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Theory of the striped superconductor

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We define a distinct phase of matter, a *pair-density wave* (PDW), in which the superconducting order parameter $\phi(\vec{r},\vec{r}\,')$ varies periodically as a function of position such that when averaged over the center of mass position, $(\vec{r}+\vec{r}\,')/2$, all components of ϕ vanish identically. Specifically, we study the simplest unidirectional PDW, the "striped superconductor," which we argue may be at the heart of a number of spectacular experimental anomalies that have been observed in the failed high-temperature superconductor $La_{2-x}Ba_xCuO_4$. We present a solvable microscopic model with strong electron-electron interactions which supports a PDW ground state. We also discuss, at the level of Landau theory, the nature of the coupling between the PDW and other order parameters and the origins and some consequences of the unusual sensitivity of this state to quenched disorder.

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I. A DISTINCT PHASE OF MATTER

Superconductivity which arises from the pairing of electrons in time-reversed states is well understood as a weak-coupling "Fermi-surface" instability—a consequence of arbitrarily weak effective attractive interactions.[1] In this paper we explore another type of superconducting (SC) state, a pair-density wave (PDW), which is a distinct state of matter and which does not occur under generic circumstances in the weak-coupling limit—it requires interactions in excess of a critical strength. The PDW spontaneously breaks the global gauge symmetry, in precisely the same way as a conventional superconductor does. Thus, the order parameter is a charge $2e$ complex scalar field, ϕ. However, it also spontaneously breaks some of the translational and point-group symmetries of the host crystal, in the same way as a conventional charge-density wave (CDW). In a PDW, for fixed $\vec{r}-\vec{r}\,'$, the order parameter

$$\Delta(\vec{r},\vec{r}\,') \equiv \langle \psi_\uparrow^\dagger(\vec{r})\psi_\downarrow^\dagger(\vec{r}\,')\rangle \tag{1.1}$$

is a periodic function of the center of mass coordinate, $\vec{R} \equiv (\vec{r}+\vec{r}\,')/2$. Here, $\psi_\sigma^\dagger(\vec{r})$ is the electron creation operator at position $\vec{r}$ with spin polarization σ. We will focus on the case in which translational symmetry is broken only in one direction, i.e., a unidirectional PDW or equivalently a striped superconductor.

Accompanying the PDW, there is, as we will show below, induced CDW order with half the period of the PDW. However, the PDW differs from a state of coexisting superconducting and CDW order,[2] which has also been previously called a pair-density wave.[3] Unlike the pair-density-wave state of Ref. 2, the average value of the superconducting order parameter vanishes for the PDW state we discuss here. This is a defining symmetry property of this state. The PDW is more closely analogous to the Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state[4] which arises, under appropriate circumstances, when the electron gas is partially polarized by an applied magnetic field. However, explicit time-reversal symmetry breaking is an essential ingredient of the FFLO state and is responsible for lifting the degeneracy ("nesting") between time-reversed pairs of quasiparticle states. In the absence of quenched disorder, the PDW state preserves time-reversal symmetry.[5–10]

In general, there is no necessary relation between spin-density wave (SDW) and PDW order—whatever relation there is derives from common microscopic physics rather than from general symmetry conditions. However, the two orders can be linked if one postulates a larger emergent SO(5) symmetry[11] which unifies the superconducting and SDW orders. In this context, an early speculation[12] concerning the existence of a unidirectional SO(5) spiral-density-wave state is particularly interesting.[13,14] Such a state would consist of interleaving SDW and superconducting spirals, both with the same period. The fact that this state is a spiral implies that the phase of ϕ varies as a function of position, and hence that there are equilibrium currents. Thus, the SO(5) spiral, although similar to a state of coexisting PDW and collinear SDW orders, is not the same. We have not found any microscopic model that supports a spiral-PDW phase—whether coexisting with SDW order, or not. However, it is an interesting state and deserves further study.[15,16] Many features of the superconducting state depend only on the existence of a charge $2e$ order parameter. This includes zero resistance, the Meissner effect, flux quantization, and, in three dimensions (3D), the existence of a finite critical current. Therefore, these properties are expected in a PDW (in the absence of quenched disorder).

