+^* + r_-)\lambda_e}}{r_+ + r_-^*}\right)$$

$$+ |A_2|^2[\Delta^2 + (\epsilon - \xi_p)(\epsilon - \xi_p^*)]\frac{e^{i(p_+ - p_+^*)L_x} - e^{i(p_+ - p_+^*)\lambda_e}}{p_+ - p_+^*} + A_2 B_2^*[\Delta^2 + (\epsilon - \xi_p)(\epsilon - \xi_p^*)]\frac{e^{i(p_+ + p_+^*)L_x} - e^{i(p_+ + p_+^*)\lambda_e}}{p_+ + p_+^*}$$

$$+A_2C_2^*[\Delta^2+(\epsilon-\xi_p)(\epsilon+\xi_p^*)]\frac{e^{i(p_+-p_-^*)L_x}-e^{i(p_+-p_-^*)\lambda_e}}{p_+-p_-^*}+A_2D_2^*[\Delta^2+(\epsilon-\xi_p)(\epsilon+\xi_p^*)]\frac{e^{i(p_++p_-^*)L_x}-e^{i(p_++p_-^*)\lambda_e}}{p_++p_-^*}$$

$$+B_2A_2^*[\Delta^2+(\epsilon-\xi_p)(\epsilon-\xi_p^*)]\frac{e^{-i(p_++p_+^*)L_x}-e^{-i(p_++p_+^*)\lambda_e}}{-p_+-p_+^*}+|B_2|^2[\Delta^2+(\epsilon-\xi_p)(\epsilon-\xi_p^*)]\frac{e^{i(p_+^*-p_+)L_x}-e^{i(p_+^*-p_+)\lambda_e}}{p_+^*-p_+}$$

$$+B_2C_2^*[\Delta^2+(\epsilon-\xi_p)(\epsilon+\xi_p^*)]\frac{e^{-i(p_++p_-^*)L_x}-e^{-i(p_++p_-^*)\lambda_e}}{-p_+-p_-^*}+B_2D_2^*[\Delta^2+(\epsilon-\xi_p)(\epsilon+\xi_p^*)]\frac{e^{i(p_-^*-p_+)L_x}-e^{i(p_-^*-p_+)\lambda_e}}{p_-^*-p_+}$$

$$+C_2A_2^*[\Delta^2+(\epsilon+\xi_p)(\epsilon-\xi_p^*)]\frac{e^{i(p_--p_+^*)L_x}-e^{i(p_--p_+^*)\lambda_e}}{p_--p_+^*}+C_2B_2^*[\Delta^2+(\epsilon+\xi_p)(\epsilon-\xi_p^*)]\frac{e^{i(p_-+p_+^*)L_x}-e^{i(p_-+p_+^*)\lambda_e}}{p_-+p_+^*}$$

$$+|C_2|^2[\Delta^2+(\epsilon+\xi_p)(\epsilon+\xi_p^*)]\frac{e^{i(p_--p_-^*)L_x}-e^{i(p_--p_-^*)\lambda_e}}{p_--p_-^*}+C_2D_2^*[\Delta^2+(\epsilon+\xi_p)(\epsilon+\xi_p^*)]\frac{e^{i(p_-+p_-^*)L_x}-e^{i(p_-+p_-^*)\lambda_e}}{p_-+p_-^*}$$

$$+D_2A_2^*[\Delta^2+(\epsilon+\xi_p)(\epsilon-\xi_p^*)]\frac{e^{-i(p_-+p_+^*)L_x}-e^{-i(p_-+p_+^*)\lambda_e}}{-p_--p_+^*}+D_2B_2^*[\Delta^2+(\epsilon+\xi_p)(\epsilon-\xi_p^*)]\frac{e^{i(p_+^*-p_-)L_x}-e^{i(p_+^*-p_-)\lambda_e}}{p_+^*-p_-}$$

$$+D_2C_2^*[\Delta^2+(\epsilon+\xi_p)(\epsilon+\xi_p)]\frac{e^{-i(p_-+p_-^*)L_x}-e^{-i(p_-+p_-^*)\lambda_e}}{-p_--p_-^*}+|D_2|^2[\Delta^2+(\epsilon+\xi_p)(\epsilon+\xi_p^*)]\frac{e^{i(p_-^*-p_-)L_x}-e^{i(p_-^*-p_-)\lambda_e}}{p_-^*-p_-} \tag{3.15}$$

In general, not all of the constants $A_2$, $B_2$, $C_2$, and $D_2$ can be simultaneously nonzero. This arises upon considering the $u$, $v$ solutions (3.8) for $x\to\infty$. If $p_+$ and/or $p_-$ has a nonzero imaginary part, the coefficient of the appropriate term must be set to zero for the $u$, $v$ solutions to be bounded as $x\to\infty$.

Regarding $\epsilon$ (the excitation energy), $\xi_t$ (a measure of transverse momentum), and $V_e$ (the effective potential strength) as the independent state-naming variables, we consider the consequences of the following possibilities for

$$\frac{p_\pm}{k_F}=\left(\frac{-\xi_t\pm(\epsilon^2-\Delta^2)^{1/2}}{E_F}\right)^{1/2}.$$

*1.* $0\le\epsilon<\Delta$, *any* $\xi$, *and* $\epsilon\ge\Delta$, $-\xi_t+(\epsilon^2-\Delta^2)^{1/2}<0$

For this case, both $p_+$ and $p_-$ have imaginary parts so that, necessarily, $B_2=D_2=0$; but then Eqs. (3.14) are inconsistent unless the determinant of the coefficient vanishes. If $\xi_t$ and $V_e$ are fixed, there exist solutions only for certain discrete values of $\epsilon$ given by the vanishing of the determinant here.

Since $p_\pm$ are complex, these states are localized near the surface as determined by $e^{ip_\pm x}$ in $u$, $v$. Generally the distances here are ~5000 Å into the sample. BCS[11] used infinite metals so, of course, they could not obtain any solutions where $p_+$ or $p_-$ had any imaginary part. Furthermore, these surface states exist for excitation energies less than the gap width $\Delta$, a fact which will have important consequences for absorption at low temperatures ($kT\ll\Delta$).

Equations (3.14) are sufficiently complicated that results can only be obtained numerically. We have examined, therefore, many cases numerically using a computer and find a few general features to be described below.

First, surface states exist only for $V_e<0$ and only for $\epsilon>\Delta-|V_e|$; i.e., $V_e$ must appear to be a potential well and the possible states must lie above the bottom of the well. The preceding is true irrespective of how $V_e$ and $\lambda_e$ are chosen.

Second, surface states exist only for $-20\Delta\lesssim\xi_t\lesssim\Delta$, which depends on $\lambda_e$. The small values of $\xi_t$ indicate a particular direction of quasiparticle travel to be discussed later. The spectrum used in further calculations is shown in Fig. 1.

It might be thought that these surface states depend critically on the surface at $x=0$ being perfectly flat and smooth. However, as long as the surface irregularities are small compared to a penetration depth of the $H_{dc}$, this is not so. For then one could make wave packets of sinusoids which would match the boundary conditions, out of sinusoids whose momenta were all peaked sharply about the correct value for a flat surface. In such a case, all the surface-state energies would still be nearly the same as before. Thus small scale ($\lesssim 10$ Å) irregularities are no problem.

*2.* $-\xi_t\pm(\epsilon^2-\Delta^2)^{1/2}>0$, $\epsilon\le\Delta$

In this realm both $p_+$ and $p_-$ are purely real, and all the constants $A_2$, $B_2$, $C_2$, and $D_2$ can be nonzero, which leaves a dilemma. There are more unknown constants than constraining equations. Two approaches are possible to resolve the problem. We can put the system in a large box of length $L_x$ and require $u$, $v$ to vanish at $x=L_x$ just

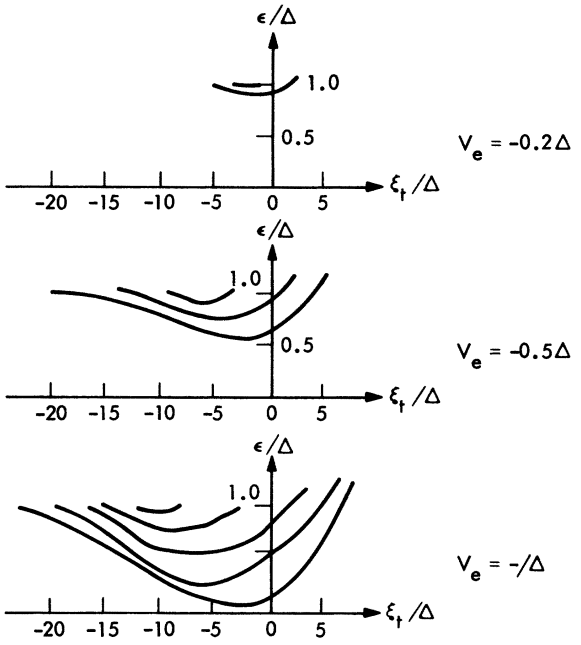


FIG. 1. Typical surface-state spectrum; $\lambda_e = 2\lambda = 1000$ Å and $\Delta/E_f = 10^{-4}$.

as we imposed for $u$, $v$ at $x=0$ surface. This yields a situation like the prior case—solutions exist only if the determinant of the coefficients vanishes. However, since $L_x$ is very large these states are very closely spaced in energy and for $L_x \to \infty$ form a continuum. All this is well defined, but extremely difficult to handle numerically. Clearly the results should be essentially independent of the size of the box for $L_x \gg \lambda$, and yet in solving for the $u$, $v$ functions and the locations of the states in energy, slight changes in $L_x$ radically affect the numbers involved through vast oscillating terms $\sim e^{ip_\pm L_x}$.

Fortunately another approach is possible that is more easily implemented. We imagine a scattering process. If $e^{ipx}$ represents a quasiparticle incident upon the $x=0$ surface from deep inside the metal in the state with $x$-component momentum $p$, then $e^{-ipx}$ is the reflected quasiparticle; the transverse momentum is conserved, and the momentum normal to the surface reverses sign upon collision. Thus it is natural to consider two cases. A quasiparticle can be incident upon the surface in either state $p_+$ or $p_-$. By a change of variables to spherical momentum coordinates, it can be shown that $p_+$ states lie above the Fermi momentum, while $p_-$ states lie below the Fermi momentum. For each case, there is some amplitude for the quasiparticle to be reflected in its original state with only the direction altered, and there is some amplitude to find the quasiparticle in the other momentum state also with altered direction. This is a new feature of a superconductor when $H_{dc} \neq 0$. In BCS theory where $H_{dc} = 0$, a quasiparticle is always reflected in the same state as it was incident; no mixing of $p_+$ and $p_-$ states occurs. Regarding $\xi_t$, $\epsilon$, and $V_e$ as the independent variables of a state, there are two degenerate solutions which we will refer to as "$p_+$-incident" and "$p_-$-incident" continuum states. In the former, $D_2 = 0$, and in the latter, $B_2 = 0$.

