Sawtooth current-phase relation of a superconducting trilayer system described using Ohta's formalism

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(Received 21 November 2008; revised manuscript received 15 January 2009; published 17 March 2009)

The Josephson equations for a trilayer superconducting system are derived by means of Ohta's semiclassical approach. Being the middle layer not accessible by the external biasing circuit, a purely quantum-mechanical analysis is performed in this superconducting island. It is found as a strict conventional voltage-phase dependence in the overall system, while an additional half harmonic term appears in the current-phase relation. This extraterm gives a sawtooth behavior in the effective current-phase relation.

DOI: 10.1103/PhysRevB.79.094516 PACS number(s): 74.50.+r

I. INTRODUCTION

The quantum properties of superconducting junctions were first described by Josephson,[1] so that these systems were named after this author. Feynman,[2] on the other hand, was able to capture, by means of a more accessible analytical approach, the main characteristics of two weakly coupled superconductors. Feynman's model, however, failed to give a consistent account of the external bias circuit. Following Feynman's approach, Ohta[3] introduced a semiclassical model by which effects of the external circuit were introduced so that the strict voltage-phase relation could be obtained.

Lately, Carapella *et al.*[4] have presented an extension of Ohta's analysis to a superconducting trilayer system in which access to the middle electrode is allowed. The authors predicted a strict voltage-phase relation and the appearance of a second-harmonic term in the current-phase relation (CPR) for each Josephson junction (JJ). This additional term accounted for the direct coupling of the two outermost electrodes.

The latter point implies that a modified current-phase relation contains information on the proximity effects in the overall junction, revealing the nature of the weak coupling between the superconducting outer electrodes. In this way, deviations of the CPR from the conventional behavior can be considered as an important probe of fundamental quantities involved in the physics of the modified junction.

In order to obtain JJs characterized by an unconventional current-phase relations, one can consider hybrid superconducting systems in which the degrees of freedom of some interstitial subsystem (middle layer) play an active role in determining the properties of the whole junction.[5] Inside the middle layer of this modified junction, the superconducting order parameter of the external electrodes can oscillate as an effect of the competition between the normal and the superconducting solution. For appropriate thickness of the middle region, it is possible that the superconducting order parameter takes opposite signs across the two external superconducting electrodes. In the latter situation, the so-called π-junction[6,7] behavior is expected. The mentioned mechanism is observed, for instance, in the superconductor-ferromagnet-superconductor (SFS) junctions.[8–11] Indeed, inside the ferromagnetic region of a SFS junction, a competition of different solutions is present and the one with lower energy is favored.

Alternatively, the insertion of a nonequilibrium normal interstitial layer[12] or of a quantum dot,[13] implemented, for instance, by exploiting the carbon nanotube technology, can lead to interesting modification of the CPR and also to a π-junction behavior. Naturally, all the above mentioned systems can be made by using conventional superconductors, even though the most natural way to obtain a π junction is to fabricate a JJ by using superconducting electrodes with different symmetry of the order parameter (one electrode being a d-wave superconductor combined with a conventional superconductor) arranged in such way to obtain the desired π shift in the CPR.[14]

In the present work we focus our attention on modified JJs in which the interstitial layer is a superconducting subsystem (SISIS junctions). Some attention has been paid in the literature to these systems,[5,15] although a semiclassical approach, similar to what is proposed in the present paper, has never been exploited in deriving a closed simple form of the CPR of these composite systems. In this context, by not allowing the external circuit to access the middle electrode and by treating the interstitial superconducting island as a pure quantum system, two types of phase-dependent terms arise in the effective CPR of the composite system (seen as one single junction). These properties are those of a conventional *zero* junction with a half harmonic term in the CPR which is responsible for a sawtooth behavior.

The present work is thus organized as follows. In Sec. II we present the semiclassical Ohta's formalism for the two outer electrodes, leaving the time evolution of the middle layer undefined. In Sec. III we consider the middle layer as a pure quantum system, deriving the time evolution for its macroscopic wave function. In Sec. IV the effective CPR of this system is found under the assumptions of a perfectly symmetric trilayer and of a first-order approximation of the coupling energy. The maximum Josephson current and Shapiro steps are found in Sec. V. Conclusions are drawn in Sec. VI.

