Ferromagnetic Hall Effect in an Electron-Phonon Gas

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The contribution of the electron-phonon interaction to the anomalous Hall effect in ferromagnetic metals is calculated, utilizing Holstein's transport theory of the electron-phonon gas. Specifically, the transverse dc conductivity σ_{xy} is calculated using Kubo's formalism and the temperature diagrammatic technique. Results based on the (ferromagnetically) polarized itinerant-electron model are consistent with the Luttinger's theory of the anomalous Hall effect for impurity scattering. This leads to the non-Boltzmann (so-called "anomalous-velocity") contribution as well as the skew-scattering contribution of the type first obtained by Smit. However, it is in disagreement with the result of Leribaux based on the same model. A principal feature of the treatment is the explicit representation in terms of charge-carrier motion between different Wannier cells as first suggested by Smit. However, in contrast to the latter author, who took the velocity operator of this intercell motion to be the ordinary Bloch velocity, it is found that there is an additional velocity component of intercell motion arising from the electron scattering mechanism (either electron-phonon interaction, as in this case, or electron-impurity interaction as in the previous treatments of Smit or Luttinger). This additional component gives rise, in turn, to the above-mentioned anomalous-velocity correction to σ_{xy}, and hence, to the Hall coefficient.

I. INTRODUCTION

The experimental results[1,2] for the anomalous Hall effect in ferromagnetic metals and alloys can generally be described by the following formula:

$$\rho_H = E_x/J_y = R_0 B_z + 4\pi R_s M_z , \tag{1.1}$$

where ρ_H, E_x, B_z, M_z, and J_y are, respectively, the Hall resistivity, Hall field, magnetic field, magnetization, and the electrical current in the directions indicated by the subscripts in the presence of an external electric field E_y. The first term represents the ordinary Hall effect and is caused by the magnetic Lorentz force. R_0 is defined as the normal Hall coefficient. The second term has a strong dependence on the temperature and represents the anomalous Hall effect with R_s defined as anomalous Hall coefficient. R_s is usually much larger than R_0 except at low temperature. It is found experimentally[2,3] that $R_s \propto \rho^n$ for varying temperature or impurity concentration with n ranging from 1.2 to 2 (ρ is the resistivity).

The first theoretical attempt to explain the origin of the anomalous Hall effect was made by Karplus and Luttinger.[4] According to their model, the anomalous Hall current is caused by magnetically polarized conduction electrons (responsible for the ferromagnetism) which gain a transverse component of velocity through spin-orbit scattering, as they are driven by an external field in a periodic lattice potential. However, this theory has been criticized by Smit[2] on the grounds that under a perfect periodic potential, such as spin-orbit interaction, electrons are in stationary Bloch states and no acceleration (or scattering) can result. He further argued that the system will not even be in a stationary state in the absence of the scattering mechanism which can destroy the periodicity of the system and that if one introduces a scattering mechanism such as impurity or phonon scattering, those terms found by Karplus and Luttinger will be canceled out. Instead Smit[5] proposed, as a main cause for the anomalous Hall current, a "skew-scattering" mechanism whereby a conduction electron is more favorably scattered in the vicinity of an impurity into a direction given by $\vec{k}\times\vec{s}$ ($\vec{k}$ is the momentum and $\vec{s}$ is the spin of the electron), resulting in a net transverse current for polarized spins. He further showed that the Born approximation of the scattering process does not include the spin-orbit interaction to the first order and that only the higher Born orders are responsible for the skew scattering. However, Luttinger,[6] in subsequent work, improved the original Karplus-Luttinger theory by explicitly treating impurity scattering (instead of using the usual phenomenological relaxation time *Ansatz*) and applying a rigorous electron quantum-transport theory developed by Kohn and Luttinger.[7] His results show, indeed, the cancellation of the terms of Karplus and Luttinger as well as the appearance of skew-scattering contributions entering through the higher-Born-order corrections to the collision matrix in the Boltzmann equation. However, contrary to Smit's claim, there are also some other important terms [(3.26) of Luttinger[6]] which cannot be explained classically according to the Boltzmann equation.

Different types of models based entirely on the skew-scattering mechanism have been considered by Kondo[8] and Maranzana.[9] In their models the main scattering responsible for the ordinary resistivity is provided by the s-d interaction between the unpolarized itinerant s electrons and the polarized local d (or f) electrons. The skew scattering arises from the spin-orbit interaction of the d electrons within the magnetic ions in the case of Kondo and from the d-spin other (s-) orbit interaction in the case of Maranzana. The theories of these authors claim reasonably correct temperature dependence of the Hall resistivity. However, as pointed out by Maranzana,[9] the effect predicted by these theories is two orders of magnitude too small. These models are beyond the scope of this work and will not be considered any further.

This paper applies the same basic idea proposed by Luttinger[6] to a system of "ferromagnetically" polarized conduction electrons interacting with *phonons* (instead of impurities). The extension of the Kohn-Luttinger (KL) transport theory to this system is rather difficult.[10] The field-theoretic many-body approach based on Kubo's formalism[11] and Holstein's transport theory[12] of electron-phonon gas is used. The obtained results are completely consistent with Luttinger's theory as is to be expected in view of the basic similarity of the two problems. It is also found, by separating the cordinate operator into nonlocal and local parts [corresponding to $\vec{r}=\vec{g}+(\vec{r}-\vec{g})$, $\vec{g}$ is a lattice vector], that (a) only the motion of the electron between different Wannier cells contributes to the transverse current, and (b) the velocity associated with this intercell motion consists not only of the usual Bloch velocity but also of a contribution arising from the electron scattering mechanism (in our case, electron phonon; in Luttinger's case, electron impurity). This last term was omitted by Smit.

In Sec. II some basic formulations are developed and in Sec. III the quantities such as conductivity tensors, correlation functions, established in Sec. II, are evaluated by an extensive use of Holstein's theory. In Sec. IV a brief discussion is given on the relations between the results of this work and other theories.

II. BASIC FORMULATION

As a starting point for this investigation, the Fröhlich Hamiltonian for the electron-phonon system is adopted, i.e.,

$$H=\sum_{kn}\epsilon_{kn}a^\dagger_{kn}a_{kn}+\sum_q \hbar\omega_q^{(0)}b^\dagger_q b_q + \sum_{knk'n'q} V^{(0)}_{kn,k'n'}\,\delta_{k'+q,k}\,a^\dagger_{kn}a_{k'n'}(b_q+b^\dagger_{-q}). \qquad (2.1)$$

The first term annihilates and then creates an electron of wave vector $\vec{k}$ and band index n with a Bloch energy ϵ_{kn}. The Bloch state $|\vec{k}n\rangle$ includes the effect of the periodic spin-orbit interaction averaged over electron spins[6]:

$$H_{so}=\sum_i \frac{\hbar}{4m^2c^2}\,\frac{\vec{M}}{M_0}\times\nabla_i U(\vec{r}_i)\cdot\vec{p}_i, \qquad (2.2)$$

where $\vec{M}$, M_0, $U(\vec{r}_i)$, m, and $\vec{p}_i$ are magnetization, saturation magnetization of the sample, periodic lattice potential, electron mass, and electron momentum for the ith electron, respectively. The second term of (2.1) annihilates and then creates a phonon of wave vector $\vec{q}$ and "bare" frequency $\omega_q^{(0)}$. Finally, the third term describes the "bare" interaction (as indicated by a superscript zero on the matrix element) between the electron and phonon. The arrows for vectors are suppressed.

In order to relate the anomalous Hall coefficient R_s to the conductivity tensor, the anomalous part of (1.1) is separated out:

$$E_x^{(s)}/J_y=4\pi R_s M_z, \qquad (2.3)$$

where $E_x^{(s)}$ is an anomalous Hall field. Using $E_x^{(s)}=\rho J_x^{(s)}=\rho\sigma_{xy}^{(s)}E_y$ and $J_y=(1/\rho)E_y$, it follows that

$$R_s=(1/4\pi M_z)\,\rho^2\sigma_{xy}^{(s)}. \qquad (2.4)$$

In the above expression the superscript s of $\sigma_{xy}^{(s)}$ stands for the linear component in the spin-orbit coupling parameter in the Taylor expansion of σ_{xy} in powers of this parameter. The superscript s will have the same meaning for other quantities later throughout this work. For the conductivity tensor σ_{xy}, a specific form given by Holstein and Friedman[13] is used:

$$\sigma_{xy}=\frac{e^2}{i\Omega}\lim_{\omega\to 0}\frac{\mathfrak{F}_{xy}(\hbar\omega+i0)-\mathfrak{F}_{xy}(i0)}{\omega}, \qquad (2.5)$$

where Ω is the volume of the crystal and 0 is interpreted as an infinitesimally small positive quantity. $\mathfrak{F}_{xy}(\hbar\omega+i0)$ is the analytic continuation of the velocity-velocity correlation function

$$\mathfrak{F}_{xy}(\hbar\omega_r)=(1/\beta)\int_0^\beta du_2\int_0^\beta du_1\, e^{\hbar\omega_r(u_2-u_1)}\mathfrak{F}_{xy}(u_2,u_1), \qquad (2.6)$$

where

$$\beta=1/\kappa T,\quad \omega_r=2\pi i r/\hbar\beta \ (r \text{ is the integer}) \qquad (2.7)$$

and

$$\mathfrak{F}_{xy}(u_2,u_1)=\langle T e^{u_2H}v_x e^{-u_2H}e^{u_1H}v_y e^{-u_1H}\rangle = (1/Z_G)\,\mathrm{Tr}(Te^{-\beta(H-\mu\hat{N})}e^{u_2H}v_x e^{-u_2H}e^{u_1H}v_y e^{-u_1H}). \qquad (2.8)$$

Here $\langle\cdots\rangle$, Z_G, T, μ, and $\hat{N}$ are the thermodynamic average, grand partition function, Dyson's time-ordering operator, chemical potential, and particle-number operator, respectively, and

$$\vec{v}=\sum_i \frac{[\vec{r}_i, H]}{i\hbar}\equiv\frac{[\vec{r}, H]}{i\hbar}. \qquad (2.9)$$

At this point it is convenient to divide the electron coordinate operator into two parts

$$\vec{r}=\vec{r}^{\,\mathrm{I}}+\vec{r}^{\,\mathrm{II}}=\sum_i [\vec{r}_i^{\,\mathrm{I}}+\vec{r}_i^{\,\mathrm{II}}], \qquad (2.10)$$

where $\vec{r}^{\,\mathrm{I}}$ and $\vec{r}^{\,\mathrm{II}}$ are, respectively, given in Bloch representation by[4,14]

$$\langle\vec{k}n|\vec{r}_i^{\,\mathrm{I}}|\vec{k}'n'\rangle = i\frac{\partial}{\partial\vec{k}}\delta_{\vec{k},\vec{k}'}\delta_{n,n'},$$
$$\langle\vec{k}n|\vec{r}_i^{\,\mathrm{II}}|\vec{k}'n'\rangle = i\vec{J}^{nn'}(\vec{k})\delta_{\vec{k},\vec{k}'}, \qquad (2.11)$$

where

$$\vec{J}^{nn'}(\vec{k})=\int u_{kn}^*(\vec{r}_i)\frac{\partial}{\partial\vec{k}}u_{kn'}(\vec{r}_i)\,d\vec{r}_i .$$

Here $u_{kn}(\vec{r}_i)$ is the periodic part of the Bloch function. In the Wannier representation one has

$$\langle\vec{g}n|\vec{r}_i^{\,\mathrm{I}}|\vec{g}'n'\rangle=\vec{g}\,\delta_{\vec{g},\vec{g}'}\delta_{n,n'},$$
$$\langle\vec{g}n|\vec{r}_i^{\,\mathrm{II}}|\vec{g}'n'\rangle=\int\phi_n^*(\vec{r}+\vec{g}'-\vec{g})\,\vec{r}\,\phi_{n'}(\vec{r})\,d\vec{r},$$

where $\langle\vec{r}|\vec{g}n\rangle\equiv\phi_n(\vec{r}-\vec{g})$ is a Wannier function corresponding to the nth orbital and centered at a lattice site $\vec{g}$. Physically, the singular (unbounded) intraband component $\vec{r}_{\vec{g}}^{\,\mathrm{I}}$ represents the position of the $\vec{g}$th Wannier cell appropriate for the description of the nonlocal intercell motion of an electron, and the periodic regular (bounded)[15] part $\vec{r}_{\vec{g}}^{\,\mathrm{II}}$ describes the local electric polarization effect within the $\vec{g}$th Wannier cell. According to the above scheme the conductivity tensor is separated into four parts:

$$\sigma_{xy}=\sigma_{xy}^{\mathrm{I\,I}}+\sigma_{xy}^{\mathrm{I\,II}}+\sigma_{xy}^{\mathrm{II\,I}}+\sigma_{xy}^{\mathrm{II\,II}}, \qquad (2.12)$$

where, for $A=\mathrm{I},\mathrm{II}$, $B=\mathrm{I},\mathrm{II}$,

$$\sigma_{xy}^{AB}=\frac{e^2}{i\Omega}\lim_{\omega\to0}\frac{\mathfrak{F}_{xy}^{AB}(\hbar\omega+i0)-\mathfrak{F}_{xy}^{AB}(i0)}{\omega}. \qquad (2.13)$$

The velocity-velocity correlation function $\mathfrak{F}_{xy}^{AB}(\hbar\omega_r)$ is given from (2.6) and (2.8) by

$$\mathfrak{F}_{xy}^{AB}(\hbar\omega_r)=\frac{-1}{Z_G}\sum_{nm}e^{-\beta(E_n-\mu N_n)}\left(\frac{\langle n|v_x^A|m\rangle\langle m|v_y^B|n\rangle}{E_n-E_m+\hbar\omega_r}+\frac{\langle n|v_y^B|m\rangle\langle m|v_x^A|n\rangle}{E_n-E_m-\hbar\omega_r}\right), \qquad (2.14)$$

with

$$\vec{v}^{A}=\frac{[\vec{r}^{A}, H]}{i\hbar}, \qquad (2.15)$$

where the basis kets are taken to be eigenstates of the field-free Hamiltonian ($H|n\rangle=E_n|n\rangle$, $\hat{N}|n\rangle=N_n|n\rangle$). It will turn out that the last three terms of (2.12) will cancel themselves. The last term of (2.12) can be reduced to a simple form by using (2.13)–(2.15) and noting that $\vec{r}^{\,\mathrm{II}}$ has no singular matrix element[15]:

$$\sigma_{xy}^{\mathrm{II\,II}}=\frac{-e^2}{i\Omega Z_G}\lim_{\omega\to0}\frac{1}{\omega}\sum_{nm}e^{-\beta(E_n-\mu N_n)}$$
$$\times\Bigg[\bigg(\langle n|\frac{[x^{\mathrm{II}},H]}{i\hbar}|m\rangle\langle m|\frac{[y^{\mathrm{II}},H]}{i\hbar}|n\rangle\Big/(E_n-E_m+\hbar\omega+i0)+\langle n|\frac{[y^{\mathrm{II}},H]}{i\hbar}|m\rangle\langle m|\frac{[x^{\mathrm{II}},H]}{i\hbar}|n\rangle\Big/(E_n-E_m-\hbar\omega-i0)\bigg)$$
$$-\bigg(\langle n|\frac{[x^{\mathrm{II}},H]}{i\hbar}|m\rangle\langle m|\frac{[y^{\mathrm{II}},H]}{i\hbar}|n\rangle\Big/(E_n-E_m+i0)+\langle n|\frac{[y^{\mathrm{II}},H]}{i\hbar}|m\rangle\langle m|\frac{[x^{\mathrm{II}},H]}{i\hbar}|n\rangle\Big/(E_n-E_m-i0)\bigg)\Bigg]$$
$$=\frac{e^2}{i\hbar\Omega Z_G}\mathrm{Tr}\{e^{-\beta(H-\mu\hat{N})}[x^{\mathrm{II}},y^{\mathrm{II}}]\}=\frac{e^2}{i\hbar\Omega}\langle[x^{\mathrm{II}},y^{\mathrm{II}}]\rangle, \qquad (2.16)$$

where the second equality is achieved by directly subtracting the third (fourth) term from the first (second) term and then evaluating the commutators by observing the cancellations in the numerators and the denominators. An alternative more rigorous proof of (2.16) and the following (2.17) is given in Appendix A. The second and third terms of (2.12) can also be reduced to

$$\sigma_{xy}^{\mathrm{I\,II}}=\frac{e^2}{\Omega}\lim_{\omega\to0}\tilde{\mathfrak{F}}_{xy}^{\mathrm{I\,II}}(\hbar\omega+i0),$$
$$\sigma_{xy}^{\mathrm{II\,I}}=\frac{-e^2}{\Omega}\lim_{\omega\to0}\tilde{\mathfrak{F}}_{xy}^{\mathrm{II\,I}}(\hbar\omega+i0), \qquad (2.17)$$

where the velocity-coordinate correlation functions are defined by

$$\tilde{\mathfrak{F}}_{xy}^{\mathrm{I\,II}}(\hbar\omega_r)=\frac{-1}{Z_G}\sum_{nm}e^{-\beta(E_n-\mu N_n)}\times\bigg(\langle n|\frac{[x^{\mathrm{I}},H]}{i\hbar}|m\rangle\langle m|y^{\mathrm{II}}|n\rangle\Big/(E_n-E_m+\hbar\omega_r)+\langle n|y^{\mathrm{II}}|m\rangle\langle m|\frac{[x^{\mathrm{I}},H]}{i\hbar}|n\rangle\Big/(E_n-E_m-\hbar\omega_r)\bigg), \qquad (2.18a)$$

$$\tilde{\mathfrak{F}}_{xy}^{\mathrm{II\,I}}(\hbar\omega_r)=\frac{-1}{Z_G}\sum_{nm}e^{-\beta(E_n-\mu N_n)}$$

$$\times\left(\langle n|x^{\rm II}|m\rangle\langle m|\frac{[y^{\rm I},H]}{i\hbar}|n\rangle\Big/(E_n-E_m+\hbar\omega_r)\right.$$

$$\left.+\langle n|\frac{[y^{\rm I},H]}{i\hbar}|m\rangle\langle m|x^{\rm II}|n\rangle\Big/(E_n-E_m-\hbar\omega_r)\right)\;. \tag{2.18b}$$

Therefore, one is left with evaluating (2.17) and

$$\sigma_{xy}^{\rm I\,I}=\frac{e^2}{i\Omega}\lim_{\omega\to 0}\frac{\mathfrak{F}_{xy}^{\rm I\,I}(\hbar\omega+i0)-\mathfrak{F}_{xy}^{\rm I\,I}(i0)}{\omega}\;. \tag{2.19}$$

At this point we introduce, for a future purpose, a formula[7] which results from (2.11) on integrating by part,

$$\langle kn|\,[x^{\rm I},\rho]\,|k'n'\rangle=iD_x\rho_{kn,k'n'}\;, \tag{2.20a}$$

and especially for the diagonal part,

$$\langle kn|[x^{\rm I},\rho]|kn\rangle=i\frac{\partial}{\partial k_x}\rho_{kn,kn}\;, \tag{2.20b}$$

where ρ is arbitrary and $D_\alpha=\partial/\partial k_\alpha+\partial/\partial k'_\alpha$. It then follows that

$$\frac{[\vec{r}^{\rm I},H]}{i\hbar}=\sum_{knk'n'}\left(\vec{v}_{kn}\delta_{\vec{k},\vec{k}'}\delta_{n,n'}+\frac{1}{\hbar}\sum_q(\vec{D}\,V^{(0)}_{kn,k'n'})\right.$$

$$\left.\times(b_q+b_{-q}^\dagger)\,\delta_{\vec{k},\vec{k}'+\vec{q}}\right)a^\dagger_{kn}a_{k'n'}\;, \tag{2.21}$$

where $\vec{v}_{kn}=(1/\hbar)(\partial/\partial\vec{k})\epsilon_{kn}$.

