Generalized Scaling Hypothesis in Multicomponent Systems. II. Scaling Hypothesis at Critical Points of Arbitrary Order*

Alex Hankey,† T. S. Chang,‡ and H. Eugene Stanley

Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

(Received 29 January 1973)

We make the scaling hypothesis at a space of critical points of arbitrary order, using the geometric classification of critical points by order proposed in a preceding paper, part I. It follows from the definition of order that there are regions in which several scaling hypotheses are simultaneously valid, one at each space of critical points of order $\mathcal{O} \geq 2$ which is connected to the point of highest order. The successive hypotheses are conveniently framed in terms of invariants of the groups of transformations about the points of higher order. This procedure facilitates the formation of multiple-power scaling functions. We suggest that the regions of validity of later hypotheses are bounded by hypersurfaces which scale and which are therefore most easily visualized in the spaces of invariants. A general critical point of order 4 is treated as a preliminary example. We then apply the hypothesis in detail to the critical point of order 4 in a specific example: the Ising model with variable interplanar interaction. This is done in two ways: the first shows specifically the relationship to the case of a metamagnet and to scaling with a parameter for change of lattice dimension; the second explicitly demonstrates that the connectivity between the various surfaces of ordinary ($\mathcal{O}=2$) critical points is the same as for the edges of a tetrahedron. As a consequence of this scaling hypothesis, we can make predictions about the shapes of the coexistence and critical surfaces meeting at the critical point of order 4.

I. INTRODUCTION

In a preceding paper[1] (hereafter referred to as I) we introduced the geometric concept of order for a space of critical points (CRS) and also a convenient notation ${}^{\mathcal{O}}R_d$ and ${}^{p}X_d$ for critical and coexistence spaces in complex phase diagrams. Well-known examples of tricritical points, and other critical points of order 3 were discussed, and critical spaces of order 4 were shown to exist in certain model systems involving four field or fieldlike variables. The purpose of this paper is to show how to formulate a scaling hypothesis at a space of critical points of arbitrary order; to derive multiple-power scaling functions at such points; and to explicitly analyze the critical point of order[2] 4 in the three-dimensional Ising model with variable interplanar interaction in terms of invariants.[2]

To formulate a scaling hypothesis at a critical point of arbitrary order, it is first necessary to select special directions at the point analogous to the strong and weak directions described in I. As discussed in Sec. IV of I, the problem of finding the principal directions of scaling at a critical point of arbitrary order is exactly the same as that for a tricritical point (or ${}^3R_0$). Once the principal directions at a space of points of order $\mathcal{O}$ have been found, we can take the limiting orientation of the principal directions at a point of order $\mathcal{O}+1$, as it is approached along a line of points of order $\mathcal{O}$. The problem of uniqueness of this coordinate system is the same for a point of order $\mathcal{O}=3$ and its solution may be obtained by the same principles as treated in Sec. IV of I.

It is therefore possible to induce a set of coordinate systems for a thermodynamic system showing critical spaces of order 4 or more, starting at a ${}^2R_{n-2}$ and proceeding by induction up to the ${}^{\mathcal{O}}R_{n-\mathcal{O}}$. As each higher order is reached, another principal direction is defined; at an ${}^{\mathcal{O}}R_{n-\mathcal{O}}$, $\mathcal{O}$ directions out of the ${}^{\mathcal{O}}R_{n-\mathcal{O}}$ have been defined. They will be denoted by $\{x_{\mathcal{O}1}, x_{\mathcal{O}2}, \ldots, x_{\mathcal{O}\mathcal{O}}\}$. The geometric possibilities are too complex (even for $\mathcal{O}=4$) to warrant the case by case analysis done earlier for a ${}^3R_0$; however, it is clear that the consistency arguments demonstrated in I will apply equally well for any order $\mathcal{O}$.

The outline of this paper is as follows. In Sec. II, we form the scaling hypothesis and show how to make it at each ${}^{\mathcal{O}}R_d$ of successively smaller order. We thereby obtain a total group of scaling transformations which is the direct product of several subgroups.

In Sec. III we treat a general ${}^4R_0$, considering a particular ${}^3R_1$ and ${}^2R_2$. We schematically plot the crossover surfaces and show how they may be expected to be expressible as functional relations between invariants of the groups of scaling transformations.

In Sec. IV we treat in detail the ${}^4R_0$ in the three-dimensional Ising model with variable interplanar interaction.[2] This is done in two ways. Firstly we scale with respect to the interaction parameter and thus reduce the model to the familiar metamagnet. Secondly we scale with respect to the variable

TABLE I. Steps in the formation of a scaling hypothesis in terms of invariants.

Step 1. Hypothesize that the Gibbs potential is given by an invariant equation under the direct-product group $\bar{\mathcal{G}}_\beta \equiv \bar{\mathcal{G}}_\alpha \otimes \mathcal{G}_\beta$ ($\beta < \alpha$) in terms of the principal directions (or variables) of scaling in the invariant space of the group $\bar{\mathcal{G}}_\alpha$ which is a direct product of a subset of the groups $\mathcal{G}_\gamma (\mathcal{O} \geq \gamma \geq \alpha)$.

Step 2. Form absolute invariants of the group $\bar{\mathcal{G}}_\beta$ from the principal variables of scaling in the invariant space of $\bar{\mathcal{G}}_\alpha$ by scaling each variable with respect to one of the variables.