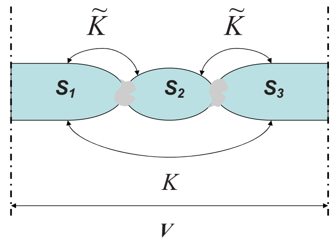

FIG. 1. (Color online) Schematic representation of a superconducting trilayer system, where the coupling constants $\widetilde{K}$ and K and are shown.

II. SEMICLASSICAL ANALYSIS

Let us consider the superconducting trilayer system shown in Fig. 1. Let us assume that, differently from what done by Carapella *et al.*,[4] the classical measuring system is not allowed to access to the middle layer, so that we are forced to develop a single-junction model for the whole superconducting trilayer system. Indeed, being not able to measure the properties of single junctions, we do not attempt to write the CPR for each one of these elements. Instead, we would like to obtain the voltage-phase and current-phase relations for the overall system. Furthermore, considering that only the outer electrodes are connected with the outer classical measuring system, we treat the inner superconducting island (the middle layer) as a pure quantum system.

We start by writing a portion of the semiclassical Hamiltonian for the quantum superconducting system as a quadratic form as follows:[4]

$$H_0 = \langle \psi | \hat{H}_0 | \psi \rangle = (\psi_1^* \quad \psi_2^* \quad \psi_3^*) \begin{pmatrix} E_1 & -\widetilde{K} & -K \\ -\widetilde{K} & E_2 & -\widetilde{K} \\ -K & -\widetilde{K} & E_3 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \tag{1}$$

where the wave function ψ_k and the constant energy E_k are relative to the kth superconductor S_k in Fig. 1 $(k=1,2,3)$, while the constants $\widetilde{K}$ and K are the coupling terms between adjacent layers and the two outer junction electrodes, respectively. In this way, we may write

$$H_0 = E_1|\psi_1|^2 + E_2|\psi_2|^2 + E_3|\psi_3|^2 - \widetilde{K}(\psi_1^*\psi_2 + \psi_1\psi_2^*) - K(\psi_1^*\psi_3 + \psi_1\psi_3^*) - \widetilde{K}(\psi_2^*\psi_3 + \psi_2\psi_3^*). \tag{2}$$

As stated before, we may now treat the outer electrodes as semiclassical system. In this way, we start by writing the superconducting wave functions ψ_1 and ψ_3 as follows:

$$\psi_1 = \sqrt{N_1}e^{-i\theta_1}; \quad \psi_3 = \sqrt{N_3}e^{-i\theta_3}, \tag{3}$$

where N_k is the density of Cooper pairs and θ_k is the superconducting phase of the kth electrode $(k=1,3)$. By substituting the above relation in Eq. (2), we have

$$H_0 = N_1E_1 + N_3E_3 - 2K\sqrt{N_1N_3}\cos(\theta_3 - \theta_1) + E_2|\psi_2|^2 - 2\widetilde{K}\Re\{\psi_2^*(\psi_1 + \psi_3)\}, \tag{4}$$

where the symbol $\Re$ stands for the real part of the number in parenthesis and where the quantity ψ_2 is left undefined for the moment being. We need to say, however, that ψ_2 can be expressed, as we shall show in Sec. III, in terms of ψ_1 and ψ_3, and thus of the variables N_k and θ_k $(k=1,3)$. Let us now introduce the energy of the external circuit in the semiclassical Hamiltonian by defining the complete Hamiltonian H as follows:

$$H = H_0 - \int I dV = H_0 - W_{\mathrm{ex}}. \tag{5}$$

Following Ohta, we define $E_C=-2K\sqrt{N_1N_3}\cos(\theta_3-\theta_1)+E_2|\psi_2|^2-2\widetilde{K}\Re\{\psi_2^*(\psi_1+\psi_3)\}$, so that, the complete Hamiltonian can be written as $H=N_1E_1+N_3E_3+(E_C-W_{\mathrm{ex}})$, where the parentheses are inserted for convenience. We now recognize that the quantities θ_k and $\hbar N_k$ $(k=1,3)$ are conjugated variables, so that Hamilton's equations for these quantities are