With these basic formulations established, one proceeds to Sec. III to evaluate the correlation functions and the conductivity tensors.

III. HALL CONDUCTIVITY

In this section the correlation functions and the conductivity tensors are evaluated to the lowest order (i.e., zeroth order) in electron-phonon interaction and first order in spin-orbit coupling, using diagrammatic many-body theory. The prescription to the diagrammatic rule is detailed in the work of Holstein[12]; it will be used without repeating it here.

In Sec. III A, $\sigma_{xy}^{\rm I\,I}$ is evaluated to the first order in spin-orbit interaction. In Sec. III B we show to the lowest order in electron-phonon interaction that the contribution to the conductivity tensor from the polarization part vanishes, i.e., $\sigma_{xy}^{p}\equiv\sigma_{xy}^{\rm I\,II}+\sigma_{xy}^{\rm II\,I}+\sigma_{xy}^{\rm II\,II}=0$. This is a consequence of a more general transport property; in dc limit, σ_{xy}^{p} arises solely from the intracell motion (i.e., $\sigma_{xy}^{p}\propto d\langle x^{\rm II}\rangle/dt$) and vanishes to all orders of interactions for arbitrary interaction mechanisms.[16(a)] One can also show that σ_{xy}^{p} represents the contribution from the "dipole driving" term (i.e., the term proportional to $\vec{r}^{\rm II}\cdot\vec{E}$ in the total Hamiltonian). These problems will be treated elsewhere.[16(b)] Finally, in Sec. III C we study the temperature dependence of the Hall effect in high- and low-temperature regimes.

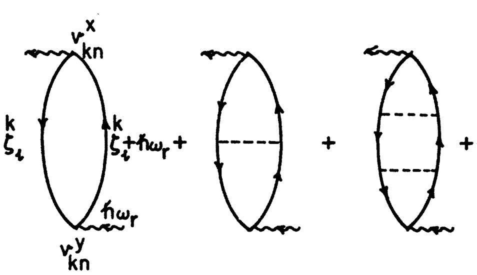


FIG. 1. Contribution of the ladder diagrams to the velocity correlation function.

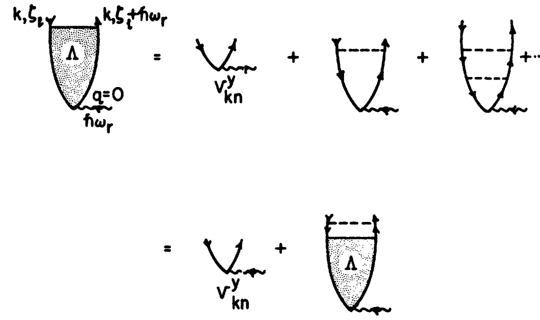


FIG. 2. Diagrammatic definition of the EF vertex.

A. Evaluation of $\sigma_{xy}^{\rm II(s)}$

It is well known[12] that the most important contribution to the velocity correlation function $\mathfrak{F}_{xy}^{\rm I\,I}(\hbar\omega_r)$ [represented by $\mathfrak{F}_{xy}(\hbar\omega_r)$ hereafter] arises from the series of ladder diagrams shown in Fig. 1, where the solid lines represent the full electron propagators and the dotted lines indicate the full phonon propagators. The external field lines are shown by wiggly lines. Each member of the ladder series shown in the figure is equally important; due to the occurrence of two overlapping resonances of the electron propagators introduced in pairs by an addition of each rung, the contribution of each extra rung turns out to be of zeroth order in electron-phonon interaction. Therefore, a summation of the series is necessary. However, as will be shown, this ladder series contribution which is of lowest order (i.e., of order V^{-2} in electron-phonon interaction) does not include the spin-orbit interaction to the first order, so that one has to go to the next-higher-order (i.e., zeroth-order) terms.

For this purpose it is useful, following Holstein,[12] to carry out the correction of the external-field (EF) vertex. This quantity, independent of the spin-orbit interaction parameter to the first order, leads to the Holstein-Boltzmann equation (3.13) and will be a fundamental tool for the investigation. As schematically shown in Fig. 2, the EF vertex

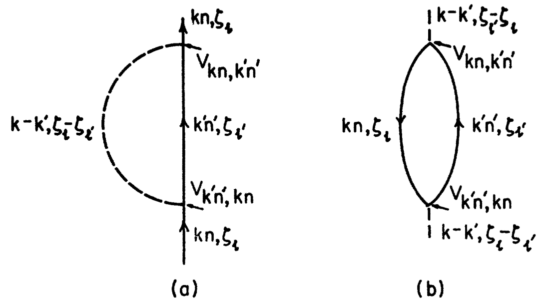


FIG. 3. Diagrams representing the dominant contributions to (a) the electron self-energy part and (b) the phonon self-energy part.

is given by an integral equation, for vanishing EF momentum,

$$\Lambda_{kn}(\zeta_l,\ \zeta_l+\hbar\omega_r)=v^y_{kn}+\frac{1}{\beta}\sum_{k'n'l'}|V_{k'n',kn}|^2 D_{kk'}(\zeta_l-\zeta_{l'})\,S_{k'n'}(\zeta_{l'})\,S_{k'n'}(\zeta_{l'}+\hbar\omega_r)\times\Lambda_{k'n'}(\zeta_{l'},\ \zeta_{l'}+\hbar\omega_r). \tag{3.1}$$

Definitions of the symbols are as follows: (i) $V_{k'n',kn}$ is a renormalized[17] electron-phonon interaction given by $V_{k'n',kn}=(\omega^{(0)}_{k'k}/\omega_{k'k})\,V^{(0)}_{k'n',kn}$, where $\omega^{(0)}_{k'k}=\omega^{(0)}(|\vec{k}'-\vec{k}|)$, and $\omega_{k'k}=\omega(|\vec{k}'-\vec{k}|)$ is a renormalized phonon frequency. (ii) $D_{kk'}(\zeta_l-\zeta_{l'})$ is the renormalized phonon propagator with the phonon self-energy part taken to the lowest order in electron-phonon interaction

$$D_{kk'}(\zeta_l-\zeta_{l'})=\frac{2\hbar\omega_{kk'}}{(\hbar\omega_{kk'})^2-(\zeta_l-\zeta_{l'})^2}=\sum_{\pm}\frac{1}{\hbar\omega_{kk'}\pm(\zeta_l-\zeta_{l'})}, \tag{3.2}$$

where $\sum_\pm$ means summing on both signs and

$$\zeta_l=\mu+[(2l+1)/\beta]\pi i\quad (l \text{ is the integer}). \tag{3.3}$$

(iii) $S_{kn}(\zeta_l)$ is a full electron propagator given by

$$S_{kn}(\zeta_l)=1/[\zeta_l-\epsilon_{kn}-G_{kn}(\zeta_l)], \tag{3.4}$$

where $G_{kn}(\zeta_l)$ represents the self-energy. (iv) EF frequency parameter ω_r is given by (2.7). At this point, the EF vertex part of (2.21) should be renormalized as

$$\frac{[\vec{r}^{\,I},\,H]}{i\hbar}=\sum_{knk'n'}\left(\vec{v}_{kn}\,\delta_{\vec{k},\vec{k}'}\,\delta_{n,n'}+\frac{1}{\hbar}\sum_q(\vec{D}\,V_{kn,k'n'})\times(b_q+b^\dagger_{-q})\,\delta_{\vec{k},\vec{k}+\vec{q}}\right)a^\dagger_{kn}a_{k'n'}. \tag{3.5}$$

The dominant contribution to the electron and phonon self-energy comes from the diagrams shown in Fig. 3. The following analytic properties of the fermion propagator and the self-energy part will be used:

$S_{kn}(\zeta)$ is analytic everywhere on the complex ζ plane except for a branch cut on $\mathrm{Im}\,\zeta=0$ axis;

$S_{kn}(\zeta)$ goes to zero as $1/\zeta$ for an infinitely large value of $|\zeta|$;

and (3.6)

$G_{kn}(x\pm i0)=M_{kn}(x)\mp i\Gamma_{kn}(x)$ for real variable x, where $M_{kn}(x)$ and $\Gamma_{kn}(x)$ are real functions of x.

The summation on l' in (3.1) can be replaced by integration introducing a factor $\pm(\beta/2\pi i)[f^{(\mp)}(\zeta')+N_{kk'}]$ in conjunction with an integration contour Γ shown in Fig. 4, with $f^{(\mp)}(\zeta')$ and $N_{kk'}$ given by

$$f^{(\mp)}(\zeta')=1/(e^{\pm\beta(\zeta'-\mu)}+1), \tag{3.7}$$

$$N_{kk'}=1/(e^{\beta\hbar\omega_{kk'}}-1). \tag{3.8}$$

Noting that the residue of the phonon poles vanishes, one can deform the integral path Γ into Γ_0 by using the analytic properties (3.6) and by assuming that $\Lambda_{kn}(\zeta,\ \zeta+\hbar\omega_r)$ is analytic everywhere except for the branch cuts along $\mathrm{Im}\,\zeta=0$ and $\mathrm{Im}(\zeta+\hbar\omega_r)=0$ axes as can be proved *a posteriori*. Thus (3.1) can be rewritten

$$\Lambda_{kn}(\zeta_l,\ \zeta_l+\hbar\omega_r)=v^y_{kn}+\frac{1}{2\pi i}\sum_{k'n'\pm}\int_{\Gamma_0}d\zeta'\,\frac{\pm[f^{(\mp)}(\zeta')+N_{kk'}]\,|V_{k'n',kn}|^2\Lambda_{k'n'}(\zeta',\ \zeta'+\hbar\omega_r)}{[\hbar\omega_{kk'}\pm(\zeta_l-\zeta')][\zeta'-\epsilon_{k'n'}-G_{k'n'}(\zeta')][\zeta'+\hbar\omega_r-\epsilon_{k'n'}-G_{k'n'}(\zeta'+\hbar\omega_r)]} \tag{3.9}$$

The result of this integration (followed by analytic continuation to the real axis in the ζ plane) taken to the lowest order in the smallness parameter $\Gamma/E_F=\hbar/2\tau E_F=\lambda/l$ (E_F is the Fermi energy, τ is the relaxation time, λ is the de Broglie wavelength, l is the mean free path)[18] is given for real values of z by Holstein[12] as

$$\Lambda_{kn}(z-i0,\ z+\hbar\omega+i0)=v^y_{kn}+\sum_{k'n'\pm}\frac{|V_{k'n',kn}|^2\Lambda_{k'n'}(\epsilon_{k'n'}-i0,\ \epsilon_{k'n'}+\hbar\omega+i0)}{\hbar\omega-[M_{k'n'}(\epsilon_{k'n'}+\hbar\omega)-M_{k'n'}(\epsilon_{k'n'})]+i[\Gamma_{k'n'}(\epsilon_{k'n'})+\Gamma_{k'n'}(\epsilon_{k'n'}+\hbar\omega)]}$$
$$\times\left[\mathcal{P}\frac{1}{z-\epsilon_{k'n'}\pm\hbar\omega_{kk'}}\,[f^{(\mp)}(\epsilon_{k'n'})-f^{(\mp)}(\epsilon_{k'n'}+\hbar\omega)]+2\pi i\delta(z-\epsilon_{k'n'}\pm\hbar\omega_{kk'})\left(\frac{f^{(\mp)}(\epsilon_{k'n'})+f^{(\mp)}(\epsilon_{k'n'}+\hbar\omega)}{2}+N_{kk'}\right)\right], \tag{3.10}$$

$$\Lambda_{kn}(z+i\eta 0,\ z+\hbar\omega+i\eta 0)=\mathcal{V}^{y}_{kn}+\sum_{k'n'\pm}\frac{|V_{k'n',kn}|^2\Lambda_{k'n'}(\epsilon_{k'n'}-i0,\ \epsilon_{k'n'}+\hbar\omega+i0)}{\hbar\omega-[M_{k'n'}(\epsilon_{k'n'}+\hbar\omega)-M_{k'n'}(\epsilon_{k'n'})]+i[\Gamma_{k'n'}(\epsilon_{k'n'})+\Gamma_{k'n'}(\epsilon_{k'n'}+\hbar\omega)]}$$
$$\times\left(\mathcal{P}\frac{1}{z-\epsilon_{k'n'}\pm\hbar\omega_{kk'}}-i\pi\eta\delta(z-\epsilon_{k'n'}\pm\hbar\omega_{kk'})\right)[f^{(\mp)}(\epsilon_{k'n'})-f^{(\mp)}(\epsilon_{k'n'}+\hbar\omega)], \quad (3.11)$$

where $\eta=\pm 1$ and $\delta(z)$ is Dirac's δ function. $\mathcal{P}$ indicates the principal part. Once the quantity $\Lambda_{kn}(z-i0,\ z+\hbar\omega+i0)$ is found, this can be put into (3.11) for the evaluation of $\Lambda_{kn}(z+i0,\ z+\hbar\omega+i\eta 0)$. It is to be noted that when $\hbar\omega\to 0$, Λ_{kn} are of zeroth order in electron-phonon interaction, because M_{kn}, Γ_{kn} are of order V^2. By setting

$$\Phi_{kn}(\omega)=\frac{i\hbar\Lambda_{kn}(\epsilon_{kn}-i0,\ \epsilon_{kn}+\hbar\omega+i0)}{\hbar\omega-[M_{kn}(\epsilon_{kn}+\hbar\omega)-M_{kn}(\epsilon_{kn})]+[\Gamma_{kn}(\epsilon_{kn})+\Gamma_{kn}(\epsilon_{kn}+\hbar\omega)]}, \quad (3.12)$$

and by evaluating the self-energy part to the order given in Fig. 3, one transforms (3.10) into

$$-i\omega\Phi_{kn}(\omega)=\mathcal{V}^{y}_{kn}+\frac{2\pi}{\hbar}\sum_{k'n'\pm}|V_{k'n',kn}|^2[\Phi_{k'n'}(\omega)-\Phi_{kn}(\omega)]\left[\delta(\epsilon_{kn}-\epsilon_{k'n'}\pm\hbar\omega_{kk'})\left(\frac{f^{(\mp)}(\epsilon_{k'n'})+f^{(\mp)}(\epsilon_{k'n'}+\hbar\omega)}{2}+N_{kk'}\right)\right.$$
$$\left.+\frac{i}{2\pi}\mathcal{P}\frac{1}{\epsilon_{kn}-\epsilon_{k'n'}\pm\omega_{kk'}}[f^{(\mp)}(\epsilon_{k'n'}+\hbar\omega)-f^{(\mp)}(\epsilon_{k'n'})]\right], \quad (3.13)$$

known as the Holstein-Boltzmann transport equation,[19] which contains all the information on the transport properties. This equation will be the basic tool for subsequent work.