Step 3. Find the new principal directions of scaling in the invariant space of $\bar{\mathcal{G}}_\beta$ by forming linear combinations of a basis set of absolute invariants of the group $\bar{\mathcal{G}}_\beta^s$.

Quantity	Notation
Variables transformed by a group $\bar{\mathcal{G}}_\alpha$	$x_{\alpha i}$
Invariants of the group $\bar{\mathcal{G}}_\alpha$	$y_{\alpha i}$
Quantities parametrizing a critical space of order β ($<\alpha$)	$k_{\alpha i}$
Rotation matrix to obtain principal variables of scaling at critical space of order β ($<\alpha$)	${}^\alpha R_{ij}$

Equation connecting the above quantities: $x_{\beta i} = {}^\alpha R_{ij}(y_{\alpha i} - k_{\alpha i})$

$\tau \equiv T - T_c$ ($\mathcal{R} = 0$) which is tangent to the lines of tricritical points, and thus demonstrate that the connectivity between the various ${}^2R_2$ is the same as for the edges of a tetrahedron, a result that has been previously obtained by quite different arguments.

II. FORMATION OF SCALING HYPOTHESIS

The selection of principal directions proceeds step-by-step starting at a critical point of lowest order—a ${}^2R_{n-2}$. In contrast the scaling hypothesis starts at the critical point of highest order, and states the scaling hypothesis for critical spaces of lower order in terms of "invariants" of groups of transformations about spaces of higher order. The complete system of scaling equations that are valid at a particular point can be set up in the following series of steps.

Step 0. Determine principal directions of scaling $x_{\mathcal{O}1}, x_{\mathcal{O}2}, \ldots, x_{\mathcal{O}\mathcal{O}}$ (as outlined above), by choosing an ordered set of critical subspaces ${}^iR_{n-i}$ of all orders $i < \mathcal{O}$ such that ${}^{\mathcal{O}}R_{n-\mathcal{O}} \subset {}^iR_{n-i} \subset {}^jR_{n-j}$ for all $j < i < \mathcal{O}$.

Step 1. Make the hypothesis that at the critical space of order $\mathcal{O}$ the Gibbs potential G is given by an equation $G = \mathfrak{F}(x_{\mathcal{O}1}, x_{\mathcal{O}2}, \ldots, x_{\mathcal{O}\mathcal{O}}; x_{\mathcal{O},\mathcal{O}+1}, \ldots, x_{\mathcal{O}n})$ which is invariant under the one-parameter continuous group of transformations $\mathcal{G}_{\mathcal{O}}$:

$$\mathcal{G}_{\mathcal{O}}: x'_{\mathcal{O}i} = \lambda^{a_{\mathcal{O}i}} x_{\mathcal{O}i} \quad (i = 0, 1, \ldots, n) . \tag{2.1}$$

where $x_{\mathcal{O}0} \equiv G$, $a_{\mathcal{O}0} = 1$, and $a_{\mathcal{O}i} = 0$ for $i > \mathcal{O}$.

Step 2. Re-express the variables $x_{\mathcal{O}1}, x_{\mathcal{O}2}, \ldots, x_{\mathcal{O}n}$ as absolute invariants $y_{\mathcal{O}i}$ of the group $\mathcal{G}_{\mathcal{O}}$ by scaling with respect to a suitable variable.[3] In this section we assume it to be $x_{\mathcal{O}\mathcal{O}}$:

$$y_{\mathcal{O}0} \equiv x_{\mathcal{O}0}/x_{\mathcal{O}\mathcal{O}}^{1/a_{\mathcal{O}\mathcal{O}}} , \tag{2.2a}$$

$$y_{\mathcal{O}i} \equiv x_{\mathcal{O}i}/x_{\mathcal{O}\mathcal{O}}^{a_{\mathcal{O}i}/a_{\mathcal{O}\mathcal{O}}} , \quad i = 1, 2, \ldots, \mathcal{O} - 1 \tag{2.2b}$$

$$y_{\mathcal{O},j-1} \equiv x_{\mathcal{O}j} , \quad j = \mathcal{O} + 1, \mathcal{O} + 2, \ldots n . \tag{2.2c}$$

But theorem 2 of I assures us that $y_{\mathcal{O}0}$ can be expressed as a function[4] of the invariants $(y_{\mathcal{O}1}, y_{\mathcal{O}2}, \ldots, y_{\mathcal{O},n-1})$.

Step 3. Near the critical space of order $\mathcal{O}$, the equations of the critical spaces of order $\mathcal{O} - 1$ are given by points $(k_{\mathcal{O}1}, k_{\mathcal{O}2}, \ldots, k_{\mathcal{O},n-1})$ in the space of invariants $(y_{\mathcal{O}1}, y_{\mathcal{O}2}, \ldots, y_{\mathcal{O},n-1})$. The principal directions of scaling at the ${}^{\mathcal{O}-1}R_d$ are determined by the geometry as linear combinations of the variables $(y_{\mathcal{O}1}, y_{\mathcal{O}2}, \ldots, y_{\mathcal{O},\mathcal{O}-1})$:

$$x_{\mathcal{O}-1,i} = \sum_{j=1}^{\mathcal{O}-1} {}^{\mathcal{O}}R_{ij}(y_{\mathcal{O}j} - k_{\mathcal{O}j}) , \quad i = 1, 2, \ldots, \mathcal{O} - 1. \tag{2.3}$$

We also define

$$x_{\mathcal{O}-1,0} \equiv y_{\mathcal{O}0} ,$$
$$x_{\mathcal{O}-1,i} \equiv y_{\mathcal{O}i} , \quad i = \mathcal{O}, \mathcal{O}+1, \ldots, n-1. \tag{2.3a}$$

Step 3+1. Now make the scaling hypothesis at the ${}^{\mathcal{O}-1}R_d$ in exactly the same fashion as in step 1, introducing a group of transformations $\mathcal{G}_{\mathcal{O}-1}$ which transforms the $x_{\mathcal{O}-1,i}$ as in Eq. (2.1). One can then proceed to steps 2 and 3 as before. In general a repetition of a cycle of steps 1, 2, and 3 is needed for the scaling hypothesis at each order of CRS. The steps are summarized in Table I.

