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# Energy-Diffusion Equation in Hot-Electron Problems: Disturbance of the Polar-Optical-Phonon Distribution

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In polar semiconductors, the emission of polar optical phonons by electrons with $\mathcal{E} > \hbar\,\omega_0$ is a major energy-relaxation process. At low temperatures, the large number of emitted phonons disturbs the polar mode distribution. The energy-diffusion equation for carrier flow in energy space can be used to solve the energy distribution function of the carriers. This function is then used to solve for the disturbed phonon distribution. A phonon quasitemperature is introduced to aid in the calculation of the disturbed phonon density and the hot-electron energy distribution function. At high values of the electric field, tailing of the energy distribution function occurs primarily due to the phonon disturbance. The theories are applied to $n$-type InSb. All optical-phonon-emission and -absorption processes are considered, even at low temperatures since the absorption processes are important for determining a balance in the distribution function. Scattering by acoustic deformation potential, piezoelectric, and ionized impurities are also considered.

One of the major problems in the study of hot electrons in semiconductors is to find the distribution function of the carriers in momentum space. Traditionally, this has been accomplished by solving the Boltzmann transport equation either by assuming a Legendre function expansion for the distribution function or by assuming a known form for the distribution function and evaluating various parameters.[1] Such attempts are limited in scope, however, by streaming of the carriers in high-mobility semiconductors and additionally are made somewhat difficult due to nonparabolicity of the energy bands. A different approach, due primarily to Kurosawa,[2] is possible which eases these problems somewhat. Although equivalent to the Boltzmann equation, this approach is based upon the diffusion of carriers in energy space. In the case where the scattering is mainly elastic, Levinson has shown that the diffusion approach is equivalent to a kinetic approach based upon the Fokker-Planck equation.[3]

There have been attempts to solve the Boltzmann equation by statistical methods. One fairly successful method, the Monte Carlo technique, was also formulated by Kurosawa,[4] and has been used extensively by Boardman and others.[5] However, this method suffers somewhat from not being able to ascertain *a priori* the effects which individual processes will produce on the distribution function. It also suffers from the lack of direct inclusion of electron correlation effects, since it is basically a one-electron approximation to the solution. Although several attempts have been made to overcome this deficiency,[6] results are still limited, especially at low values of the electric field and in material with high carrier concentrations. In this paper, an approach based upon the diffusion equations will be utilized.

An electron in the conduction band of a semiconductor behaves equivalently to a particle undergoing Brownian motion in energy space if processes involving large energy changes do not occur. The diffusion equation is modified by an additional term for cases in which large energy exchanges occur. The statistical behavior of the electrons, given by the distribution function, is determined by the drift and diffusion of the carriers in the energy space, under the influence of external forces and collisions. Dumke used the diffusion approach to compute the generation rate of electrons for avalanche breakdown in InSb and InAs.[7] The application of the diffusion equation approach has been particularly useful in calculations of hot electrons in strong magnetic fields by Kurosawa and coworkers.[8-10] A common result of these calculations is that the energy distribution function exhibits a sharp break in slope at $\hbar\omega_0$, the energy of the polar optical phonon, and decreases rapidly with energy for energies above this value. The break in the distribution function at $\hbar\omega_0$ arises primarily from the onset of emission of polar optical phonons by carriers with energy $\mathcal{E} > \hbar\omega_0$.

The rather sharp breaks that occur in the calculated energy distribution functions indicate that a considerable number of polar phonons are being emitted by the electrons, especially since the emission of these phonons is the dominant energy and momentum relaxing process for the electrons, even at low temperatures.[11] One might then reasonably ask what happens to these phonons. The emitted phonons can either decay or be reabsorbed by an electron. The decay process arises primarily from a three-phonon interaction in which the polar phonon is annihilated and two acoustic phonons are created.[12] However, at low temperatures this is rather a slow process due to the low state densities of the two phonon distributions. Similarly, the probability of phonon absorption is also low due to the low state density of the polar phonon distribution. As a consequence, for either of these decay processes, the polar phonon distribution is driven out of equilibrium due to the phonon emission by the electrons. Experiments verify that the phonon distribution is, in fact, disturbed by the electrons.[13] All of the previously mentioned calculations suffer by the neglect of this disturbance of the phonon distribution. The diffusion approach is not alone in this respect, however, as calculations based upon the Boltzmann equation similarly ignore the effect on the phonon distribution. Moreover, these calculations suffer from the disregard of the phonon absorption transition. Although the probability for absorption of a polar phonon is very low at low temperatures, equilibrium is maintained by a delicate balance between emission and absorption.

In this paper, the effect of a nonequilibrium phonon distribution function is investigated. The energy distribution function of the electrons is strongly affected by the disturbance of the phonon distribution. The phonon distribution itself is determined by a balance between phonon emission and absorption by the electrons and by decay of the polar phonons into the acoustic modes. The calculations are carried out using the diffusion equations developed by Kurosawa.[2]

## I. DIFFUSION EQUATION IN ENERGY SPACE

The energy distribution function $f(\mathcal{E})$ and the electron particle flow in energy space $I(\mathcal{E})$ are related by the diffusion equation[2]

$$I(\mathcal{E}) = -(D_T + D_F)w(\mathcal{E})\frac{df(\mathcal{E})}{d\mathcal{E}} - \frac{D_T}{k_B T_0}\,w(\mathcal{E}) ,$$

or

$$\frac{df(\mathcal{E})}{d\mathcal{E}} = -\frac{f(\mathcal{E})}{k_B T_0(1 + D_F/D_T)} - \frac{I(\mathcal{E})}{w(\mathcal{E})(D_F + D_T)} , \qquad (1)$$

where $T_0$ is the lattice temperature, $w(\mathcal{E})$ is the density of states at energy $\mathcal{E}$, and $D_F$ and $D_T$ are the energy diffusion constants due to external forces and relaxing collisions, respectively. In fact, $D_F$ and $D_T$ are not constants at all, but are functions of energy and depend strongly upon the particular scattering type involved. In addition, the electron flow in energy space $I(\mathcal{E})$ is related to large energy exchange collisions through the continuity equation

$$\begin{aligned} \frac{dI(\mathcal{E})}{d\mathcal{E}} &= \frac{d[w(\mathcal{E})f(\mathcal{E})]}{dt}\bigg|_{\substack{\text{large }\Delta\mathcal{E}\\ \text{collisions}}} \\ &= W_{\text{em}}(\mathcal{E} + \hbar\omega_0)\, W(\mathcal{E} + \hbar\omega_0) f(\mathcal{E} + \hbar\omega_0) \\ &\quad - W_{\text{em}}(\mathcal{E}) w(\mathcal{E}) f(\mathcal{E}) \\ &\quad + W_{\text{abs}}(\mathcal{E} - \hbar\omega_0)\, w(\mathcal{E} - \hbar\omega_0) f(\mathcal{E} - \hbar\omega_0) \\ &\quad - W_{\text{abs}}(\mathcal{E}) w(\mathcal{E}) f(\mathcal{E}). \qquad (2) \end{aligned}$$

$W_{\text{em}}$ and $W_{\text{abs}}$ are the probability for emission and absorption, respectively, of a longitudinal polar optical phonon by the electrons. The two absorption transitions are not normally treated in the diffusion approach,[7-10] but it is shown in Appendix B that these terms are necessary for an adequate treatment of the phonon balance with the electrons.

The right-hand side of (2) is just the time rate of change of the total electron population in the energy shell between $\mathscr{E}$ and $\mathscr{E}+d\mathscr{E}$ due to the large-energy-exchanging polar scattering. The first and third terms represent the flow of particles into the energy shell by emission and absorption processes of electrons in shells a distance $\hbar\omega_0$ removed, while the second and fourth terms represent scattering of electrons out of the energy shell due to phonon emission and absorption, respectively. For a later discussion on the importance of the absorption processes, it is convenient to rewrite (2) as

$$\frac{dI(\mathscr{E})}{d\mathscr{E}}=-\Gamma_1-\Gamma_2\,, \tag{2$'$}$$

where

$$\Gamma_1=W_{\mathrm{em}}(\mathscr{E})w(\mathscr{E})f(\mathscr{E}) - W_{\mathrm{abs}}(\mathscr{E}-\hbar\omega_0)w(\mathscr{E}-\hbar\omega_0)f(\mathscr{E}-\hbar\omega_0), \tag{3}$$

$$\Gamma_2=W_{\mathrm{abs}}(\mathscr{E})w(\mathscr{E})f(\mathscr{E}) - W_{\mathrm{em}}(\mathscr{E}+\hbar\omega_0)w(\mathscr{E}+\hbar\omega_0)f(\mathscr{E}+\hbar\omega_0). \tag{4}$$

The processes in $\Gamma_1$ occur only for $\mathscr{E}>\hbar\omega_0$, since $W_{\mathrm{em}}(\mathscr{E})=0$ and $w(\mathscr{E}-\hbar\omega_0)=0$ for $\mathscr{E}<\hbar\omega_0$.