One is now ready to embark on the evaluation of the correlation function. As already discussed, the most important contribution comes from the diagram of Fig. 5(a), which is written

$$\mathfrak{F}^{(a)}_{xy}(\hbar\omega_r)=\frac{-1}{\beta}\sum_{knl}\frac{\mathcal{V}^{x}_{kn}\Lambda_{kn}(\zeta_l,\ \zeta_l+\hbar\omega_r)}{[\zeta_l-\epsilon_{kn}-G_{kn}(\zeta_l)][\zeta_l+\hbar\omega_r-\epsilon_{kn}-G_{kn}(\zeta_l+\hbar\omega_r)]}. \quad (3.14)$$

The l summation can be performed by introducing a factor $(\beta/2\pi i)f^{(-)}(\zeta)$ in connection with an integration along the contour Γ in Fig. 4 on the ζ plane. As before, the contour Γ is then deformed into Γ_0. The final result neglecting the terms of order Γ/E_F or higher (shown in Ref. 12) is given by

$$\mathfrak{F}^{(a)}_{xy}(\hbar\omega+i0)=-\sum_{kn}\left(\mathcal{V}^{x}_{kn}\frac{[f^{(-)}(\epsilon_{kn})-f^{(-)}(\epsilon_{kn}+\hbar\omega)]\Lambda_{kn}(\epsilon_{kn}-i0,\ \epsilon_{kn}+\hbar\omega+i0)}{\hbar\omega-[M_{kn}(\epsilon_{kn}+\hbar\omega)-M_{kn}(\epsilon_{kn})]+i[\Gamma_{kn}(\epsilon_{kn})+\Gamma_{kn}(\epsilon_{kn}+\hbar\omega)]}\right.$$
$$\left.-\int_{-\infty}^{\infty}\frac{1}{2}\delta'(z-\epsilon_{kn})[\Lambda_{kn}(z-i0,\ z+\hbar\omega-i0)f^{(-)}(z+\hbar\omega)+\Lambda_{kn}(z+i0,\ z+\hbar\omega+i0)f^{(-)}(z)]\right), \quad (3.15)$$

where the prime on the δ function indicates taking a derivative with respect to the argument. It is shown by Karplus and Luttinger[4] that ϵ_{kn} does not include the spin-orbit interaction to the first order. This is also true for $|V_{k'n',kn}|^2$ and, therefore, for M_{kn} and Γ_{kn},[20] as can be shown,[21] by using the following conjugation properties[22] of the matrix elements derived by Luttinger,[6] *assuming inversion* symmetry,[23]

$$(\overline{V}_{k'n',kn})^*=\delta_n\delta_{n'}\overline{V}_{k'n',kn},$$
$$(V^{(s)}_{k'n',kn})^*=-\delta_n\delta_{n'}V^{(s)}_{k'n',kn}, \quad (3.16)$$

where $\delta_n=\pm 1$, depending on the band, and $\overline{V}_{k'n',kn}$ and $V^{(s)}_{k'n',kn}$ are zeroth and first order in spin-orbit coupling, respectively. Therefore, as long as one restricts the self-energy part to the order shown in Fig. 3, the EF vertex Λ_{kn} and the correlation function in (3.15) do not contain spin-orbit interaction to the first order. Thus one can conclude that the diagram of Fig. 6(a) is not responsible for the anomalous Hall effect. Especially, there is no contribution of order V^{-2}. Dis-

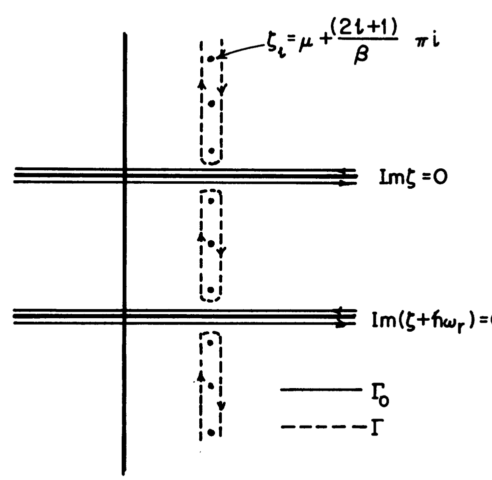


FIG. 4. Integration contours for Γ and Γ_0.

cussion of the effect of the higher-order self-energy correction will come later.

Proceeding to the next-higher-order diagrams (b) and (c) exhibited in Fig. 5, one seeks the zeroth-order contributions. Using (3.5), one computes

$$\mathcal{F}_{xy}^{(b)}(\hbar\omega_r)=\frac{1}{\hbar\beta^2}\sum_{knk'n'll'\pm}\left(\frac{V_{kn,k'n'}D_xV_{k'n',kn}}{\zeta_{l'}-\epsilon_{k'n'}-G_{k'n'}(\zeta_{l'})}+\frac{V_{k'n',kn}D_xV_{kn,k'n'}}{\zeta_{l'}+\hbar\omega_r-\epsilon_{k'n'}-G_{k'n'}(\zeta_{l'}+\hbar\omega_r)}\right)$$

$$\times\frac{\Lambda_{kn}(\zeta_l,\zeta_l+\hbar\omega_r)}{[\zeta_l-\epsilon_{kn}-G_{kn}(\zeta_l)][\zeta_l+\hbar\omega_r-\epsilon_{kn}-G_{kn}(\zeta_l+\hbar\omega_r)][\hbar\omega_{kk'}\pm(\zeta_l-\zeta_{l'})]}$$

$$=\frac{1}{\hbar\beta^2}\sum_{knk'n'l\pm}\frac{\beta}{2\pi i}\int_{\Gamma_0}d\zeta'\frac{\pm[f^{(\mp)}(\zeta')+N_{kk'}]}{\hbar\omega_{kk'}\pm(\zeta_l-\zeta')}$$

$$\times\left(\frac{V_{kn,k'n'}D_xV_{k'n',kn}}{\zeta'-\epsilon_{k'n'}-G_{k'n'}(\zeta')}+\frac{V_{k'n',kn}D_xV_{kn,k'n'}}{\zeta'+\hbar\omega_r-\epsilon_{k'n'}-G_{k'n'}(\zeta'+\hbar\omega_r)}\right)\frac{\Lambda_{kn}(\zeta_l,\zeta_l+\hbar\omega_r)}{[\zeta_l-\epsilon_{kn}-G_{kn}(\zeta_l)][\zeta_l+\hbar\omega_r-\epsilon_{kn}-G_{kn}(\zeta_l+\hbar\omega_r)]}$$

$$=\frac{\hbar^{-1}}{2\pi i\beta}\sum_{knk'n'l\pm}\int_{-\infty}^{\infty}d\zeta'\left[\frac{[f^{(\mp)}(\zeta')+N_{kk'}]}{\pm\hbar\omega_{kk'}+\zeta_l-\zeta'}V_{kn,k'n'}D_xV_{k'n',kn}+\frac{[f^{(\mp)}(\zeta'-\hbar\omega_r)+N_{kk'}]}{\pm\hbar\omega_{kk'}+\zeta_l+\hbar\omega_r-\zeta'}V_{k'n',kn}D_xV_{kn,k'n'}\right]$$

$$\times\left(\frac{1}{\zeta'-\epsilon_{k'n'}-G_{k'n'}(\zeta'-i0)}-\frac{1}{\zeta'-\epsilon_{k'n'}-G_{k'n'}(\zeta'+i0)}\right)\frac{\Lambda_{kn}(\zeta_l,\zeta_l+\hbar\omega_r)}{[\zeta_l-\epsilon_{kn}-G_{kn}(\zeta_l)][\zeta_l+\hbar\omega_r-\epsilon_{kn}-G_{kn}(\zeta_l+\hbar\omega_r)]}\,. \tag{3.17}$$

In the second equality of (3.17) use has been made of the fact that the factor $[f^{(\mp)}(\zeta')+N_{kk'}]$ vanishes at the poles, $\zeta'=\zeta_l\mp\hbar\omega_{kk'}$, of the phonon propagators (as in Ref. 12). For (3.17) one performs the k' summation first instead of attempting ζ' integration. Then, owing to the slowly varying nature of the quantity in the large square brackets as a function of k', one can approximate, to the lowest order in the electron-phonon interaction,[24]

$$\frac{1}{\zeta'-\epsilon_{k'n'}-G_{k'n'}(\zeta'-i0)}-\frac{1}{\zeta'-\epsilon_{k'n'}-G_{k'n'}(\zeta'+i0)}=2\pi i\delta(\zeta'-\epsilon_{k'n'}). \tag{3.18}$$

Hence

$$\mathcal{F}_{xy}^{(b)}(\hbar\omega_r)=\frac{1}{\hbar\beta}\sum_{\substack{knk'n'\\ l\pm}}\frac{[f^{(\mp)}(\epsilon_{k'n'})+N_{kk'}]\,\Lambda_{kn}(\zeta_l,\zeta_l+\hbar\omega_r)}{[\zeta_l-\epsilon_{kn}-G_{kn}(\zeta_l)][\zeta_l+\hbar\omega_r-\epsilon_{kn}-G_{kn}(\zeta_l+\hbar\omega_r)]}$$

$$\times\left(\frac{V_{kn,k'n'}D_xV_{k'n',kn}}{\pm\hbar\omega_{kk'}+\zeta_l-\epsilon_{k'n'}}+\frac{V_{k'n',kn}D_xV_{kn,k'n'}}{\pm\hbar\omega_{kk'}+\zeta_l+\hbar\omega_r-\epsilon_{k'n'}}\right)$$

$$=\frac{1}{\hbar\beta}\sum_{knk'n'\pm}\frac{\beta}{2\pi i}\int_{\Gamma_0}d\zeta\,\frac{[f^{(\mp)}(\epsilon_{k'n'})+N_{kk'}]\,\Lambda_{kn}(\zeta,\zeta+\hbar\omega_r)}{[\zeta-\epsilon_{kn}-G_{kn}(\zeta)][\zeta+\hbar\omega_r-\epsilon_{kn}-G_{kn}(\zeta+\hbar\omega_r)]}$$

$$\times\left(\frac{V_{kn,k'n'}D_xV_{k'n',kn}}{\pm\hbar\omega_{kk'}+\zeta-\epsilon_{k'n'}}+\frac{V_{k'n',kn}D_xV_{kn,k'n'}}{\pm\hbar\omega_{kk'}+\zeta_l+\hbar\omega_r-\epsilon_{k'n'}}\right)f^{(-)}(\zeta)$$

$$=\frac{\hbar^{-1}}{2\pi i}\sum_{\substack{knk'n'\\ \pm}}\int_{-\infty}^{\infty}d\zeta\,\frac{[f^{(\mp)}(\epsilon_{k'n'})+N_{kk'}]f^{(-)}(\zeta)}{\zeta+\hbar\omega_r-\epsilon_{kn}-G_{kn}(\zeta+\hbar\omega_r)}$$

$$\times\left(\frac{\Lambda_{kn}(\zeta-i0,\zeta+\hbar\omega_r)\,V_{kn,k'n'}D_xV_{k'n',kn}}{[\zeta-\epsilon_{kn}-G_{kn}(\zeta-i0)][\pm\hbar\omega_{kk'}+\zeta-i0-\epsilon_{k'n'}]}-(-i0\to+i0)\right)$$

$$+\frac{\hbar^{-1}}{2\pi i}\sum_{\substack{knk'n'\\ \pm}}\int_{-\infty}^{\infty}d\zeta\,\frac{[f^{(\mp)}(\epsilon_{k'n'})+N_{kk'}]f^{(-)}(\zeta)V_{k'n',kn}D_xV_{kn,k'n'}}{[\zeta+\hbar\omega_r-\epsilon_{kn}-G_{kn}(\zeta+\hbar\omega_r)][\pm\hbar\omega_{kk'}+\zeta+\hbar\omega_r-\epsilon_{k'n'}]}$$

$$\times\left(\frac{\Lambda_{kn}(\zeta-i0,\zeta+\hbar\omega_r)}{\zeta-\epsilon_{kn}-G_{kn}(\zeta-i0)}-(-i0\to+i0)\right)$$

$$+\frac{\hbar^{-1}}{2\pi i}\sum_{\substack{knk'n'\\ \pm}}\int_{-\infty}^{\infty}d\zeta\,\frac{[f^{(\mp)}(\epsilon_{k'n'})+N_{kk'}]f^{(-)}(\zeta)\,V_{kn,k'n'}D_xV_{k'n',kn}}{[\zeta-\hbar\omega_r-\epsilon_{kn}-G_{kn}(\zeta-\hbar\omega_r)][\pm\hbar\omega_{kk'}+\zeta-\hbar\omega_r-\epsilon_{k'n'}]}$$

$$\times\left(\frac{\Lambda_{kn}(\zeta-\hbar\omega_r,\,\zeta-i0)}{\zeta-\epsilon_{kn}-G_{kn}(\zeta-i0)}-(-i0\to+i0)\right)$$

$$+\frac{\hbar^{-1}}{2\pi i}\sum_{\substack{knk'n'\\ \pm}}\int_{-\infty}^{\infty}d\zeta\,\frac{[f^{(\mp)}(\epsilon_{k'n'})+N_{kk'}]f^{(-)}(\zeta)}{[\zeta-\hbar\omega_r-\epsilon_{kn}-G_{kn}(\zeta-\hbar\omega_r)]}$$

$$\times\left(\frac{\Lambda_{kn}(\zeta-\hbar\omega_r,\,\zeta-i0)V_{k'n',kn}D_xV_{kn,k'n'}}{[\zeta-\epsilon_{kn}-G_{kn}(\zeta-i0)][\pm\hbar\omega_{kk'}+\zeta-i0-\epsilon_{k'n'}]}-(-i0\to+i0)\right),\tag{3.19}$$

where the relation $f^{(-)}(\zeta-\hbar\omega_r)=f^{(-)}(\zeta)$ has been used for the third and fourth terms. Therefore,

$$\mathfrak{F}_{xy}^{(b)}(\hbar\omega+i0)=\frac{\hbar^{-1}}{2\pi i}\sum_{\substack{knk'n'\\ \pm}}\int_{-\infty}^{\infty}d\zeta\,\frac{[f^{(\mp)}(\epsilon_{k'n'})+N_{kk'}][f^{(-)}(\zeta)-f^{(-)}(\zeta+\hbar\omega)]\Lambda_{kn}(\zeta-i0,\,\zeta+\hbar\omega+i0)}{[\zeta-\epsilon_{kn}-G_{kn}(\zeta-i0)][\zeta+\hbar\omega-\epsilon_{kn}-G_{kn}(\zeta+\hbar\omega+i0)]}$$

$$\times\left(\frac{V_{knk'n'}D_xV_{k'n'kn}}{\pm\hbar\omega_{kk'}+\zeta-i0-\epsilon_{k'n'}}+\frac{V_{k'n'kn}D_xV_{knk'n'}}{\pm\hbar\omega_{kk'}+\zeta+\hbar\omega+i0-\epsilon_{k'n'}}\right)$$

$$-\frac{\hbar^{-1}}{2\pi i}\sum_{\substack{knk'n'\\ \pm}}\int_{-\infty}^{\infty}d\zeta\,\frac{[f^{(\mp)}(\epsilon_{k'n'})+N_{kk'}]f^{(-)}(\zeta)\,\Lambda_{kn}(\zeta+i0,\,\zeta+\hbar\omega+i0)}{[\zeta-\epsilon_{kn}-G_{kn}(\zeta+i0)][\zeta+\hbar\omega-\epsilon_{kn}-G_{kn}(\zeta+\hbar\omega+i0)]}$$

$$\times\left(\frac{V_{knk'n'}D_xV_{k'n'kn}}{\pm\hbar\omega_{kk'}+\zeta+i0-\epsilon_{k'n'}}+\frac{V_{k'n'kn}D_xV_{knk'n'}}{\pm\hbar\omega_{kk'}+\zeta+\hbar\omega+i0-\epsilon_{k'n'}}\right)$$

$$+\frac{\hbar^{-1}}{2\pi i}\sum_{\substack{knk'n'\\ \pm}}\int_{-\infty}^{\infty}d\zeta\,\frac{[f^{(\mp)}(\epsilon_{k'n'})+N_{kk'}]f^{(-)}(\zeta+\hbar\omega)\,\Lambda_{kn}(\zeta-i0,\,\zeta+\hbar\omega-i0)}{[\zeta-\epsilon_{kn}-G_{kn}(\zeta-i0)][\zeta+\hbar\omega-\epsilon_{kn}-G_{kn}(\zeta+\hbar\omega-i0)]}$$

$$\times\left(\frac{V_{knk'n'}D_xV_{k'n'kn}}{\pm\hbar\omega_{kk'}+\zeta-i0-\epsilon_{k'n'}}+\frac{V_{k'n'kn}D_xV_{knk'n'}}{\pm\hbar\omega_{kk'}+\zeta+\hbar\omega-i0-\epsilon_{k'n'}}\right).\tag{3.20}$$

To the lowest order, one has[24]

$$\frac{1}{[\zeta-\epsilon_{kn}-G_{kn}(\zeta\pm i0)][\zeta+\hbar\omega-\epsilon_{kn}-G_{kn}(\zeta+\hbar\omega+i0)]}=-\frac{\partial}{\partial\zeta}\,\mathcal{P}\,\frac{1}{\zeta-\epsilon_{kn}}\pm i\pi\frac{\partial}{\partial\zeta}\,\delta(\zeta-\epsilon_{kn})\tag{3.21}$$

and

$$\frac{1}{[\zeta-\epsilon_{kn}-G_{kn}(\zeta-i0)][\zeta+\hbar\omega-\epsilon_{kn}-G_{kn}(\zeta+\hbar\omega+i0)]}$$