However, other properties are very different. For instance, in an s-wave superconductor, there is a full gap in the quasiparticle spectrum. For a d-wave superconductor, there are typically nodal points on the Fermi surface which support gapless quasiparticle excitations, but the density of states still vanishes at the Fermi energy, even in the presence of coexisting CDW or most forms of SDW order.[17] In contrast, in a PDW portions of the Fermi surface are typically ungapped, which as in the case of many CDWs, results in the reconstruction but incomplete gapping of the underlying Fermi surface.[17,18] In a conventional superconductor, the anisotropy of the superconducting coherence length is determined by

TABLE I. Summary of the different phases discussed in the text and their order parameters, as well as their sensitivity to quenched nonmagnetic disorder. A ✓ specifies that in a particular phase, the corresponding order parameter is nonzero. See text for details.

	Δ_0	Δ_Q	ρ_Q	ρ_{2Q}	Sensitivity to disorder
PDW	0	✓	0	✓	Fragile
CDW	0	0	✓	✓	Fragile
CDW′	0	0	0	✓	Fragile
SC	✓	0	0	0	Robust
SC+CDW	✓	✓	✓	✓	Becomes equivalent to SC

the anisotropy of the single-electron effective mass; in a PDW, depending on the precise character of the translation symmetry breaking, the anisotropy of the superconducting state can be parametrically larger than the normal-state resistivity anisotropy.

Perhaps the most significant feature of a PDW superconductor is its anomalous sensitivity to (nonmagnetic) quenched disorder, in stark comparison to a conventional superconductor. Specifically, although the disorder potential does not couple directly to the superconducting order parameter, it produces random variations in the magnitude and *sign* of the local superfluid density. Hence quenched disorder can readily drive a PDW into a superconducting *XY* glass phase. (A similar effect was recently predicted[19] in a *d*-wave superconductor near the critical point of the superconductor to metal transition.)

Table I characterizes the PDW, CDW, and uniform SC phases by their order parameters and their sensitivity to quenched disorder. Here, Δ_0 and Δ_Q are, respectively, the uniform SC and PDW (modulated SC) order parameters (where Q is the ordering wave vector) and ρ_Q and ρ_{2Q} are the two relevant CDW order parameters. By definition, the pure PDW phase has nonzero Δ_Q and a vanishing Δ_0. It is thus different from a phase of coexisting SC and CDW order, which has a nonzero Δ_0 and Δ_Q (as well as a subsidiary ρ_Q order). (In various places in Refs. 2, as an aid to intuition, the SC+CDW and/or the CDW states have also been referred to as "Cooper-pair crystals.") As mentioned above, another main characteristic of the PDW (as opposed to a uniform SC) is its sensitivity to disorder. Even weak disorder destroys the long-range PDW order, turning it into a glassy state. The properties of these phases and their inter-relations will be derived from an order-parameter Landau theory in Sec. V.

In a recent publication[20] we argued that the remarkable properties of $La_{2-x}Ba_xCuO_4$ (LBCO) near $x=1/8$ filling can be explained as a consequence of the symmetries of a striped superconducting state. The main purpose of the present paper is to develop the theory of this state.

This paper is organized as follows. In Sec. II we give an overview of the physics the striped superconductor $La_{2-x}Ba_xCuO_4$ evidenced in recent experiments.[21] In Sec. III we discuss microscopic mechanisms that give rise to a PDW. In Sec. IV we present several solvable models that shed light on the physical origin of a PDW superconducting state. Here we investigate the role of different model junctions separating two superconducting regions, including an empty barrier (a trivial insulator) and an antiferromagnetic insulating barrier. In both cases, under certain conditions, we find that a π phase shift between phases of the two superconductors is favored. We then build on these insights to construct a model with a striped superconducting ground state. We discuss qualitatively the quasiparticle spectrum of a PDW state. In Sec. V we present a Landau theory of this state and discuss in detail the symmetry-dictated couplings between the PDW order parameter and the nematic CDW and SDW order parameter characteristic of a stripe state. In a recent publication, Agterberg and Tsunetsugu[22] discussed a Landau theory for a pair-density wave which complements the results presented in this section. In Sec. VI we present a statistical-mechanical model that describes a layer decoupled striped superconductor, which is proposed to describe the phenomenology of $La_{2-x}Ba_xCuO_4$. In Sec. VII we discuss the implications of this theory. Further details of the Landau theory are presented in Appendix A. The case of coexisting uniform and striped *d*-wave superconductivity is discussed in Appendix B.