The diffusion coefficient for the electrons in energy space due to an external electric field $E$ is given by[2]

$$D_F=\frac{2(eE)^2}{3m^*}\,\mathscr{E}\tau(\mathscr{E})=\frac{2(eE)^2}{3m_c}\left(\frac{\mathscr{E}}{1+2\mathscr{E}/\mathscr{E}_G}\right)\tau(\mathscr{E})\,, \tag{5}$$

where $m^*$ is the conductivity effective mass and $m_c$ is the band-edge mass of the electrons. The second equality in (5) takes account of the nonparabolicity of the conduction band, expressed in Kane's hyperbolic form[14]

$$\mathscr{E}=\tfrac{1}{2}\mathscr{E}_G\left[(1+2\hbar^2k^2/m_c\mathscr{E}_G)^{1/2}-1\right], \tag{6}$$

where $\mathscr{E}_G$ is the conduction-valence band gap (the zero of energy is taken as the conduction-band edge). The scattering time $\tau(\mathscr{E})$ is the total scattering time due to all collision processes. In this study, scattering due to acoustic phonons interacting through both piezoelectric and deformation potentials, ionized impurity, and polar optical phonons is considered. The diffusion constant for relaxing collisions $D_T$ arises from the effect of collisions that are nearly elastic. Thus,

$$D_T=D_{T,\mathrm{ac}}+D_{T,\mathrm{pe}}+D_{T,\mathrm{imp}} \tag{7}$$

accounts for the effects of acoustic deformation potential, piezoelectric, and impurity scattering, respectively. The individual terms are derived in Appendix A.

In the absence of large energy exchange collisions, i.e., in the absence of polar-optical-mode scattering, the flow in energy space must be zero. Then, the energy distribution function is found from (1) to be

$$f(\mathscr{E})=f(0)\exp\left(-\int_0^{\mathscr{E}}\frac{d\mathscr{E}}{k_BT_0(1+D_F/D_T}\right), \tag{8}$$

which properly reduces to a Maxwellian in small electric fields. If the ratio $D_F/D_T$ is energy independent, it is possible to define an effective electron temperature from (14) as

$$T_e=T_0(1+D_F/D_T)\,. \tag{9}$$

However, such a defined temperature is not constant unless the ratio $D_F/D_T$ is independent of energy. In some cases, (8) may be integrated in closed forms to provide the proper energy distribution function. In particular, for acoustic-mode deformation potential scattering,

$$\frac{D_F}{D_T}=\frac{\mathscr{E}_0}{\mathscr{E}}\,,$$

where

$$\mathscr{E}_0=(eE\,l_{\mathrm{ac}})^2/4m_cv_s^2$$

in the parabolic case, where $l_{\mathrm{ac}}$ is the mean free path of the electrons for acoustic scattering. Then, (8) may be integrated to yield

$$f(\mathscr{E})=f(0)\,(1+x/x_0)^{x_0}e^{-x}\,, \tag{10}$$

where $x=\mathscr{E}/k_BT_0$, $x_0=\mathscr{E}_0/k_BT_0$, a result previously obtained by Davydov[15] and Yamashita and Watanabe[16] by solving the Boltzmann transport equation, a considerably more complicated process. In the more general case of multiple scattering processes, the integral usually cannot be solved in closed form, but is readily evaluated with a computer.

When large-energy exchange processes are involved, Eqs. (1) and (2′) must be solved together to provide the solution. Kurosawa[2] has proposed using a Runge-Kutta-Gill numerical approach on the equations, much as one would for the single equation of (1) when $I(\mathscr{E})=0$. In general, this cannot be done however. This becomes obvious by considering only the case of optical phonon emission. The equations become

$$\frac{df}{d\mathscr{E}}=\frac{f(\mathscr{E})}{k_{BT0}(1+D_F/D_T}-\frac{I(\mathscr{E})}{w(\mathscr{E})(D_F+D_T)}, \tag{11}$$

$$\frac{dI}{d\mathscr{E}}=-W_{\mathrm{em}}(\mathscr{E})w(\mathscr{E})f(\mathscr{E})\,. \tag{12}$$

$I(\mathscr{E})$ is a decreasing function and, if $I(0)=0$ is assumed, is a negative function. Thus this causes $df/d\mathscr{E}$ to lie above the value that would occur in the absence of the polar-mode interactions, clearly an unphysical result. The fact is that the system of equations (11), (12) constitutes an absolutely unstable system and the solution diverges. This is also true of the more general set (1, ) (2′) for $\mathcal{E} > \hbar\omega_0$. But this is true of any diffusion equation. In particle flow, this problem is handled by disregarding the growing solution by proper utilization of the physical requirement that

$$n_0 = \int_0^\infty f(\mathcal{E}) w(\mathcal{E}) d\mathcal{E} \tag{13}$$

or

$$\lim_{\mathcal{E}\to\infty} f(\mathcal{E}) = 0 \ . \tag{14}$$

Solving the set of equations (1), (2′) by a direct numerical approach constitutes choosing two initial conditions $f(0)$, $df(0)/d\mathcal{E}$ [through a choice of $I(0)$]. If (14) is to hold, we are not free to choose two initial conditions. The set of equations can be rearranged to a generalized form of the Sturm-Liouville equation as

$$\frac{d}{d\mathcal{E}}\ A(\mathcal{E})\frac{df}{d\mathcal{E}}\ + B(\mathcal{E}) f(\mathcal{E}) = \Gamma_1(\mathcal{E}) + \Gamma_1(\mathcal{E}) \ , \tag{15}$$

where

$$A(\mathcal{E}) = w(\mathcal{E})(D_F + D_T) \ , \quad B(\mathcal{E}) = (D_T / k_B T_0) w(\mathcal{E}) \ .$$

Such a second-order differential equation can be constrained by, at most, two boundary conditions, and (14) constitutes one of these. Thus, one is not free to choose both $f(0)$ and $I(0)$. The constraints (13) and (14) select the proper solution of the two independent solutions of (15) and also govern the initial value $f(0)$. Thus, we must apply more convential techniques to the matrix differential equation

$$\begin{bmatrix} \dfrac{df}{d\mathcal{E}} \\[2ex] \dfrac{dI}{d\mathcal{E}} \end{bmatrix} = \begin{bmatrix} -\alpha(\mathcal{E}) & -\beta(\mathcal{E}) \\[2ex] -\gamma(\mathcal{E}) & 0 \end{bmatrix} \begin{bmatrix} f(\mathcal{E}) \\[2ex] I(\mathcal{E}) \end{bmatrix} + \begin{bmatrix} 0 \\[2ex] 1 \end{bmatrix} [\delta_1 f(\mathcal{E} - \hbar\omega_0) + \delta_2 f(\mathcal{E} + \hbar\omega_0)] \ , \tag{16}$$

where

$$\alpha(\mathcal{E}) = +[k_B T_0 (1 + D_F/D_T)]^{-1} \ ,$$

$$\beta(\mathcal{E}) = [w(\mathcal{E})(D_F + D_T)]^{-1} \ ,$$

$$\gamma(\mathcal{E}) = [W_{\mathrm{em}}(\mathcal{E}) + W_{\mathrm{abs}}(\mathcal{E})] w(\mathcal{E}) \ ,$$

$$\delta_1(\mathcal{E}) = W_{\mathrm{abs}}(\mathcal{E} - \hbar\omega_0) w(\mathcal{E} - \hbar\omega_0),$$

$$\delta_2(\mathcal{E}) = W_{\mathrm{em}}(\mathcal{E} + \hbar\omega_0) w(\mathcal{E} + \hbar\omega_0) \ .$$