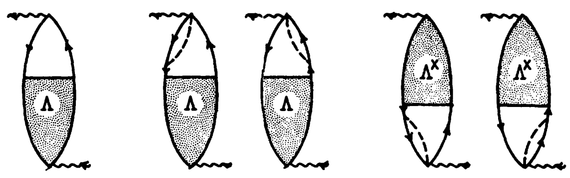
$$=\frac{2\pi i\delta(\zeta-\epsilon_{kn})}{\hbar\omega-[M_{kn}(\zeta+\hbar\omega)-M_{kn}(\zeta)]+i[\Gamma_{kn}(\zeta)+\Gamma_{kn}(\zeta+\hbar\omega)]}-\frac{\partial}{\partial\zeta}\,\mathcal{P}\,\frac{1}{\zeta-\epsilon_{kn}},\tag{3.22}$$

from which it is clear that the first term of (3.22) is of the lowest order. Therefore, maintaining only this term,[25] one obtains to the first order in ω,

$$\mathfrak{F}_{xy}^{(b)}(\hbar\omega+i0)=\sum_{\substack{knk'n'\\ \pm}}\frac{1}{i\hbar^2}[f^{(-)}(\epsilon_{kn})-f^{(-)}(\epsilon_{kn}+\hbar\omega)][f^{(\mp)}(\epsilon_{k'n'})+N_{kk'}]\,\phi_{kn}^{(-2)}\left(\frac{V_{knk'n'}D_xV_{k'n'kn}}{\pm\hbar\omega_{kk'}+\epsilon_{kn}-\epsilon_{k'n'}-i0}+\text{c.c.}\right),\tag{3.23}$$

where $\phi_{kn}^{(-2)}\equiv\Phi_{kn}(0)$ [see (3.12)]. The superscript is to indicate the order in electron-phonon interaction strength. Therefore, from (2.19), one obtains

$$\sigma_{xy}^{\mathrm{II}(b)}=\frac{e^2}{\hbar\Omega}\sum_{\substack{kn\\ k'n'\\ \pm}}f^{(-)\prime}(\epsilon_{kn})[f^{(\mp)}(\epsilon_{k'n'})+N_{kk'}]\,\phi_{kn}^{(-2)}\left(\frac{V_{kn,k'n'}D_xV_{k'n',kn}}{\pm\hbar\omega_{kk'}+\epsilon_{kn}-\epsilon_{k'n'}-i0}+\text{c.c.}\right),\tag{3.24}$$



(a) (b) (c)

FIG. 5. Important diagrams contributing to the correlation function. The EF vertex Λ^x has the same structure as Λ with the external field vertex replaced by $\mathfrak{V}^x_{kn}$.

where the prime on the fermion occupation function means differentiation with respect to the argument. As already mentioned ϵ_{kn} and $\phi^{(-2)}_{kn}$ are independent of the spin-orbit interaction to the first order, so that the effect of the spin-orbit coupling is contained only in $V_{kn,k'n'}D_x V_{k'n',kn}$. More explicitly, it can be shown[26] using (3.16) that $\{V_{kn,k'n'}D_x V_{k'n',kn}\}^{(s)}$ is purely imaginary. It follows then from (3.24) that

$$\sigma^{II(b,s)}_{xy} = -\frac{e^2}{\Omega}\sum_{kn} f^{(-)\prime}(\epsilon_{kn})\,\omega^x_{kn}\,\phi^{(-2)}_{kn}, \qquad (3.25)$$

where we define the anomalous velocity as

$$\omega^{\alpha}_{kn} = \frac{2\pi}{\hbar}\sum_{k'n'\pm}[f^{(\mp)}(\epsilon_{k'n'}) + N_{kk'}]$$
$$\times \operatorname{Im}\{V_{kn,k'n'}D_\alpha V_{k'n',kn}\}\,\delta(\epsilon_{kn} - \epsilon_{k'n'} \pm \hbar\omega_{kk'}) \qquad (3.26)$$

and the transport function $\phi^{(-2)}_{kn}$ is defined by

$$0 = \mathfrak{V}^y_{kn} + \frac{2\pi}{\hbar}\sum_{kn\pm}|V_{k'n',kn}|^2(\phi^{(-2)}_{k'n'} - \phi^{(-2)}_{kn})$$
$$\times[f^{(\mp)}(\epsilon_{k'n'}) + N_{kk'}]\,\delta(\epsilon_{kn} - \epsilon_{k'n'} \pm \hbar\omega_{kk'}). \qquad (3.27)$$

It is understood that the quantities such as ϵ_{kn}, $|V_{k'n',kn}|^2$, and $\phi^{(-2)}_{kn}$ are taken to the zeroth order in spin-orbit interaction without introducing new symbols for these. The above equations (3.25)–(3.27) determine the zeroth-order contribution of the diagrams given in Fig. 5(b).

We proceed next to the evaluation of the diagrams given in Fig. 5(c). This contribution can be written

$$\mathfrak{F}^{(c)}_{xy}(\hbar\omega_r) = \frac{-1}{\beta}\sum_{knl}\frac{\mathfrak{V}^x_{kn}\tilde{\Lambda}_{kn}(\zeta_l, \zeta_l + \hbar\omega_r)}{[\zeta_l - \epsilon_{kn} - G_{kn}(\zeta_l)][\zeta_l + \hbar\omega_r - \epsilon_{kn} - G_{kn}(\zeta_l + \hbar\omega_r)]}, \qquad (3.28)$$

where we define

$$\tilde{\Lambda}_{kn}(\zeta_l, \zeta_l + \hbar\omega_r) = \tilde{\mathfrak{V}}^y_{kn}(\zeta_l, \zeta_l + \hbar\omega_r)$$
$$+\frac{1}{\beta}\sum_{\substack{k'n'l\\ \pm}}\frac{|V_{k'n',kn}|^2\tilde{\Lambda}_{kn}(\zeta_{l'}, \zeta_{l'} + \hbar\omega_r)}{[\hbar\omega_{kk'} \pm (\zeta_l - \zeta_{l'})][\zeta_{l'} - \epsilon_{k'n'} - G_{k'n'}(\zeta_{l'})][\zeta_{l'} + \hbar\omega_r - \epsilon_{k'n'} - G_{k'n'}(\zeta_{l'} + \hbar\omega_r)]} \qquad (3.29)$$

and

$$\tilde{\mathfrak{V}}^y_{kn}(\zeta_{l'}, \zeta_l + \hbar\omega_r) = \frac{-1}{\hbar\beta}\sum_{\substack{k'n'l'\\ \pm}}\frac{1}{\hbar\omega_{kk'} \pm (\zeta_l - \zeta_{l'})}\left(\frac{V_{k'n',kn}D_y V_{kn,k'n'}}{\zeta_{l'} - \epsilon_{k'n'} - G_{k'n'}(\zeta_{l'})} + \frac{V_{kn,k'n'}D_y V_{k'n',kn}}{\zeta_{l'} + \hbar\omega_r - \epsilon_{k'n'} - G_{k'n'}(\zeta_{l'} + \hbar\omega_r)}\right). \qquad (3.30)$$

The diagrammatic relations leading to (3.28)–(3.30) are shown schematically in Fig. 6. One can immediately evaluate (3.28) and (3.29) to the lowest order in the smallness parameter Γ/E_F[18] utilizing the analogy between these and (3.1) and (3.14). It then follows that

$$\mathfrak{F}^{(c)}_{xy}(\hbar\omega + i0) = -\sum_{kn}\mathfrak{V}^x_{kn}\left(\frac{\tilde{\Lambda}_{kn}(\epsilon_{kn} - i0, \epsilon_{kn} + \hbar\omega + i0)[f^{(-)}(\epsilon_{kn}) - f^{(-)}(\epsilon_{kn} + \hbar\omega)]}{\hbar\omega - [M_{kn}(\epsilon_{kn} + \hbar\omega) - M_{kn}(\epsilon_{kn})] + i[\Gamma_{kn}(\epsilon_{kn}) + \Gamma_{kn}(\epsilon_{kn} + \hbar\omega)]}\right.$$
$$\left. - \int_{-\infty}^{\infty}\frac{1}{2}\delta'(z - \epsilon_{kn})[\tilde{\Lambda}_{kn}(z - i0, z + \hbar\omega - i0)f^{(-)}(z + \hbar\omega) + \tilde{\Lambda}_{kn}(z + i0, z + \hbar\omega + i0)f^{(-)}(z)]\,dz\right) \qquad (3.31)$$

and

$$\tilde{\Lambda}_{kn}(z - i0, z + \hbar\omega + i0) = \tilde{\mathfrak{V}}^y_{kn}(z - i0, z + \hbar\omega + i0)$$
$$+\sum_{\substack{k'n'\\ \pm}}\frac{|V_{k'n',kn}|^2\tilde{\Lambda}_{k'n'}(\epsilon_{k'n'} - i0, \epsilon_{k'n'} + \hbar\omega + i0)}{\hbar\omega - [M_{k'n'}(\epsilon_{k'n'} + \hbar\omega) - M_{k'n'}(\epsilon_{k'n'})] + i[\Gamma_{k'n'}(\epsilon_{k'n'} + \hbar\omega) + \Gamma_{k'n'}(\epsilon_{k'n'})]}$$
$$\times\left[\mathcal{P}\frac{1}{z - \epsilon_{k'n'} \pm \hbar\omega_{kk'}}[f^{(\mp)}(\epsilon_{k'n'}) - f^{(\mp)}(\epsilon_{k'n'} + \hbar\omega)] + 2\pi i\delta(z - \epsilon_{k'n'} \pm \hbar\omega_{kk'})\left(\frac{f^{(\mp)}(\epsilon_{k'n'}) + f^{(\mp)}(\epsilon_{k'n'} + \hbar\omega)}{2} + N_{kk'}\right)\right]. \qquad (3.32)$$

The latter can be put into a transport equation analogous to (3.13):

$$-i\omega\Psi_{kn}(\omega)=\tilde{\mathfrak{v}}^{y}_{kn}(\epsilon_{kn}-i0,\epsilon_{kn}+\hbar\omega+i0)+\frac{2\pi}{\hbar}\sum_{\substack{k'n'\\ \pm}}|V_{k'n',kn}|^2[\Psi_{k'n'}(\omega)-\Psi_{kn}(\omega)]$$
$$\times\left[\delta(\epsilon_{kn}-\epsilon_{k'n'}\pm\hbar\omega_{kk'})\left(\frac{f^{(\mp)}(\epsilon_{k'n'})+f^{(\mp)}(\epsilon_{k'n'}+\hbar\omega)}{2}+N_{kk'}\right)\right.$$
$$\left.+\frac{i}{2\pi}\mathcal{P}\frac{1}{\epsilon_{kn}-\epsilon_{k'n'}\pm\hbar\omega_{kk'}}[f^{(\mp)}(\epsilon_{k'n'}+\hbar\omega)-f^{(\mp)}(\epsilon_{k'n'})]\right], \qquad (3.33)$$

where

$$\Psi_{kn}(\omega)=\frac{i\hbar\tilde{\Lambda}_{kn}(\epsilon_{kn}-i0,\epsilon_{kn}+\hbar\omega+i0)}{\hbar\omega-[M_{kn}(\epsilon_{kn}+\hbar\omega)-M_{kn}(\epsilon_{kn})]+i[\Gamma_{kn}(\epsilon_{kn})+\Gamma_{kn}(\epsilon_{kn}+\hbar\omega)]}. \qquad (3.34)$$

One evaluates (3.30) directly obtaining

$$\tilde{\mathfrak{v}}^{y}_{kn}(\zeta_l,\zeta_l+\hbar\omega_r)=\frac{-1}{\hbar\beta}\sum_{\substack{k'n'\\ \pm}}\frac{\beta}{2\pi i}\int_{\Gamma_0}d\zeta\,\frac{\pm[f^{(\mp)}(\zeta')+N_{kk'}]}{\hbar\omega_{kk'}\pm(\zeta_l-\zeta')}$$
$$\times\left(\frac{V_{k'n'kn}D_yV_{knk'n'}}{\zeta'-\epsilon_{k'n'}-G_{k'n'}(\zeta')}+\frac{V_{knk'n'}D_yV_{k'n'kn}}{\zeta'+\hbar\omega_r-\epsilon_{k'n'}-G_{k'n'}(\zeta'+\hbar\omega_r)}\right)$$
$$=\frac{-\hbar^{-1}}{2\pi i}\sum_{\substack{k'n'\\ \pm}}\int_{-\infty}^{\infty}d\zeta'[f^{(\mp)}(\zeta')+N_{kk'}]\left(\frac{V_{k'n'kn}D_yV_{knk'n'}}{\pm\hbar\omega_{kk'}+\zeta_l-\zeta'}+\frac{V_{knk'n'}D_yV_{k'n'kn}}{\pm\hbar\omega_{kk'}+\zeta_l+\hbar\omega_r-\zeta'}\right)$$
$$\times\left(\frac{1}{\zeta'-\epsilon_{k'n'}-G_{k'n'}(\zeta'-i0)}-\frac{1}{\zeta'-\epsilon_{k'n'}-G_{k'n'}(\zeta'+i0)}\right),$$

which yields in view of (3.19)

$$\tilde{\mathfrak{v}}^{y}_{kn}(\epsilon_{kn}-i0,\epsilon_{kn}+\hbar\omega+i0)=\frac{-1}{\hbar}\sum_{\substack{k'n'\\ \pm}}[f^{(\mp)}(\epsilon_{k'n'})+N_{kk'}]\left(\frac{V_{k'n'kn}D_yV_{knk'n'}}{\pm\hbar\omega_{kk'}+\epsilon_{kn}-\epsilon_{k'n'}-i0}+\frac{V_{knk'n'}D_yV_{k'n'kn}}{\pm\hbar\omega_{kk'}+\epsilon_{kn}-\epsilon_{k'n'}+\hbar\omega+i0}\right). \qquad (3.35)$$

Finally, using (2.19), (3.31)–(3.35), and $\tilde{\mathfrak{v}}^{y}_{kn}(\epsilon_{kn}-i0,\ \epsilon_{kn}+i0)=\omega^{y}_{kn}$ [defined in (3.26)], one obtains

$$\sigma^{\mathrm{II}(c)}_{xy}=\frac{e^2}{\Omega}\sum_{kn}\mathfrak{v}^{x}_{kn}f^{(-)\prime}(\epsilon_{kn})\psi^{y(s)}_{kn}, \qquad (3.36)$$

where $\psi^{y(s)}_{kn}\equiv-\Psi_{kn}(0)$ satisfies a transport equation

$$0=\omega^{y}_{kn}+\frac{2\pi}{\hbar}\sum_{\substack{k'n'\\ \pm}}|V_{k'n',kn}|^2[f^{(\mp)}(\epsilon_{k'n'})+N_{kk'}]$$
$$\times(\psi^{y(s)}_{k'n'}-\psi^{y(s)}_{kn})\,\delta(\epsilon_{kn}-\epsilon_{k'n'}\pm\hbar\omega_{kk'}). \qquad (3.37)$$

Note that ω^{y}_{kn} and thus $\psi^{y(s)}_{kn}$ are first order in spin-orbit coupling and $\psi^{y(s)}_{kn}$ is zeroth order in electron-phonon interaction. Therefore, the anomalous part of the conductivity tensor is given by

$$\sigma^{\mathrm{II}(c,s)}_{xy}=\frac{e^2}{\Omega}\sum_{kn}\mathfrak{v}^{x}_{kn}f^{(-)\prime}(\epsilon_{kn})\psi^{y(s)}_{kn}. \qquad (3.38)$$

Thus, the zeroth-order contribution from the diagrams in Fig. 5(c) is determined from (3.37) and (3.38). Finally, combining (3.25) and (3.38),

$$\sigma^{\mathrm{II}(s)}_{xy}=\frac{e^2}{\Omega}\sum_{kn}f^{(-)\prime}(\epsilon_{kn})[\mathfrak{v}^{x}_{kn}\psi^{y(s)}_{kn}-\omega^{x}_{kn}\phi^{(-2)}_{kn}]. \qquad (3.39)$$

This is the total anomalous velocity contribution to the lowest order. It can be shown that (3.39) is independent of the phase of the Bloch states.

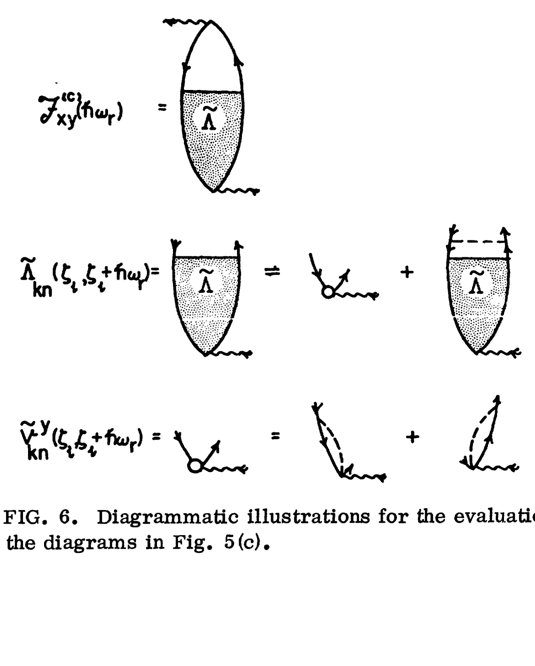


FIG. 6. Diagrammatic illustrations for the evaluation of the diagrams in Fig. 5(c).