II. STRIPED SUPERCONDUCTOR IN $La_{2-x}Ba_xCuO_4$

$La_{2-x}Ba_xCuO_4$ is currently the most promising candidate experimental system as a realization of a striped superconductor.[20,21] Considerable indirect evidence in favor of the existence of PDW order in this material has recently been gathered and compiled in Ref. 23. The putative PDW order has the same period as the unidirectional (spin-stripe) SDW order which is known, from neutron-scattering studies, to exist in this material.[24,25] Strong evidence that a striped SC can be induced in underdoped $La_{2-x}Sr_xCuO_4$ by the application of a transverse magnetic field, which is also known to induce spin-stripe order, has also recently been presented in Ref. 26.

The behavior of $La_{2-x}Ba_xCuO_4$ is very striking and rather complex. We will not elaborate upon it here (see Ref. 23). However, there are two qualitative features of the data on which we would like to focus: (1) With the onset of stripe-spin order at 42 K,[27,28] there is a large (in magnitude) and strongly temperature-dependent enhancement of the anisotropy of the resistivity and other properties, such that below 42 K the in-plane charge dynamics resembles those of a superconductor, while in the *c* direction the system remains poorly metallic. The most extreme illustration of this occurs in the temperature range $10<T<16$ K, in which the in-plane direction resistivity is immeasurably small, while the *c*-axis resistivity is in the $1-10$ mΩ range, so the resistivity anisotropy ratio is consistent with infinity. (2) Despite the fact that many clear signatures of superconductivity onset at temperatures in excess of 40 K, and that angle-resolved photoemission spectroscopy (ARPES) has inferred a "gap"[29,30] of the order of 20 meV, the fully superconducting state (i.e., the Meissner effect and zero resistance in all directions) only occurs below a critical temperature of 4 K. It is very difficult to imagine a scenario in which a strong conventional superconducting order develops locally on such high scales but fully orders only at such low temperatures in a system that is three dimensional nongranular in structure and is not subjected to an external magnetic field.

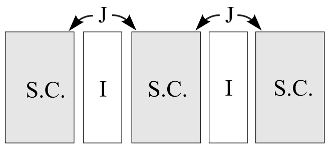

FIG. 1. Schematic model of a striped superconductor consisting of an array of superconducting regions (SC) separated by correlated insulator regions (i). The Josephson coupling between neighboring superconducting regions is J. If $J<0$, the system forms a striped superconducting (or PDW) state.

We will see that both these qualitative features of the problem are natural consequences of the assumed existence of a PDW. Indeed, analogous features have long been known to be a feature of the spin ordering in the same family of materials.[25] Specifically, unidirectional spin-stripe order is observed to occur under a number of circumstances in the 214 family of cuprate superconductors. However, when it occurs: (1) Although the in-plane correlation length can be very long, in the range of 100–400 Å, the interplane correlation length is never more than a few angstroms, a degree of anisotropy that cannot be reasonably explained simply[31] on the basis of the anisotropy in the magnitude of the exchange couplings. (2) Despite the presence of long correlation lengths, true long-range spin-stripe order has never been reported. It will be made clear below that interlayer decoupling due to the geometry of the stripe order and the rounding of the ordering transitions by quenched disorder can be understood as arising from closely analogous considerations applied to incommensurate SDW and PDW orders.[20]