When a ${}^2R_{n-2}$ is reached, and the last of the $(\mathcal{O} - 1)$ scaling hypotheses has been made, an invariance is achieved under an $(\mathcal{O} - 1)$ parameter continuous group of transformations:

$$\mathcal{G}_{\text{tot}} = \mathcal{G}_{\mathcal{O}} \otimes \mathcal{G}_{\mathcal{O}-1} \otimes \cdots \otimes \mathcal{G}_2 . \tag{2.4}$$

The validity of such a large group of transformations will be limited to a fairly small region close to all the appropriate ${}^iR_{n-i}$.

However, we have not exhausted *all* the possibilities near a ${}^{\mathcal{O}}R_{n-\mathcal{O}}$ because in general it is possible to pass directly from the ${}^{\mathcal{O}}R_{n-\mathcal{O}}$ to an ${}^{\mathcal{O}'}R_{n-\mathcal{O}'}$ with $(\mathcal{O} - \mathcal{O}') \neq 1$ and to remain outside the range of validity of the groups $\mathcal{G}_{\mathcal{O}-1}, \ldots, \mathcal{G}_{\mathcal{O}'+1}$. This indicates that there may be regions where any particular subgroup (and only that subgroup) of the total group $\mathcal{G}_{\text{tot}}$ is valid.

III. EXAMPLE: A GENERAL ${}^4R_0$

To give a little feeling for what is involved when $\mathcal{O} > 3$, we offer in Fig. 1 a schematic representation

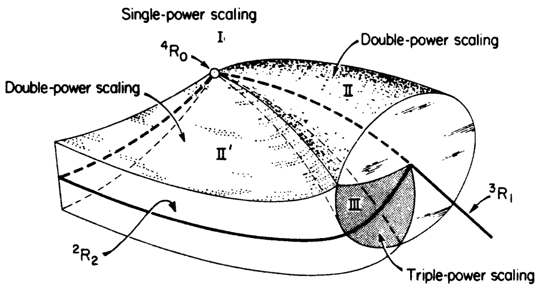

FIG. 1. Part of the region around a $^4R_0$ on a three-dimensional subspace of the total space. Only one $^3R_1$ and one $^2R_2$ are depicted. Boundaries of regions where various scaling laws are valid are indicated by surfaces.

of a three-dimensional projection of the four-dimensional field space in the vicinity of a $^4R_0$. There are several different regions where different groups of transformations are valid.

Region I. Only the group $\mathcal{G}_4$ is valid and the singular part of Gibbs potential G is an invariant equation under $\mathcal{G}_4$.

Region II. The singular part of the Gibbs potential is an invariant equation under the two-parameter group $\mathcal{G}_4 \otimes \mathcal{G}_3$ since this region is close to both the $^4R_0$ and to a critical line $^3R_1$. Region II is bounded by a crossover surface defined by an equation connecting the invariants of the group $\mathcal{G}_4^s$:

$$f^{\mathrm{II}}(y_{41},\ y_{42},\ y_{43}) = 0 \quad . \tag{3.1}$$

Region II′. The singular part of the Gibbs potential is an invariant equation under the two-parameter continuous group of transformations $\mathcal{G}_4 \otimes \mathcal{G}_2$. Region II′ is bounded by another crossover surface defined by a different relation between the invariants of $\mathcal{G}_4^s$:

$$f^{\mathrm{II}'}(y_{41},\ y_{42},\ y_{43}) = 0 \quad . \tag{3.2}$$

Region III. The singular part of the Gibbs potential is an invariant equation under the three-parameter continuous group of transformations $\mathcal{G}_4 \otimes \mathcal{G}_3 \otimes \mathcal{G}_2$. The boundary surface of this region is partly the surface defined by f^{II} in Eq. (3.1) and also a surface defined by an equation connecting the invariants under the group $\mathcal{G}_4^s \otimes \mathcal{G}_3^s$:

$$f^{\mathrm{III}}(y_{31},\ y_{32}) = 0 \quad . \tag{3.3}$$

IV. ANALYSIS OF ISING MODEL WITH VARIABLE INTERPLANAR INTERACTION

A. The Scaling Hypothesis at the $^4R_0$

In Sec. II of I, it was demonstrated that the 3d Ising model with variable interplanar interaction [Hamiltonian given by Eq. (2.2) of I] provides an excellent example of a critical point of order 4. This aspect of the model has also been treated by Harbus *et al.*,[2] who showed that combining the scaling hypothesis for change of lattice dimension when $\mathcal{R} > 0$, with that for $\mathcal{R} < 0$ leads to a general hypothesis for the critical point of order 4 of the form