Were it not for the $\delta(\mathcal{E})$ terms on the right-hand side of (16), (16) could be solved in a straightforward manner by assuming a solution of the form

$$\exp\left(\int_0^{\mathcal{E}} \lambda(\mathcal{E}') d\mathcal{E}'\right) \ ,$$

and solving for $\lambda(\mathcal{E})$ as the eigenvalues of the matrix of the first term on the right-hand side. Only the eigenvalue which gives $f(\mathcal{E})$ satisfying (14) is retained. The terms involving $\delta(\mathcal{E})$ constitute memory and prediction effects and complicate matters. However, these terms generally have only a limited effect on $f(\mathcal{E})$ and can be taken into account by an iterative technique. In general, the retained root is of the form

$$\lambda(\mathcal{E}) = -\tfrac{1}{2}\alpha(\mathcal{E})\{1 + [1 + 4\ \beta(\mathcal{E})\gamma'(\mathcal{E})/\alpha(\mathcal{E})]^{1/2}\} \ , \tag{17}$$

where $\gamma'(\mathcal{E})$ is modified to include the effects of the additional factors in the second term on the right of (16). A similar result with constant factors was utilized by Dumke.[7]

It is exceedingly important not to disregard the absorption transitions, even at low temperatures, as it is just these transitions that are necessary to balance the emission of phonons when the distribution function is a Maxwellian. In Appendix B, the reasoning leading to this conclusion is presented in a more detailed manner.

The effect of the various contributions to $\Gamma_1$, $\Gamma_2$ and hence to $f(\mathcal{E})$ are shown in Figs. 1 and 2 for lattice temperatures of 4.2 and 77 °K, respectively. The matric differential equation (16) is solved in an iterative method. As expected, at low temperatures, only the emission interaction has a significant effect of the shape of $f(\mathcal{E})$ for a relatively large value of electric field, 20 V/cm in this case. Of interest is the fact that use of a constant scattering rate for $W_{\mathrm{em}}$ leads to considerable error in the results. In each case the natural logarithm of $f(\mathcal{E})$ is plotted as a function of electron energy. This scheme is followed throughout this paper.

The equations we are solving are basically a set in one dimension, that dimension being the energy. The result is that the energy distribution function represents an average over the more normal carrier distribution function in momentum space, where the average is carried out over a surface of constant energy.[17] Basically, the energy distribution then has the symmetry of the constant energy surfaces in the semiconductor. In the present case of InSb, this means that the energy

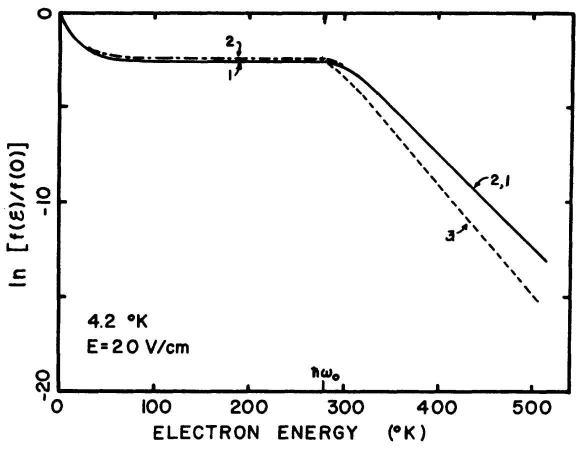

FIG. 1. Effect of the various optical phonon processes on the energy distribution function at 4.2 °K lattice temperature. The various curves include the following effects: (1), all emission and absorption processes; (2), only $W_{em}(\mathcal{E})$; (3), only $W_{em}(\mathcal{E})$ with a constant value of $W_{em}$ assumed. At this electric field, the phonon absorption processes are not effective, the slight difference in curves (1) and (2) being due to $W_{em}(\mathcal{E}+\hbar\omega_0)$.

distribution function basically is just the spherically symmetric part of the total carrier distribution. Since there is considerable streaming of the total distribution along the electric field direction, it is expected that there will be a correction term arising in (13) from non-spherically-symmetric terms. For this reason, the energy distribution functions calculated here are normalized to $f(0)=1$.

## II. DISTURBANCE OF THE PHONON DISTRIBUTION

At low temperatures, the total lattice energy content is comparable to the energy transferred by the hot electrons to the lattice,[18] particularly in the case where the dominant energy relaxation mechanism is the interaction of the electrons with the polar optical phonons. Because of this, it is expected that the phonon distribution will be disturbed in the hot-electron case. The disturbed phonon distribution can be observed by changes in the electron mobility with time, and has been experimentally observed both in polar[13] and nonpolar[19] semiconductors (in the latter case, interaction with nonpolar optical phonons is the dominant energy relaxing mechanism). At low temperatures, the dominant energy relaxation mechanism in polar materials is the emission of polar optical phonons by the electrons. These emitted phonons are sufficiently large in number to disturb the equilibrium distribution, given by the Bose-Einstein distribution

$$n_{q0}=(e^{\hbar\omega_0/k_BT_0}-1)^{-1}\,. \tag{18}$$

These phonons are dissipated either by reabsorption by the electrons or by decay into acoustic modes of the lattice. The total time rate of change of the optical phonon density is given by

$$\frac{\partial n_q}{\partial t}=-\frac{n_q-n_{q0}}{\tau_p}+\frac{\partial n_q}{\partial t}\bigg|_e\,, \tag{19}$$

where the last term on the right is the rate of change of $n_q$ due to the electron-phonon interaction, $n_q$ is the phonon density, $n_{q0}$ is the equilibrium density, and $\tau_p$ is the polar phonon lifetime for decay into acoustic modes.

The time rate of change of $n_q$ due to the electron-phonon interaction may be calculated by summing the relevant phonon-emission and -absorption processes over the electron distribution.[1] This is basically the approach that will be followed here, although the details of the calculation are different from those previously utilized for acoustic phonon cases. The present calculation is more adaptable for the polar interaction due to the narrow range of $q$'s involved and the independence of $\omega_0$ on $q$. The energy loss rate of electrons at energy $\mathcal{E}$ to optical phonons is just

$$\frac{d\mathcal{E}}{dt}=-\hbar\omega_0[W_{em}(\mathcal{E})-W_{abs}(\mathcal{E})]\,. \tag{20}$$

If we average this over the distribution function, the average rate of change of carrier energy is just

$$\left\langle\frac{d\mathcal{E}}{dt}\right\rangle=-\frac{\hbar\omega_0}{n_0}\int[W_{em}(\mathcal{E})-W_{abs}(\mathcal{E})]w(\mathcal{E})f(\mathcal{E})d\mathcal{E}. \tag{21}$$

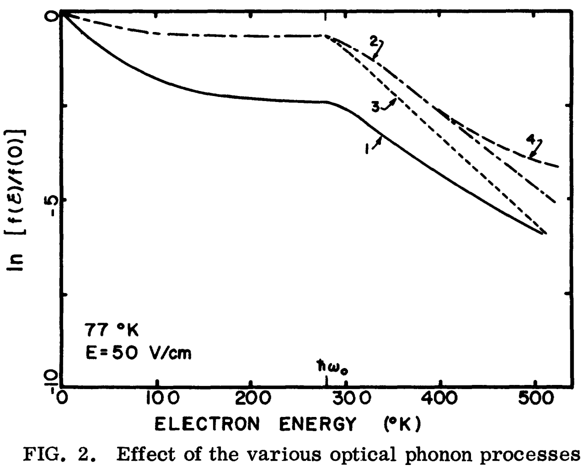

FIG. 2. Effect of the various optical phonon processes on the energy distribution function for a lattice temperature of 77 °K. The various curves include the following processes: (1), all emission and absorption processes; (2), only $W_{em}(\mathcal{E})$; (3), only $W_{em}(\mathcal{E})=\text{constant}$ (4), only $W_{em}(\mathcal{E})$ and $W_{abs}(\mathcal{E}-\hbar\omega_0)$ to show the effect of electrons entering the distribution at high energies due to phonon absorption. This latter effect accounts for the tailing of curves (1), (4).

The distribution function involved here is essentially averaged over a constant energy surface, so that from the discussion of Sec. I, it is recognized as just the energy distribution function in which we are already interested.