$$\hbar \dot{N}_k = -\frac{\partial H}{\partial \theta_k}, \tag{6a}$$

$$\dot{\theta}_k = \frac{1}{\hbar}\frac{\partial H}{\partial N_k}. \tag{6b}$$

where $k=1,3$ and the dot stands for derivative with respect to time. We notice that, for charge conservation, $\dot{N}=\dot{N}_1+\dot{N}_2+\dot{N}_3=0$. Under the assumption of a thermal bath, we may set $\dot{N}_1=\dot{N}_2=\dot{N}_3=0$ as in Ref. 4. By energy conservation, we may also write $\dot{H}=(\dot{E}_C-\dot{W}_{\mathrm{ex}})=0$, so that the quantity (E_C-W_{ex}) is a constant. By these assumptions, Eq. (6a) becomes an identity and Eq. (6b) simply gives

$$\dot{\theta}_k = \frac{E_k}{\hbar}, \tag{7}$$

for $k=1,3$. When we apply a voltage difference V across the first and third electrode, we may set $E_1=-eV/\hbar$, $E_3=eV/\hbar$, so that the strict Josephson voltage-phase relation is obtained,

$$\dot{\theta}_3 - \dot{\theta}_1 = \frac{2eV}{\hbar}. \tag{8}$$

In order to obtain the first Josephson equation, or CPR, by energy conservation we write $\dot{E}_C=\frac{\partial E_C}{\partial \theta_1}\dot{\theta}_1+\frac{\partial E_C}{\partial \theta_3}\dot{\theta}_3=\dot{W}_{\mathrm{ex}}=IV$, so that

$$I = \frac{4eK}{\hbar}\sqrt{N_1N_3}\sin(\theta_3-\theta_1) + \frac{e}{\hbar}\left(\frac{\partial E_R}{\partial \theta_3} - \frac{\partial E_R}{\partial \theta_1}\right). \tag{9}$$

where $E_R=E_2|\psi_2|^2-2\widetilde{K}\Re\{\psi_2^*(\psi_1+\psi_3)\}$. Notice the appearance of an extraterm in the CPR in Eq. (9), besides the usual term given by the first addendum. Naturally, in the case the middle layer is normal, we set $\psi_2=0$ in E_R, so that Eq. (9)

reduces to the usual current-phase relation of a single Josephson junction.

III. QUANTUM MECHANICAL ANALYSIS FOR THE MIDDLE LAYER

In this section we give a pure quantum-mechanical evaluation of the quantity ψ_2 to be utilized in Eq. (9) to fully evaluate the effective CPR of the superconducting trilayer system represented in Fig. 1.

We start by restating that no evaluation of the time evolution of the wave function ψ_2 in Eqs. (6a) and (6b) has been made yet. The only assumption we have put forth is that the function ψ_2 can be expressed in terms of ψ_1 and ψ_3, and thus of the variables N_k and θ_k $(k=1,3)$. We treat the wave function ψ_2, indeed, on a different footing with respect to ψ_1 and ψ_3, by assuming that it evolves, according to Schroedinger equation, as follows:

$$\hat{H}_0|\psi_2\rangle = i\hbar\frac{\partial|\psi_2\rangle}{\partial t} \Rightarrow \left(i\hbar\frac{\partial}{\partial t} - E_2\right)\psi_2 = -\tilde{K}(\psi_1 + \psi_3). \tag{10}$$

We might thus give a complete formal solution to Eq. (10) as follows. First determine the homogeneous solution, given by

$$\psi_2^{(H)} = Ae^{(-iE_2/\hbar)t}. \tag{11}$$

The particular solution can be found by solving the following associated Green's problem:

$$\left(i\hbar\frac{\partial}{\partial t} - E_2\right)G(t) = \delta(t), \tag{12}$$

where $G(t)$ is found to be

$$G(t) = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{(\hbar\omega - E_2)} = -\frac{i}{\hbar}\theta(t)e^{-i(E_2/\hbar)t}, \tag{13}$$

$\theta(t)$ being the Heavyside step function. In this way, the particular solution $\psi_2^{(P)}$ can be written formally as follows:

$$\psi_2^{(P)}(t) = -\tilde{K}\int_{-\infty}^{+\infty}[\psi_1(\tau) + \psi_3(\tau)]G(t-\tau)d\tau. \tag{14}$$

The general solution is thus $\psi_2 = \psi_2^{(H)} + B\psi_2^{(P)}$. We shall evaluate the wave function under the following simplifying conditions:

(a) the trilayer is perfectly symmetric, so that E_2 is taken as the average value between E_1 and E_3, giving $E_2=0$ and $N_1=N_3=N$;

(b) only first-order terms in the coupling constants are retained in E_R.