$$G(\lambda^{a_H} H_2, \lambda^{a_{\mathcal{R}}}\mathcal{R}, \lambda^{a_H} H, \lambda^{a_\tau}\tau) = \lambda G(H_2, \mathcal{R}, H, \tau) \ , \tag{4.1}$$

i.e., $G = F(H_2, \mathcal{R}, H, \tau)$ is an invariant equation under the group of scaling transformations $\mathcal{G}_4$:

$$\mathcal{G}_4 \begin{cases} G' = \lambda G, \quad H_2' = \lambda^{a_H} H_2, \quad \mathcal{R}' = \lambda^{a_{\mathcal{R}}}\mathcal{R} \ , \\ \\ \qquad H' = \lambda^{a_H} H, \quad \tau' = \lambda^{a_\tau}\tau \ . \end{cases} \tag{4.2}$$

where H and H_2 have equal scaling power and $a_\tau < a_{\mathcal{R}} < a_H$ for reasons given in Ref. 2. Equation (4.1) is precisely of the form predicted for a critical point of order 4 with the principal directions of scaling given by

$$x_{41} \equiv H_2, \quad x_{42} \equiv \mathcal{R}, \quad x_{43} \equiv H, \quad x_{44} \equiv \tau . \tag{4.3}$$

We list the directions x_{4i} in the order shown because when referring to the critical surface $^2R_2$ of Fig. 2 of I, H_2 is a direction pointing out of the coexistence volume bounded by the $^2R_2$, $\mathcal{R}$ is a direction in the coexistence volume but pointing out of the $^2R_2$, H is a direction parallel to the $^2R_2$ at the Néel point but not parallel to the tricritical lines $^3R_1$, and τ, the "tangent variable," is parallel to the $^3R_1$.

Since the field space is now four dimensional, it is now peculiarly advantageous to use the method of invariants outlined in Secs. V and VI of I and Sec. II of this paper. The space of invariants of $\mathcal{G}_4^s$ is only three dimensional, and the $^3R_1$ will become points in this space.

There are two advantageous ways of choosing a basis set of invariants. The first was used in I and forms invariants by scaling with respect to the tangent variable τ. It is shown below (IV C) that this procedure enables one to plot the lines of tricritical points as points at the corners of a tetrahedron in the space of invariants. The $^2R_2$ become the six lines joining the vertices of the tetrahedron. First, however, we analyze the situation in the space of invariants formed by scaling with respect to $\mathcal{R}$, because this reduces the problem to a form similar to the already familiar tricritical point of a metamagnet which has been fully treated in I.

B. Analysis in the Space of Invariants of $\mathcal{G}_4$ Formed by Scaling with Respect to $\mathcal{R}$

We form invariants by scaling with respect to $\mathcal{R}$,

$$y_{40} \equiv x_{40} / |x_{42}|^{1/a_{42}} \equiv G_{\mathcal{R}} ,$$

$$y_{41} \equiv x_{41} / |x_{42}|^{a_{41}/a_{42}} \equiv H_{2\mathcal{R}} ,$$

$$y_{42} \equiv x_{43} / |x_{42}|^{a_{43}/a_{42}} \equiv H_{\mathcal{R}} ,$$

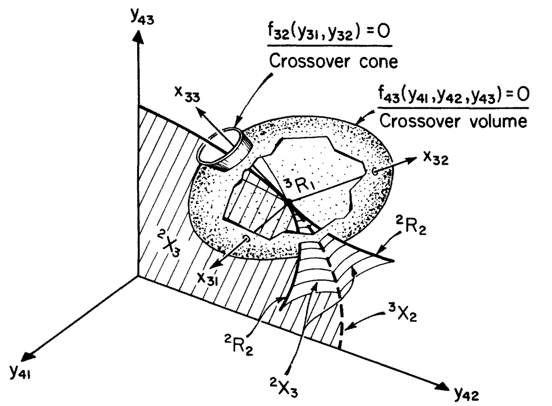

FIG. 2. Plot of the region around one of four $^3R_1$ in the space of invariants (y_{41}, y_{42}, y_{43}) of $\mathcal{G}_4$. The boundaries of regions of validity for double- and triple-power scaling are indicated.

$$y_{43} \equiv x_{44} / |x_{42}|^{a_{44}/a_{42}} \equiv \tau_{\mathcal{R}} . \tag{4.4a}$$

The scaling hypothesis of Eq. (4.1) requires that

$$y_{40}=f_3(y_{41}, y_{42}, y_{43}) \quad \text{or} \quad G_{\mathcal{R}}=f_3(H_{2\mathcal{R}}, H_{\mathcal{R}}, \tau_{\mathcal{R}}) . \tag{4.4b}$$

A representation of the various critical spaces close to the $^4R_0$ in the invariant (y_{41}, y_{42}, y_{43})-space is given in Fig. 2 (for $\mathcal{R}<0$).[5] That figure is similar to Fig. 14 of I, which shows the situation for constant $\mathcal{R}(<0)$. This similarity is clearly to be expected because one can always choose a particular value of $\mathcal{R}\neq 0$ in the variables of Fig. 2.

The various coexistence volumes $^2X_3$ map into surfaces in Fig. 2 with a conformation close to the tricritical point that is entirely analogous to the conformation in Fig. 14 of I. The $^2X_3$ end in $^2R_2$, which are the same as before, in regions that are close to the $^4R_0$.