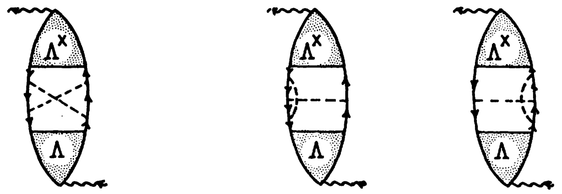


FIG. 7. Diagrams responsible for the skew scattering contribution to the anomalous Hall effect.

There are still other important diagrams to be considered such as exhibited in Fig. 7. These diagrams will lead to the corrections to the collision matrix of second order in electron-phonon interaction (relative to the principal one) in the Boltzmann equation; in the presence of the time-reversal-breaking mechanism provided by spin-orbit interaction, these diagrams give rise to the skew-scattering contribution to the transverse current. One evaluates these diagrams in a way illustrated in Fig. 8, retaining only those parts which are first order in spin-orbit coupling. One thereby obtains the following set of integral equations:

$$\mathfrak{F}_{xy}^{\mathrm{col}\,(s)}(\hbar\omega_r)=\frac{-1}{\beta}\sum_{\substack{kn\\ l}}\frac{v_{kn}^{x}\Lambda_{kn}^{(2,s)}(\zeta_l,\zeta_l+\hbar\omega_r)}{[\zeta_l-\epsilon_{kn}-G_{kn}(\zeta_l)][\zeta_l+\hbar\omega_r-\epsilon_{kn}-G_{kn}(\zeta_l+\hbar\omega_r)]}, \tag{3.40}$$

$$\Lambda_{kn}^{(2,s)}(\zeta_l,\zeta_l+\hbar\omega_r)=\Upsilon_{kn}^{(2,s)}(\zeta_l,\zeta_l+\hbar\omega_r)$$
$$+\frac{1}{\beta}\sum_{k'n'l}\frac{|V_{k'n',kn}|^2\Lambda_{k'n'}^{(2,s)}(\zeta_{l'},\zeta_{l'}+\hbar\omega_r)}{[\hbar\omega_{kk'}\pm(\zeta_l-\zeta_{l'})][\zeta_{l'}-\epsilon_{k'n'}-G_{k'n'}(\zeta_{l'})][\zeta_{l'}+\hbar\omega_r-\epsilon_{k'n'}-G_{k'n'}(\zeta_{l'}+\hbar\omega_r)]}, \tag{3.41}$$

and

$$\Upsilon_{kn}^{(2,s)}(\zeta_l,\zeta_l+\hbar\omega_r)=\frac{1}{\beta^2}\sum_{k'n'l'}\frac{\Lambda_{k'n'}(\zeta_{l'},\zeta_{l'}+\hbar\omega_r)}{[\zeta_{l'}-\epsilon_{k'n'}-G_{k'n'}(\zeta_{l'})][\zeta_{l'}+\hbar\omega_r-\epsilon_{k'n'}-G_{k'n'}(\zeta_{l'}+\hbar\omega_r)]}$$
$$\times\sum_{\substack{k''n''k'''n'''\\ l''l'''\\ t=0,1,u=0,1}}\Bigg(\frac{\{V_{kn,k'''n'''}V_{k'''n''',k'n'}V_{k'n',k''n''}V_{k''n'',kn}\delta_{\vec{k}+\vec{k}',\vec{k}''+\vec{k}'''}\}^{(s)}\delta_{l+l',l''+l'''}}{[\zeta_{l''}-\epsilon_{k''n''}-G_{k''n''}(\zeta_{l''})][\zeta_{l'''}+\hbar\omega_r-\epsilon_{k'''n'''}-G_{k'''n'''}(\zeta_{l'''}+\hbar\omega_r)][\hbar\omega_{kk''}+(-1)^u(\zeta_l-\zeta_{l''})][\hbar\omega_{kk'''}+(-1)^t(\zeta_l-\zeta_{l'''})]}$$
$$+\frac{\{V_{kn,k'n'}V_{k'n',k'''n'''}V_{k'''n''',k''n''}V_{k''n'',kn}\delta_{\vec{k}+\vec{k}''',\vec{k}'+\vec{k}''}\}^{(s)}\delta_{l+l''',l'+l''}}{[\zeta_{l''}-\epsilon_{k''n''}(\zeta_{l''})][\zeta_{l'''}-\epsilon_{k'''n'''}-G_{k'''n'''}(\zeta_{l'''})][\hbar\omega_{kk'}+(-1)^t(\zeta_l-\zeta_{l'})][\hbar\omega_{kk''}+(-1)^u(\zeta_l-\zeta_{l''})]}$$
$$+\frac{\{V_{kn,k''n''}V_{k''n'',k'''n'''}V_{k'''n''',k'n'}V_{k'n',kn}\delta_{\vec{k}+\vec{k}''',\vec{k}'+\vec{k}''}\}^{(s)}\delta_{l+l''',l'+l''}}{[\zeta_{l''}+\hbar\omega_r-\epsilon_{k''n''}-G_{k''n''}(\zeta_{l''}+\hbar\omega_r)][\zeta_{l'''}+\hbar\omega_r-\epsilon_{k'''n'''}-G_{k'''n'''}(\zeta_{l'''}+\hbar\omega_r)][\hbar\omega_{kk'}+(-1)^t(\zeta_l-\zeta_{l'})][\hbar\omega_{kk''}+(-1)^u(\zeta_l-\zeta_{l''})]}\Bigg), \tag{3.42}$$

$\Lambda_{kn}^{(2,s)}$, $\Upsilon_{kn}^{(2,s)}$ being second order in electron-phonon interaction and first order in spin-orbit coupling. The superscript col stands for the contribution from the correction of the collison matrix. One easily recognizes that (3.40) and (3.41) have the same formal structure as (3.28) and (3.29). Therefore, following a similar procedure leading to (3.38), one obtains

$$\sigma_{xy}^{\mathrm{I\,Icol}\,(s)}=-\frac{e^2}{\Omega}\sum_{kn}v_{kn}^{x}f^{(-)\prime}(\epsilon_{kn})\phi_{kn}^{(0,s)}, \tag{3.43}$$

where

$$\phi_{kn}^{(0,s)}=\frac{\hbar\Lambda_{kn}^{(2,s)}(\epsilon_{kn}-i0,\epsilon_{kn}+i0)}{2\Gamma_{kn}(\epsilon_{kn})}, \tag{3.44}$$

and

$$0=\Upsilon_{kn}^{(2,s)}(\epsilon_{kn}-i0,\epsilon_{kn}+i0)+\frac{2\pi}{\hbar}\sum_{\substack{k'n'\\ \pm}}|V_{k'n',kn}|^2[f^{(\mp)}(\epsilon_{k'n'})+N_{kk'}](\phi_{k'n'}^{(0,s)}-\phi_{kn}^{(0,s)})\times\delta(\epsilon_{kn}-\epsilon_{k'n'}\pm\hbar\omega_{kk'}). \tag{3.45}$$

It is shown in Appendix B that

$$\Upsilon_{kn}^{(2,s)}(\epsilon_{kn}-i0,\epsilon_{kn}+i0)=\sum_{k'n'}\mathfrak{L}_{kn,k'n'}^{(4,s)}\phi_{k'n'}^{(-2)}, \tag{3.46}$$

where $\phi_{kn}^{(-2)}$ are given by (3.27) and the fourth-order collision matrix $\mathfrak{L}_{kn,k'n'}^{(4,s)}$ is given by

$$\mathfrak{L}_{kn,k'n'}^{(4,s)}=-\frac{(2\pi)^2}{\hbar}\sum_{\substack{k''n''k'''n'''\\ t=0,1,u=0,1}}(-1)^{t+u}\Big[-\mathrm{Im}(V_{kn,k'''n'''}V_{k'''n''',k'n'}V_{k'n',k''n''}V_{k''n'',kn})\delta_{\vec{k}+\vec{k}',\vec{k}''+\vec{k}'''}$$
$$\times\Big([f^{(-)}(\epsilon_{k'n'})-f^{(-)}(\epsilon_{k'''n'''})][f^{(-)}(\epsilon_{k''n''})+N(\epsilon_{k'n'}-\epsilon_{k'''n'''})]$$

$$\times\,\delta(\epsilon_{kn}+\epsilon_{k'n'}-\epsilon_{k''n''}-\epsilon_{k'''n'''})\,\delta[(-1)^u\hbar\omega_{kk''}+\epsilon_{kn}-\epsilon_{k''n''}]$$

$$\times\,\mathcal{P}\frac{1}{(-1)^t\hbar\omega_{kk'''}+\epsilon_{kn}-\epsilon_{k'''n'''}}+[f^{(-)}(\epsilon_{k'n'})+N^{(t)}_{kk'''}]\{f^{(-)}[\epsilon_{k'n'}-(-1)^t\hbar\omega_{kk'''}]+N^{(u)}_{kk''}\}$$

$$\times\,\delta[\epsilon_{kn}+(-1)^t\hbar\omega_{kk'''}-\epsilon_{k'''n'''}]\,\delta[(-1)^u\hbar\omega_{kk''}+\epsilon_{k'''n'''}-\epsilon_{k'n'}]\,\mathcal{P}\frac{1}{\epsilon_{kn}+\epsilon_{k'n'}-\epsilon_{k''n''}-\epsilon_{k'''n'''}}\Bigg)$$

$$+\,\mathrm{Im}(V_{kn,k'n'}V_{k'n',k'''n'''}V_{k'''n''',k''n''}V_{k''n'',kn})\,\delta_{\vec{k}+\vec{k}''',\vec{k}'+\vec{k}''}$$

$$\times\Big([f^{(-)}(\epsilon_{k'n'})-f^{(-)}(\epsilon_{k'''n'''})][f^{(-)}(\epsilon_{k''n''})+N(\epsilon_{k'''n'''}-\epsilon_{k'n'})]$$

$$\times\,\delta(\epsilon_{kn}+\epsilon_{k'''n'''}-\epsilon_{k'n'}-\epsilon_{k''n''})\,\delta[(-1)^u\hbar\omega_{kk''}+\epsilon_{kn}-\epsilon_{k''n''}]$$

$$\times\,\mathcal{P}\frac{1}{(-1)^t\hbar\omega_{kk'}+\epsilon_{kn}-\epsilon_{k'n'}}-[f^{(-)}(\epsilon_{k'''n'''})+N^{(t)}_{kk'}]\{f^{(-)}[\epsilon_{k'''n'''}-(-1)^t\hbar\omega_{kk'}]+N^{(u)}_{kk''}\}\,\delta[\epsilon_{kn}+(-1)^t\hbar\omega_{kk'}-\epsilon_{k'n'}]$$

$$\times\,\delta[(-1)^u\hbar\omega_{kk''}+\epsilon_{k'n'}-\epsilon_{k'''n'''}]\,\mathcal{P}\frac{1}{\epsilon_{kn}+\epsilon_{k'''n'''}-\epsilon_{k'n'}-\epsilon_{k''n''}}\Bigg)$$

$$+\,\mathrm{Im}(V_{kn,k'n'}V_{k'n',k'''n'''}V_{k'''n''',k''n''}V_{k''n'',kn})\,\delta_{\vec{k}+\vec{k}''',\vec{k}'+\vec{k}''}$$

$$\times[f^{(-)}(\epsilon_{k'n'})+N^{(t)}_{kk'}][f^{(-)}(\epsilon_{k''n''})+N^{(u)}_{kk''}]\,\delta[(-1)^t\hbar\omega_{kk'}+\epsilon_{kn}-\epsilon_{k'n'}]$$

$$\times\,\delta[(-1)^u\hbar\omega_{kk''}+\epsilon_{kn}-\epsilon_{k''n''}]\,\mathcal{P}\frac{1}{\epsilon_{kn}+\epsilon_{k'''n'''}-\epsilon_{k'n'}-\epsilon_{k''n''}}\Bigg]\,, \qquad (3.47)$$

where

$$N^{(t)}_{kk'}=\frac{1}{\exp[(-1)^t\beta\hbar\omega_{kk'}]-1}\equiv N((-1)^t\hbar\omega_{kk'})\,. \qquad (3.48)$$

Using the additional property

$$\sum_{k'n'}\mathfrak{L}^{(4,s)}_{kn,k'n'}=0 \qquad (3.49)$$

shown at the end of Appendix B, one finally obtains

$$0=\sum_{k'n'}\mathfrak{L}^{(4,s)}_{kn,k'n'}(\phi^{(-2)}_{k'n'}-\phi^{(-2)}_{kn})+\sum_{k'n'}\mathfrak{L}^{(2)}_{kn,k'n'}(\phi^{(0,s)}_{k'n'}-\phi^{(0,s)}_{kn}), \qquad (3.50)$$

where

$$\mathfrak{L}^{(2)}_{kn,k'n'}=\frac{2\pi}{\hbar}\sum_{\pm}|V_{kn,k'n'}|^2[f^{(\mp)}(\epsilon_{k'n'})+N_{kk'}]\times\delta(\epsilon_{kn}-\epsilon_{k'n'}\pm\hbar\omega_{kk'}). \qquad (3.51)$$

As discussed in Sec. I, as well as in the text between Eqs. (3.39) and (3.40), Eq. (3.50) constitutes a correction to the collision kernel of the Boltzmann equation (3.27) of second order in the electron-phonon interaction. Thus the skew-scattering effect manifests itself in the third Born order (i.e., fourth order in electron-phonon interaction) of the collision matrix, whereas for the impurity-scattering problem, the skew-scattering

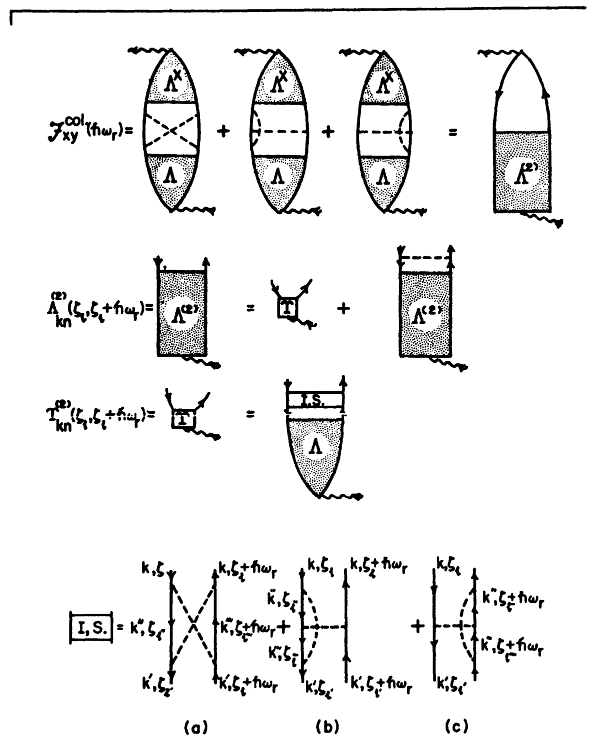


FIG. 8. Diagrammatic illustrations for the evaluation of the diagrams in Fig. 7. I. S. stands for the fourth-order irreducible scattering part.

term begins to appear in the second Born order (i.e., third order in impurity potential) as noted by Luttinger.[6] It should, however, be stated that in the next order, Luttinger obtains corrections explicitly analogous to those expressed by our equations (3.50) and (3.51). These constitute the skew-scattering contribution of interactions of the electron with two impurity-scattering centers.

At this point one has to extend the self-energy of the electron propagator to the fourth order in electron-phonon interaction as schematically shown in Fig. 9. This quantity is given by

$$G_{kn}^{\prime\prime(s)}(\zeta_l) \equiv \frac{1}{\beta} \sum_{\substack{k'n'k''n''k'''n''' \\ l'l''l''' \\ t=0,1,u=0,1}} \frac{\{V_{kn,k'n'}V_{k'n',k'''n'''}V_{k'''n''',k''n''}V_{k''n'',kn}\,\delta_{\vec{k}+\vec{k}''',\vec{k}'+\vec{k}''}\}^{(s)}}{[\hbar\omega_{kk'}+(-1)^t(\zeta_l-\zeta_{l'})][\hbar\omega_{kk''}+(-1)^u(\zeta_l-\zeta_{l''})][\zeta_{l'''}-\epsilon_{k'''n'''}-G_{k'''n'''}(\zeta_{l'''})]}$$
$$\times \frac{\delta_{l+l''',l'+l''}}{[\zeta_{l'}-\epsilon_{k'n'}-G_{k'n'}(\zeta_{l'})][\zeta_{l''}-\epsilon_{k''n''}-G_{k''n''}(\zeta_{l''})]}\,, \tag{3.52}$$

where the superscript s indicates that we are interested only in the part which is linear in spin-orbit interaction. The right-hand side of (3.52) follows from the fact that only the electron-phonon interaction matrices contain the spin-orbit interaction linearly for the reason already discussed in the text between Eqs. (3.15) and (3.16). The quantity on the right-hand side of (3.52) is readily seen to vanish, by noting that, under the interchange of the singly primed dummy indices with the doubly primed ones, the summand remains invariant except that $\{\cdots\}^{(s)}$ changes to its complex conjugate. However, $\{\cdots\}^{(s)}$ is purely imaginary in view of (3.16). Therefore, the right-hand side of (3.52) is equal to its negative value, meaning that

$$G_{kn}^{\prime\prime(s)}(\zeta_l)=0\quad . \tag{3.53}$$

Therefore, there is no contribution from this fourth-order self-energy part. Bearing in mind that the self-energy part represents the total scattering out rate, (3.53) is consistent with (3.49) which states that the total scattering in rate also vanishes.