The above picture does not depend upon the type of particle being scattered. They can be either electronlike or holelike excitations; indeed, our argument applies equally well to both $u(x)$ and $v(x)$ solutions. Holes or antiparticle quasiparticles exist both above and below the Fermi momentum, and electrons or particle quasiparticles do so, too. This latter fact holds in superconductors whether $H_{dc}$ is zero or not. For normal metals, though, we have $u \sim e^{\pm ip_- x}$, which says that electron excitations exist only above the Fermi momentum, while hole excitations exist only below the Fermi momentum.

For these plane-wave-like scattering states, the normalization condition (3.15) is particularly simple; in the limit of $L_x \to \infty$, only diagonal terms in region 2 contribute substantially. Thus for $p_+$ incident states,

$$(|A_2|^2 + |B_2|^2)[\Delta^2 + (\epsilon - \xi_p)^2] + |C_2|^2[\Delta^2 + (\epsilon + \xi_p)^2] = \Delta^2/L_x \ , \quad (3.16a)$$

and for $p_-$ incident states,

$$|A_2|^2[\Delta^2 + (\epsilon - \xi_p)^2] + (|C_2|^2 + |D_2|^2) \times [\Delta^2 + (\epsilon + \xi_p)^2] = \Delta^2/L_x \ . \quad (3.16b)$$

The sum over states is also simple:

$$\sum \to \frac{L_x}{2\pi} \sum_{+,-} \int_{-\infty}^{\infty} dp_\pm \ , \quad (3.17)$$

which will later be transformed to a more useful independent variable.

3. $\epsilon \geq \Delta, -\xi_t - (\epsilon^2 - \Delta^2)^{1/2} < 0, -\xi_t + (\epsilon^2 - \Delta)^{1/2} > 0$

In this realm $p_+$ is purely real, so $e^{\pm ip_+ x}$ are perfectly acceptable solutions, while $p_-$ is complex and hence $e^{-ip_- x}$ must be discarded, so $D_2 = 0$. BCS also could not have considered this case, since it depends on a surface being present and a nonzero $V_e$.

This case is very similar to the $p_+$ scattering state considered except that for $p_+$ incident, the $p_-$ reflected solution is very strongly localized near the surface, so all can be applied except that the normalization condition simplifies even more than before to

$$(|A_2|^2+|B_2|^2)[\Delta^2+(\epsilon-\xi_p)^2]=\Delta^2/L_x \quad . \qquad (3.18)$$

Since the preceding derivation is rather mathematical, a more intuitive picture is desirable. According to BCS theory, the superconducting ground state is made up of overlapping paired electrons. Each pair has an average length, coherence distance $\xi_0$, and binding energy $2\Delta$. In this model we might imagine that static magnetic fields penetrating the surface cause the following modifications: First, consider pairs far from the surface and the static field. They feel no fields, so their pairing should be unaltered from the field-free case. However, pairs near the surface are quite a different matter, since the electron-electron attraction via the phonons can be perturbed by the magnetic field. Now for typical Pippard superconductors, penetration depths are much less than pair lengths. Hence pairs aligned perpendicular to the surface have only a small fraction of their lengths in the surface region where the field is strong. Accordingly, their binding energies are essentially unaltered. But pairs that are very nearly parallel to the surface ($|\xi_t| \lesssim 10\Delta$) have a field acting over their entire length. These pairs, the most important ones for interaction with radiation, cannot maintain their field-free status. If they are to remain at the surface, they must "quantize" their binding energies (surface states) or be expelled from the surface region to the model's interior where no fields allow $2\Delta$ binding energy (continuum states with small amplitudes in the surface region).

## IV. FORMULAS FOR SURFACE IMPEDANCE

In order to calculate surface impedances, it is necessary to solve Maxwell's equations. In the $\nabla\cdot\vec{A}=0$ gauge, Maxwell's equations for the vector potential read

$$\nabla^2\vec{A}+\omega^2\vec{A}=-4\pi\vec{j}(\vec{A}) \quad , \qquad (4.1a)$$

$$\vec{H}=\nabla\times\vec{A} \quad , \qquad (4.1b)$$

$$\vec{E}=-i\omega\vec{A} \quad , \qquad (4.1c)$$

for sinusoidally varying fields, $A\sim e^{i\omega t}$. The displacement current is negligible in a metal for frequencies of interest, since $\omega^2 A \ll \nabla^2 A$, so it can be ignored.

From the work in Sec. II, we know that $\vec{j}(\vec{A})$ is of the form $\vec{j}(\vec{r})=\int Q(\vec{r},\vec{r}')\vec{A}(\vec{r}')\,d\vec{r}'$. Thus for the planar geometry at hand, Eq. (4.1a) becomes

$$\frac{\partial^2 A(x)}{\partial x^2}=-4\pi\int_0^\infty Q(x,x')A(x')\,dx' \quad , \qquad (4.2)$$

an integrodifferential equation very difficult to solve. The only cases that have been solved are (i) the normal metal at room temperature where $Q(x,x')\sim\delta(x-x')$, and (ii) the anomalous skin effect in normal metals and homogeneous type-I superconductors, where $Q(x,x')\sim Q(x-x')$.

The surface impedance $Z$ is defined by

$$Z\equiv 4\pi\frac{\vec{E}(0)}{\int\vec{j}\,d\vec{r}}=-i4\pi\omega\frac{A(0)}{(\partial A/\partial x)|_0} \quad , \qquad (4.3)$$

where the second form applies to our geometry. $|_0$ means evaluated at $x=0^+$, just inside the surface.

It is too difficult to get $Q(x,x')$, so we must develop some other approach for finding $Z$. First, multiply both sides of Eq. (4.2) by $A^*(x)$ and integrate over $x$. Then do $\int_0^\infty A^*(\partial^2A/\partial x^2)\,dx$, integrating by parts. Eliminating $\int_0^\infty A^*(\partial^2A/\partial x^2)\,dx$ and substituting for $Z^*$, we find

$$Z^*=\frac{i4\pi\omega}{|(\partial A/\partial x)|_0|^2}\left(4\pi\int_0^\infty Q(x,x')A^*(x)A(x')\,dx'\,dx-\int_0^\infty\left|\frac{\partial A}{\partial x}\right|^2 dx\right) \quad . \qquad (4.4)$$

We shall now take the following new point of view in calculating surface impedances. Approximate the $A$ field distribution by an exponential $e^{-x/\delta}$, with one parameter $\delta$, just as for the static field distribution. ($\delta$ is a complex number and a function of $\vec{H}_{dc}$ and $\omega$!) As we have a simple explicit form for $A(x)$, we can do the spatial integrations first in Eq. (4.4) before any other operations, and hence determine $Z$ without knowing $A(x,x')$. What is $\delta$? We obtain $\delta$ by requiring that the $Z^*$ calculated from Eq. (4.4) be consistent with the $Z^*$ found from Eq. (4.3), which is

$$Z^*=-i4\pi\omega\delta \quad . \qquad (4.5)$$

If the self-consistent $\delta$ is found, we trivially know the surface impedance.

The $\mathrm{Re}(Z)=R$ is called the surface resistance and the $\mathrm{Im}(Z)=X$ is the surface reactance. $R$ is a measure of power dissipated by the metal and $X$ is roughly similar to a field-penetration depth.

For superconductors with $T<0.9T_c$ and $\omega<2\Delta$, we have $R\ll X$, so that $\mathrm{Im}(\delta)\ll\mathrm{Re}(\delta)$. As a consequence, it is really necessary to be concerned with self-consistency only for the surface reactance. Once the correct self-consistent $\mathrm{Re}(\delta)$ is known, we can calculate directly the surface resistance from Eq. (4.4) using for $\delta$ the $\mathrm{Re}(\delta)$. Actually, the case for our problem is even better as far as surface resistance is concerned. The changes in surface resistance due to static magnetic fields are so large ($R_{H_{dc}\neq 0}/R_{H_{dc}=0}>1$) and the changes in surface reactance sufficiently small ($\sim 10\%$), that using $\mathrm{Re}(\delta)$ for $H_{dc}=0$ is adequate to obtain answers both within 20% and of the right sign. This will become more obvious after further formal developments. The point here is that

maintaining self-consistency is less of a problem than might be expected initially.

We now continue to evaluate Eq. (4.4). In Sec. II we split $Q$ into a paramagnetic term $Q_1$ and a diamagnetic term $-Ne^2/m$. For the latter, the integrations are immediately done, and one finds that

$$Z^* = i\,16\pi^2\omega\,|\delta|^2\int_0^\infty dx\,dx'\,e^{-x/\delta^*}e^{-x'/\delta}Q_1(x,x') - i4\pi\omega\,\frac{|\delta|^2}{\delta+\delta^*}\left(\frac{4e^2K_F^3}{3\pi m} - \frac{1}{|\delta|^2}\right) . \quad (4.6)$$

Hence, the diamagnetic term contributes only to the surface reactance. (The two last terms are both important.)

The paramagnetic contribution is proportional to $\int_0^\infty dx\,dx'\,e^{-x/\delta^*}e^{-x'/\delta}Q_1(x,x')$. Referring to Eq. (2.10b), we see that this integral is composed of the sum of the square of the magnitude of two quantum-mechanical matrix elements:

$$M_{lm}^a = \int_0^\infty [u_l(x)u_m^*(x) + v_l(x)v_m^*(x)]e^{-x/\delta^*}\,dx , \quad (4.7a)$$

$$M_{lm}^b = \int_0^\infty [u_l(x)v_m(x) - u_m(x)v_l(x)]e^{-x/\delta^*}\,dx . \quad (4.7b)$$

Note that $M_{ml}^b = -M_{lm}^b$. $M_{lm}^a$ is the matrix element for processes scattering a quasiparticle from one state $l$ to another state $m$. $M_{lm}^b$ is the matrix element for destruction of a quasiparticle and its antiquasiparticle from states $l$ and $m$. $M_{lm}^{b*}$ is the same as before, except that it now refers to creation of a pair.