We can now proceed entirely by analogy with the scaling hypothesis made at tricritical points in I, bearing in mind that the variables being used are not H_2, H, T, but the invariant variables $H_{2\mathcal{R}}, H_{\mathcal{R}}, \tau_{\mathcal{R}}$.

The directions of the second scaling hypothesis for the "invariant space metamagnet" (Fig. 2) can be chosen in an analogous fashion to the ordinary metamagnet of I. The direction x_{31} is chosen in the y_{4j} direction and points out of the coexistence volume $^2X_3$, the direction x_{32} lies in the $y_{42}-y_{43}$ plane and is not parallel to the critical surface $^2R_2$, and the direction x_{33} is parallel to the critical surface $^2R_2$. More precisely,

$$x_{31}=y_{41}, x_{3j}=R_{ji}y_{4i} \quad (i,j=2,3) \tag{4.5}$$

and R_{ji} is an appropriate nonsingular linear transformation and summation over i is implied. The scaling hypothesis now requires the invariant part of the Gibbs potential $x_{30}\equiv y_{40}=G/|x_{42}|^{1/a_{42}}\equiv F_3(x_{31}, x_{32}, x_{33})$ to be an invariant equation under the group $\mathcal{G}_3$:

$$\mathcal{G}_3:\ x_{3i}^{\lambda}=\lambda^{a_{3i}}x_{3i} \text{ with } a_{30}=1 \ (i=0,1,2,3). \tag{4.6}$$

The result of the simultaneous validity of the groups $\mathcal{G}_3$ and $\mathcal{G}_4$ (i.e., the product group $\mathcal{G}_3\otimes\mathcal{G}_4$) is that the Gibbs potential is expressible as a "double-power scaling function"[4]

$$y_{30}=f_2(y_{31}, y_{32}) , \tag{4.7}$$

where

$$y_{3i}\equiv x_{3i}/x_{33}^{a_{3i}/a_{33}} \quad (i=0,1,2). \tag{4.8}$$

Equation (4.7) is valid within a double crossover region, bounded by the region of validity of $\mathcal{G}_4$ and also a crossover surface in the y_{4i}-invariant space—given by

$$f_{43}(y_{41}, y_{42}, y_{43})=0 . \tag{4.9}$$

Within this region all thermodynamic functions are representable in the space y_{31}, y_{32} of Fig. 3. Here appropriate functions are invariant under the product group $\mathcal{G}_4\otimes\mathcal{G}_3$.

Figure 3 is entirely analogous to Fig. 15 of I, and, as in that case, the three lines of critical points in Fig. 2 map into three points.

The $^2R_2$ are thus predicted to be representable in regions close to both the $^3R_1$ and $^4R_0$ by the equations

$$y_{31}=k_{31}^i , \quad y_{32}=k_{32}^i, \tag{4.10}$$

where $i=1, 2, 3$. For the $^2R_2$ shown in Fig. 2 of I, we have $y_{31}=0$ and $y_{32}=-k_{32}$. Similarly the equation of the $^3X_2$ is $y_{31}=0$, $y_{32}=\text{const}$. Appropriate regions of other $^2X_3$ will map into lines in the y_{31}, y_{32} plane. In particular the $^2X_3$ in the $H_2=0$ hyperplane is restricted to an appropriate range of y_{32} for $y_{31}=0$.

Near each $^2R_2$, a third group of scaling equations

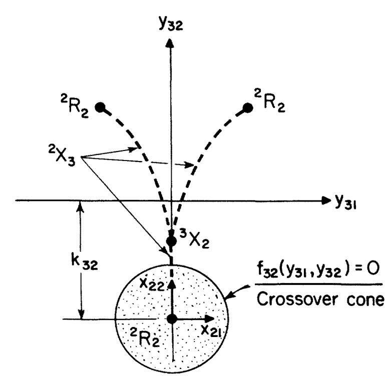

FIG. 3. Plot of the three $^2R_2$ of Fig. 2 in the space (y_{31}, y_{32}), invariants of the group $\mathcal{G}_4\otimes\mathcal{G}_3$. The crossover cone of Fig. 2 has become the circle in Fig. 3.

is valid. For each ${}^2R_2$, we pick two directions x_{21} and x_{22}, the strong and weak directions of Griffiths and Wheeler, in the y_{3i} invariant plane, and define $x_{20} \equiv y_{30} = x_{30}/x_{33}^{1/a_{33}}$. We now require $x_{20} = F_2(x_{21}, x_{22})$ to be an invariant equation under the group $\mathcal{G}_2$ of transformations:

$$\mathcal{G}_2: \quad x'_{2i} = \lambda^{a_{2i}} x_{2i}, \quad i = 0, 1, 2 \tag{4.11}$$

where $a_{20} = 1$. This implies that the Gibbs potential near $a\,{}^4R_0$, $a\,{}^3R_1$, and $a\,{}^2R_2$ is expressible as a *triple-power scaling function*:

$$x_{20}/x_{22}^{1/a_{22}} = F_1(x_{21}/x_{22}^{a_{21}/a_{22}}) . \tag{4.12}$$

Thus all the data from the appropriate four-dimensional region should collapse into a line. The crossover cone for the validity of the triple-power scaling function should also be representable in the space (y_{31}, y_{32}) of Fig. 3 as

$$f_{32}(y_{31}, y_{32}) = 0 . \tag{4.13}$$

We illustrate this result for the special case of $H_2 = 0$ (for $\mathcal{R} < 0$). For the $H_2 = 0$ hyperplane, the invariant mapping of the various CRS and CXS are given in Fig. 4. Within the appropriate region (a wedge within a circle), Eq. (4.12) applies. Since $x_{21} = 0$ for this case, all data collapse to a point