In polar semiconductors at low temperatures, the polar-mode interaction with the electrons is essentially the only energy relaxation process for the hot electrons. Therefore, the energy lost by the electrons is dumped into the phonons, and[1]

$$\frac{d\mathcal{E}}{dt} = -\frac{1}{n_0 V}\sum_{\vec{q}} \hbar\omega_0 \frac{\partial n_q}{\partial t}\bigg|_e , \tag{22}$$

where $V$ is the volume of the crystal. In the polar interaction, only a very narrow range of $q$ is involved . Moreover, the phonon frequency may be taken to be independent of the wave vector $\vec{q}$. The important quantity for describing the interaction is the number of phonons rather than their detailed distribution. For this reason, we shall write the phonon distribution as

$$n_q = \bar{n}_q S(\vec{q}), \tag{23}$$

where $\bar{n}_q$ is independent of $q$ and defined in analogy to (30) as

$$\bar{n}_q = (e^{\hbar\omega_0/k_B T_p} - 1)^{-1} . \tag{24}$$

The quantity $T_p$ is a phonon quasitemperature describing the number of phonons in the disturbed distribution, much as an electron quasitemperature is used to describe a hot-electron distribution function. The shape factor is normalized so that

$$\frac{1}{V}\sum_{\vec{q}} S(\vec{q}) = 1. \tag{25}$$

To be perfectly correct, the form of the shape factor must be included in the evaluation of the electron scattering rates $W_{\text{em}}$, $W_{\text{abs}}$. However, precisely the same range of $q$ is involved in these interactions as is important for $S(\vec{q})$. This range of $\vec{q}$ is very small and centered about the value $q_0 = (2m_c\omega_0/\hbar)^{1/2}$. As a general approximation, it might be assumed that $S(\vec{q})$ is constant over the allowed range of $q$ and zero outside this range. Then it will not affect the scattering probabilities. The actual variation of $S(\vec{q})$ will only contribute a higher order correction term that will not appreciably affect the results. The important part is thus $\bar{n}_q$ which is independent of $q$ and this replaces $n_q$ in (B1) and (B2). This will not be the case for phonons in which $\omega$ is a function of $q$. Then, (22) can be rewritten as

$$\left\langle \frac{d\mathcal{E}}{dt} \right\rangle = -\frac{\hbar\omega_0}{n_0}\,\frac{\partial n_q}{\partial t}\bigg|_e$$

and

$$\frac{\partial \bar{n}_q}{\partial t}\bigg|_e = \int_0^\infty [W_{\text{em}}(\mathcal{E}) - W_{\text{abs}}(\mathcal{E})]\,w(\mathcal{E})f(\mathcal{E})\,d\mathcal{E}. \tag{26}$$

Equation (26) is useful in calculating $f(\mathcal{E})$ by computer techniques, in that it can be evaluated as $f(\mathcal{E})$ is carried through the range of energy without appreciably increasing the computer time.

Equation (19) can be solved as a function of $\bar{n}_q$ rather than $n_q$, and this gives

$$\bar{n}_q - n_{q0} - \tau_p \frac{\partial \bar{n}_q}{\partial t}\bigg|_e = 0. \tag{27}$$

As will be shown in Sec. III, $\tau_p$ can be written as

$$\tau_p = C_p/\bar{n}_q ,$$

so that

$$\bar{n}_q = \frac{n_{q0}}{2}\left[1 + \left( + \frac{4C_p}{n_{q0}^2}\,\frac{\partial \bar{n}_q}{\partial t}\bigg|_e \right)^{1/2}\right] . \tag{28}$$

The term in parentheses actually is a function of $\bar{n}_q$ so that the solution to (28) is not as straightforward as implied. However, it can be obtained iteratively, with an evaluation of $f(\mathcal{E})$ included within the iteration.

## III. OPTICAL PHONON LIFETIME

The polar optical phonons can not effectively transfer their energy to the surface, because of the extremely small value of the group velocity of these phonons. These phonons can decay, however, into the acoustic modes of the lattice through a three-phonon interaction involving the anharmonic terms of the crystal potential.[12] To a normal system with crystal potential energy, a cubic term in the strain can be added as

$$V_{pp'} = \frac{1}{3!}\iiint \nabla_1 u\, \nabla_2 u\, \nabla_3 u \,\vdots\, \mathfrak{A}\; d^3 r , \tag{29}$$

where $\mathfrak{A}$ is a Cartesian tensor of the sixth rank, and corresponds to the third-order elastic strain. The subscripts on the gradient operators are to distinguish the three separate crystal momentum terms. Fourier transforming (41), the expression for $V_{pp}$ can be carried into wave coordinates and yields

$$V_{pp} = \frac{1}{3!(NV)^{1/2}} \sum_{\vec{q},\vec{q}',\vec{q}''} \delta_{0,\vec{q}-\vec{q}'-\vec{q}''}\, \xi_{\vec{q}}\, \xi_{\vec{q}'}\, \xi_{\vec{q}''} \,\vdots\, \mathfrak{F} , \tag{30}$$

where

$$[\mathfrak{F}]_{\alpha\gamma\rho} = -i(N)^{1/2}\sum_{\beta,\delta,\sigma} q_\beta q_\delta q_\sigma \mathfrak{A}^{\beta\delta\sigma}_{\alpha\gamma\rho} .$$

The wave amplitudes $\xi_{\vec{q}'}$, $\xi_{\vec{q}''}$ can further be expressed in terms of creation and annihilation operators $a^\dagger_{\vec{q}'}$ and $a_{\vec{q}'}$ , respectively, in terms as

$$\xi_{\vec{q}'} = -i\left(\frac{\hbar}{2\omega_{q'}}\right)^{1/2} (a^{\dagger}_{\vec{q}'} - a_{\vec{q}'}) \frac{\vec{e}_{q'}}{(D)^{1/2}} , \tag{31}$$

and similarly for $\xi_{\vec{q}''}$. The mode frequency is just $\omega_{q'} = v_s q'$, $\vec{e}_{\vec{q}'}$ is a polarization vector, and $D$ is the mass density of the crystal. For $\xi_{\vec{q}}$, $\vec{q}$, however, the polarization of the lattice must be introduced. These terms arise from $\nabla_1 u$, the lattice wave momentum, so that the conjugate momentum corresponding to the lattice polarization must be used. The polarization is just[20]

$$P = \left(\frac{\hbar}{2\gamma\omega_0}\right)^{1/2} \sum_{\vec{q}} \vec{e}_{\vec{q}} (a_{\vec{q}} + a^{\dagger}_{\vec{q}}) , \tag{32}$$

where

$$\frac{1}{\gamma} = \omega_0^2 \left(\frac{1}{\epsilon_\infty} - \frac{1}{\epsilon_0}\right) . \tag{33}$$

The conjugate momentum is then[12]

$$\gamma \dot{p} = -i\left(\frac{\hbar\gamma\omega_0}{2}\right)^{1/2} \sum_{\vec{q}} \vec{e}_{\vec{q}} (a^{\dagger}_{\vec{q}} - a_{\vec{q}}), \tag{34}$$

and (30) becomes

$$V_{pp'} = -\frac{1}{3!} \frac{1}{(V)^{1/2}} \sum_{\vec{q},\vec{q}',\vec{q}''} \delta_{0,\vec{q}-\vec{q}'-\vec{q}''} \left(\frac{\hbar^3\gamma\omega_0}{8\omega_{q'}\omega_{q''}}\right)^{1/2} \times \frac{1}{D} (a^{\dagger}_{\vec{q}} - a_{\vec{q}}) (a^{\dagger}_{\vec{q}'} - a_{\vec{q}'}) (a^{\dagger}_{\vec{q}''} - a_{\vec{q}''}) \times \vec{e}_{\vec{q}} \vec{e}_{\vec{q}'} \vec{e}_{\vec{q}''} \,\vdots\, \mathcal{A}\, \vec{q}' \vec{q}'' .$$

In the following, we need to retain only the term involving the annihilation of a polar phonon and the creation of two acoustic phonons;

$$V_{pp'} = \left(\frac{\gamma\hbar^3\omega_0}{8\omega_{q'}\omega_{q''} D^2 V}\right)^{1/2} \times a_{-\vec{q}} a^{\dagger}_{\vec{q}'} a^{\dagger}_{\vec{q}''} \mathcal{A}\, \vec{q}' \vec{q}'' \delta_{0,\vec{q}-\vec{q}'-\vec{q}''} . \tag{35}$$

The corresponding matrix element is just

$$|M_{pp'}|^2 = |\langle f | V_{pp'} | i \rangle|^2 = \left(\frac{\gamma\hbar^3\omega_0 v_s^2}{8c_l^2 v}\right) q' q'' n_q (n_{q'} + 1) \times (n_{q''} + 1) \mathcal{A}^2 \delta_{0,\vec{q}-\vec{q}'-\vec{q}''} , \tag{36}$$

where we have used the facts that $\omega_{q'} = v_s q'$ and $D v_s^2 = c_l$.