Below we calculate $M_{12}^a = \int_0^\infty [u_1(x)u_2^*(x) + v_1(x)v_2^*(x)]e^{-x/\delta^*}\,dx$ for the most general $u, v$'s:

$$\begin{aligned}
M_{12}^a = \{&A_{11}A_{12}^*[\Delta^2 + (\epsilon_1 - \xi_{r_1} - V_e)(\epsilon_2 - \xi_{r_2}^* - V_e)]I_1(r_{+1}, r_{+2}) + C_{11}C_{12}^*[\Delta^2 + (\epsilon_1 + \xi_{r_1} - V_e)(\epsilon_2 + \xi_{r_2}^* - V_e)]I_1(r_{-1}, r_{-2})\\
&+A_{11}C_{12}^*[\Delta^2 + (\epsilon_1 - \xi_{r_1} - V_e)(\epsilon_2 + \xi_{r_2}^* - V_e)]I_1(r_{+1}, r_{-2}) + C_{11}A_{12}^*[\Delta^2 + (\epsilon_1 + \xi_{r_1} - V_e)(\epsilon_2 - \xi_{r_2}^* - V_e)]I_1(r_{-1}, r_{+1})\\
&+A_{21}A_{22}^*[\Delta^2 + (\epsilon_1 - \xi_{p_1})(\epsilon_2 - \xi_{p_2}^*)]I_2(p_{+1}, p_{+2}) + C_{21}C_{22}^*[\Delta^2 + (\epsilon_1 + \xi_{p_1})(\epsilon_2 + \xi_{p_2}^*)]I_2(p_{-1}, p_{-2})\\
&+A_{21}C_{22}^*[\Delta^2 + (\epsilon_1 - \xi_{p_1})(\epsilon_2 + \xi_{p_2}^*)]I_2(p_{+1}, p_{-2}) + C_{21}A_{22}^*[\Delta^2 + (\epsilon_1 + \xi_{p_1})(\epsilon_2 - \xi_{p_2}^*)]I_2(p_{-1}, p_{+2})\\
&+B_{21}B_{22}^*[\Delta^2 + (\epsilon_1 - \xi_{p_1})(\epsilon_2 - \xi_{p_2}^*)]I_2(-p_{-1}, -p_{+2}) + D_{21}D_{22}^*[\Delta^2 + (\epsilon_1 + \xi_{p_1})(\epsilon_2 + \xi_{p_2}^*)]I_2(-p_{-1}, -p_{-2})\\
&+B_{21}D_{22}^*[\Delta^2 + (\epsilon_1 - \xi_{p_1})(\epsilon_2 + \xi_{p_2}^*)]I_2(-p_{+1}, -p_{-2}) + D_{21}B_{22}^*[\Delta^2 + (\epsilon_1 + \xi_{p_1})(\epsilon_2 - \xi_{p_2}^*)]I_2(-p_{-1}, -p_{+2})\\
&+A_{21}B_{22}^*[\Delta^2 + (\epsilon_1 - \xi_{p_1})(\epsilon_2 - \xi_{p_2}^*)]I_2(p_{+1}, -p_{+2}) + A_{21}D_{22}^*[\Delta^2 + (\epsilon_1 - \xi_{p_1})(\epsilon_2 + \xi_{p_2}^*)]I_2(p_{+1}, -p_{-2})\\
&+B_{21}A_{22}^*[\Delta^2 + (\epsilon_1 - \xi_{p_1})(\epsilon_2 - \xi_{p_2}^*)]I_2(-p_{+1}, p_{+2}) + B_{21}C_{22}^*[\Delta^2 + (\epsilon_1 - \xi_{p_1})(\epsilon_2 - \xi_{p_2}^*)]I_2(-p_{+1}, p_{-2})\\
&+C_{21}B_{22}^*[\Delta^2 + (\epsilon_1 + \xi_{p_1})(\epsilon_2 - \xi_{p_2}^*)]I_2(p_{-1}, -p_{+2}) + C_{21}D_{22}^*[\Delta^2 + (\epsilon_1 + \xi_{p_1})(\epsilon_2 + \xi_{p_2}^*)]I_2(p_{-1}, -p_{-2})\\
&+D_{21}A_{22}^*[\Delta^2 + (\epsilon_1 + \xi_{p_1})(\epsilon_2 - \xi_{p_2}^*)]I_2(-p_{-1}, p_{+2}) + D_{21}C_{22}^*[\Delta^2 + (\epsilon_1 + \xi_{p_1})(\epsilon_2 + \xi_{p_2}^*)]I_2(-p_{-1}, p_{-2})\}/\Delta^2 , \quad (4.8)
\end{aligned}$$

where

$$I_1(r_1, r_2) = \frac{(2/\delta^*) - ie^{-\lambda_e/\delta^*}\{[r_1 - r_2^* - (i/\delta^*)]e^{i(r_1 - r_2^*)\lambda_e} - [r_1 - r_2^* + (i/\delta^*)]e^{-i(r_1 - r_2^*)\lambda_e}\}}{(1/\delta^*)^2 + (r_1 - r_2^*)^2}$$

$$- \frac{(2/\delta^*) - ie^{-\lambda_e/\delta^*}\{[r_1 + r_2^* - (i/\delta^*)]e^{i(r_1 + r_2^*)\lambda_e} - [r_1 + r_2^* + (i/\delta^*)]e^{-i(r_1 + r_2^*)\lambda_e}\}}{(1/\delta^*)^2 + (r_1 + r_2^*)^2} , \quad (4.9a)$$

$$I_2(p_1, p_2) = i\,\frac{e^{i[p_1 - p_2^* + (i/\delta^*)]\lambda_e}}{p_1 - p_2^* + i/\delta^*} . \quad (4.9b)$$

Next, we calculate $M_{lm}^b = \int_0^\infty [u_l(x)v_m(x) - u_m(x)v_l(x)]e^{-x/\delta^*}\,dx$ for the most general $u$, $v$'s:

$$\begin{aligned}
M_{lm}^b = \{&A_{1l}A_{1m}[(\epsilon_m - \xi_{r_m}) - (\epsilon_l - \xi_{r_l})]I_3(r_{+l}, r_{+m}) + C_{1l}C_{1m}[(\epsilon_m + \xi_{r_m}) - (\epsilon_l + \xi_{r_l})]I_3(r_{-l}, r_{-m})\\
&+A_{1l}C_{1m}[(\epsilon_m + \xi_{r_m}) - (\epsilon_l - \xi_{r_l})]I_3(r_{+l}, r_{-m}) + C_{1l}A_{1m}[(\epsilon_m - \xi_{r_m}) - (\epsilon_l - \xi_{r_l})]I_3(r_{-l}, r_{+m})\\
&+A_{2l}A_{2m}[(\epsilon_m - \xi_{p_m}) - (\epsilon_l - \xi_{p_l})]I_4(p_{+l}, p_{+m}) + C_{2l}C_{2m}[(\epsilon_m + \xi_{p_m}) - (\epsilon_l + \xi_{p_l})]I_4(p_{-l}, p_{-m})\\
&+A_{2l}C_{2m}[(\epsilon_m + \xi_{p_l}) - (\epsilon_l - \xi_{p_l})]I_4(p_{+l}, p_{-m}) + C_{2l}A_{2m}[(\epsilon_m - \xi_{p_m}) - (\epsilon_l + \xi_{p_l})]I_4(p_{-l}, p_{+m})\\
&+B_{2l}B_{2m}[(\epsilon_m - \xi_{p_m}) - (\epsilon_l - \xi_{p_l})]I_4(p_{+l}, -p_{+m}) + D_{2l}D_{2m}[(\epsilon_m + \xi_{p_m}) - (\epsilon_l + \xi_{p_l})]I_4(-p_{-l}, -p_{-m})\\
&+B_{2l}D_{2m}[(\epsilon_m + \xi_{p_m}) - (\epsilon_l - \xi_{p_l})]I_4(-p_{+l}, -p_{-m}) + D_{2l}B_{2m}[(\epsilon_m - \xi_{p_m}) - (\epsilon_l + \xi_{p_l})]I_4(-p_{-l}, -p_{+m})\\
&+A_{2l}B_{2m}[(\epsilon_l - \xi_{p_l}) - (\epsilon_m - \xi_{p_m})]I_4(p_{+l}, -p_{+m}) + A_{2l}D_{2m}[(\epsilon_l - \xi_{p_l}) - (\epsilon_m + \xi_{p_m})]I_4(p_{+l}, -p_{-m})
\end{aligned}$$

$$+B_{2l}A_{2m}[(\epsilon_l-\xi_{p_l})-(\epsilon_m-\xi_{p_m})]I_4(-p_{+l},p_{+m})+B_{2l}C_{2m}[(\epsilon_l-\xi_{p_l})-(\epsilon_m+\xi_{p_m})]I_4(-p_{+l},p_{-m})$$
$$+C_{2l}B_{2m}[(\epsilon_l+\xi_{p_l})-(\epsilon_m-\xi_{p_m})]I_4(p_{-l},-p_{+m})+C_{2l}D_{2m}[(\epsilon_l+\xi_{p_l})-(\epsilon_m+\xi_{p_m})]I_4(p_{-l},-p_{-m})$$
$$+D_{2l}A_{2m}[(\epsilon_l+\xi_{p_l})-(\epsilon_m-\xi_{p_m})]I_4(-p_{-l},p_{+m})+D_{2l}C_{2m}[(\epsilon_l+\xi_{p_l})-(\epsilon_m+\xi_{p_m})]I_4(-p_{-l},p_{-m})\}/\Delta\ , \quad (4.10)$$

$$I_4(p_1,p_2)=i\,\frac{e^{i[p_1+p_2+(i/\delta^*)]\lambda_e}}{p_1+p_2+i/\delta^*}\ , \quad (4.11a)$$

$$I_3(r_1,r_2)=\frac{2/\delta^*-ie^{-\lambda_e/\delta}[(r_1+r_2-i/\delta^*)e^{i(r_1+r_2)\lambda_e}-(r_1+r_2+i/\delta^*)e^{-i(r_1+r_2)\lambda_e}]}{(1/\delta^*)^2+(r_1+r_2)^2}$$
$$-\frac{2/\delta^*-ie^{-\lambda_e/\delta}[(r_1-r_2-i/\delta^*)e^{i(r_1-r_2)\lambda_e}-(r_1-r_2+i/\delta^*)e^{-i(r_1+r_2)\lambda_e}]}{(1/\delta^*)^2+(r_1-r_2)^2}\ . \quad (4.11b)$$

In terms of these matrix elements $Z^*$ is just

$$Z^*=i32\pi^2\,\frac{e^2}{m^2}\,\omega|\delta|^2\sum_{l,m}(K_y^l\cos\theta+K_z^l\sin\theta)^2\left[[f_T(\epsilon_l)-f_T(\epsilon_m)]\,\frac{|M_{lm}^a|^2}{\epsilon_m-\epsilon_l-\omega+i\delta}\right.$$
$$\left.+\frac{1}{2}[1-f_T(\epsilon_l)-f_T(\epsilon_m)]|M_{lm}^b|^2\left(\frac{1}{\epsilon_m+\epsilon_l+\omega-i\delta}+\frac{1}{\epsilon_m+\epsilon_l-\omega+i\delta}\right)\right]-i4\pi\omega\,\frac{|\delta|^4}{\delta+\delta^*}\left(\frac{4e^2K_F^3}{3\pi m}-\frac{1}{|\delta|^2}\right)\ . \quad (4.12)$$

The $u$, $v$ amplitudes are invariant under the change $K_z\to -K_z$. Thus $\int_{-\infty}^{\infty}dK_z^l$ contained in $\sum_l$ causes the field-particle coupling cross term $K_y^lK_z^l\cos\theta\sin\theta$ contribution to vanish in Eq. (4.12). This just expresses an obvious symmetry—the surface impedance is the same for field linear polarization of $\pm\theta$ about the direction of $A_{dc}$. Because of this,

$$(K_y^l\cos\theta+K_z^l\sin\theta)^2\to(K_y^l)^2\cos^2\theta+(K_z^l)^2\sin^2\theta\ ,$$

and we can derive the complete angular dependence of the surface impedance by calculating $Z$ for just two angles, 0 and $\frac{1}{2}\pi$; i.e.,

$$Z(\theta)=Z(\theta=0)\cos^2\theta+Z(\theta=\tfrac{1}{2}\pi)\sin^2\theta\ . \quad (4.13)$$

It should be noted that this relation is independent of the long series of approximations made above. We could have derived this long ago.