$$x_{20} = C x_{22}^{1/a_{22}} , \tag{4.14}$$

which can be rewritten (from definitions of x_{20}, x_{30}, y_{40})

$$G = C x_{42}^{1/a_{42}} x_{33}^{1/a_{33}} x_{22}^{1/a_{22}} . \tag{4.15}$$

This "triple-power scaling law" is a special case of the general triple-power scaling function. It is valid within the wedge between the curves $y_{32} = x_{32} x_{33}^{-a_{32}/a_{33}} =$ const. and the circle $f_{43}(y_{42}, y_{43}) = 0$ as shown in Fig. 4.

In conclusion, the systematic application of the heirarchy of scaling hypotheses gives predictions of the shapes of the CRS and CXS within the various crossover regions of the product groups. Finally, if we set $y_{41} = y_{42} = 0$ in Eq. (4.4b) we obtain the scaling hypothesis about the critical point $T_c(\mathcal{R} = 0)$. If $\mathcal{R}$ is allowed to vary we simply obtain the well-known double-power scaling law with a parameter[6] for $\mathcal{R} > 0$. This double-power law is valid under the product group $\mathcal{G}_4 \otimes \mathcal{G}_2$ where $\mathcal{G}_2$ is the symmetry group about the line $T_c(\mathcal{R})$ of Fig. 2 of I.

This is an explicit example of a case where a product group $\mathcal{G}_j \otimes \mathcal{G}_i$ with $|j - i| \neq 1$ may be appropriate.

C. Analysis Using the Tangent Variable to Form Invariants

We now form a different set of functionally independent absolute invariants $\bar{y}_{4i}$ of $\mathcal{G}_4$ by scaling with respect to the tangent variable $x_{44} \equiv \tau - T_{\mathrm{NO}}$:

$$\bar{y}_{4i} = x_{4i}/|x_{44}|^{a_{4i}/a_{44}}, \quad i = 0, 1, \ldots 3 . \tag{4.16}$$

The scaling hypothesis of Eq. (4.1) requires that

$$\bar{y}_{40} = f_3(\bar{y}_{41}, \bar{y}_{42}, \bar{y}_{43}) . \tag{4.17}$$

In the space of invariant field variables $(\bar{y}_{41}, \bar{y}_{42}, \bar{y}_{43})$, the four ${}^3R_1$ again map into points: two in the $\bar{y}_{41} = 0$ plane for $\bar{y}_{42} < 0$ and two in the $\bar{y}_{43} = 0$ plane for $\bar{y}_{42} > 0$. These four points form the corners of a tetrahedron, as in Fig. 5. In this representation we are explicitly able to show the way that the ${}^2R_2$ connect the different ${}^3R_1$.

In Fig. 5, the curves joining the vertices of the tetrahedron are the scaled representation of the ${}^2R_2$. Each ${}^3R_1$ is joined by these curves to each of the other ${}^3R_1$. Similarly, the ${}^4X_1$ on the T axis of Fig. 2 becomes the point at the origin of Fig. 5 and the four ${}^3X_2$ become curves joining the origin to the ${}^3R_1$. Similarly, for the ${}^2X_3$ which are not depicted in Fig. 5.

The curve joining the two ${}^3R_1$ for $\bar{y}_{42} < 0$ is the ${}^2R_2$ on top of the "mountain" in Fig. 2 of I. The

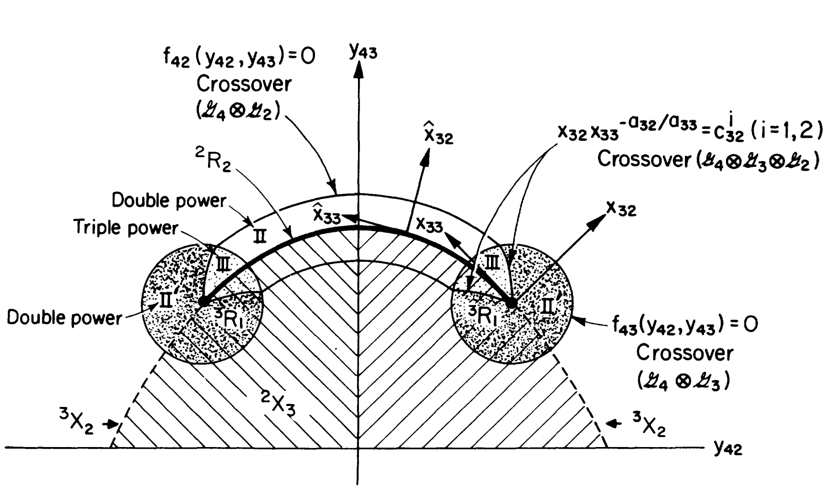

FIG. 4. Full (y_{43}, y_{42}) plane of Fig. 2 showing both ${}^3R_1$ for $y_{41} = 0$. They are connected by a ${}^2R_2$ near which there is a region where the groups $\mathcal{G}_4$ and $\mathcal{G}_2$ are valid, but $\mathcal{G}_3$ is not. The crossover surfaces and the groups whose ranges of validity are thereby limited, are indicated.