Under this assumption we may write

$$E_R = -2\tilde{K}\sqrt{N_2}\Re\{(\psi_1 + \psi_3)\} = -2\tilde{K}\sqrt{N_2 N}(\cos\theta_1 + \cos\theta_3), \tag{15}$$

where we have set $|\psi_2|^2 = |A|^2 = N_2$, since only the constant term A needs to be retained in ψ_2 because of (b).

IV. EFFECTIVE CURRENT-PHASE RELATION

We can now evaluate the effective CPR under the assumptions set forth in Sec. III, namely, a perfectly symmetric trilayer and a first-order approximation of the coupling energy E_R. By Eq. (15), we can rewrite Eq. (9) as follows:

$$I = \frac{4eKN}{\hbar}\sin(\theta_3 - \theta_1) + \frac{2e\tilde{K}\sqrt{N_2 N}}{\hbar}(\sin\theta_3 - \sin\theta_1). \tag{16}$$

By setting $\phi = \theta_3 - \theta_1$ and recalling simple trigonometric identities, we have

$$I = \frac{4e\tilde{K}\sqrt{NN_2}}{\hbar}\left[\sqrt{\frac{N}{N_2}}\frac{K}{\tilde{K}}\sin\phi + \cos\theta_A\sin\left(\frac{\phi}{2}\right)\right], \tag{17}$$

where $\theta_A = \frac{\theta_1+\theta_3}{2}$ is the average superconducting phase of the two outer electrodes. Let us now notice that this average phase is a constant, given that $\dot{\theta}_1 + \dot{\theta}_3 = 0$. Also notice that until now no special constraint on θ_A has been imposed, so that, in principle, we could set $-1 \le \cos\theta_A \le +1$. However, let us consider the energy of the superconducting system in Eq. (4). Under the hypothesis put forth in Sec. III, we could write

$$H_0 = -2KN\cos\phi - 4\tilde{K}\sqrt{NN_2}\cos\theta_A\cos\left(\frac{\phi}{2}\right). \tag{18}$$

By choosing the value of θ_A for which the above relation becomes an extremal functional, we have $\cos\theta_A = \pm 1$. We further require that the dynamical state of the system be univocally determined, so that we choose, between the above two extremal function, the one giving the minimum of H_0 with respect to θ_A. In this way, by setting the second derivative of H_0 with respect to θ_A greater than zero, we have

$$\cos\theta_A\cos\left(\frac{\phi}{2}\right) > 0. \tag{19}$$

This means that $\frac{\cos\theta_A\cos(\frac{\phi}{2})}{|\cos\theta_A\cos(\frac{\phi}{2})|} = 1$ and thus, by setting $I_0 = \frac{4e\tilde{K}\sqrt{NN_2}}{\hbar}$, $\beta = \sqrt{\frac{N}{N_2}}\frac{K}{\tilde{K}} > 0$ and $\alpha = \mathrm{sign}[\cos\frac{\phi}{2}]$, we may rewrite Eq. (17), representing the effective CPR of the superconducting trilayer system, as follows:

$$I = I_0\left[\beta\sin\phi + \alpha\sin\left(\frac{\phi}{2}\right)\right], \tag{20}$$

where the constant parameter β can be much less than one. The normalized current $f = \frac{I}{I_0}$ versus the superconducting phase ϕ is shown in Fig. 2 for $\beta = 0.2$. The CPR in Eq. (20) thus gives the effective behavior of the trilayer system under an external bias term not affecting the middle layer. We notice that the first term is the direct tunneling term (Cooper pairs going from the first to the third superconducting element in the trilayer) while the second term takes account of the tunneling between adjacent layers. In this way, the parameter β may be much lower than $|\alpha| = 1$. Notice also that