B. Evaluation of $\sigma_{xy}^{\mathrm{II\,I}(s)}$ and $\sigma_{xy}^{\mathrm{I\,II}(s)}$

The lowest-order (i.e., zeroth-order) contribution to $\sigma_{xy}^{\mathrm{II\,I}}$ and $\sigma_{xy}^{\mathrm{I\,II}}$ comes, respectively, from the diagrams of Figs. 10(a) and 10(b). The diagrams are the same as that given in Fig. 5(a), except for the upper and lower vertices being replaced by $J_x^{nn}(\vec{k})$ and $J_y^{nn}(\vec{k})$, respectively. In this case, however, we are interested in that part of the correlation function which is of zeroth order in ω rather than the term of first order in ω. Using (3.14) and (3.15), one obtains

$$\tilde{\mathfrak{F}}_{xy}^{\mathrm{II\,I}}(\hbar\omega_r)=\frac{-1}{\beta}\sum_{knl}\frac{iJ_x^{nn}(\vec{k})\Lambda_{kn}(\zeta_l,\,\zeta_l+\hbar\omega_r)}{[\zeta_l-\epsilon_{kn}-G_{kn}(\zeta_l)][\zeta_l+\hbar\omega_r-\epsilon_{kn}-G_{kn}(\zeta_l+\hbar\omega_r)]}\,, \tag{3.54}$$

$$\tilde{\mathfrak{F}}_{xy}^{\mathrm{II\,I}}(\hbar\omega+i0)=-\sum_{kn} iJ_x^{nn}(\vec{k})\Bigg\{\frac{[f^{(-)}(\epsilon_{kn})-f^{(-)}(\epsilon_{kn}+\hbar\omega)]\Lambda_{kn}(\epsilon_{kn}-i0,\epsilon_{kn}+\hbar\omega+i0)}{\hbar\omega-[M_{kn}(\epsilon_{kn}+\hbar\omega)-M_{kn}(\epsilon_{kn})]+i[\Gamma_{kn}(\epsilon_{kn})+\Gamma_{kn}(\epsilon_{kn}+\hbar\omega)]}$$
$$-\int_{-\infty}^{\infty}\frac{1}{2}\,\delta'(z-\epsilon_{kn})[\Lambda_{kn}(z-i0,\,z+\hbar\omega-i0)f^{(-)}(z+\hbar\omega)+\Lambda_{kn}(z+i0,\,z+\hbar\omega+i0)f^{(-)}(z)]\,dz\Bigg\}. \tag{3.55}$$

The first term in the curly bracket of (3.55) vanishes in the limit $\omega\to 0$. Noting from (3.11) that $\lim_{\omega\to 0}\Lambda_{kn}(z\pm i0,\,z+\hbar\omega\pm i0)=\mathfrak{v}_{kn}^{y}$, one simplifies (3.55) as

$$\lim_{\omega\to 0}\tilde{\mathfrak{F}}_{xy}^{\mathrm{II\,I}}(\hbar\omega+i0)=\frac{1}{2}\sum_{kn} iJ_x^{nn}(\vec{k})\mathfrak{v}_{kn}^{y}\times\int_{-\infty}^{\infty}d\zeta\,\delta'(\zeta-\epsilon_{kn})f^{(-)}(\zeta)$$
$$=-\sum_{kn} iJ_x^{nn}(\vec{k})\mathfrak{v}_{kn}^{y}f^{(-)\prime}(\epsilon_{kn})=\sum_{kn}f^{(-)}(\epsilon_{kn})\,i\frac{\partial}{\hbar\partial k_y}J_x^{nn}(\vec{k})\,. \tag{3.56}$$

Therefore, from (2.17),

$$\sigma_{xy}^{\mathrm{II\,I}}=\frac{e^2}{i\hbar\Omega}\sum_{kn}f^{(-)}(\epsilon_{kn})\frac{\partial}{\partial k_y}J_x^{nn}(\vec{k})\,. \tag{3.57}$$

One also obtains, in a similar way,

$$\sigma_{xy}^{\mathrm{I\,II}}=-\frac{e^2}{i\hbar\Omega}\sum_{kn}f^{(-)}(\epsilon_{kn})\frac{\partial}{\partial k_x}J_y^{nn}(\vec{k})\,. \tag{3.58}$$

As shown by Karplus and Luttinger, the quantity $\vec{J}^{nn}(\vec{k})$ is proportional to the spin-orbit interaction (to the first order in this parameter). Therefore, $\sigma_{xy}^{\mathrm{II\,I}}$ and $\sigma_{xy}^{\mathrm{I\,II}}$ ostensibly give contributions to the

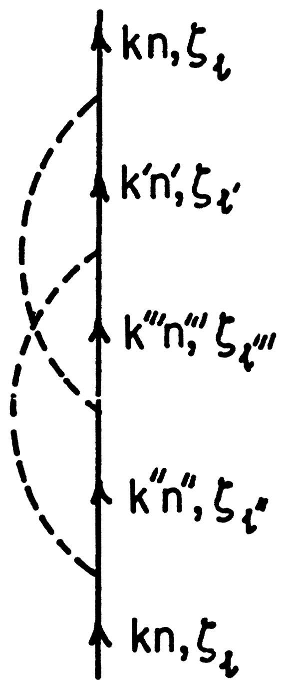


FIG. 9. Fourth-order electron self-energy part.

anomalous Hall current. The quantity $\sigma_{xy}^{\text{I II}}+\sigma_{xy}^{\text{II II}}$ can be shown to give the result obtained by Karplus and Luttinger [their (2.14)]. It will now be shown, however, that this is exactly canceled by $\sigma_{xy}^{\text{II I}}$. To show this, one evaluates (2.16) to the zeroth order in electron-phonon interaction; introducing the identity[4]

$$\langle kn|[x_l^{\text{II}}, y_l^{\text{II}}]|k'n'\rangle$$

$$=\delta_{\vec{k},\vec{k}'}\left(\frac{\partial}{\partial k_x}J_y^{nn'}(\vec{k})-\frac{\partial}{\partial k_y}J_x^{nn'}(\vec{k})\right), \quad (3.59)$$

which can be obtained by using integration by part and closure property, one obtains

$$\sigma_{xy}^{\text{II II}}=\frac{e^2}{i\hbar\Omega}\sum_{kn}f^{(-)}(\epsilon_{kn})\left(\frac{\partial}{\partial k_x}J_y^{nn}(\vec{k})-\frac{\partial}{\partial k_y}J_x^{nn}(\vec{k})\right). \quad (3.60)$$

This cancels (3.57) and (3.58) exactly. This means that only the nonlocal intercell motion discussed in Sec. III A contributes to the transverse current. As already mentioned, one can prove generally that the contribution to the polarization part of the conductivity arises solely from the intracell motion and vanishes to all orders in electron-phonon interaction in dc limit [i.e., $(d/dt)\langle x^{\text{II}}\rangle \propto \sigma_{xy}^{p}\equiv\sigma^{\text{I II}}+\sigma^{\text{II I}}+\sigma^{\text{II II}}=0$].[16a] One can also show that σ_{xy}^{p} represents the contribution from the dipole driving part $(\vec{r}^{\,\text{II}}\cdot\vec{E})$ of the external field.[16b] Therefore, the effect of the polarization caused by the intracell motion does not play any direct role in producing a transverse current. However, it should be mentioned that the combined effect of the polarization and the scattering is very important (at least, for the case of slowly varying impurity potential), because it is recognized to enhance the free-electron value of the Hall current by a large factor ($\sim 10^4$ for the impurity-scattering case) when the band effect is introduced.[3,5,16a]

C. Temperature Dependence of Hall Effect

At low temperature ($T\ll\Theta_D$, Θ_D is the Debye temperature) one employs a small-angle-scattering approximation[6]

$$V_{k'n',kn}\approx V_{k'k}[\delta_{n,n'}+(\vec{k}-\vec{k}')\cdot\vec{J}^{n'n}(\vec{k})]$$

$$\approx V_{k'k}[\delta_{n,n'}+(\vec{k}-\vec{k}')\cdot\vec{J}^{n'n}(\vec{k}')], \quad (3.61)$$

where $V_{k'k}$ is a Fourier component of the potential and a function of $\vec{k}'-\vec{k}$. One substitutes this in (3.26) and finds

$$\omega_{kn}^{\alpha}=\frac{2\pi}{i\hbar}\sum_{k'\pm}[f^{(\mp)}(\epsilon_{k'n})+N_{kk'}]|V_{kk'}|^2\delta(\epsilon_{kn}-\epsilon_{k'n}\pm\hbar\omega_{kk'})$$

$$\times\left(\vec{k}\cdot\frac{\partial}{\partial k_\alpha}\vec{J}^{nn}(\vec{k})-\vec{k}'\cdot\frac{\partial}{\partial k'_\alpha}\vec{J}^{nn}(\vec{k}')\right), \quad (3.62)$$

where use has been made of the fact that $\vec{J}^{nn}(\vec{k})$ is purely imaginary. Therefore, from (3.37),

$$\psi_{kn}^{y(s)}=-i\vec{k}\cdot\frac{\partial}{\partial k_y}\vec{J}^{nn}(\vec{k}). \quad (3.63)$$

To evaluate the second term of (3.39), it is convenient to put (3.39) in a more symmetric form. This can be achieved by evaluating the diagrams in Fig. 5(b) upside down and following the same procedure as in the evaluation of the diagrams in Fig. 5(c). The result is given by

$$\sigma_{xy}^{\text{I I}(s)}=\frac{e^2}{\Omega}\sum_{kn}f^{(-)\prime}(\epsilon_{kn})(v_{kn}^{x}\psi_{kn}^{y(s)}-v_{kn}^{y}\psi_{kn}^{x(s)}), \quad (3.39')$$

where $\psi_{kn}^{x(s)}$ satisfies the same type of transport equation as (3.37) with ω_{kn}^{y} replaced by ω_{kn}^{x}. Hence

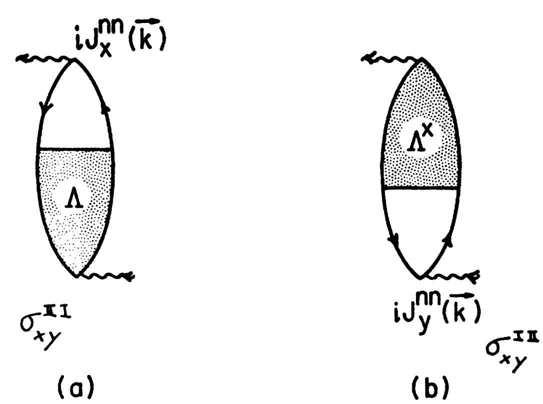


FIG. 10. Important diagrams contributing to (a) $\sigma_{xy}^{\text{II I}}$ and (b) $\sigma_{xy}^{\text{I II}}$.

$$\sigma_{xy}^{\mathrm{I\,I}(s)}=\frac{e^2}{i\Omega}\sum_{kn} f^{(-)}(\epsilon_{kn})\left(\frac{\partial}{\partial k_x}\, J_y^{nn}(\vec{k})-\frac{\partial}{\partial k_y}\, J_x^{nn}(\vec{k})\right) . \quad (3.64)$$

This is identical with the result obtained by Luttinger for the case of small-angle impurity scattering. Finally, from (2.4) one concludes for the anomalous velocity contribution at low temperature that $R_s \propto \rho^2$. At high temperature ($T \gg \Theta_D$) one notes from (3.26), (3.27), and (3.37) that the quantity in (3.39) becomes independent of temperature. Therefore, in view of (2.4) one obtains $R_s \propto \rho^2$.

So far we have discussed only the anomalous velocity contribution. For the skew-scattering contribution one can estimate the temperature dependence only for the high-temperature regime ($T \gg \Theta_D$), where one finds from (3.43) and (3.51) that $R_s \propto \rho^2 + a\rho$ (a is a constant).

IV. CONCLUSION

The central result is given by (3.39) and (3.43), which arises solely from the intercell motion of the electron. These contributions correspond, respectively, to (3.36) and (3.32) of Luttinger.[6] As already discussed, (3.39) is an anomalous velocity contribution and (3.43) is a skew-scattering contribution.

The result of this work disagrees with that of Leribaux[27] based on the same model. He uses Kubo's formalism and Fujita and Abe's[28] diagrammatic technique. His result corresponds to (3.59) of this work. He does not consider the anomalous velocity contribution (3.39), which is nonetheless important. It appears that the rest of the discrepancy is due to other diagrams left out in his treatment.

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APPENDIX A

In this Appendix a proof of (2.16) and (2.17) is given. One begins with rewriting (2.6) as

$$\mathfrak{F}_{xy}(\hbar\omega_r)=\int_0^\beta du\, e^{\hbar\omega_r u}\langle T e^{uH} v_x e^{-uH} v_y\rangle$$
$$=\int_{-\epsilon}^{\beta-\epsilon} du\, e^{\hbar\omega_r u}\langle T e^{uH} v_x e^{-uH} v_y\rangle \quad (\mathrm{A1})$$

for an arbitrary value of ϵ given by $0<\epsilon<\beta$. The first equality is due to the fact that $\mathfrak{F}_{xy}(u_2, u_1)$ is a function of u_2-u_1 only [i.e., $\mathfrak{F}_{xy}(u_2,u_1)\equiv\mathfrak{F}_{xy}(u_2-u_1)$], and the second equality follows from the periodicity of the correlation function $\mathfrak{F}_{xy}(u)$ with the period β. Define

$$\vec{r}(u)=e^{uH}\vec{r}e^{-uH} ,$$

then

$$e^{uH}v_x e^{-uH}=e^{uH}\frac{[x,H]}{i\hbar}e^{-uH}$$
$$=-\frac{1}{i\hbar}\frac{d}{du}x(u)\equiv-\frac{1}{i\hbar}x'(u)$$

and similarly

$$v_y=\frac{[y,H]}{i\hbar}=-\frac{1}{i\hbar}y'(0) .$$

Using these and defining a step function $\theta(u)$ such that

$$\theta(u)=1,\quad u\geqq 0$$
$$\theta(u)=0,\quad u<0$$

one can rewrite (A1) as

$$\mathfrak{F}_{xy}(\hbar\omega_r)=-\frac{1}{\hbar^2}\int_{-\epsilon}^{\beta-\epsilon} du\, e^{\hbar\omega_r u}\langle\theta(u)x'(u)y'(0)$$
$$+\theta(-u)y'(0)x'(u)\rangle . \quad (\mathrm{A2})$$

To prove (2.16) one computes $\mathfrak{F}_{xy}^{\mathrm{II\,II}}(\hbar\omega_r)$ by integrating by parts:

$$\mathfrak{F}_{xy}^{\mathrm{II\,II}}(\hbar\omega)=-\frac{1}{\hbar^2}\int_{-\epsilon}^{\beta-\epsilon} du\, e^{\hbar\omega_r u}\left\langle\theta(u)\frac{d}{du}x^{\mathrm{II}}(u)y^{\mathrm{II}\prime}(0)+\theta(-u)y^{\mathrm{II}\prime}(0)\frac{d}{du}x^{\mathrm{II}}(u)\right\rangle$$
$$=-\frac{1}{\hbar^2}\left[e^{\hbar\omega_r u}\langle Tx^{\mathrm{II}}(u)y^{\mathrm{II}\prime}(0)\rangle\right]_{-\epsilon}^{\beta-\epsilon}+\frac{\hbar\omega}{\hbar^2}\int_{-\epsilon}^{\beta-\epsilon} e^{\hbar\omega_r u}\langle Tx^{\mathrm{II}}(u)y^{\mathrm{II}\prime}(0)\rangle\, du$$
$$+\frac{1}{\hbar^2}\int_{-\epsilon}^{\beta-\epsilon} du\, e^{\hbar\omega_r u}\langle\delta(u)x^{\mathrm{II}}(u)y^{\mathrm{II}\prime}(0)-\delta(u)y^{\mathrm{II}\prime}(0)x^{\mathrm{II}}(u)\rangle . \quad (\mathrm{A3})$$

The first term vanishes due to the periodicity. Using $\mathrm{Tr}(AB)=\mathrm{Tr}(BA)$, one rewrites (A3) as

$$\mathfrak{F}_{xy}^{\mathrm{II\,II}}(\hbar\omega_r)=\frac{\omega_r}{\hbar}\int_{-\epsilon}^{\beta-\epsilon} du\, e^{\hbar\omega_r u}\langle Tx^{\mathrm{II}}(0)\, e^{-uH}y^{\mathrm{II}\prime}(0)\, e^{uH}\rangle$$
$$+\frac{1}{\hbar^2}\langle x^{\mathrm{II}}(0)y^{\mathrm{II}\prime}(0)-y^{\mathrm{II}\prime}(0)x^{\mathrm{II}}(0)\rangle . \quad (\mathrm{A4})$$

Noting that

$$e^{-uH}y^{\mathrm{II}\prime}(0)\, e^{uH}=e^{-uH}[H, y^{\mathrm{II}}(0)]\, e^{uH}$$
$$=-\frac{d}{du}y^{\mathrm{II}}(-u) ,$$

one obtains

$$\mathfrak{F}_{xy}^{\mathrm{II\,II}}(\hbar\omega_r) = -\frac{\omega_r}{\hbar}\int_{-\epsilon}^{\beta-\epsilon} du\, e^{\hbar\omega_r u}\left\langle \theta(u)x^{\mathrm{II}}(0)\frac{d}{du}y^{\mathrm{II}}(-u) + \theta(-u)\frac{d}{du}y^{\mathrm{II}}(-u)x^{\mathrm{II}}(0)\right\rangle + \frac{1}{\hbar^2}\langle [x^{\mathrm{II}}(0), y^{\mathrm{II}\prime}(0)]\rangle . \tag{A5}$$

Integrating by parts and using the periodic property, one finds

$$\begin{aligned}\mathfrak{F}_{xy}^{\mathrm{II\,II}}(\hbar\omega_r) &= -\frac{\omega_r}{\hbar}\left[e^{\hbar\omega_r u}\langle Tx^{\mathrm{II}}(0)y^{\mathrm{II}}(-u)\rangle\right]_{-\epsilon}^{\beta-\epsilon} + \frac{\omega_r}{\hbar}\hbar\omega_r\int_{-\epsilon}^{\beta-\epsilon} du\, e^{\hbar\omega_r u}\langle Tx^{\mathrm{II}}(0)y^{\mathrm{II}}(-u)\rangle \\ &\quad + \frac{\omega_r}{\hbar}\int_{-\epsilon}^{\beta-\epsilon} du\, e^{\hbar\omega_r u}\langle \delta(u)x^{\mathrm{II}}(0)y^{\mathrm{II}}(-u) - \delta(u)y^{\mathrm{II}}(-u)x^{\mathrm{II}}(0)\rangle + \frac{1}{\hbar^2}\langle [x^{\mathrm{II}}(0), y^{\mathrm{II}\prime}(0)]\rangle \\ &= \omega_r^2\int_{-\epsilon}^{\beta-\epsilon} du\, e^{\hbar\omega_r u}\langle Tx^{\mathrm{II}}(u)y^{\mathrm{II}}(0)\rangle + \frac{\omega_r}{\hbar}\langle [x^{\mathrm{II}}(0), y^{\mathrm{II}}(0)]\rangle + \frac{1}{\hbar^2}\langle [x^{\mathrm{II}}(0), y^{\mathrm{II}\prime}(0)]\rangle . \end{aligned} \tag{A6}$$

The first integral does not diverge for $\omega_r \to i0$. Therefore one obtains, using (2.13),

$$\sigma_{xy}^{\mathrm{II\,II}} = (e^2/i\hbar\Omega)\langle [x^{\mathrm{II}}, y^{\mathrm{II}}]\rangle . \tag{A7}$$

which proves (2.16) (QED).