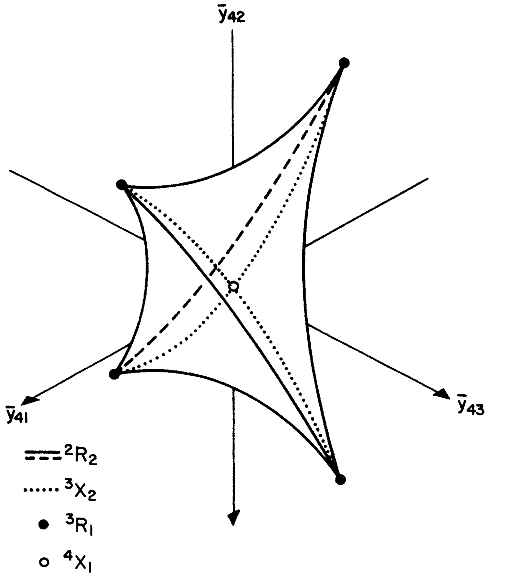

FIG. 5. A schematic representation of the complete system of CRS and CXS near the $^4R_0$, obtained by scaling with respect to the tangent variable. Two of the $^3R_1$ for $\bar{y}_{42}>0$ are in the plane $\bar{y}_{43}=0$ while those for $\bar{y}_{42}<0$ (also shown in Fig. 4) are in the plane $\bar{y}_{41}=0$. The total system is topologically equivalent to a tetrahedron. Not shown are the various $^2X_3$ which join the $^3X_2$ to the $^2R_2$. The values of $\bar{y}_{4i}$ are mapped as $\bar{y}_{4i}/(\bar{y}_{4i}+c_i)$ where c_i are constants to ensure that all pertinent points are mapped within a finite region.

other curves leading away from these two $^3R_1$ (for $\bar{y}_{42}<0$) to the other two $^3R_1$ (for $\bar{y}_{42}$) are the wings of Fig. 14 of I, which extend into the region $H_2\neq 0$.

Thus the wings for $\mathcal{R}<0$ are joined by continuously connected surfaces to the wings for $\mathcal{R}>0$ and are thus sections of the same $^2R_2$. There are in reality six $^2R_2$: one for $\mathcal{R}>0$, one for $\mathcal{R}<0$, and four "wings."

Now one can proceed to make a scaling hypothesis about the $^3R_1$ as before. First we choose a suitable set of variables which are linear combinations of the $\bar{y}_{4i}$:

$$\bar{x}_{3j}=R_{ji}(\bar{y}_{4i}-k_{4i}) \ , \qquad (4.18)$$

where $i, j=1, 2, 3$ and summation over i is implied. Then the Gibbs function will be expected to be given by an invariant equation under the group $\mathcal{G}_3$ of transformations

$$\bar{x}'_{3j}=\lambda^{a_{3j}}x_{3j}, \quad j=0, 1, 2, 3 \qquad (4.19)$$

and $a_{30}=1$. However this new transformation must inevitably give the same results as when the scaling was performed with respect to the groups $\mathcal{G}_4\otimes\mathcal{G}_3$ in Sec. IV B. The analysis will be the same as for that case.

V. SCALING LAWS AND SCALING FUNCTIONS

The consequences of the existence of valid equations of scaling for critical points of higher order are very similar to those for a $^3R_0$.

In a region where a direct product of r groups of transformations are valid, it is possible to form scaling functions with simultaneous scaling by r variables. This will give data collapsing by r dimensions, and this should make the testing of the scaling hypothesis in the vicinity of a critical subspace of any order quite possible: the maximum value for r is $(\mathcal{O}-1)$ because in the vicinity of a critical point of order $\mathcal{O}$ there should be regions where a direct product of $(\mathcal{O}-1)$ groups of transformations are valid (e.g., the crossover region near a $^3R_0$ of I and region III in Fig. 4). Now it should be possible to plot all the data from the $\mathcal{O}$-dimensional region in terms of one variable.

There should be larger regions where one group $\mathcal{G}_i$ is not valid and the data should be expressible in terms of two appropriately scaled variables. Then data collapse onto a surface, which should be easy to verify.

For each set of scaling powers $a_{\mathcal{O}i}$ at a critical subspace of order $\mathcal{O}$, there will exist a different set of exponents and scaling laws relating them. For a point of order $\mathcal{O}$, there will therefore be $\mathcal{O}$ independent exponents, and if more than $\mathcal{O}$ are defined, there must be scaling laws relating them.

In a system like the one considered in Sec. IV, there will therefore be four independent exponents at the $^4R_0$, three at each of the four $^3R_1$, and two at each of the $^2R_2$, giving 28 possible exponents before the symmetry is taken into account. The symmetries show that the $^3R_1$ are all equivalent because we have reflections $H\to -H$ and $H_2\to -H_2$ and the pseudoreflection $\mathcal{R}\to-\mathcal{R}$, $H\to H_2$. These symmetries show that there are only two sorts of $^2R_2$. Thus there are only $4+3+2+2=11$ scaling powers we need to know to evaluate every possible exponent for the system.

In fact for this system we shall see that we know seven scaling powers certainly and we can estimate two more. The only pair that we do not know are the scaling powers for the "wings." (see Ref. 6 of I for definition of "wings").

At the $^4R_0$ the exponents are the same as for the two-dimensional Ising model, and this gives $a_{41}=a_{43}=\frac{15}{16}$, $a_{42}=\frac{7}{8}$, and $a_{44}=\frac{1}{2}$, where the first two are equal by the pseudoreflection symmetry and exponent a_{42} is evaluated from our knowledge of the crossover exponent $\gamma_4\equiv a_{42}/a_{44}=\varphi_4$.