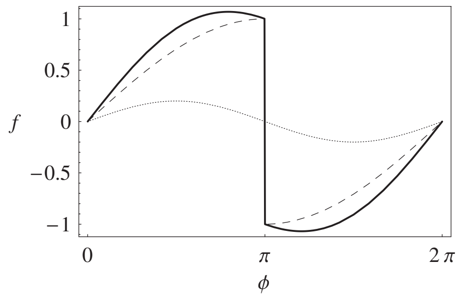

FIG. 2. Normalized current as a function of the superconducting phase difference across a trilayer system (full line) with $\beta=0.2$ as compared to the current-phase relation for $\beta=0$ (dashed line). The dotted line represents the direct tunneling contribution $\beta \sin\phi$ to the CPR.

the half harmonic term corresponds to the ordinary phase difference in the work by Carapella *et al.*[4]

Finally, by Eq. (18) and (20) we can define an effective potential for the junction as follows:

$$U_{\mathrm{eff}} = -\frac{I_0\Phi_0}{2\pi}\left[\beta\cos\phi + 2\alpha\cos\left(\frac{\phi}{2}\right)\right], \qquad (21)$$

where Φ_0 is the elementary flux quantum. Representation of the normalized effective potential $u_{\mathrm{eff}}=\frac{2\pi U_{\mathrm{eff}}}{I_0\Phi_0}$ is given in Fig. 3 for $\beta=0.1$, where one can readily verify that the value of the energy barrier is $\Delta u_{\mathrm{eff}}=2(1+\beta)$.

V. MAXIMUM JOSEPHSON CURRENT AND SHAPIRO STEPS

Having evaluated the effective CPR in the previous section, under the assumptions of symmetric layers, retaining the coupling constants only to first order, we here look at the maximum Josephson current in the case $\beta \ll 1$, by still keeping the value of β finite, and at the Shapiro steps in the *I-V* characteristics.

Let us then start by considering Eq. (20) and look for the maximum of the function $f(\phi)$ in its period. We thus notice that only the positive branch of $f(\phi)$ needs to be considered, so that, by defining $x=\cos(\frac{\phi}{2})$ and by setting $f'(\phi)=0$, we have

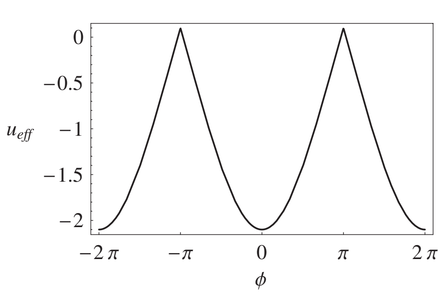

FIG. 3. Normalized effective potential u_{eff} for the superconducting trilayer system as a function of the superconducting phase difference ϕ for $\beta=0.1$.

$$4\beta x^2 + x - 2\beta = 0, \qquad (22)$$

The only admissible solution to the above equation in this limit is the following: $x=\cos(\frac{\phi}{2})\approx 2\beta$, from which $\sin(\frac{\phi}{2}) =\sqrt{1-4\beta^2}$ gives the maximum of $f(\phi)$ in Eq. (20). In this way, $\sin\phi=2\sin(\frac{\phi}{2})\cos(\frac{\phi}{2})=4\beta\sqrt{1-4\beta^2}$. By finally substituting in Eq. (20), we have

$$f_{\max} = \frac{I_{\max}}{I_0} = (1+4\beta^2)\sqrt{1-4\beta^2} \approx 1 + 2\beta^2, \qquad (23)$$

which gives a slight increase of the maximum Josephson current with respect to the value $f_{\max}=1$, obtainable for $\beta=0$.

Let us now turn our attention to Shapiro steps in a trilayer with the same characteristics as above. Differently from what was done by Carapella *et al.*,[4] we here assume that the whole trilayer system is driven by an external voltage $V(t)=V_0+V_1\cos(\omega_r t)$, where V_0 is the dc component and V_1 the amplitude of the oscillating part, having angular frequency ω_r. If we normalize this voltage V with respect to RI_0, where R is the overall resistance parameter of the superconducting trilayer system and make a rescaling of the time variable, defining $\tau=\frac{2\pi RI_0}{\Phi_0}t$, then we may write

$$v(\tau) = \frac{V(\tau)}{RI_0} = \frac{d\phi}{d\tau} \Rightarrow \phi(\tau) = \phi_0 + v_0\tau + a\sin(\tilde{\omega}_r\tau). \qquad (24)$$