III. MICROSCOPIC CONSIDERATIONS

At first, the notion that a PDW phase could be stable sounds absurd. Intuitively, the superconducting state can be thought of as the condensed state of charge $2e$ bosons. However, in the absence of magnetic fields, the ground state of a bosonic fluid is always nodeless, independent of the strength of the interactions, and therefore cannot support a state in which the superconducting order parameter changes sign. Thus, for a PDW state to arise, microscopic physics at scales less than or on the order of the pair size ξ_0 must be essential. This physics reflects an essential difference between superfluids of paired fermions and preformed bosons.[32]

Our goal in this section is to shed some light on the mechanism by which strongly interacting electrons can form a superconducting ground state with alternating signs of the order parameter. We will consider the case of a unidirectional (striped) superconductor, but the same considerations apply to more general forms of PDW order. We will not discuss the origin of the pairing which leads to superconductivity. Likewise, we will not focus on the mechanism of translation symmetry breaking by the density wave, as that is similar to the physics of CDW and SDW formation. Our focus is on the *sign alternation* of ϕ. Thus, in much of this discussion, we will adopt the model shown schematically in Fig. 1, in which we have alternating stripes of superconductor and correlated insulator. The system looks like an array of extended superconductor-insulator-superconductor (SIS) junctions, and we will primarily be concerned with computing the Josephson coupling across the insulating barriers. If the effective Josephson coupling is positive, then a uniform phase (normal) superconducting state is favored, but if the coupling is negative (favoring a π junction), then a striped superconducting phase is found.

A. Previous results

So long as time-reversal symmetry is neither spontaneously nor explicitly broken, the Josephson coupling J between two superconductors must be real. If it is positive, as is the usual case (for reasons that will be expanded upon in later sections), the energy is minimized by the state in which the phase difference across the junction is zero; if it is negative, a phase difference of π is preferred, leading to a "π junction." π junctions have been shown, both theoretically and experimentally, to occur for two distinct reasons: they can be a consequence of strong correlation effects[32–34] or magnetic ordering[35,36] in the junction region between two superconductors, or due to the internal structure (e.g., d-wave symmetry) of the superconductors themselves.[37,38]

In the present paper, we will build on the first set of ideas. Until now, π junctions have been confined to systems in which the Josephson tunneling is dominated by a single impurity site or quantum dot. It is nontrivial to extend this mechanism to the situation in which there is an extended barrier in which J is proportional to the cross-sectional "area" of the junction.[39] Indeed, as we shall see below, under most circumstances, even in the case in which the Josephson tunneling through an isolated impurity would produce a negative J, for a barrier consisting of an extended area of identical impurities, J will typically be positive. One thing that we achieve below is to obtain a proof that there are circumstances in which J of such an extended barrier is negative and to elucidate the conditions under which this occurs.

In the context of the cuprates, there have been several studies looking for a striped superconducting state in the t-J or Hubbard models. On the one hand, density matrix renormalization group (DMRG) calculations by White and Scalapino[40] consistently failed to find evidence in support of any sort of spontaneously occurring π junctions. On the other hand, a number of variational Monte Carlo calculations have concluded that the striped superconductor is either the ground state of such a model,[41] under appropriate circumstances, or at least close in energy to the true ground state.[42,43] These variational calculations are certainly encouraging in the sense that they suggest that there is no large energetic reason to rule out the existence of spontaneously occurring PDW order in strongly correlated electronic systems. However, since no such state has yet been observed in DMRG or other "unbiased" studies of these models, we believe that the mechanism of formation of these states and indeed whether they occur at all in physically reasonable microscopic models remain as unsolved problems. It is these problems that we aim to address.

B. π junctions from d-wave symmetry

In the case in which the preferred superconducting state has a nontrivial internal pair structure, as in a d-wave super-

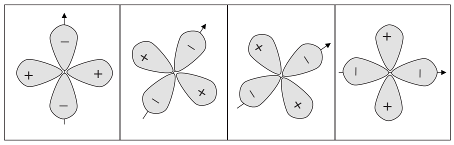

FIG. 2. An example of a π junction due to a rotating domain walls in a system with a d-wave order parameter. From left to right, the crystallographic axes rotate by $\pi/6$ across each domain wall. After three domain walls, the crystal axis has returned to itself, but the order parameter has changed sign.

conductor, it is well known both theoretically and experimentally that π junctions can be achieved by suitable orientation of the crystal fields across grain boundaries. This is *not* the physics we have in mind. However, since it is simple and well understood, we start with a highly artificial model in which d-wave symmetry gives rise to a striped superconductor.