For the $^3R_1$, we know the scaling powers from our knowledge of the exponents γ and γ_{st} (describing the divergence of the functions χ and χ_{st}), and also from the crossover φ_3 describing the shape of the $^2R_2$ for $H_2=0$ approaching the $^3R_1$.

Work by Harbus and Stanley[7] estimates the scaling power a_{32} to be $\frac{2}{3}$. Tentative calculations[8] based on the shape of the phase boundary and the amplitude function of the staggered susceptibility along the phase boundary give rough estimates of $a_{31} \cong \frac{11}{12}$ and $a_{33} \cong \frac{1}{3}$.

Of the ${}^2R_2$, two have the familiar exponents of the three-dimensional Ising model, giving[6] $a_{21} = \frac{5}{6}$ and $a_{22} = \frac{8}{15}$ while the other four ${}^2R_2$ are "wings" and possess exponents about which we know nothing at all.[9]

The scaling laws are obtained by eliminating the a_{ij} between expressions for different exponents. Exponents defined for paths of approach to a set of equivalent ${}^{\mathcal{O}}R_d$ (e.g., the ${}^3R_1$ above) will thus be related by scaling laws, but equalities between exponents for different sets of ${}^{\mathcal{O}}R_d$ (i.e., not related by symmetry) should be regarded as fortuitous and not as scaling laws. This will remain true for the critical spaces of any system.

VI. SUMMARY AND CONCLUSIONS

We have shown above how to make a scaling hypothesis at a critical point of arbitrary order, in the case where the dimensionality of the spaces of critical points of order $\mathcal{O}$ is $\mathcal{O}$ less than the total number of dimensions available; i.e., Eq. (1.1) of paper I is satisfied. We have also shown how the choice of variables for hypotheses at points of lower order is decided by the groups of transformations about points of higher order. In the example given in Sec. IV it was shown that there could be regions where different direct products of groups of scaling transformations were valid, e.g. $\mathcal{G}_4 \otimes \mathcal{G}_3$ and $\mathcal{G}_4 \otimes \mathcal{G}_2$.

In the particular example treated, there was a very high degree of symmetry about the temperature axis, and instead of having different scaling powers for every CRS (giving 28 possible exponents) only nine independent exponents were found—two at each of two sorts of ${}^2R_2$, three at the four ${}^3R_1$, and only two at the ${}^4R_0$ because of the symmetry and the fact $\gamma_4 = \varphi_4$.

The same symmetry that reduced the number of exponents from 28 to 9 was operative in making the lines of tricritical points intersect on the $T=0$ axis. In a four-dimensional space there is nothing to force lines of tricritical points to intersect except a symmetry [10]; it is of interest to note that in complex fluid mixtures[11] critical points of higher order can occur when the Hamiltonians do not possess symmetries. The order $\mathcal{O}$ for such a critical point is equal to the number of phases in equilibrium there as is the case here but the number of scaling fields at such a critical point is usually not equal to $\mathcal{O}$.

ACKNOWLEDGMENTS

We wish to thank the participants at the MIT Summer School on Phase Transitions and Critical Phenomena, June, 1972, for helpful suggestions. The authors profitted from discussions with F. Harbus, R. B. Griffiths, J. C. Wheeler, and E. K. Riedel.

*Supported by the National Science Foundation, the Office of Naval Research, and the Air Force Office of Scientific Research.

†Supported by a Lindemann Fellowship. Present address: Stanford Linear Accelerator Center, Stanford, Calif. 94305.

‡Also at North Carolina State University, Raleigh, N.C. 27607.

[1]T. S. Chang, A. Hankey, and H. E. Stanley, Phys. Rev. B 8, 346 (1973).

[2]F. Harbus, A. Hankey, H. E. Stanley, and T. S. Chang Phys. Rev. B (to be published).

[3]Usually this is the "tangent" variable with smallest $a_{\mathcal{O}r}$, but it may be advantageous to scale with respect to a different variable. See, for example, Sec. IV B.

[4]Scaling functions may change their values or functional forms as the scaled variables change signs.

[5]Clearly, the result for $\mathcal{R} > 0$ will be the same as for $\mathcal{R} < 0$ with y_{41} and y_{42} interchanged.

[6]A. Hankey and H. E. Stanley, Phys. Rev. B **6**, 3515 (1972), and references contained therein.

[7]F. Harbus and H. E. Stanley, Phys. Rev. Lett. **29**, 58 (1972).

[8]F. Harbus, H. E. Stanley, and T. S. Chang, Proceedings of the Thirteenth International Conference on Low Temperature Physics, Boulder, Colo., August, 1972 (unpublished).

[9]There has been a suggestion by P. Kortman [Phys. Rev. Lett. **29**, 1449 (1972)] that the exponents for the wings are classical, but this may not be justified.

[10]Tricritical points possess a reflection symmetry in the strong variable $[x_s \to -x_s]$ and this *almost* forces the three lines to intersect—but not quite, e.g., the He^4 phase diagram. Another operative fact here is that x_s is also a strong direction for the wings.

[11]R. B. Griffiths and B. Widom, Phys. Rev. A (to be published); A. Hankey, T. S. Chang, and H. E. Stanley, Stanford Linear Accelerator Center Technical Report, May, 1973 (unpublished).