The phonon relaxation time is related to the matrix element (36) through the relation

$$\frac{1}{\tau_p} = \Gamma_{p0\to\mathrm{ac}} = \frac{2\pi}{\hbar} \sum_{\vec{q}',\vec{q}''} |M_{pp'}|^2 \delta(\mathcal{E}_{\vec{q}} - \mathcal{E}_{\vec{q}''} - \mathcal{E}_{\vec{q}'}) , \tag{37}$$

where the last δ function ensures the conservation of energy. The summation over $q''$ can readily be carried out with the Kronecker δ function in (36) to yield

$$\Gamma_{p0\to\mathrm{ac}} = \frac{2\pi}{\hbar} \sum \left(\frac{\gamma\hbar^3\omega_0 v_s^2}{8c_l^2 v}\right) q'(q - q') \times n_q (n_{q'} + 1)(n_{q'-q} + 1) \mathcal{A}^2 \tag{38}$$

subject to the condition that

$$q_0' - q \leq q' \leq q_0' + q, \quad q \ll q_0' , \tag{39}$$

where

$$q_0' = \omega_0 / 2v_s . \tag{40}$$

The two extremes on $q$ from (39) arise from the case where all three wave vectors are collinear. The summation can be carried over to an integral using

$$\sum \to \iiint \frac{V}{8\pi^3} q'^2 \sin\theta \, d\theta \, d\phi \, dq' .$$

The integration over $\phi$ is straightforward yielding $2\pi$, while the integration over $\theta$ involves the energy-conserving δ function. This yields a multiplicative factor of $(\hbar v_s)^{-1}$ plus the limits on $q'$ given in (39). Thus, we have

$$\frac{1}{\tau_p} = \Gamma_{p0\to\mathrm{ac}} = \frac{1}{2\pi}\left(\frac{\gamma\hbar\omega_0 v_s \mathcal{A}^2}{8c_l^2}\right) \times \int_{q_0'+q}^{q_0'+q} (q')^3 (q - q') n_q (n_{q'} + 1)^2 dq' = \frac{\hbar\omega_0^4 \mathcal{A}^2}{128\pi c_l^2 v_2^3} \left(\frac{1}{\epsilon_\infty} - \frac{1}{\epsilon_0}\right) q n_q (n_{q_0'} + 1). \tag{41}$$

The value of $q$ varies only slightly from $q_0 = (2m_c\omega_0/\hbar)^{1/2}$, so that this value can be used in (41).

There are very little data available on the third-order elastic constants of the III-V compounds, especially in the case of InSb. However, the average elastic constant $A$ can be obtained by extrapolating from the data for Ge and Si. The most reliable value for germanium is about $2.67 \times 10^{12}$ dyn/cm$^2$ at 77 °K obtained by Drabble and Fendley.[21] Similarly, a value of $2.84 \times 10^{12}$ dyn/cm$^2$ was obtained for silicon.[22] These values primarily are determined by using ultrasonic waves in the presence of a uniaxial stress. A value of $\mathcal{A}$ of $2.7 \times 10^{12}$ dyn/cm$^2$ was adopted for the present work.

The phonon decay rate of the polar mode phonons into acoustic phonons is plotted in Fig. 3 as a function of the lattice temperature. Since $\Gamma_{p0\to\mathrm{ac}}$ depends strongly on the number of optical and acoustic phonons, the decay rate is very low at low temperatures. At 77 °K, a value of 1.6 μsec is obtained for equilibrium phonon distributions.

## IV. RESULTS

The set of diffusion equations (1), (2′) for the energy distribution function $f(\mathcal{E})$ are solved using

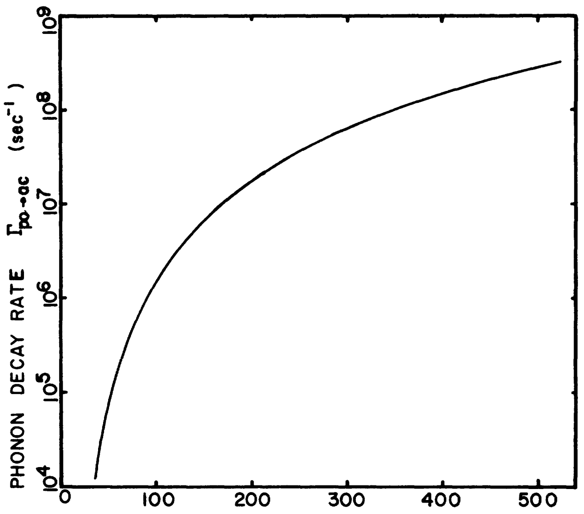

FIG. 3. Decay rate of the polar optical phonons into acoustic modes of the lattice through phonon-phonon interactions. The interaction arises through the anharmonic lattice potential. In the plot, it is assumed that both phonon distributions are in equilibrium.

an iterative technique to properly include the terms in $f(\mathcal{E} \pm \hbar\omega_0)$. These equations were solved on a computer, and subjected to the additional constraints of determining the equilibrium phonon quasitemperature. The constants used were for InSb and are shown in Table I. Although the constants are given in cgs units, the problem was solved in mks units since all of the above equations are expressed in mks units.

The effect of the disturbance of the phonon distribution on the distribution function is shown in Figs. 4 and 5. In Fig. 4, the case of InSb at 77 °K, for an applied electric field of 50 V/cm is shown. The solid curve takes account of the disturbed polar phonon density. The major effect of the increased absorption transitions appears as the tailing of $f(\mathcal{E})$ at high energy. The rise of $f(\mathcal{E})$ below $\hbar\omega_0$ occurs primarily from the $W_{\mathrm{em}}(\mathcal{E} \pm \hbar\omega_0)$ terms being increased due to the band tailing. In Fig. 5, the case for a lattice temperature of 4.2 °K and an applied electric field of 10 V/cm is shown. Again, the solid curve takes account of the disturbed phonon density. In this case, however, the absorption transitions $W_{\mathrm{abs}}(\mathcal{E})$ are ineffective until the phonon distribution is disturbed. In the disturbed case, this process accounts for the decrease of $f(\mathcal{E})$ below $\hbar\omega_0$ from the equilibrium case.

The energy distribution function for various values of the applied electric field at 4.2 °K is shown in Fig. 6. For values of the electric field of 0.1 V/cm or less the energy distribution function is a Maxwellian at the lattice temperature. As the electric field is raised, the diffusion terms cause an increase in $f(\mathcal{E})$ and phonon emission becomes important for $\mathcal{E} > \hbar\omega_0$. Although it is not apparent from these plots, both the energy dis-

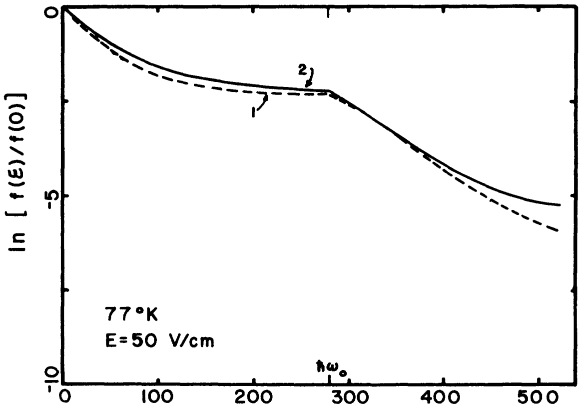

FIG. 4. Effect of a nonequilibrium phonon distribution on the energy distribution function in InSb at 77 °K. The energy distribution function with an equilibrium phonon density is shown in the dashed curve (1) and for a disturbed phonon density in the solid curve (2).