One can prove (2.17) in a similar way. From (A2) one computes

$$\begin{aligned}\mathfrak{F}_{xy}^{\mathrm{II\,I}}(\hbar\omega_r) &= \frac{-1}{\hbar^2}\int_{-\epsilon}^{\beta-\epsilon} du\, e^{\hbar\omega_r u}\langle \theta(u)x^{\mathrm{II}\prime}(u)y^{\mathrm{I}\prime}(0) + \theta(-u)y^{\mathrm{I}\prime}(0)x^{\mathrm{II}\prime}(u)\rangle = -\frac{1}{\hbar^2}\left[e^{\hbar\omega_r u}\langle Tx^{\mathrm{II}}(u)y^{\mathrm{I}\prime}(0)\rangle\right]_{-\epsilon}^{\beta-\epsilon} \\ &\quad + \frac{1}{\hbar^2}\hbar\omega_r\int_{-\epsilon}^{\beta-\epsilon} du\, e^{\hbar\omega_r u}\langle Tx^{\mathrm{II}}(u)y^{\mathrm{I}\prime}(0)\rangle + \frac{1}{\hbar^2}\int_{-\epsilon}^{\beta-\epsilon} du\, e^{\hbar\omega_r u}\langle \delta(u)x^{\mathrm{II}}(u)y^{\mathrm{I}\prime}(0) - \delta(u)y^{\mathrm{I}\prime}(0)x^{\mathrm{II}}(u)\rangle \\ &= \frac{\omega_r}{\hbar}\int_{-\epsilon}^{\beta-\epsilon} du\, e^{\hbar\omega_r u}\langle Tx^{\mathrm{II}}(u)y^{\mathrm{I}\prime}(0)\rangle + \frac{1}{\hbar^2}\langle [x^{\mathrm{II}}(0), y^{\mathrm{I}\prime}(0)]\rangle . \end{aligned} \tag{A8}$$

Hence

$$\sigma_{xy}^{\mathrm{II\,I}} = -\frac{e^2}{\Omega}\lim_{\omega\to 0}\tilde{\mathfrak{F}}_{xy}^{\mathrm{II\,I}}(\hbar\omega + i0) , \tag{A9}$$

where

$$\tilde{\mathfrak{F}}_{xy}^{\mathrm{II\,I}}(\hbar\omega_r) = \frac{-1}{i\hbar}\int_{-\epsilon}^{\beta-\epsilon} du\, e^{\hbar\omega_r u}\langle Tx^{\mathrm{II}}(u)y^{\mathrm{I}\prime}(0)\rangle . \tag{A10}$$

This is identical to (2.18b). The proof for $\sigma_{xy}^{\mathrm{I\,II}}$ is also similar and will be omitted.

APPENDIX B

In this appendix we evaluate (3.42) to the second order in electron-phonon interaction as illustrated diagrammatically in Fig. 8 and we derive (3.47). Define

$$\begin{aligned}\langle\cdots\rangle_a^{(s)} &= \{V_{kn,k'''n'''}V_{k'''n''',k'n'}V_{k'n',k''n''}V_{k''n'',kn}\delta_{k+k',k''+k'''}\}^{(s)} , \\ \langle\cdots\rangle_b^{(s)} &= \{V_{kn,k'n'}V_{k'n',k'''n'''}V_{k'''n''',k''n''}V_{k''n'',kn}\delta_{k+k''',k'+k''}\}^{(s)} , \\ \langle\cdots\rangle_c^{(s)} &= \{V_{kn,k''n''}V_{k''n'',k'''n'''}V_{k'''n''',k'n'}V_{k'n',kn}\delta_{k+k''',k'+k''}\}^{(s)} ,\end{aligned} \tag{B1}$$

where the subscripts a, b, and c refer to the irreducible scattering parts denoted as a, b, c in Fig. 8. Then one has

$$\begin{aligned}\Upsilon_{kn}^{(2,s)}(\zeta_l, \zeta_l+\hbar\omega_r) &= \frac{1}{\beta^2}\sum_{k'n'l'}\frac{\Lambda_{k'n'}(\zeta_{l'}, \zeta_{l'}+\hbar\omega_r)}{[\zeta_{l'}-\epsilon_{k'n'}-G_{k'n'}(\zeta_{l'})][\zeta_{l'}+\hbar\omega_r-\epsilon_{k'n'}-G_{k'n'}(\zeta_{l'}+\hbar\omega_r)]} \\ &\times \sum_{\substack{k''n''k'''n''' \\ l''l'''}}\sum_{\substack{t=0,1 \\ u=0,1}}\left(\frac{\langle\cdots\rangle_a^{(s)}\delta_{l+l',l''+l'''}}{[\zeta_{l''}-\epsilon_{k''n''}-G_{k''n''}(\zeta_{l''})][\zeta_{l'''}+\hbar\omega_r-\epsilon_{k'''n'''}-G_{k'''n'''}(\zeta_{l'''}+\hbar\omega_r)][\hbar\omega_{kk''}+(-1)^u(\zeta_l-\zeta_{l''})][\hbar\omega_{kk'''}+(-1)^t(\zeta_l-\zeta_{l'''})]}\right. \\ &+ \frac{\langle\cdots\rangle_b^{(s)}\delta_{l+l''',l'+l''}}{[\zeta_{l''}-\epsilon_{k''n''}-G_{k''n''}(\zeta_{l''})][\zeta_{l'''}-\epsilon_{k'''n'''}-G_{k'''n'''}(\zeta_{l'''})][\hbar\omega_{kk'}+(-1)^t(\zeta_l-\zeta_{l'})][\hbar\omega_{kk''}+(-1)^u(\zeta_l-\zeta_{l''})]} \\ &+ \left.\frac{\langle\cdots\rangle_c^{(s)}\delta_{l+l''',l'+l''}}{[\zeta_{l''}+\hbar\omega_r-\epsilon_{k''n''}-G_{k''n''}(\zeta_{l''}+\hbar\omega_r)][\zeta_{l'''}+\hbar\omega_r-\epsilon_{k'''n'''}-G_{k'''n'''}(\zeta_{l'''}+\hbar\omega_r)][\hbar\omega_{kk'}+(-1)^t(\zeta_l-\zeta_{l'})][\hbar\omega_{kk''}+(-1)^u(\zeta_l-\zeta_{l''})]}\right) .\end{aligned} \tag{B2}$$

One performs the l' summation by introducing a factor $(\beta/2\pi i)[f^{(-)}(\zeta')-f^{(-)}(\epsilon_{k'''n'''})]$ and then integrating around the contour Γ shown in Fig. 4. One then obtains

$$\Upsilon_{kn}^{(2,s)}(\zeta_l,\ \zeta_l+\hbar\omega_r)=\frac{1}{\beta^2}\sum_{k'n'k''n''k'''n'''l''}\ \sum_{\substack{t=0,1\\u=0,1}}(-1)^{t+u}\frac{\beta}{2\pi i}\int_\Gamma d\zeta'\frac{[f^{(-)}(\zeta')-f^{(-1)}(\epsilon_{k'''n'''})]\Lambda_{k'n'}(\zeta',\ \zeta'+\hbar\omega_r)}{[\zeta'-\epsilon_{k'n'}-G_{k'n'}(\zeta')][\zeta'+\hbar\omega_r-\epsilon_{k'n'}-G_{k'n'}(\zeta''+\hbar\omega_r)]}$$
$$\times\left(\frac{\langle\cdots\rangle_a^{(s)}}{[\zeta_{l''}-\epsilon_{k''n''}][(-1)^u\hbar\omega_{kk''}+(\zeta_l-\zeta_{l''})][(-1)^t\hbar\omega_{kk'''}+(\zeta_{l''}-\zeta')][\zeta'-\zeta_{l''}+\zeta_l-\epsilon_{k'''n'''}+\hbar\omega_r]}\right.$$
$$+\frac{\langle\cdots\rangle_b^{(s)}}{[\zeta_{l''}-\epsilon_{k''n''}][(-1)^u\hbar\omega_{kk''}+(\zeta_l-\zeta_{l''})][(-1)^t\hbar\omega_{kk'}+(\zeta_l-\zeta')][\zeta'+\zeta_{l''}-\zeta_l-\epsilon_{k'''n'''}]}$$
$$\left.+\frac{\langle\cdots\rangle_c^{(s)}}{[\zeta_{l''}+\hbar\omega_r-\epsilon_{k''n''}][(-1)^u\hbar\omega_{kk''}+(\zeta_l-\zeta_{l''})][-1)^t\hbar\omega_{kk'}+(\zeta_l-\zeta')][\zeta'+\zeta_{l''}-\zeta_l+\hbar\omega_r-\epsilon_{k'''n'''}]}\right),\tag{B3}$$

where the unimportant self-energy parts $G_{k''n''}$ and $G_{k'''n'''}$ are neglected, maintaining only $G_{k'n'}(\zeta')$ and $G_{k'n'}(\zeta'+\hbar\omega_r)$. Noting that the residues of the fermion poles (related with the energy ϵ_{kn}) vanish, one can deform the contour Γ of (B3) into Γ_0 of Fig. 4, taking the contributions from the phonon poles. Changing the variable $\zeta'+\hbar\omega_r\to\zeta'$ for the part of the contour Γ_0 along $\mathrm{Im}(\zeta'+\hbar\omega_r)=0$, one obtains, using (3.49)

$$\Upsilon_{kn}^{(2,s)}(\zeta_l,\ \zeta_l+\hbar\omega_r)=\frac{1}{2\pi i\beta}\sum_{\substack{k'n'k''n''k'''n'''\\t=0,1,u=0,1}}(-1)^{t+u}\int_{-\infty}^{\infty}d\zeta'[f^{(-)}(\zeta')-f^{(-)}(\epsilon_{k'''n'''})]$$
$$\times\left[\left(\frac{\Lambda_{k'n'}(\zeta'-i0,\ \zeta'+\hbar\omega_r)}{[\zeta'-\epsilon_{k'n'}-G_{k'n'}(\zeta'-i0)][\zeta'+\hbar\omega_r-\epsilon_{k'n'}-G_{k'n'}(\zeta'+\hbar\omega_r)]}-(-i0\to+i0)\right)\right.$$
$$\times\left(\frac{\langle\cdots\rangle_a^{(s)}}{[\zeta_{l''}-\epsilon_{k''n''}][(-1)^u\hbar\omega_{kk''}+(\zeta_l-\zeta_{l''})][(-1)^t\hbar\omega_{kk'''}+(\zeta_{l''}-\zeta')][\zeta_l+\zeta'-\zeta_{l''}+\hbar\omega_r-\epsilon_{k'''n'''}]}\right.$$
$$+\frac{\langle\cdots\rangle_b^{(s)}}{[\zeta_{l''}-\epsilon_{k''n''}][(-1)^u\hbar\omega_{kk''}+(\zeta_l-\zeta_{l''})][(-1)^t\hbar\omega_{kk'}+(\zeta_l-\zeta')][\zeta'+\zeta_{l''}-\zeta_l-\epsilon_{k'''n'''}]}$$
$$\left.+\frac{\langle\cdots\rangle_c^{(s)}}{[\zeta_{l''}+\hbar\omega_r-\epsilon_{k''n''}][(-1)^u\hbar\omega_{kk''}+(\zeta_l-\zeta_{l''})][(-1)^t\hbar\omega_{kk'}+(\zeta_l-\zeta')][\zeta'+\zeta_{l''}-\zeta_l+\hbar\omega_r-\epsilon_{k'''n'''}]}\right)$$
$$+\left(\frac{\Lambda_{k'n'}(\zeta'-\hbar\omega_r,\ \zeta'-i0)}{[\zeta'-\hbar\omega_r-\epsilon_{k'n'}-G_{k'n'}(\zeta'-\hbar\omega_r)][\zeta'-\epsilon_{k'n'}-G_{k'n'}(\zeta'-i0)]}-(-i0\to+i0)\right)$$
$$\times\left(\frac{\langle\cdots\rangle_a^{(s)}}{[\zeta_{l''}-\epsilon_{k''n''}][(-1)^u\hbar\omega_{kk''}+(\zeta_l-\zeta_{l''})][(-1)^t\hbar\omega_{kk'''}+\zeta_{l''}+\hbar\omega_r-\zeta'][\zeta_l-\zeta_{l''}+\zeta'-\epsilon_{k'''n'''}]}\right.$$
$$+\frac{\langle\cdots\rangle_b^{(s)}}{[\zeta_{l''}-\epsilon_{k''n''}][(-1)^u\hbar\omega_{kk''}+(\zeta_l-\zeta_{l''})][(-1)^t\hbar\omega_{kk'}+\zeta_l+\hbar\omega_r-\zeta'][\zeta'-\hbar\omega_r+\zeta_{l''}-\zeta_l-\zeta_{k'''n'''}]}$$
$$\left.\left.+\frac{\langle\cdots\rangle_c^{(s)}}{[\zeta_{l''}+\hbar\omega_r-\epsilon_{k''n''}][(-1)^u\hbar\omega_{kk''}+\zeta_l-\zeta_{l''}][(-1)^t\hbar\omega_{kk'}+\zeta_l+\hbar\omega_r-\zeta'][\zeta'+\zeta_{l''}-\zeta_l-\epsilon_{k'''n'''}]}\right)\right]+\frac{1}{\beta}\sum_{\substack{k'n'k''n''k'''n'''\\l''}}\ \sum_{\substack{t=0,1\\u=0,1}}(-1)^{t+u}$$
$$\times\left(\frac{\langle\cdots\rangle_a^{(s)}[N_{kk'''}^{t}+f^{(-)}(\epsilon_{k'''n'''})]\Lambda_{k'n'}[(-1)^t\hbar\omega_{kk'''}+\zeta_{l''},\ (-1)^t\hbar\omega_{kk'''}+\zeta_{l''}+\hbar\omega_r][\zeta_{l''}-\epsilon_{k''n''}]^{-1}[(-1)^u\hbar\omega_{kk''}+\zeta_l-\zeta_{l''}]^{-1}}{\{(-1)^t\hbar\omega_{kk'''}+\zeta_{l''}-\epsilon_{k'n'}-G_{k'n'}[(-1)^t\hbar\omega_{kk'''}+\zeta_{l''}]\}\{(-1)^t\hbar\omega_{kk'''}+\zeta_{l''}+\hbar\omega_r-\epsilon_{k'n'}-G_{k'n'}[(-1)^t\hbar\omega_{kk'''}+\zeta_{l''}+\hbar\omega_r]\}[\zeta_l+\hbar\omega_r+(-1)^t\hbar\omega_{kk'''}-\epsilon_{k'''n'''}]}\right.$$
$$+\frac{\langle\cdots\rangle_b^{(s)}[N_{kk'}^{(t)}+f^{(-)}(\epsilon_{k'''n'''})]\Lambda_{k'n'}[(-1)^t\hbar\omega_{kk'}+\zeta_l,\ (-1)^t\hbar\omega_{kk'}+\zeta_l+\hbar\omega_r][\zeta_{l''}-\epsilon_{k''n''}]^{-1}[(-1)^u\hbar\omega_{kk''}+\zeta_l-\zeta_{l''}]^{-1}}{\{(-1)^t\hbar\omega_{kk'}+\zeta_l-\epsilon_{k'n'}-G_{k'n'}[(-1)^t\hbar\omega_{kk'}+\zeta_l]\}\{(-1)^t\hbar\omega_{kk'}+\zeta_l+\hbar\omega_r-\epsilon_{k'n'}-G_{k'n'}[(-1)^t\hbar\omega_{kk'}+\zeta_l+\hbar\omega_r]\}[(-1)^t\hbar\omega_{kk'}+\zeta_{l''}-\epsilon_{k'''n'''}]}$$
$$\left.+\frac{\langle\cdots\rangle_c^{(s)}[N_{kk'}^{(t)}+f^{(-)}(\epsilon_{k'''n'''})]\Lambda_{k'n'}[(-1)^t\hbar\omega_{kk'}+\zeta_l,\ (-1)^t\hbar\omega_{kk'}+\zeta_l+\hbar\omega_r][\zeta_{l''}+\hbar\omega_r-\epsilon_{k''n''}]^{-1}[(-1)^u\hbar\omega_{kk''}+\zeta_l-\zeta_{l''}]^{-1}}{\{(-1)^t\hbar\omega_{kk'}+\zeta_l-\epsilon_{k'n'}-G_{k'n'}[(-1)^t\hbar\omega_{kk'}+\zeta_l]\}\{(-1)^t\hbar\omega_{kk'}+\zeta_l+\hbar\omega_r-\epsilon_{k'n'}-G_{k'n'}[(-1)^t\hbar\omega_{kk'}+\zeta_l+\hbar\omega_r]\}[(-1)^t\hbar\omega_{kk'}+\zeta_{l''}+\hbar\omega_r-\epsilon_{k'''n'''}]}\right)\tag{B4}$$