The sum over states, $\sum_{l,m}$, in Eq. (4.12) is actually more than a double sum; this is so, since a surface state requires three numbers to fix it—$\xi_t$, $V_e$, and the particular branch number—while a continuum state requires four numbers—$\xi_t$, $V_e$, $\epsilon$, and $p_\pm$ incident. In any case, it is always possible to pull out from the sum

$$\sum_{K_y^l,K_z^l}\to\int_{-\infty}^{\infty}\frac{dK_y^l}{2\pi}\int_{-\infty}^{\infty}\frac{dK_z^l}{2\pi}$$

and express these integrals in terms of $\xi_t$ and $\varphi(V_e=V_0\cos\varphi)$, which are useful in subsequent work. This is trivially done using the definitions of $\xi_t$ and $\varphi$ in Sec. III, and the result is

$$\sum_{l,m}\to\frac{m}{2\pi^2}\int_{-E_F}^{\infty}d\xi_t\int_0^{\pi}d\varphi\sum_{l,m}\ ,$$

where $\sum_{l,m}$ on the right-hand side refers to the remaining variables necessary to specify the state considered. If only surface states are considered, then $\int_0^{\pi}d\varphi\to\int_{\pi/2}^{\pi}d\varphi$, since there are no surface states for $V_e>0$. Putting all this together, we find $Z^*$:

$$Z^*=i\,\frac{16e^2\omega|\delta|^2K_F^2}{m}\int_{-E_F}^{\infty}d\xi_t\int_0^{\pi}d\varphi\sum_{l,m}\left(1+\frac{\xi_t}{E_F}\right)(\cos^2\theta\cos^2\varphi+\sin^2\theta\sin^2\varphi)\left[[f_T(\epsilon_l)-f_T(\epsilon_m)]|M_{lm}^a|^2\right.$$
$$\left.\times\frac{1}{\epsilon_m-\epsilon_l-\omega+i\delta}+\frac{1}{2}[1-f_T(\epsilon_l)-f_T(\epsilon_m)]|M_{lm}^b|^2\left(\frac{1}{\epsilon_m+\epsilon_l+\omega-i\delta}+\frac{1}{\epsilon_m+\epsilon_l-\omega+i\delta}\right)\right]$$
$$-i4\pi\omega\,\frac{|\delta|^4}{\delta+\delta^*}\left(\frac{4e^2K_F^3}{3\pi m}-\frac{1}{|\delta|^2}\right)\ . \quad (4.14)$$

For processes involving surface states, it is most useful to switch from the integration variable $\varphi$ to the variable $V_e$. The substitution made is

$$\frac{1}{V_0^2}\int_{-V_0}^{0}dV_e\left(\cos^2\theta\,\frac{V_e^2}{(V_0^2-V_e^2)^{1/2}}\right.$$

$$+\sin^2\theta(V_0^2 - V_e^2)^{1/2}\Big) \quad . \tag{4.15}$$

Finally, for continuum states, the sum over states is just

$$\sum_{+,-} \int_{-\infty}^{\infty} \frac{dp_{\pm}}{2\pi}$$

assuming unit volume. $\sum_{+,-}$ means the sum over $p_+$ and $p_-$ incident states. However, the more useful variable is $\epsilon$. Thus

$$\sum_l \to \frac{1}{\pi} \sum_{+,-} \int_{\Delta}^{\infty} \frac{\partial p_{\pm}^l}{\partial \epsilon_l}\, d\epsilon_l \tag{4.16}$$

can be used in Eq. (4.14) whenever considering continuum states.

Assembling all these bits and pieces, we can now write the surface-impedance contributions for the various processes in a form that can be directly evaluated numerically on a computer. Let $Z_S$ be the surface impedance of the superconductor, and $Z_N$ denote surface impedances of normal metals. Write the total $Z_S$ as $Z_{S_1} + Z_{S_2} + \cdots$. $Z_S$ will be in $\Omega$ if $E_F$ is in eV, $\delta$ is in Å, and we use the following:

$$\frac{e^2}{\hbar c}\,\frac{1}{\hbar^2 c^3} = 5.62\times 10^{-8}\ \Omega/(\text{Å eV})^2 \quad .$$

The familiar identity $1/(x+i\delta) \to \mathcal{P}(1/x) - i\pi\delta(x)$ is used in the reduction, where $\mathcal{P}$ denotes principal part of the integration.

We merely state the formulas below and discuss the consequences later.

*1. Transitions between Surface States*

We have

$$Z_{S_1}^{*} = i8\,\frac{e^2}{\hbar c}\,\frac{1}{\hbar^2 c^3}\,\frac{\omega}{\Delta}\,\frac{\Delta}{E_F}\,E_F^2|\delta|^2 \sum_{\substack{\text{br}\\ l,m}} \int_{-V_0/\Delta}^{0} d\left(\frac{V_e}{\Delta}\right) \int_{-\infty}^{\infty} d\left(\frac{\xi_t}{\Delta}\right) \Bigg\{ \cos^2\theta\, \frac{(V_e/\Delta)^2}{(V_0/\Delta)^2[(V_0/\Delta)^2 - (V_e/\Delta)^2]^{1/2}}$$

$$+ \sin^2\theta \left[\left(\frac{V_0}{\Delta}\right)^2 - \left(\frac{V_e}{\Delta}\right)^2\right]^{1/2} \Bigg\} \left[f_T\left(\frac{\epsilon_l}{\Delta}\right) - f_T\left(\frac{\epsilon_m}{\Delta}\right)\right] |M_{lm}^a|^2 \left[P\left(\frac{1}{\epsilon_m/\Delta - \epsilon_l/\Delta - \omega/\Delta}\right) - i\pi\delta\left(\frac{\epsilon_m}{\Delta} - \frac{\epsilon_l}{\Delta} - \frac{\omega}{\Delta}\right)\right], \tag{4.17a}$$

where br stands for branches.

We shall need later

$$\int_{\omega_1/\Delta}^{\omega_2/\Delta} R_{S_1}\, d\left(\frac{\omega}{\Delta}\right) \cong R_{S_1}\left(\frac{\omega_2}{\Delta} - \frac{\omega_1}{\Delta}\right) \quad ,$$

so that

$$R_{S_1} \cong \left(\frac{1}{\omega_2/\Delta - \omega_1/\Delta}\right) 8\pi\,\frac{e^2}{\hbar c}\,\frac{1}{\hbar^2 c^3}\,\frac{\Delta}{E_F}\,E_F^2|\delta|^2 \sum_{\substack{\text{br}\\ l,m}} \int_{-V_0/\Delta}^{0} d\left(\frac{V_e}{\Delta}\right) \int_{-\infty}^{\infty} d\left(\frac{\xi_t}{\Delta}\right)$$

$$\times \left\{ \cos^2\theta\, \frac{(V_e/\Delta)^2}{(V_0/\Delta)^2[(V_0/\Delta)^2 - (V_e/\Delta)^2]^{1/2}} + \sin^2\theta \left[\left(\frac{V_0}{\Delta}\right)^2 - \left(\frac{V_e}{\Delta}\right)^2\right]^{1/2} \right\}$$

$$\times \left(\frac{\epsilon_m}{\Delta} - \frac{\epsilon_l}{\Delta}\right) \left[f_T\left(\frac{\epsilon_l}{\Delta}\right) - f_T\left(\frac{\epsilon_m}{\Delta}\right)\right] |M_{lm}^a|^2 \Big|_{\omega_1 \le \epsilon_m - \epsilon_l \le \omega_2} \quad , \tag{4.17b}$$

which is further discussed in Sec. V.

We calculate the matrix element $M_{lm}^a$ for integrals (4.17) as follows: Solve the set of simultaneous equations (3.14) with $B_2 = D_2 = 0$ twice, once for $A_{1l}$, $C_{1l}$, $A_{2l}$, $C_{2l}$, and again for $A_{1m}$, $C_{1m}$, $A_{2m}$, $C_{2m}$, where $l$ and $m$ refer to two surface states with the same $\xi_t$ and $V_e$ but with different energies, $\epsilon_l$ and $\epsilon_m$. These coefficients $A_1$, $C_1$, etc. must satisfy the normalization condition (9.15). $M_{lm}^a$ is then trivially found from formula (4.8) with $B_{2l,m} = D_{2l,m} = 0$.

*2. Transition between Surface States and Continuum States*

We have

$$Z_{S_2}^{*} = i\,\frac{8}{\pi}\,\frac{e^2}{\hbar c}\,\frac{1}{\hbar^2 c^3}\,\frac{\omega}{\Delta}\,\frac{\Delta}{E_F}\,E_F^2|\delta|^2 \int_{-V_0/\Delta}^{0} d\left(\frac{V_e}{\Delta}\right) \int_{-\infty}^{\infty} d\left(\frac{\xi_t}{\Delta}\right) \left[\sum_{\substack{\text{br}\\ l}} \sum_{\substack{+,-\\ m}} \int_{1}^{\infty} \frac{\partial(p_{\pm}^m/K_F)}{\partial(\epsilon_m/\Delta)}\, d\left(\frac{\epsilon_m}{\Delta}\right)\right.$$

$$+\sum_{\substack{\mathrm{br}\\ m}}\sum_{\substack{+,-\\ l}}\int_1^\infty \frac{\partial(p_\pm^l/K_F)}{\partial(\epsilon_l/\Delta)}\,d\left(\frac{\epsilon_l}{\Delta}\right)\left[f_T\left(\frac{\epsilon_l}{\Delta}\right)-f_T\left(\frac{\epsilon_m}{\Delta}\right)\right]|M_{lm}^a|^2\left[P\left(\frac{1}{\epsilon_m/\Delta-\epsilon_l/\Delta-\omega/\Delta}\right)-i\pi\delta\left(\frac{\epsilon_m}{\Delta}-\frac{\epsilon_l}{\Delta}-\frac{\omega}{\Delta}\right)\right]$$
$$\times\left\{\cos^2\theta\,\frac{(V_e/\Delta)^2}{(V_0/\Delta)^2[(V_0/\Delta)-(V_e/\Delta)^2]^{1/2}}+\sin^2\theta\left[\left(\frac{V_0}{\Delta}\right)^2-\left(\frac{V_e}{\Delta}\right)^2\right]^{1/2}\right\}\,. \qquad (4.18)$$

Since the energy of continuum states is always higher than the energy of surface states, the $\delta$-function term contributing to the resistive part contains only the $\sum_{\mathrm{br},l}\sum_{+,-,m}\int\cdots$ terms. The reactance, though, picks up contributions from both continuum and surface states as final states in the interaction and keeps all sums.