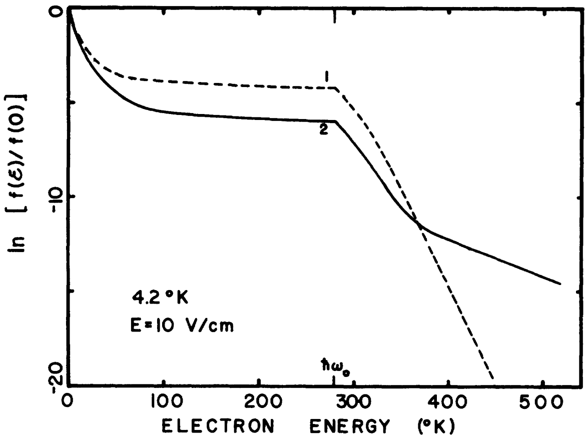

FIG. 5. Effect of a nonequilibrium phonon distribution on the energy distribution function in InSb at 4.2 °K. The energy distribution function for an equilibrium phonon density is shown in the dashed curve (1) and for a disturbed phonon density in the solid curve (2).

TABLE I. Values of the constants of InSb used in the calculations.

| | |
|---|---|
| $m_c = 0.013 m_0$ | $c_l = 8.0 \times 10^{11}$ dyn/cm$^2$ |
| $\epsilon_0 = 17.8$ | $\bar{r} = -8.0 \times 10^{-3}$ cm/V |
| $\mathcal{E}_1 = 7.2$ eV | $v_s = 3.5 \times 10^5$ cm/sec |
| $n_0 = 1.0 \times 10^{14}$/cm$^2$ | $\omega_0 = 3.67 \times 10^{13}$/sec |
| $\mathcal{E}_G = 0.22$ eV | $\mathfrak{a} = 2.7 \times 10^{12}$ dyn/cm$^2$ |
| $\epsilon_\infty = 15.7$ | |

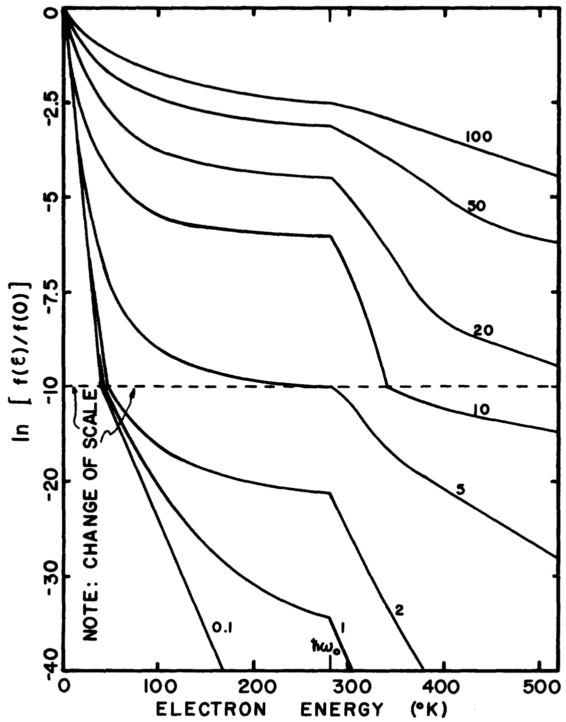

FIG. 6. Energy distribution function for InSb at 4.2 °K. The parameter is the applied electric field in V/cm. The curves break at $\hbar\omega_0$ due to the onset of optical phonon emission. Note that the vertical scale changes at $\ln[f(\mathcal{E})/f(0)] = -10$.

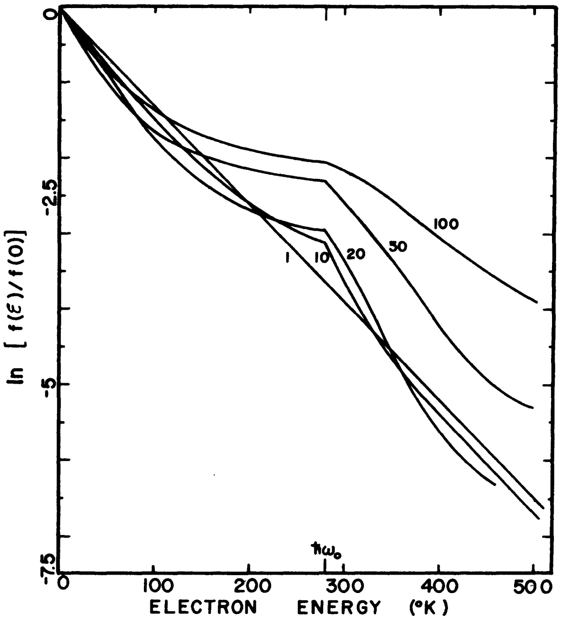

FIG. 7. Energy distribution function for InSb at 77 °K. The parameter is the applied electric field in V/cm.

tribution function $f(\mathcal{E})$ and its derivative are continuous at $\mathcal{E} = \hbar\omega_0$, as required by (1), (2′) [$W_{\mathrm{em}}$, $W_{\mathrm{abs}}(\mathcal{E} - \hbar\omega_0) = 0$ through the logarithm term]. At higher values of the electric field, absorption begins to play a strong role in the shape of $f(\mathcal{E})$ and leads to strong tailing in energy. At 100 V/cm, the variation of $f(\mathcal{E})$ is almost smooth throughout the energy range. In no case is there the sharp falloff of $f(\mathcal{E})$, for $\mathcal{E} > \hbar\omega_0$, that was reported in previous work.[8–10]

The energy distribution function at 77 °K for various values of the electric field is shown in Fig. 7. At this higher temperature, the distribution function remains Maxwellian for values of the applied electric field up to 1 V/cm, primarily due to the increased scattering owing to the increased polar-mode phonon density. For higher values of the electric field, the shape of the curves differs from the case at 4.2 °K. Generally, these curves lie below the Maxwellian for $\mathcal{E} < \hbar\omega_0$ due to the increased carrier density near $\mathcal{E} = 0$ due to the strong emission terms of $\mathcal{E} \gtrsim \hbar\omega_0$. In addition, absorption plays a role in depleting the states of $\mathcal{E} < \hbar\omega_0$. Again, however, strong tailing occurs at high values of the electric field.

The phonon quasitemperature $T_p$, defined in (24), is plotted in Fig. 8 as a function of applied electric field. At 4.2 °K, the phonon population is disturbed for fields above about 0.2 V/cm, while at 77 °K, the phonon density is disturbed only for electric fields above 20 V/cm. Above about 50

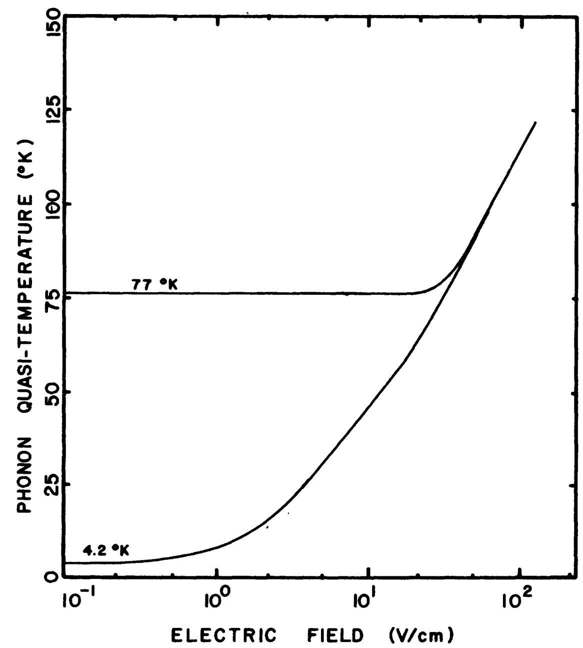

FIG. 8. Phonon quasitemperature, defined in Eq. (24), determining the disturbed optical phonon density as a function of carrier heating by the electric field. The curves here show this dependence for lattice temperatures of 4.2 and 77 °K.

V/cm, the phonon quasitemperature is the same regardless of the lattice temperature in the range of interest here. This is understandable in that the major energy exchange process for the electrons is through interactions with the optical phonons, so that an equilibrium is reached between the energy input from the electric field and the energy output to the phonons, a requirement that the phonon quasitemperature be determined by the electric field and not the initial lattice temperature.

## V. CONCLUSIONS

The strong falloff of $f(\mathcal{E})$ for $\mathcal{E} > \hbar\omega_0$ reported by others[8-10] using the diffusion equation approach to the energy distribution function is not observed in these calculations. Whether this anomaly arises from a convergence error in solution or an improper modification of Eqs. (1), (2′) by these authors to allow a convergent solution by Runge-Kutta-Gill techniques is unknown. As pointed out in Sec. I, care must be exercised in the application of diffusion equations to physical problems to ensure that a physical result is obtained.