where $v_0=\frac{V_0}{RI_0}$, $a=\frac{2\pi V_1}{\Phi_0\omega_r}$ and $\tilde{\omega}_r=\frac{\Phi_0}{2\pi RI_0}\omega_r$. The current can be expressed, in terms of its Fourier components, as follows:

$$I = I_0\,\mathrm{Im}\left\{\left(\beta+\frac{8}{3\pi}\right)e^{i\phi} - \frac{8}{\pi}\sum_{k=2}^{\infty}\frac{(-1)^k k}{4k^2-1}e^{ik\phi}\right\}. \qquad (25)$$

By expressing the superconducting phase ϕ as in Eq. (24), we have

$$I = I_0\,\mathrm{Im}\left\{\left(\beta+\frac{8}{3\pi}\right)e^{i\phi_0}e^{iv_0\tau}e^{ia\sin\tilde{\omega}_r\tau} - \frac{8}{\pi}\sum_{k=2}^{\infty}\frac{(-1)^k k}{4k^2-1}e^{ik\phi_0}e^{ikv_0\tau}e^{ika\sin\tilde{\omega}_r\tau}\right\}. \qquad (26)$$

By substituting in Eq. (26) the expression $e^{i\lambda\sin\tilde{\omega}_r\tau} =\sum_n J_n(\lambda)e^{in\tilde{\omega}_r\tau}$, where $J_n(\lambda)$ are Bessel functions of integer order n, we have

$$I = I_0\,\mathrm{Im}\left\{\left(\beta+\frac{8}{3\pi}\right)e^{i\phi_0}\sum_l J_l(a)e^{i(v_0+l\tilde{\omega}_r)\tau} - \frac{8}{\pi}\sum_{k=2}^{\infty}\frac{(-1)^k k}{4k^2-1}e^{ik\phi_0}\sum_m J_m(ka)e^{i(kv_0+m\tilde{\omega}_r)\tau}\right\} \qquad (27)$$

As it is well known,[16] the *I-V* characteristics will show Shapiro current steps in correspondence with null values of the coefficients $(v_0+l\tilde{\omega}_r)$, with l integer, and $(kv_0+m\tilde{\omega}_r)$, with m and k integers and $k\geq 2$, in the exponent of the oscillating terms in Eq. (27). In this way, Shapiro steps at $v_{0,n}=n\tilde{\omega}_r$ (n

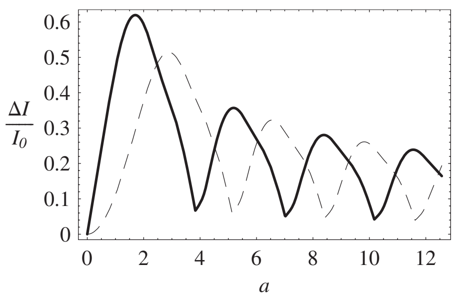

FIG. 4. Approximate normalized semiamplitudes $\frac{\Delta I_n}{I_0}$ of Shapiro current steps appearing in the I-V characteristics of a trilayer system having $\beta=0.1$ as a function of $a=\frac{2\pi V_1}{\Phi_0\omega_r}$ for $n=1$ (full line) and $n=2$ (dashed line).

$=0,1,2,\ldots)$ are obtained by setting $l=-n$ in the first sum and $\frac{m}{k}=-n$ (with n nonnegative integer) in the second, so that

$$I_n = I_0 \,\mathrm{Im}\left\{ (-1)^n J_n(a)\left(\beta+\frac{8}{3\pi}\right)e^{i\phi_0} - \frac{8}{\pi}\sum_{k=2}^{\infty}\frac{(-1)^{k(n+1)}k}{4k^2-1}e^{ik\phi_0}J_{nk}(ka)\right\}, \quad (28)$$

where we have used the following property of Bessel functions $J_{-k}(x)=(-1)^k J_k(x)$. The semiamplitude ΔI_n of current steps appearing at the voltage $v_{0,n}=n\tilde{\omega}_r$ is given by maximizing I_n with respect to ϕ_0, so that

$$\frac{\Delta I_n}{I_0} = \max_{\phi_0\in[0,2\pi]} \mathrm{Im}\left\{ (-1)^n J_n(a)\left(\beta+\frac{8}{3\pi}\right)e^{i\phi_0} - \frac{8}{\pi}\sum_{k=2}^{\infty}\frac{(-1)^{k(n+1)}k}{4k^2-1}e^{ik\phi_0}J_{nk}(ka)\right\}. \quad (29)$$