We determined $M_{lm}^a$ for integrals (4.18) as follows: If $l$ is a surface state, then solve Eqs. (3.14) with $B_2=D_2=0$ for $A_{1l}$, $C_{1l}$, $C_{2l}$ normalized according to Eq. (3.15). The continuum states $m$ are found by solving Eqs. (3.14) first with $D_2=0$ (for $p_+$ incident scattering states) and then with $B_2=0$ (for $p_-$ incident scattering states) normalized now according to Eqs. (3.16). This determines either the nonzero set $A_{1m}$, $C_{1m}$, $A_{2m}$, $B_{2m}$, $C_{2m}$, or $A_{1m}$, $C_{1m}$, $A_{2m}$, $C_{2m}$, $D_{2m}$, respectively, and hence $M_{lm}^a$ is again fixed by expression (4.8).

### 3. *Transitions between Continuum States*

We have

$$Z_{S_3}^*=i\,\frac{8}{\pi^2}\,\frac{e^2}{\hbar c}\,\frac{1}{\hbar^2c^3}\,|\delta|^2\,\frac{\omega}{\Delta}\,\frac{\Delta}{E_F}\,E_F^2\int_{-E_F/\Delta}^0 d\left(\frac{\xi_t}{\Delta}\right)\int_0^\pi d\varphi\int_1^\infty d\left(\frac{\epsilon_l}{\Delta}\right)\int_1^\infty d\left(\frac{\epsilon_m}{\Delta}\right)$$
$$\times\sum_{\substack{+,-\\ l,m}}\frac{\partial(p_\pm^l/K_F)}{\partial(\epsilon_l/\Delta)}\,\frac{\partial(p_\pm^m/K_F)}{\partial(\epsilon_m/\Delta)}\left(1+\frac{\Delta}{E_F}\,\frac{\xi_t}{\Delta}\right)(\cos^2\theta\cos^2\varphi+\sin^2\theta\sin^2\varphi)\left[f_T\left(\frac{\epsilon_l}{\Delta}\right)-f_T\left(\frac{\epsilon_m}{\Delta}\right)\right]|M_{lm}^a|^2$$
$$\times\left[P\left(\frac{1}{\epsilon_m/\Delta-\epsilon_l/\Delta-\omega/\Delta}\right)-i\pi\delta\left(\frac{\epsilon_m}{\Delta}-\frac{\epsilon_l}{\Delta}-\frac{\omega}{\Delta}\right)\right]\,. \qquad (4.19)$$

$M_{lm}^a$ in Eq. (4.19) is calculated from continuum scattering states solutions to Eqs. (3.14), normalized according to Eqs. (3.16) as described earlier. However, not all the nonzero terms are kept in the $M_{lm}^a$ expression (4.8); we ignore all terms which vary as $e^{i(p_l+p_m^*)\lambda_e}$, $e^{i(r_l+r_m^*)\lambda_e}$, and keep all terms that vary as $e^{i(p_l-p_m^*)\lambda_e}$, $e^{i(r_l-r_m^*)\lambda_e}$.

## V. SURFACE-IMPEDANCE THEORY FOR SUPERCONDUCTORS WITH $H_{dc}\neq 0$ AND COMPARISON TO EXPERIMENTAL DATA

We now have developed enough formalism to discuss the implications of this surface-impedance theory for $H_{dc}\neq 0$. Since everything must be calculated numerically on a computer ranging from single to quadruple integrals, our results are only for a few cases. These are done in realms where the individual processes are separated as clearly as possible from one another. The experimental data is also rather spotty, so a complete comparison of theory and experiment is impossible at this time.

Wherever numerical results of calculation are quoted, the following data have been assumed for the model which corresponds roughly to tin and tantalum: $K_F=1$ Å$^{-1}$, $E_F=5$ eV, $\Delta/E_F=10^{-4}\Rightarrow\Delta\cong 150$ GHz, $\delta(T=0)=5000$ Å, $\lambda_e=1000$ Å.

Some important experiments have been performed by Pippard[1] in tin at 9.4 GHz, Spiewak[2] in tin at 1 GHz, Richards[5] in tin at 3 GHz, Lewis[4] in tin at 24 GHz, Glosser[3] in tantalum at 9 GHz, and Sharvin and Gantmakher[6] in tin at 2 MHz.

The number measured by the experimentalist is usually the surface impedance for some $H_{dc}$, $Z_S(H_{dc})=R_S(H_{dc})+X_S(H_{dc})$, divided by the normal-state surface resistance $R_N$. For temperatures and magnetic fields of interest here ($T\lesssim 10$ °K, $H_{dc}\lesssim 1000$ G), $R_N$ is given by

$$R_N=6.75\times10^{-12}(\omega/K_F)^{2/3}\,, \qquad (5.1)$$

where $R_N$ is in $\Omega$ if $\omega$ is in rad/sec and $K_F$ is in Å$^{-1}$. This value is obtained in the theory of the anomalous skin effect for specular reflection.[7-9]

We shall begin by considering the superconducting surface resistance at low temperatures, $0<T\lesssim 0.5T_c$, and for electromagnetic radiation frequencies, $\omega\sim 0.1\Delta$. If $H_{dc}=0$, $R_S$ (and $R_S/R_N$) is vanishingly small. This is well understood, since for such temperatures there are negligibly few quasiparticles in the continuum to absorb power. If $H_{dc}\lesssim H_c\sim 500$ G ($H_c$ is the critical magnetic field where the superconductor reverts to a normal metal), however, there is a sizeable power absorption, $0.1\%\lesssim R_S/R_N\lesssim 10\%$, which depends on

every conceivable factor involved here. Thus $H_{dc}$ has changed the surface resistance by several orders of magnitude. What could be the reason? Within our model, there is only one possible candidate for an explanation—the surface states. As described in Sec. III, the $H_{dc}$ field sets up a potential well, within which there are discrete energy levels that range from $\epsilon \simeq 0$ to $\epsilon \lesssim \Delta$. They exist only for $|\xi_t| \lesssim 20\Delta$, which indicates quasiparticle trajectories even more parallel to the surface than the effective electrons in the anomalous skin effect. $|\xi_t| \sim 10\Delta$ corresponds to an angle $\sim 2°$.

Since there exist states whose excitation energies are much less than the gap, even low temperatures yield sufficient thermal populations so that a photon can kick a quasiparticle from a lower branch into another higher-energy branch, conserving transverse momentum (initial and final states both have the same $\xi_t$ and $V_e$).

Using the surface-state spectrum described in Sec. III, we have calculated the surface resistance at low temperatures for various $T$, $\omega$, $H_{dc}$, and $\theta$. (Recall that $\theta$ is the orientation of the radiation field polarization with the $y$ axis.) By confining ourselves to $T \lesssim 0.5T_c$ and $\omega \lesssim 0.2\Delta$, we can be sure that the surface resistance is totally dominated by transitions between surface states.

We must do this calculation numerically, which according to Eq. (4.17a) amounts to just a single integral over either $V_e$ or $\xi_t$, as one chooses. However, doing the integral over the $\delta(\epsilon_m - \epsilon_t - \omega)$ intoduces either $1/|\partial(\epsilon_m - \epsilon_t)/\partial V_e|$ or $1/|\partial(\epsilon_m - \epsilon_t)/\partial \xi_t|$ according to which variable is used for integration. Both of these factors are singular for a typical energy spectrum. Although the integrals are well behaved, it is too difficult to handle such a situation numerically on a computer. A more reasonable approach is to integrate Eq. (4.17a) over a narrow frequency range from $\omega_1$ to $\omega_2$, where $\omega_2 - \omega_1 \ll \Delta$, and to obtain the surface resistance per unit frequency range from a double integral over $V_e$ and $\xi_t$. We assume that the surface resistance varies continuously and smoothly with frequency so that for $\omega_2 - \omega_1 \simeq 0.01\Delta$, we are really calculating the surface resistance at each point in frequency, except for $\omega_{1,2}$ less than the minimum energy separation of the nearest pair of surface states that are $\simeq 0.01\Delta$. (Obviously, there is no absorption when $\omega$ is less than the minimum energy separation of two surface states, so for some $\omega$ there is a discontinuous change in the surface resistance to zero.) The formula then used for calculating the surface resistance is given by Eq. (4.17b). Although there is much extra work in doing a double integral, one is rewarded by obtaining the surface resistance over a wide frequency range, and not just at one frequency. Some results of this calculation are summarized graphically in Figs. 2(a)–2(c).

It should be mentioned first that the requirement of self-consistency on $\delta$, the penetration depth, is least stringent for this case since we have $\mathrm{Im}\delta \ll \mathrm{Re}\delta$. $\mathrm{Re}\delta$ changes by about 10% as $H_{dc}$ varies over its range, while $R_S/R_N$ runs from zero to about 1%. So we can use, to very high accuracy, $\delta = \mathrm{Re}\delta$ only, and using $\delta$ for $H_{dc} = 0$ merely makes an over-all consistent error in $R_S/R_N$ of about 20%. The results in Figs. 2(a)–2(c) use $\delta$ at $H_{dc} = 0$ and $T = 0$. (It is well known that penetration depths are essentially independent of $T$ up to $T \sim 0.5T_c$.) Such approximations are adequate for the qualitative behavior desired.

Figure 2(a) displays results which can be most closely related to experimental data available. Glosser[3] has measured $R_S/R_N$ at $\theta = 0°$ and 90° as a function of $H_{dc}$ for $T \cong 0.2T_c$, $\omega \cong 0.07\Delta$. His results are consistent with ours in that $R_S/R_N$ in-

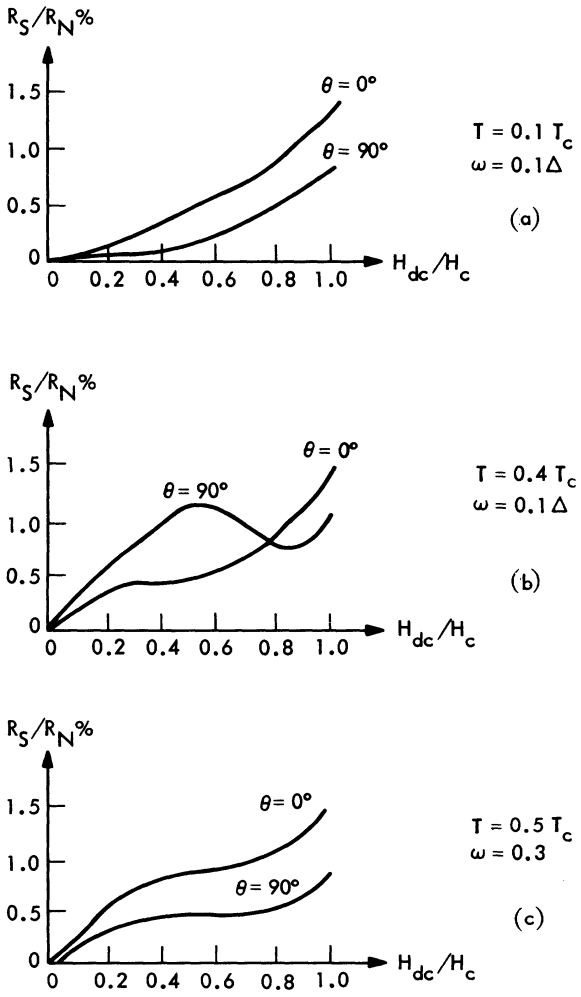


FIG. 2. Surface resistance due to surface states at low $T$.

creases monotonically with $H_{dc}$, $R_S/R_N$ is larger for $\theta = 0°$ than for $90°$, and $R_S/R_N$ is about a maximum of 1.5%.