The second term in the first large parentheses and the first term in the third large parentheses can be dropped in the limit $\omega_r\to i0$ in view of (3.21) and (3.22). Taking the limits $\omega_r\to i0$, $\zeta_l\to\epsilon_{kn}-i0$, $\zeta_l+\hbar\omega_r\to\epsilon_{kn}+i0$ whenever there is no ambiguity, approximating

$$\frac{1}{[\zeta'-\epsilon_{k'n'}-G_{k'n'}(\zeta'-i0)][\zeta'-\epsilon_{k'n'}-G_{k'n'}(\zeta'+i0)]}=\frac{\pi\delta(\zeta'-\epsilon_{k'n'})}{\Gamma_{k'n'}(\epsilon_{k'n'})},$$

and replacing $\sum_{l''}(\cdots\zeta_{l''}\cdots)$ by $(\beta/2\pi i)\int_\Gamma d\zeta''[f^{(-)}(\zeta'')+N_{kk'}^{(u)}](\cdots\zeta''\cdots)$, one obtains

$$\Upsilon_{kn}^{(2,s)}(\zeta_l,\zeta_l+\hbar\omega_r)=\frac{1}{i\hbar\beta}\sum_{k'n'k''n''k'''n'''}\ \sum_{\substack{t=0,1\\u=0,1}}(-1)^t\phi_{k'n'}^{(-2)}[f^{(-)}(\epsilon_{k'n'})-f^{(-)}(\epsilon_{k'''n'''})]\frac{\beta}{2\pi i}\int_\Gamma d\zeta''[f^{(-)}(\zeta'')+N_{kk'}^{(u)}]$$
$$\times\left[\frac{\langle\cdots\rangle_a^{(s)}}{[\zeta''-\epsilon_{k''n''}][(-1)^u\hbar\omega_{kk''}+\zeta_l-\zeta'']}\left(\frac{1}{[(-1)^t\hbar\omega_{kk'''}+\zeta''-\epsilon_{k'n'}][\zeta_l+\hbar\omega_r-\zeta''+\epsilon_{k'n'}-\epsilon_{k'''n'''}]}\right.\right.$$

$$-\frac{1}{[(-1)^t\hbar\omega_{kk'''}+\zeta''+\hbar\omega_r-\epsilon_{k'n'}][\zeta_l-\zeta''+\epsilon_{k'n'}-\epsilon_{k'''n'''}]}\Bigg)+\frac{\langle\cdots\rangle_b^{(s)}}{[\zeta''-\epsilon_{k''n''}][(-1)^u\hbar\omega_{kk''}+\zeta_l-\zeta'']}$$

$$\times\left(\frac{1}{[(-1)^t\hbar\omega_{kk'}+\zeta_l-\epsilon_{k'n'}][\zeta''-\zeta_l+\epsilon_{k'n'}-\epsilon_{k'''n'''}]}-\frac{1}{[(-1)^t\hbar\omega_{kk'}+\zeta_l+\hbar\omega_r-\epsilon_{k'n'}][\epsilon_{k'n'}-\epsilon_{k'''n'''}+\zeta''-\zeta_l-\hbar\omega_r]}\right)$$

$$+\frac{\langle\cdots\rangle_c^{(s)}}{[\zeta''+\hbar\omega_r-\epsilon_{k''n''}][(-1)^u\hbar\omega_{kk''}+\zeta_l-\zeta'']}\Bigg(\frac{1}{[(-1)^t\hbar\omega_{kk'}+\zeta_l-\epsilon_{k'n'}][\zeta''-\zeta_l+\hbar\omega_r+\epsilon_{k'n'}-\epsilon_{k'''n'''}]}$$

$$-\frac{1}{[(-1)^t\hbar\omega_{kk'}+\zeta_l+\hbar\omega_r-\epsilon_{k'n'}][\epsilon_{k'n'}-\epsilon_{k'''n'''}+\zeta''-\zeta_l]}\Bigg)\Bigg]+\frac{1}{\beta}\sum_{k'n'k''n''k'''n'''}\sum_{\substack{t=0,1\\u=0,1}}(-1)^t\frac{\beta}{2\pi i}\int_\Gamma d\zeta''\frac{[f^{(-)}(\zeta'')+N_{kk''}^{(u)}]}{(-1)^u\hbar\omega_{kk''}+\zeta_l-\zeta''}$$

$$\times\Bigg(\frac{\langle\cdots\rangle_a^{(s)}[N_{kk'''}^{(t)}+f^{(-)}(\epsilon_{k'''n'''})]\Lambda_{k'n'}[(-1)^t\hbar\omega_{kk'''}+\zeta'',(-1)^t\hbar\omega_{kk'''}+\zeta''+\hbar\omega_r][\zeta''-\epsilon_{k''n''}]^{-1}}{\{(-1)^t\hbar\omega_{kk'''}+\zeta''-\epsilon_{k'n'}-G_{k'n'}[(-1)^t\hbar\omega_{kk'''}+\zeta'']\}\{(-1)^t\hbar\omega_{kk'''}+\zeta''+\hbar\omega_r-\epsilon_{k'n'}-G_{k'n'}[(-1)^t\hbar\omega_{kk'''}+\zeta''+\hbar\omega_r]\}[\zeta_l+\hbar\omega_r+(-1)^t\hbar\omega_{kk'''}-\epsilon_{k'''n'''}]}$$

$$+\frac{\phi_{k'n'}^{(-2)}\langle\cdots\rangle_b^{(s)}2\pi\delta[(-1)^t\hbar\omega_{kk'}+\epsilon_{kn}-\epsilon_{k'n'}][N_{kk'}^{(t)}+f^{(-)}(\epsilon_{k'''n'''})]}{\hbar[\zeta''-\epsilon_{k''n''}][(-1)^t\hbar\omega_{kk'}+\zeta''-\epsilon_{k'''n'''}]}+\frac{\phi_{k'n'}^{(-2)}\langle\cdots\rangle_c^{(s)}2\pi\delta[(-1)^t\hbar\omega_{kk'}+\epsilon_{kn}-\epsilon_{k'n'}][N_{kk'}^{(t)}+f^{(-)}(\epsilon_{k'''n'''})]}{\hbar[\zeta''+\hbar\omega_r-\epsilon_{k''n''}][(-1)^t\hbar\omega_{kk'}+\zeta''+\hbar\omega_r-\epsilon_{k'''n'''}]}\Bigg),\qquad\text{(B5)}$$

where $\phi_{kn}^{(-2)}=\hbar\Lambda_{kn}(\epsilon_{kn}-i0,\ \epsilon_{kn}+i0)/2\Gamma_{kn}(\epsilon_{kn})$ as defined in Sec. III. Introducing the same kind of approximations as were employed in proceeding from (B4) to (B5) and using $\langle\cdots\rangle_b^{(s)}=-\langle\cdots\rangle_c^{(s)}$, one finds, after some lengthy algebra,

$$\Upsilon_{kn}^{(2,s)}(\epsilon_{kn}-i0,\ \epsilon_{kn}+i0)=\frac{2\pi}{\hbar}\sum_{k'n'k''n''k'''n'''}\sum_{\substack{t=0,1\\u=0,1}}(-1)^{t+u}\phi_{k'n'}^{(-2)}$$

$$\times\Bigg\{-\langle\cdots\rangle_a^{(s)}\Bigg(\frac{[f^{(-)}(\epsilon_{k'n'})-f^{(-)}(\epsilon_{k'''n'''})][f^{(-)}(\epsilon_{k''n''})+N(\epsilon_{k'n'}-\epsilon_{k'''n'''})]\delta(\epsilon_{kn}+\epsilon_{k'n'}-\epsilon_{k''n''}-\epsilon_{k'''n'''})}{[(-1)^u\hbar\omega_{kk''}+\epsilon_{k'''n'''}-\epsilon_{k'n'}-i0][(-1)^t\hbar\omega_{kk'''}+\epsilon_{k''n''}-\epsilon_{k'n'}+i0]}$$

$$+\frac{[f^{(-)}(\epsilon_{k'n'})+N_{kk'''}^{(t)}]\{f^{(-)}[\epsilon_{k'n'}-(-1)^t\hbar\omega_{kk'''}]+N_{kk''}^{(u)}\}\delta[(-1)^u\hbar\omega_{kk''}+(-1)^t\hbar\omega_{kk'''}+\epsilon_{kn}-\epsilon_{k'n'}]}{[\epsilon_{kn}+i0+(-1)^t\hbar\omega_{kk'''}-\epsilon_{k'''n'''}][(-1)^t\hbar\omega_{kk'''}+\epsilon_{k''n''}-\epsilon_{k'n'}+i0]}\Bigg)$$

$$+2\pi i\langle\cdots\rangle_c^{(s)}\ [f^{(-)}(\epsilon_{k''n''})+N_{kk''}^{(u)}][f^{(-)}(\epsilon_{k'n'})-f^{(-)}(\epsilon_{k'''n'''})]\delta[(-1)^u\hbar\omega_{kk''}+\epsilon_{kn}-\epsilon_{k''n''}]$$

$$\times\Bigg(\delta[(-1)^t\hbar\omega_{kk'}+\epsilon_{kn}-\epsilon_{k'n'}]\mathcal{P}\frac{1}{\epsilon_{kn}+\epsilon_{k'''n'''}-\epsilon_{k'n'}-\epsilon_{k''n''}}+\delta(\epsilon_{kn}+\epsilon_{k'''n'''}-\epsilon_{k'n'}-\epsilon_{k''n''})\mathcal{P}\frac{1}{(-1)^t\hbar\omega_{kk'}+\epsilon_{kn}-\epsilon_{k'n'}}\Bigg)$$

$$-2\pi i\langle\cdots\rangle_c^{(s)}\ [N_{kk'}^{(t)}+f^{(-)}(\epsilon_{k'''n'''})]\delta[(-1)^t\hbar\omega_{kk'}+\epsilon_{kn}-\epsilon_{k'n'}]\Bigg(\delta[(-1)^u\hbar\omega_{kk''}+\epsilon_{kn}-\epsilon_{k''n''}]$$

$$\times[f^{(-)}(\epsilon_{k''n''})+N_{kk''}^{(u)}]\mathcal{P}\frac{1}{(-1)^t\hbar\omega_{kk'}+\epsilon_{k''n''}-\epsilon_{k'''n'''}}-\delta[(-1)^u\hbar\omega_{kk''}+(-1)^t\hbar\omega_{kk'}+\epsilon_{kn}-\epsilon_{k'''n'''}]$$

$$\times\{f^{(-)}[\epsilon_{k'''n'''}-(-1)^t\hbar\omega_{kk'}]+N_{kk''}^{(u)}\}\mathcal{P}\frac{1}{(-1)^t\hbar\omega_{kk'}+\epsilon_{k''n''}-\epsilon_{k'''n'''}}\Bigg)\Bigg\}\ .\qquad\text{(B6)}$$

Using the detailed balance condition $f^{(-)}(\epsilon_{k_1})f^{(+)}(\epsilon_{k_2})=N(\epsilon_{k_1}-\epsilon_{k_2})\,[f^{(-)}(\epsilon_{k_2})-f^{(-)}(\epsilon_{k_1})]$, one can show that numerators in the first large parentheses are symmetric under the interchange of the indices; $k''n''\rightarrow k'''n'''$ and $t\rightarrow u$. Also, because $\langle\cdots\rangle_a^{(s)}$ is pure imaginary due to (3.16), one has $\langle\cdots\rangle_a^{(s)}\rightarrow-\langle\cdots\rangle_a^{(s)}$ under the interchange of $k''n''\rightarrow k'''n'''$. Using these properties, one can simplify (B6) to a final form given by (3.47). We have used the fact that $\langle\cdots\rangle_a^{(s)}$, $\langle\cdots\rangle_b^{(s)}$, $\langle\cdots\rangle_c^{(s)}$ are pure imaginary in replacing $\langle\cdots\rangle_a^{(s)}$ by $i\,\mathrm{Im}(V_{kn,k'''n'''}V_{k'''n''',k'n'}V_{k'n',k''n''}V_{k''n'',kn})$, etc. Also (3.49) follows from the properties $\langle\cdots\rangle_c^{(s)}\rightarrow\langle\cdots\rangle_a^{(s)}$ under the interchange of $k'n'\rightarrow k'''n'''$ and $\langle\cdots\rangle_c^{(s)}\rightarrow-\langle\cdots\rangle_c^{(s)}$ under $k'n'\rightarrow k''n''$.

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[19]For the case of nonvanishing external-field wave vector $\vec{q}$, the left-hand side of (3.13) should read $i(\vec{q}\cdot\vec{\mathfrak{V}}_{kn} - \omega)\Phi_{kn}$.
[20]Note that the electron-phonon interaction enters the electron or phonon self-energy part in the form $|V_{kn,k'n'}|^2$ to the lowest order shown in Fig. 3.
[21]From (3.16) it follows that $\{|V_{kn,k'n'}|^2\}^{(s)}$ is purely imaginary. Hence $\{|V_{kn,k'n'}|^2\}^{(s)} = 0$, QED.
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[24]For more detailed discussion of this kind of approximation, see Ref. 12.
[25]In view of (2.19) we are interested only in the terms which are first order in ω. Therefore the contribution from the last two terms of (3.20) is small due to their slowly varying nature as a function of ω.
[26]Using (3.16) one has $\{V_{kn,k'n'}D_x V_{k'n',kn}\}^{(s)*} = V^{(s)*}{}_{kn,k'n'}D_x \overline{V}^*_{k'n',kn} + \overline{V}^*_{kn,k'n'}D_x V^{(s)*}_{k'n',kn} = -(V_{kn,k'n'}D_x V_{k'n',kn})^{(s)}$
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PHYSICAL REVIEW B VOLUME 8, NUMBER 3 1 AUGUST 1973

$S=\frac{1}{2}$, XY Model on Cubic Lattices. I. Susceptibility and Fluctuation near Critical Temperature*

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For the $S=\frac{1}{2}$, XY model of a quantum lattice fluid or a ferromagnet the conventional order parameter does not commute with the Hamiltonian. As a result, the mean-square fluctuation of the order parameter and the isothermal susceptibility are not related in the usual way the general fluctuation theorem. For the above model, arguments are here presented to support the idea that as $T \to T_c$ the quantum effect due to the noncommutation becomes masked and the two quantities have the same critical behavior. This work is consistent with the exact results of Falk and Bruch who defined a certain moment of the spectral density and used inequalities to establish that *if* the moment $\to 0$ as $T \to T_c$, then the susceptibility-fluctuation ratio becomes unity thus ensuring coinciding critical behavior. The latter result applies to a large class of models including the one considered here.

I. INTRODUCTION

The three-dimensional $S=\frac{1}{2}$, XY model defined in Sec. II is one of the simplest *quantum mechanical* cooperative models. This model is thought to be useful as an approximation for certain physical systems such as liquid helium[4] near the λ transition.[1] A few of the static properties of this model near the critical point have now been calculated and their similarity to those of other cooperative models has been observed.[2]

One unique feature of the XY model, which particularly emphasizes its quantum nature, is that its order parameter M^x (or M^y) does *not* commute with the Hamiltonian (the order parameter for the Heisenberg ferromagnet commutes with its Hamiltonian).[3] The noncommuting gives rise to two interesting related consequences. (i) The order parameter has a time-dependent behavior at all wavelengths including zero wavelength, and the system thus can *relax*. The dynamical behavior of this system has proved to be quite interesting and will be treated in a later paper. (ii) One can define the mean-square fluctuation of the order parameter Y and the zero-field static isothermal susceptibility χ which are not connected in the usual way by the general "fluctuation" theorem (we shall refer to Y and χ simply as the fluctuation and susceptibility, respectively). This paper is concerned with the nature of the distinction between the fluctuation and susceptibility whose origin is thus quantum mechanical.

While the distinction between the fluctuation and susceptibility is valid generally, one suspects that