Naturally, the semiamplitude of the first even Shapiro step ($n=0$) represents the maximum Josephson current found above in the approximation $\beta \ll 1$. As an example, we may find an approximate value of ΔI_1 by retaining only the first term in the sum of Eq. (29). In this way, we write

$$\Delta I_1 \approx I_0 \max_{\phi_0\in[0,2\pi]}\left\{ -J_1(a)\left(\beta+\frac{8}{3\pi}\right)\sin\phi_0 - \frac{16}{15\pi}J_2(2a)\sin 2\phi_0\right\}. \quad (30)$$

By carrying out the calculation we find

$$\Delta I_1 \approx I_0\frac{\sqrt{1-X^2}}{4}\left[3|A| + \sqrt{A^2+32B^2}\right]. \quad (31)$$

where $A=J_1(a)(\beta+\frac{8}{3\pi})$, $B=\frac{16}{15\pi}J_2(2a)$, $X=\frac{A}{8B}(\sqrt{1+\frac{32B^2}{A^2}}-1)$. A graph of the approximate value of ΔI_1 with respect to a is represented in Fig. 4 (full line) for $\beta=0.1$. Similarly, we may find an approximate value of ΔI_2 by retaining only the first term in the sum of Eq. (29). Therefore, we have

$$\Delta I_2 \approx I_0 \max_{\phi_0\in[0,2\pi]}\left\{J_2(a)\left(\beta+\frac{8}{3\pi}\right)\sin\phi_0 - \frac{16}{15\pi}J_4(2a)\sin 2\phi_0\right\}. \quad (32)$$

By carrying out similar calculations as before we find

$$\Delta I_2 \approx I_0\frac{\sqrt{1-Y^2}}{4}\left[3|C| + \sqrt{C^2+32D^2}\right]. \quad (33)$$

where $C=-J_2(a)(\beta+\frac{8}{3\pi})$, $D=\frac{16}{15\pi}J_4(2a)$, $Y=\frac{C}{8D}(\sqrt{1+\frac{32D^2}{C^2}}-1)$. A graph of the approximate value of ΔI_2 with respect to a is represented in Fig. 4 (dashed line) for $\beta=0.1$.

VI. CONCLUSIONS

We studied the CPR of a trilayer system which can be viewed as a modified Josephson junction in which the interstitial subsystem consists of a thin superconducting region. In the framework of Otha's semiclassical model we developed a simplified theory which predicts a modified CPR for the whole system. On the basis of the obtained sawtooth CPR and by using an appropriate Fourier expansion, the maximum Josephson current and integer Shapiro steps have been computed in the linear coupling limit. Concerning the behavior of the JJ under an ac field, a modification of the first and second Shapiro step with respect to the one detected in the conventional case is observed due to the competition between the direct tunneling term (Cooper pairs going from the first to the third superconducting element in the trilayer) and the second term taking into account the tunneling between adjacent layers. Furthermore, the obtained sawtooth behavior of the CPR is consistent with the one presented in a more complicated context in Ref. 17 where a controllable π junction, obtained by inserting a quantum-dot (molecular spin) between two superconducting leads, has been analyzed. In addition, the obtained CPR is similar to the one of a superconducting quantum point contact in which the middle normal region, i.e., the so-called constriction, is characterized by linear dimensions smaller than the coherence length of the superconducting leads.[18] In the opposite limit, i.e., for long ballistic SNS junctions (linear dimensions of the middle layer greater than the coherence length), Ishii's sawtooth CPR is observed.[19]

Finally, we would like to stress that the proposed approach could be effective also in the case in which the middle layer consists of a normal metal or a quantum dot. In these cases, however, the quantum electronic states in the middle layer should be appropriately projected on the macroscopic superconducting wave function of the outer electrodes. A further interesting improvement of the present analysis can be also obtained by considering the time dependence of the macroscopic wave function of the middle subsystem (i.e., S_2) beyond the linear coupling limit considered in this work.

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