Glosser's work[3] has the only data taken on the angular ($\theta$) dependence, and then only for $H_{dc} \cong 0.9H_c$, near the critical field. In Fig. 2(b) we have obtained some very interesting results regarding the $\theta$ dependence on $R_S/R_N$ for $\omega \cong 0.1\Delta$, $T \cong 0.4T_c$ as a function of $H_{dc}$ for which there is no experimental check as yet. For $H_{dc} \cong H_c$, $R_S/R_N$ is larger for $\theta = 0°$ than for $\theta = 90°$. But for $H_{dc} \cong 0.5H_c$, the surface resistance is larger for $\theta = 90°$ than for $\theta = 0°$! Since the surface-state spectrum is critical here, a more realistic effective potential is generally needed to see if this result is true.

Next, consider the superconducting surface resistance at intermediate $T$, $0.5T_c < T < 0.8T_c$, and frequencies $\omega < 0.1\Delta$. All the investigators have observed that $H_{dc} \sim H_c$ causes the surface resistance to decrease. This suppression effect ranges in size $-20\% < [R_S(H_{dc} \sim H_c) - R_S(H_{dc} = 0)]/R_N < -1\%$.

At intermediate $T$ and $\omega \lesssim 0.1\Delta$, there seem to be three mechanisms at work to consider, but two are unimportant. First, there are transitions between surface states which increase the surface resistance, but $R_{S_1}/R_N$ is only about 1%. Then there are transitions from surface states to continuum states, but once again for small frequencies, $\omega \ll \Delta$, $R_{S_2}/R_N \sim +1\%$. This contribution is progressively smaller as $\omega$ is lesser. Finally, there are transitions of thermally excited quasiparticles between states in the continuum. (It is necessary to consider these three different mechanisms, since the Fermi factor is sufficiently large for $T \gtrsim 0.6T_c$ to guarantee occupancy of all types of states.) This latter case would exist if $H_{dc} = 0$, whereas the previous two cases depend upon $H_{dc} \neq 0$. It is the onset of continuum occupancy by thermally excited quasiparticles at $T \simeq 0.6T_c$ which starts the process whereby a superconductor gradually turns into a normal metal at $T = T_c$. Our calculations indicate that $H_{dc} \neq 0$ has such a marked effect as to account for the suppressed surface resistance solely through modifications of the continuum transitions mechanism. We have $R_{S_3}/R_N \gg R_{S_2}/R_N$, so that we may ignore the other two processes involving surface states. Let us consider the expression for $R_{S_3}/R_N$; we do the integral over $\epsilon_2$ to drop the $\delta$ function in Eq. (4.19) and are left with a nontrivial triple integral that must be evaluated numerically. There are a number of features to be considered here. Once again, there is the question of which $\delta$ to use for the self-consistency relation. We have $\mathrm{Re}\delta \gtrsim 10\,\mathrm{Im}\delta$, and $\mathrm{Re}\delta$ changes by about 10% owing to $H_{dc}$. However, $R_S$ changes by about 100%,

TABLE I. Surface-resistance data at intermediate temperatures. Comparison of theory of Glasser's data.

| $T/T_c$ (%) | $R_S(H_{dc}=0)/R_N$ (%) | | $[R_S(H_{dc}=H_{c-}) - R_S(H_{dc}=0)/R_N$ (%)] | |
|---|---|---|---|---|
| | theoret. | expt. | theoret. | expt. |
| 0.6 | 2.9 | 3 | 1.2 − 0.45 = 0.75 | 1 |
| 0.7 | 6.9 | 7 | 0.8 − 0.75 = 0.05 | 0 |
| 0.8 | 9.8 | 10 | 0.3 − 3.8 = −3.5 | −4 |

so that we are once again justified in using $\mathrm{Re}\delta$ for $H_{dc} = 0$ to deduce the proper qualitative behavior. [The reader is reminded that Eq. (4.19) describes the anomalous skin effect[7,9] in superconductors ($H_{dc} = 0$, $\Delta \neq 0$) and normal metals ($H_{dc} = 0$, $\Delta = 0$).]

Examination of the integrand in Eq. (4.19) indicates that only transverse momenta in the range $-10^3\Delta < \xi_t < 2\Delta$ are important, which in physical terms means that only quasiparticles traveling nearly parallel to the metal's surface are effective in absorption.

For continuum states, there are many terms in the matrix element $M^q_{lm}$; each term has an exponential factor which either varies as $e^{i(p_l+p_m)\lambda_e}$ or $e^{i(p_l-p_m)\lambda_e}$. The former class oscillates rapidly as $\xi_t$ runs over its range, while the latter class has slowly varying phases. Therefore, we keep only these terms, and drop the others, which make little contribution. (This cannot be done if the matrix element involves at least one surface state, since the $\xi_t$ range is two orders of magnitude less and all exponentials have roughly the same frequency.)

The triple integral is so expensive to do that only a few important cases were done; viz., $\omega = 0.1\Delta$, $T = 0.6T_c$, $0.7T_c$, $0.8T_c$, $H_{dc} = H_{c-}$. The result is in excellent agreement with the data of Glosser[3] in that the surface resistance falls monotonically with $T$ over this range and is within 30% of the measured values. Furthermore, for $H_{dc} \to 0$, our results are within 10% of the Glosser data. This serves as a check on the method.

The results are displayed in Table I for comparison with Glosser's[3] data. In the fourth column of Table I we have written our result as the sum of two numbers to emphasize that the two effects of surface-state transitions and continuum-state transitions compete. The first number is surface-state data.

The experimental data originally provided the clue to understanding the negative shifts in surface resistance. $R_S(H_{dc} = 0)$ does not start to rise substantially from zero until $T \cong 0.6T_c$. This is, however, also the same point where $R_S(H_{dc} \neq 0) - R_S(H_{dc} = 0)$, which is fairly flat with $T$ up to this point, begins to develop a negative slope. Hence, it is likely that $H_{dc}$ is modifying that mechanism

responsible for absorption when $T > 0.6T_c$; viz., continuum transitions.

But what is the reason for this negative shift in surface resistance? When $H_{dc} = 0$, there is uniform probability to find a quasiparticle anywhere in the metal. If $H_{dc} \neq 0$, though, the situation is vastly different. The probability to find quasiparticles, then, varies with space, and even the spatial behavior varies with $\epsilon$, $\xi_t$, and $\varphi$ in a complicated way. $H_{dc}$ sets up the effective potential $V_e = V_0 \cos\varphi$ described in Sec. III ($V_0 \sim H_{dc}$). The principal effect of $V_e$ is to repel quasiparticles from the surface region $\sim\lambda_e$ for certain important values of the parameters. If $|\xi_t| \lesssim 20\Delta$, $V_e > 0$, and $\epsilon \lesssim \Delta + V_e$, then $r_\pm$ are complex so that the probability of finding quasiparticles falls exponentially in the surface region. But it is the quasiparticles in the surface region for $|\xi_t| \lesssim 20\Delta$ which are the most important absorbers of energy; a significant fraction of quasiparticles no longer plays a role in interacting with the electromagnetic field. For $|\xi_t| \gtrsim 20\Delta$, $r_\pm \to p_\pm$ which are usually purely real. But this does not mean that the $u$, $v$ amplitudes are spatially invariant. Even here there is a smaller probability to find quasiparticles in the surface region, but only fractionally less—not by orders of magnitude as when $r_\pm$ are complex. This phenomenon is related to the induced static currents created by $H_{dc}$, and could probably be obtained if the $u$, $v$ functions were calculated for an infinite medium carrying a uniform current along the $y$ axis. That is, it is due to transforming the infinite medium-excitation spectrum from the coordinate system at rest with respect to the screening current to the lab frame in which the superconductor is mounted.[12]

In any case, the net effect of $H_{dc}$ is to make most of the matrix elements $M^q_{lm}$ smaller as $H_{dc}$ increases. There are no resonance effects, whereby $M^q_{lm}$ increases substantially. That this occurs is very important, for it will allow us to explain several important features of the reactance behavior without resorting to further calculations.

Finally, there is the superconducting surface resistance at high $T$, $T \gtrsim 0.8T_c$, and low frequencies $\omega \lesssim 0.1\Delta$. Here, the experimental data show that surface resistance starts to increase again with $H_{dc}$, so that at $T \sim 0.95T_c$, $H_{dc}$ once more causes $R_S(H_{dc} \neq 0) > R_S(H_{dc} = 0)$. Finally, for $T \gtrsim 0.95T_c$, we have $R_S(H_{dc} \neq 0) \to R_S(H_{dc} = 0) \to R_N$. This last realm is the easiest to understand, since $\Delta \to 0$ rapidly and the superconductor is turning into a normal metal. But why does the shift in $R_S$ change sign and become positive for sufficiently high $T$? First, there is the obvious fact that $H_{dc}$ or $V_0$ have decreasing maximum values at $T$ increases $[H_c(T) \cong H_c(T=0)(1 - T^2/T_c^2)]$. Thus the potential $V_e$ weakens for higher $T$ and is less effective in keeping quasiparticles away from the surface region. Also, quasiparticles of larger energy are more important at higher $T$, and these feel the effect of $V_e$ much less than those of small $\epsilon$. Hence, the suppression effect does not work as well at high $T$. Thus the suppression effect can no longer overwhelm the positive effect of transitions from surface states to continuum states. In fact, the latter mechanism dominates once again. We have only estimated the size of this effect, but it is sufficient ($R_{S_3}/R_N \sim 2\%$) to give a net positive shift in $R_S$ for large $H_{dc} \sim H_c$. Only one property of this mechanism is important to discuss in relation to the experimental data. The data of the experimentalists clearly show that the maximum positive shift in $R_S$ at high $T$ shifts to lower $T$ as $\omega$ increases. Lewis[4] summarizes all the cases and finds a nice monotonic behavior. For any $\omega$, only surface states within $\omega$ of the lower edge of the continuum can contribute. As $\omega$ grows, then more surface states absorb, giving a larger surface resistance. But further, higher $\omega$'s use surface states from deeper in the gap, and these have higher population differences at any given $T$, which also makes $R_S$ grow. Thus the higher $\omega$'s begin to make a certain size $R_{S_2}$ at lower $T$ than the smaller $\omega$'s. When added to $R_{S_3}$, the net result is as observed.