In general, it is not expected that an extremely sharp break should be observed in the distribution function at the optical phonon energy. As pointed out in Sec. I, both the distribution function and its derivative are continuous at the optical phonon energy, even though other scattering processes begin to occur at this energy. Support for a solution that does not show the extremely sharp breaks observed by Yamada *et al.*,[8-10] is also present in the Monte Carlo calculations that have been carried out so far.[23] Although the curves of $f_0(\mathcal{E})$ that are shown by Fawcett and Ruch[23] are for higher values of the electric field, the results in general agree with those that are presented in this paper.

The disturbance of the polar phonon density by the carrier heating is extremely important, especially at low lattice temperatures. The disturbed polar phonon distribution plays a strong role in energy relaxation of the carriers as well as significantly affecting the shape of the energy distribution function. The calculations here yield an effective phonon temperature that results in a phonon relaxation time that agrees within an order of magnitude of the value extrapolated from the experimental data of Kranzer and Gornik.[13]

## APPENDIX A: CALCULATION OF THERMAL DIFFUSION COEFFICIENTS

Disregarding the optical phonons for $D_T$, the motion of electrons in energy space is composed of a series of small, mutually independent and essentially almost elastic displacements due to scattering. The diffusion coefficient $D_T$, can then be written as[8]

$$D_T(\mathcal{E}) = \frac{1}{2} \sum_i (\Delta \mathcal{E}_i)^2 W_i(\mathcal{E}) = \sum_i D_{Ti}(\mathcal{E}) . \qquad \text{(A1)}$$

The individual terms of $D_T$ are related to the transition probability of the particular collision by

$$D_{Ti}(\mathcal{E}) = \frac{1}{2} \sum_{\mathcal{E}'} (\Delta\mathcal{E})^2 W_i(\mathcal{E}, \mathcal{E}') . \qquad \text{(A2)}$$

The matrix element for piezoelectric scattering can be calculated in a straightforward manner from the piezoelectric potential

$$V = e\bar{r}S/q\epsilon , \qquad \text{(A3)}$$

where $S$ is the lattice strain tensor $\nabla u \sim iqu$. The matrix element for equipartition of the phonons is

$$|M|^2 = \frac{e^2 k_B T_0}{\epsilon^2 q^2 V} \left\langle \frac{r^2}{c_l} \right\rangle , \qquad \text{(A4)}$$

where the bar in (A3) and brackets in (A4) denote an appropriate average. Then

$$W_{pe} = \frac{2\pi}{\hbar} \left( \frac{e^2 k_B T_0}{\epsilon^2 q^2 V} \right) \left\langle \frac{r^2}{c_l^2} \right\rangle \delta(\mathcal{E} - \mathcal{E}' \pm \hbar v_s q)\, \delta(\vec{k} - \vec{k}' \pm \vec{q}) \qquad \text{(A5)}'$$

and

$$D_{T,pe}(\mathcal{E}) = \frac{2\pi}{\hbar} \left( \frac{e^2 k_B T_0}{\epsilon^2 V} \right) \left\langle \frac{r^2}{c_l} \right\rangle \sum_{\vec{q}} \frac{1}{2} (\hbar v_s q)^2 \frac{1}{q^2} , \qquad \text{(A6)}$$

where $\hbar v_s q$ is the average energy exchange. The summation can be changed to an integral,[1] and

$$D_{T,pe}(\mathcal{E}) = \frac{m^* v_s^2}{\pi\hbar} \left( \frac{e^2 k_B T_0}{\epsilon^2} \right) \left\langle \frac{r^2}{c_l} \right\rangle k . \qquad \text{(A7)}$$

Using $k$ from (6) and $m^* = m_c(1 + 2\mathcal{E}/\mathcal{E}_G)$ ,

$$D_{T,pe}(\mathcal{E}) = \frac{2^{1/2} m_c^{3/2} v_s^2 e^2 k_B T_0}{\pi\hbar^2 \epsilon^2} \left\langle \frac{r^2}{c_l} \right\rangle \times \left[ \mathcal{E} \left( 1 + \frac{\mathcal{E}}{\mathcal{E}_G} \right) \right]^{1/2} \left( 1 + \frac{2\mathcal{E}}{\mathcal{E}_G} \right) \qquad \text{(A8)}$$

$$= \frac{2 m_c v_s^2}{\tau_{pe}} \mathcal{E} \left( 1 + \frac{\mathcal{E}}{\mathcal{E}_G} \right) , \qquad \text{(A9)}$$

where the last form arises from using the value of $\tau_{pe}$ from Hutson,[24] modified for nonparabolicity;

$$\frac{1}{\tau_{pe}} = \frac{m_c^{1/2} e^2 k_B T_0}{2^{1/2} \pi\hbar^2 \epsilon^2} \left\langle \frac{r^2}{c_l} \right\rangle \times \left[ \mathcal{E} \left( 1 + \frac{\mathcal{E}}{\mathcal{E}_G} \right) \right]^{-1} \left( 1 + \frac{2\mathcal{E}}{\mathcal{E}_G} \right) . \qquad \text{(A10)}$$

The matrix element for acoustic deformation potential scattering is familiar and is given for the equipartition case by

$$|M|^2 = \frac{\mathcal{E}_1^2 k_B T_0}{2VDc_l^2} \delta(\mathcal{E} - \mathcal{E}' \pm \hbar v_s q)\, \delta(\vec{k} - \vec{k}' \pm q) . \qquad \text{(A11)}$$

The diffusion coefficient is then

$$D_{T,\mathrm{ac}}(\mathcal{E})=\frac{\mathcal{E}_1^2 k_B T_0}{2VDc_l^2}\sum_{\vec{q}}\frac{1}{2}(\hbar v_s q)^2 \quad . \tag{A12}$$

Carrying the sum over $q$ into an integral gives

$$\begin{aligned}D_{T,\mathrm{ac}}(\mathcal{E})&=\frac{2v_s^2\mathcal{E}_1^2 m^* k_B T_0 k^3}{3\pi\hbar c_l}\\&=\frac{2^{5/2}v_s^2\mathcal{E}_1^2 m_c^{5/2}k_B T_0}{3\pi\hbar^4 c_l}\\&\quad\times\left[\mathcal{E}\left(1+\frac{\mathcal{E}}{\mathcal{E}_G}\right)\right]^{3/2}\left(1+\frac{2\mathcal{E}}{\mathcal{E}_G}\right)\\&=\frac{4m_c v_s^2}{3\tau_{\mathrm{ac}}}\,\mathcal{E}\left(1+\frac{\mathcal{E}}{\mathcal{E}_G}\right),\end{aligned} \tag{A13}$$

where

$$\frac{1}{\tau_{\mathrm{ac}}}=\frac{2^{1/2}\mathcal{E}_1^2 m_c^{3/2}k_B T_0}{\pi\hbar^4 c_l}\left[\mathcal{E}\left(1+\frac{\mathcal{E}}{\mathcal{E}_G}\right)\right]^{1/2}\left(1+\frac{2\mathcal{E}}{\mathcal{E}_G}\right). \tag{A14}$$

The calculation for the thermal diffusion coefficient for ionized impurity scattering is somewhat more complicated since the energy exchange is zero. We can calculate $D_T$ in a heuristic manner though. Equation (15) defines an effective electron temperature which would be valid if impurity scattering were the only scattering event present. Thus,

$$T_e=T_0(1+D_F/D_T)\ . \tag{A15}$$

The average energy gain of the electrons is given by

$$e\mu E^2\tau\ , \tag{A16}$$

where $\mu$ is the electron mobility. This energy gain determines the electron temperature through

$$k_B(T_e-T_0)=e\mu E^2\tau\ . \tag{A17}$$

Combining (A15) and (A17) yields

$$T_0\frac{D_F(\mathcal{E})}{D_T(\mathcal{E})}=\frac{e\mu(\mathcal{E})E^2\tau(\mathcal{E})}{k_B}=\frac{e^2E^2}{m_c k_B}\left(\frac{\tau^2(\mathcal{E})}{1+2\mathcal{E}/\mathcal{E}_G}\right)\ . \tag{A18}$$

Now, $D_F(\mathcal{E})$ is given from (5) as

$$D_F(\mathcal{E})=\frac{2(eE)^2}{3m_c}\left(\frac{\mathcal{E}}{1+2\mathcal{E}/\mathcal{E}_G}\right)\ . \tag{A19}$$

Equations (A18) and (A19) can be combined to yield

$$D_T(\mathcal{E})=(2k_B T_0/3\tau_{\mathrm{imp}})\,\mathcal{E}\ , \tag{A20}$$

and the lifetime is[25]

$$\frac{1}{\tau_{\mathrm{imp}}}=\frac{e^4 N_I}{16\pi\epsilon^2(2m_c)^{1/2}\mathcal{E}^{3/2}}\times\ln\left(\frac{8\mathcal{E}(1+\mathcal{E}/\mathcal{E}_G)m_c\lambda_D^2}{\hbar^2}\right)\frac{1+2\mathcal{E}/\mathcal{E}_G}{(1+\mathcal{E}/\mathcal{E}_G)^{3/2}}\ , \tag{A21}$$

where $\lambda_D$ is the Debye length.