Our next order of business is the surface-reactance behavior of superconductors when $H_{dc} \neq 0$. While surface-resistance processes are susceptible to a simple physical interpretation, surface-reactance phenomena are more elusive. Surface resistance is just another name for power absorption; a photon annihilates, kicking a quasiparticle from one state to another, conserving energy in the process. Surface reactance was shown to be a measure of field penetration and depends primarily on processes that always violate energy conservation, although staying within the bounds of the Heisenberg uncertainty principle. Nevertheless, we have devised a picture which simplifies our understanding of surface reactance and is consistent with the predictions of the formalism heretofore developed.

Consider only virtual processes. Suppose that the initial configuration of quasiparticles changes to another arrangement in the presence of some $A$ field at a particular frequency $\omega$. If the total energy of the final state is greater than the energy of the initial state plus the energy $\hbar\omega$, then the effect of such processes is to reduce the penetration depth of the $A$ field. Conversely, if the final state has less energy than the initial state plus $\hbar\omega$, then the penetration depth increases. Crudely speaking, we can imagine that if there is insufficient energy to make the transition from one

state to another, the extra energy is extracted from the $A$ field so that the penetration depth is decreased; whereas, if there is more than enough energy, the surplus energy is returned to the $A$ field, increasing the penetration depth.

Let us view this in terms of the two particular processes—scattering and pair creation. For scattering, the relevant terms in the $j(A)$ relation are $[f_T(\epsilon_2)-f_T(\epsilon_1)]/(\epsilon_2-\epsilon_1-\omega)$. If $\epsilon_2>\epsilon_1$, the thermal factors favor a net transition of quasiparticles upward from $\epsilon_1$ to $\epsilon_2$. When $\epsilon_2>\epsilon_1+\omega$, the generated current causes a field which is in such a direction as to oppose the original $A$ field. For $\epsilon_2<\epsilon_1+\omega$ ($\epsilon_2>\epsilon_1$), the opposite holds and the original $A$ field is enhanced. If $\epsilon_2<\epsilon_1$ ($\epsilon_2$ always less than $\epsilon_1+\omega$, of course), the current opposes the field. Although this transition is energetically favorable, it is statistically undesirable, since the final state has higher population probability than the initial state. Thus, for $\epsilon_2<\epsilon_1$, one might say that it is an antiparticle-type quasiparticle which is making the transition and such a particle causes a current that is opposite to particle-type quasiparticle currents.

Note that in the limit of $\omega\to 0$, all the currents always oppose the field so that the penetration depth is decreased for any transition.

For pair processes, we are concerned with the terms $-[1-f_T(\epsilon_l)-f_T(\epsilon_m)][1/(\epsilon_l+\epsilon_m+\omega)+1/(\epsilon_l+\epsilon_m-\omega)]$ in the $\vec{j}(\vec{A})$ relation. Here, $\epsilon_l$ and $\epsilon_m$ refer to the energies of a particle- and antiparticle-type quasiparticle either before or after interaction with the $A$ field. When $\epsilon_l+\epsilon_m>\omega$, the only case of interest at present, virtual creation of particle- and antiparticle-type quasiparticles into states of energy $\epsilon_l$ and $\epsilon_m$ always tends to make currents which further reduce the $A$ field. Virtual annihilation of the two quasiparticles from states $\epsilon_l$ and $\epsilon_m$ would yield current tending to increase the $A$ field if it were possible to have greater population probability in $\epsilon_l$ and $\epsilon_m$ than in the "vacuum state"—but this cannot be. Thus creation and annihilation always tend to screen the field from the interior when $\omega<\epsilon_l+\epsilon_m$.

In discussing surface resistance, we carefully checked the necessity of maintaining self-consistency on $\delta$ and concluded that this was not too important. That, however, is not true when considering surface-reactance changes; we must be very careful here. Suppose using Eq. (4.19) we found that $X_S\sim\delta^2(1+H_{dc})$; it would not imply that $X_S$ shifts positive with $H_{dc}$ because $X_S\sim\delta$ from Eq. (4.5) and $\delta\sim 1/(1+H_{dc})$, which means $X_S\sim 1/(1+H_{dc})$. $X_S$ shifts negative with $H_{dc}$! This type of situation generally prevails for superconductors.

The discussion of surface resistance for $\omega<\Delta$ above required consideration only of the terms involving scattering of a single quasiparticle from one state to another; the pair terms could make no effects. However, for surface-reactance or penetration-depth problems, it is the pair terms which are the most important.

Consider the low temperatures $T\lesssim 0.5T_c$. All the Fermi factors $f_T(\epsilon)$ are much less than one for continuum states $\epsilon\gtrsim\Delta$. Hence, only the pair terms contribute substantially. If $\omega<2\Delta$, only the reactive part of this remains. That is sufficient to explain the Meissner effect—the expulsion of the $A$ field from the superconductor's interior. The penetration depth of a field due to the pair mechanism has a weak frequency dependence. In fact AGK have shown that at $T=0$, $\delta$ varies by about 7% for $0\le\omega<2\Delta$. For $\omega\lesssim 0.1\Delta$, we have $\epsilon_l+\epsilon_m\gg\omega$, so that $\omega=0$ results are perfectly adequate in this low-temperature realm. All this, so far, applies irrespective of $H_{dc}$. A static magnetic field affects only the matrix elements $M^b_{lm}$: The effect, though, is identical to what happened to continuum states at intermediate $T$ when surface resistance was discussed—$H_{dc}$ reduces the size of most matrix elements. Thus the current of virtual pair processes tending to expel the $A$ field is reduced and so the surface reactance or penetration depth increase as $H_{dc}$ rises. Another way to say this is that $H_{dc}$ creates an effective potential $V_e$ which shoves quasiparticles into the metal's interior. Hence, the $\vec{A}$ field must penetrate further to interact with them. This is what every experimentalist has seen. The data at 700 kHz, 2 MHz, 1, 3, 8.8, and 24.5 GHz in various metals show a positive shift of surface reactance for low $T$ with no evident frequency behavior.

When $T\gtrsim 0.5T_c$, the surface-reactance shifts with $H_{dc}$ are much more complicated. At 700 kHz and 2 MHz, the shift is positive. At 1 and 3 GHz, the shift is negative. At 8.8 and 24.5 GHz, the shift is again positive. This pattern, too, has an explanation based strictly upon the reduction of matrix elements through $H_{dc}$.

The pair term works as at low $T$, causing frequency-independent positive-reactance shifts, so this can be regarded as a constant background. In addition, $T\gtrsim 0.5T_c$ makes the Fermi factors large enough so that reactance effects caused by scattering quasiparticles from one level to another are important. This was discussed qualitatively above. It was noted that for $\omega=0$, the virtual transitions lead only to further flux expulsion; so if the matrix element is squashed by $H_{dc}$, then $\vec{A}$ penetrates further. The mechanism is modified for $\omega\neq 0$. Transitions from $\epsilon_1$ to $\epsilon_2$, where $\epsilon_1<\epsilon_2<\epsilon_1+\omega$ lead to "screening" currents that suck the $\vec{A}$ field further into the metal, as shown earlier. That contribution becomes more important as $\omega$ increases. If $H_{dc}$ suppresses the matrix element,

then the penetration depth or surface reactance shifts negative, which is what the experiments at 1 and 3 GHz show. When $\omega$ is further increased, though, the processes tend to cancel; there are as many transitions above resonance as below, which in turn means that the pair processes again dominate, and the surface reactance shifts positive. This corroborates the data at 8.8 and 24.5 GHz.

Throughout the reactance analysis, we have consistently ignored the effects of surface states. Their total contributions are much too small to make any difference.

All the reactance changes can be explained by reference only to the modification of continuum-state wave functions.

## VI. GAP SELF-CONSISTENCY

A critical requirement for this theory is that the gap function $\Delta(x)$ be independent of $H_{dc}$ and spatially constant. This can be checked by seeing if the $u$, $v$ wave functions for a step potential and constant gap are self-consistent. Of course, within a few angstroms of the surface, the gap rapidly falls to zero since $u$, $v$ do, but this is not important. We are concerned with the changes over 1000-Å distances.

The self-consistency relation was last encountered in Sec. III, and we found that

$$\Delta(x) = V_{BCS} \sum_n u_n(x) v_n^*(x)[1 - 2f_T(\epsilon_n)] \quad . \tag{6.1}$$

$V_{BCS}$, the electron-electron interaction, is independent of $H_{dc}$ and $T$ and so can be ignored here.

Whereas the expressions for surface impedance involved only excitations traveling nearly parallel to the surface, the gap relation Eq. (6.1) is strongly sensitive to excitations traveling in every direction. Hence, surface states make very little effect here; only continuum states are important and we make little error in considering continuum states in the approximation $-10^4\Delta < \xi_t < 10\Delta$, where $r_+ \cong r_- \cong p_+ \cong p_-$, all purely real. The $\xi_t$ range is wide, since we are no longer trying to couple to a radiation field; the electron-electron interaction is mediated by an isotropic phonon-exchange mechanism.

With these simplifications in mind, it is useful to consider the product $v^*(x)u(x)$ for various regions and various types of scattering states.

### 1. $x \geq \lambda_e$, $p_+$ Incident Scattering States

We have

$$\begin{aligned} v^*(x)u(x) = {} & A_2A_2^*(\epsilon - \xi_p) + B_2B_2^*(\epsilon - \xi_p) + C_2C_2^*(\epsilon + \xi_p) + B_2A_2^*(\epsilon - \xi_p)e^{-i2p_+x} + C_2A_2^*(\epsilon - \xi_p)e^{i(p_- - p_+)x} \\ & + A_2B_2^*(\epsilon - \xi_p)e^{i2p_+x} + C_2B_2^*(\epsilon - \xi_p)e^{i(p_+ + p_-)x} + A_2C_2^*(\epsilon + \xi_p)e^{i(p_+ - p_-)x} + B_2C_2^*(\epsilon + \xi_p)e^{-i(p_+ + p_-)x} \quad . \end{aligned} \tag{6.2}$$

Integration over the fast-oscillating terms will give a negligible contribution, so Eq. (6.2) is simplified to

$$\begin{aligned} v^*(x)u(x) = {} & A_2A_2^*(\epsilon - \xi_p) + B_2B_2^*(\epsilon - \xi_p) \\ & + C_2C_2^*(\epsilon + \xi_p) + C_2A_2^*(\epsilon - \xi_p)e^{i(p_- - p_+)x} \\ & + A_2C_2^*(\epsilon + \xi_p)e^{i(p_+ - p_-)x} \quad . \end{aligned} \tag{6.3}$$

Around $x \sim \lambda_e$, the last two terms are very slowly varying. But deep inside the metal $x \ggg \lambda_e$, even the last two terms are fastly varying; thus they can be neglected, implying that the equilibrium gap is spatially constant as expected and determined in part by the terms

$$v^*(x)u(x) = (A_2A_2^* + B_2B_2^*)(\epsilon - \xi_p) + C_2C_2^*(\epsilon + \xi_p) \quad . \tag{6.4}$$

### 2. $x \geq \lambda_e$, $p_-$ Incident Scattering States

Using the same approximations as before for $x \ggg \lambda_e$, we have

$$v^*(x)u(x) = A_2A_2^*(\epsilon - \xi_p) + (C_2C_2^* + D_2D_2^*)(\epsilon + \xi_p) \quad . \tag{6.5}$$

### 3. $x \leq \lambda_e$, $p_\pm$ Incident

Using continuum states for the surface region and disregarding fast-oscillating terms, which requires staying several angstroms away from the surface, gives

$$\begin{aligned} v^*(x)u(x) = {} & 2A_1A_1^*(\epsilon - \xi_r - V_e) + 2C_1C_1^*(\epsilon + \xi_r - V_e) \\ & + C_1A_1^*(\epsilon - \xi_r - V_e)[e^{i(r_- - r_+)x} + e^{i(r_+ - r_-)x}] \\ & + A_1C_1^*(\epsilon + \xi_r - V_e)[e^{i(r_+ - r_-)x} + e^{i(r_- - r_+)x}] \quad . \end{aligned} \tag{6.6}$$

In the surface region for the range of integration concerned, the exponentials can sometimes be expanded and only the lowest-order terms retained; hence,

$$\begin{aligned} v^*(x)u(x) \cong {} & 2\{[A_1A_1^* + C_1A_1^*](\epsilon - \xi_r - V_e) \\ & + [C_1C_1^* + A_1C_1^*](\epsilon + \xi_r - V_e)\} \quad , \end{aligned} \tag{6.7}$$

which is spatially constant. Let us consider this last transformation carefully.

For $\xi_t < -10\Delta$,

$$r_+ - r_- \cong p_+ - p_- \cong K_F(\Delta/E_F)^{1/2}(\Delta/-\xi_t) \quad .$$

Whereas most superconductors have penetration

depths, $\lambda \sim 500$–$1000$ Å, the parameter $\Delta/E_F$ has a range of a couple orders of magnitude. Typically, $\Delta/E_F \sim 10^{-4}$ for tin and $\Delta/E_F \sim 10^{-5}$ for aluminum. Consider the exponentials in Eq. (6.6). The phase varies most rapidly at $x \sim \lambda$ and least rapidly as $x \to 0$. At $x \sim \lambda$, the phase varies in tin from about 1 to 0.1 rad as $\xi_t$ runs its full range; and $A_1 C_1^\dagger$, $C_1 A_1^\dagger$ are largest when the phase is largest (near small $\xi_t$). So the neglect of spatial variation in Eq. (6.7) is somewhat questionable. That was for tin; but in aluminum the phase varies from about $\frac{1}{3}$ to 0.03 rad, and it is a good approximation to neglect spatial behavior. The smaller $\Delta/E_F$ and the smaller $\lambda$, the less is the maximum range of the phase; but furthermore, for smaller $\Delta/E_F$ and $\lambda$, the smaller is the range of $\xi_t$ over which $A_1 C_1^\dagger$, $C_1 A_1^\dagger$ are nonzero, so the spatial variation terms can be forgotten irrespective of the behavior. Conversely, superconductors with large $\Delta/E_F$ and $\lambda$ have large phase variations and large ranges of $\xi_t$, where $A_1 C_1^\dagger$, $C_1 A_1^\dagger$ are nonzero. Thus it it likely that the gap is really constant, independent of $H_{dc}$ for superconductors with $\Delta/E_F \lesssim 10^{-5}$, $\lambda \sim 10^3$ Å, and that the gap has strong spatial variations due to $H_{dc}$ if $\Delta/E_F \gtrsim 10^{-3}$, $\lambda \gtrsim 10^3$ Å.

From the above discussion, one might suspect that $\Delta/E_F$ is of some fundamental significance in the theory of superconductors. In BCS theory, the coherence length, $\xi_0 \sim k_F/m\Delta$, so that $\Delta/E_F \sim 1/k_F\xi_0$; typically $\xi_0 \sim 3\times10^3$ Å in tin and $\xi_0 \sim 2 \times 10^4$ Å in aluminum. When the trend in gap variation for various $\Delta/E_F$ values quoted above is translated into the coherence-length language, the behavior is immediately comprehensible. $\Delta$ varies little if $\xi_0 \gg \lambda$, and $\Delta$ varies strongly if $\xi_0 \ll \lambda$. In the former case, the $\vec{A}(x)$ field is strong only in a region much smaller than the size of a pair, so the pairing is almost totally unaffected and $\Delta$ remains unchanged from its unperturbed value. In the latter case, $\vec{A}(x)$ is uniform across a pair so pairing is modified in proportion to the strength of $\vec{A}(x)$, and hence $\Delta$ has strong spatial variations where $\vec{A}(x)$ has strong spatial variations. This *result* is not new. Caroli[13] has shown, using the Landau-Ginzburg (LG) equations, that $\Delta$ does indeed have this behavior with $H_{dc}$, $\xi_0$, and $\lambda$. However, the LG equations only apply to situations where $\xi_0 \ll \lambda$, which is the same as requiring local electrodynamics to be true. Our surface-impedance theory is primarily concerned with the opposite limit, $\xi_0 \gg \lambda$, nonlocal electrodynamics. Hence, Caroli's results are only useful for $\xi_0 \ll \lambda$. The self-consistent method used here does not suffer from any restrictions on the relative sizes of $\xi_0$ and $\lambda$, and so provides an independent test of how various perturbations affect the gap $\Delta$.

That the LG equations yield the correct qualitative behavior here is still quite a mystery. Numerical calculations are in progress to check the gap self-consistency.

## VII. FINAL DISCUSSION

Maki[14] and Garfunkel[12] have made surface-impedance theories incorporating static fields. Despite radically different formalisms, both theories try to account for the phenomena of interest by shifting the energy of the quasiparticles from $\epsilon_{\vec{p}} \to \epsilon_{\vec{p}} - \vec{p}\cdot\vec{v}_s$, where $\vec{v}_s = e\vec{A}$ is the screening current velocity. That is, they introduce spatial anisotropy, but maintain spatial homogeneity. We obtain their results by letting our $\lambda_e \to \infty$. Missing, of course, are the states confined to or expelled from the surface region.

Koch[15,16] has measured the derivative of the surface resistance with respect to magnetic field strength, $dR/dH_{dc}$, versus $H_{dc}$, and concluded that his data support the existence of surface states. But this is questionable. In the first place, the published data give no scale for $dR/dH_{dc}$, so comparison with theory is not possible. But even if their data had appropriate units, transitions from surface states to continuum states would *not* produce the results they[15,16] claim to see. Koch assumes that for some $H_{dc}$ there are surface states only at a single energy $\epsilon(H_{dc})$, and so if the radiation energy $\omega$ is less than $\Delta - \epsilon(H_{dc})$, no absorption occurs. However, a realistic spectrum, as we have used and he purports to use, shows that for some $H_{dc}$ there are a continuum of surface-state energies. Examining Eq. (4.18) in the light of this remark does not reveal the singularities and properties described by Koch. His data must be considered still open to interpretation.

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# Theory of Electron-Spin Resonance in Gapless Superconductors

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The transverse dynamic spin susceptibility for conduction electrons in dirty gapless superconductors, in particular, in the vortex state of type-II superconductors, is calculated. It is shown that in the gapless region the dynamic susceptibility consists of two terms, the regular term and the anomalous term. In the low-frequency region of experimental interest, the regular term reduces to the static spin susceptibility, which is determined, for example, by the Knight-shift measurement in superconductors, while the anomalous term has a pole, which is associated with a resonance of the spin of conduction electrons. The resonance linewidth $T_2^{-1}$ is determined from the imaginary part of the resonance frequency. It is shown that $T_2^{-1}$ behaves quite differently in the superconducting state depending on whether $T_2^{-1}$ in the normal state is primarily due to the spin-orbit scattering or due to the exchange scattering from the magnetic impurities.

## I. INTRODUCTION

Dynamical spin susceptibility for conduction electrons has been studied extensively both theoretically and experimentally, since it provides useful means to study the interaction between the spin of conduction electrons and the impurities.

On the contrary, there appears no relevant calculation for superconductors, partially because in bulk type-I superconductors magnetic fields are expulsed from inside of the bulk except in the thin-skin layer at the surface, which makes the use of the resonance technique extremely difficult.[1] However, in the case of type-II superconductors in the high-field region or thin films, where magnetic fields are considered almost uniform in the specimen, we expect that the electron-spin-resonance technique can be used to study the spin-scattering mechanism of conduction electrons from magnetic and/or nonmagnetic impurities.

In this work we would like to report the calculation of the dynamical spin susceptibility for conduction electrons in dirty superconductors in high magnetic fields. We consider that the superconductor is either a bulk and in the vortex state or a very thin film so that the magnetic field in the specimen is almost uniform. Furthermore, we assume that the spin-relaxation rate due to the impurities is small.

In most of the calculations, however, we consider a superconducting thin film in the presence of parallel magnetic fields, since in the gapless region the result obtained can be easily generalized to describe the spin susceptibility of the type-II superconductors in the vortex state, if we reinterpret appropriately the pair-breaking parameter in the theory.[2]

It is shown that in the gapless region the dynamical susceptibility has a similar expression as the one in the normal state. Besides the static part, which is related to the $g$ shift of the impurity spin, the dynamical susceptibility has a contribution from the anomalous region which contains a complex pole.[3] The real and the imaginary part of the energy corresponding to the pole is interpreted in terms of the resonance frequency and the linewidth $T_2^{-1}$ for the transverse spin.[3] The linewidth $T_2^{-1}$ behaves quite differently in the superconducting state depending on whether $T_2^{-1}$ is primarily due to the spin-orbit scattering or due to the exchange scattering from magnetic impurities. In the former case, $T_2^{-1}$ decreases rapidly in the superconducting state, while in the latter case $T_2^{-1}$ increases. Therefore the measurement of $T_1^{-1}$ in the superconducting state provides certainly useful means to distinguish the two contributions to $T_2^{-1}$.

## II. FORMULATION

We will recapitulate here some of the properties of a superconducting thin film in a parallel magnetic field,[4] which are necessary for the calculation of the dynamical susceptibility. In the presence of a magnetic field the properties of conductions electrons are described by the Hamiltonian

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_1+\mathcal{H}_2\ , \tag{1}$$

where