The reciprocal relaxation times in Eqs. (A10), (A14), (A21) still must be modified by the overlap integral corrections to the matrix elements. Using Kane's wave functions,[14] the overlap integral is just

$$I_{kk'}=(a_k a_{k'}+c_k c_{k'}\cos\theta)^2\ , \tag{A22}$$

where

$$a_k=\left(\frac{1+\mathcal{E}/\mathcal{E}_G}{1+2\mathcal{E}/\mathcal{E}_G}\right)^{1/2}, \tag{A23}$$

$$c_k=\left(\frac{\mathcal{E}/\mathcal{E}_G}{1+2\mathcal{E}/\mathcal{E}_G}\right)^{1/2}. \tag{A24}$$

In the scattering processes of interest, the collisions are elastic so that $\mathcal{E}=\mathcal{E}'$. In the piezoelectric interaction, small-angle scattering is accentuated so that $\cos\theta\approx 1$, and

$$I_{\mathrm{pe}}\cong 1. \tag{A25}$$

For acoustic scattering, the scattering angle is easily determined and averaged with $1-\cos\theta$ for relaxation effects. In this case

$$I_{\mathrm{ac}}^2\approx\frac{1}{1+2\mathcal{E}/\mathcal{E}_G}\left(1+\frac{2\mathcal{E}/\mathcal{E}_G}{1+2\mathcal{E}/\mathcal{E}_G}\right), \tag{A26}$$

for $\mathcal{E}\ll\mathcal{E}_G$. Similarly, the value for impurity scattering is found to be

$$I_{\mathrm{imp}}\approx 1 \tag{A27}$$

for $\mathcal{E}\ll\mathcal{E}_G$.

## APPENDIX B: EVALUATION OF THE IMPORTANCE OF THE ABSORPTION TRANSITIONS

The importance of the absorption terms can be estimated by consideration of the terms $\Gamma_1$, $\Gamma_2$ from (3), (4), respectively. The probability of polar phonon emission or absorption is given by

$$W_{\mathrm{em}}(\mathcal{E})=W(\mathcal{E})\ln\left(\frac{1+\{1-(\hbar\omega_0/\mathcal{E})[1+(\mathcal{E}-\hbar\omega_0)/\mathcal{E}_G]\}^{1/2}}{1-\{1-(\hbar\omega_0/\mathcal{E})[1+(\mathcal{E}-\hbar\omega_0)/\mathcal{E}_G]\}^{1/2}}\right)\left(1+\frac{2(\mathcal{E}-\hbar\omega_0)}{\mathcal{E}_G}\right)(n_q+1)I_{kk'}\ , \tag{B1}$$

$$W_{\mathrm{abs}}(\mathcal{E})=W(\mathcal{E})\ln\left(\frac{\{1+(\hbar\omega_0/\mathcal{E})[1+(\mathcal{E}+\hbar\omega_0)/\mathcal{E}_G]\}^{1/2}+1}{\{1+(\hbar\omega_0/\mathcal{E})[1+(\mathcal{E}+\hbar\omega_0)/\mathcal{E}_G]\}^{1/2}-1}\right)\left(1+\frac{2(\mathcal{E}+\hbar\omega_0)}{\mathcal{E}_G}\right)(n_q)I_{kk'}\ , \tag{B2}$$

where

$$W(\mathcal{E})=\frac{m_c e^2\omega_0}{(2m_c)^{1/2}\hbar[\mathcal{E}(1+\mathcal{E}/\mathcal{E}_G)]^{1/2}}\left(\frac{1}{\epsilon_\infty}-\frac{1}{\epsilon_0}\right), \tag{B3}$$

$\omega_0$ is the longitudinal polar optical phonon frequency, $\epsilon_\infty$ is the high-frequency dielectric constant, $\epsilon_0$ is the low-frequency dielectric constant, $I_{kk'}$ is the overlap integral discussed below, and $n_q$ is the optical phonon occupation number given by the Bose-Einstein distribution

$$n_q=(e^{\hbar\omega_0/k_BT_0}-1)^{-1}. \tag{B4}$$

The density of states $w(\mathcal{E})$ is

$$w(\mathcal{E})=\frac{1}{2\pi^2}\left(\frac{2m_c}{\hbar^2}\right)^{3/2}\mathcal{E}^{1/2}(1+\mathcal{E}/\mathcal{E}_G)^{1/2} \tag{B5}$$

so that $W(\mathcal{E})w(\mathcal{E})$ is a constant independent of energy. Thus, we may write $\Gamma_1$ as

$$\Gamma_1=W_{\mathrm{em}}(\mathcal{E})w(\mathcal{E})f(\mathcal{E})\left[1-Q(\mathcal{E})\frac{f(\mathcal{E}-\hbar\omega_0)}{f(\mathcal{E})}\left(\frac{n_q}{n_q+1}\right)\right], \tag{B6}$$

where $Q(\mathcal{E})$ is the ratio of the ln functions from (B1) and (B2). In the parabolic case

$$Q(\mathcal{E})=\ln\left(\frac{(\mathcal{E})^{1/2}+(\mathcal{E}-\hbar\omega_0)^{1/2}}{(\mathcal{E})^{1/2}-(\mathcal{E}-\hbar\omega_0)^{1/2}}\right)\Big/\ln\left(\frac{(\mathcal{E}-\hbar\omega_0+\hbar\omega_0)^{1/2}+(\mathcal{E}-\hbar\omega_0)^{1/2}}{[(\mathcal{E}-\hbar\omega_0)+\hbar\omega_0]^{1/2}-(\mathcal{E}-\hbar\omega_0)^{1/2}}\right)=1.$$

Now $(n_q+1)/n_q$ is just $e^{\hbar\omega_0/k_BT_0}$ so that (B6) will vanish in the parabolic case if $f(\mathcal{E})$ falls off as a Maxwellian at the lattice temperature $T_0$. A similar result obtains for $\Gamma_2$. In the nonparabolic case, $f(\mathcal{E})$ is modified only very slightly by the additional factors in $Q(\mathcal{E})$ and still is nearly Maxwellian for $\Gamma_1$, $\Gamma_2$ to vanish. The absorption transitions must be retained if proper convergence to a Maxwellian at low electric fields is to occur.

Further, only after the diffusion effects drive $f(\mathcal{E})$ away from Maxwellian does $I(\mathcal{E})$ begin to play a major role in the dependence of $f(\mathcal{E})$ on energy. The overlap integral differs from 1 in the nonparabolic case due to an admixture of $p$-symmetry wave functions. It can be evaluated using Kane's wave functions,[14] and for optical-mode scattering is given by

$$I_{kk'}(\mathcal{E})=1-\left[\left(\frac{\mathcal{E}(\mathcal{E}\pm\hbar\omega_0+\mathcal{E}_G)}{(\mathcal{E}+\mathcal{E}_G)(2\mathcal{E}\pm 2\hbar\omega_0+\mathcal{E}_G)}\right)^{1/2}+\left(\frac{(\mathcal{E}+\mathcal{E}_G)(\mathcal{E}\pm\hbar\omega_0)}{(2\mathcal{E}+\mathcal{E}_G)(2\mathcal{E}\pm 2\hbar\omega_0+\mathcal{E}_G)}\right)^{1/2}\right]^2, \tag{B7}$$

where the upper sign is for phonon absorption and the lower sign is for phonon emission.

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