We consider the case of a d-wave superconductor in a square lattice, with a strong crystal-field coupling which locks the lobes of the pair-wave function along the crystallographically defined x and y directions, as shown in Fig. 2. We imagine we have made a striped array in which the crystallographic axis rotates by $\pi/6$ between neighboring strips. Since rotation by $\pi/2$ is a symmetry, every third strip is identical. However, the d-wave order parameter changes sign under this rotation. Consequently, the ground state is a striped superconductor in which the period of the superconducting order parameter is twice the period of the lattice structure.

This is a very contrived example, in which the origin of the PDW does not require any strong-coupling effects—it is derived from the internal structure of the pair-wave function. For the point of principle, i.e., to establish the viability of a PDW as a phase of matter, this analysis is useful. However, for practical applications, a less contrived mechanism is needed.

IV. SOLVED MODELS

A. Model of a single junction

To start with, we consider a system consisting of three strips: a "left" superconductor described by a mean-field Hamiltonian with gap function $\Delta_L(\vec{k})$ [not to be confused with the order parameter of Eq. (1.1)] and quasiparticle energy $E_L(\vec{k})=\sqrt{[\epsilon_L(\vec{k})]^2+|\Delta_L(\vec{k})|^2}$, a "right" superconductor which is also described by a mean-field Hamiltonian with the index L replaced by R, and between them a strongly correlated insulating " barrier" region. Here $\vec{k}$ is either a 1D vector (if this is a line junction) or a two-dimensional (2D) vector if this is a planar junction. Note that we have assumed the left and right superconductors are thin in the transverse direction—otherwise we would have to include a transverse band index as well. We assume that both superconductors, by themselves, preserve time-reversal symmetry, so in both cases the phase of the superconducting order (modulo π) can be defined, so that $\Delta_\alpha(\vec{k})e^{-i\theta_\alpha}$ is real.

The three decoupled subsystems are thus described by the Hamiltonian

$$H_0 = H_L + H_B + H_R, \tag{4.1}$$

where, in the discussion below, we will consider several different examples for the insulating barrier. However, in all cases, we will assume that the superconducting Hamiltonians have the quadratic forms

$$H_\alpha = \sum_{\vec{k}\sigma} \epsilon_\alpha(\vec{k}) a^\dagger_{\alpha,\vec{k},\sigma} a_{\alpha,\vec{k},\sigma} + \sum_{\vec{k}\sigma} [\Delta_\alpha(\vec{k}) a^\dagger_{\alpha,\vec{k},\sigma} a^\dagger_{\alpha,-\vec{k}\sigma} + \text{H.c.}] \tag{4.2}$$

with $\alpha=L$ or R in which

$$a_{\alpha,\vec{k},\sigma} = N^{-1/2} \sum_{\vec{r}} e^{i\vec{k}\cdot\vec{r}} c_{\alpha,\vec{r},\sigma}, \tag{4.3}$$

where $c^\dagger_{\alpha,\vec{r},\sigma}$ creates an electron of spin polarization σ at position $\vec{r}$ on, respectively, the left superconductor, the barrier, the right superconductor for $\alpha=L$, B, and R and $N=\Sigma_{\vec{r}}1$ is the area of the junction.

The three subsystems are coupled together by a single-particle hopping term,

$$H' = -\sum_{\vec{r},\sigma} [t_L c^\dagger_{L,\vec{r},\sigma} c_{\vec{r},\sigma} + t_R c^\dagger_{R,\vec{r},\sigma} c_{\vec{r},\sigma} + \text{H.c.}] \tag{4.4}$$

(with the convention that $c_{\vec{r},\sigma} \equiv c_{B,\vec{r},\sigma}$ are the electron annihilation operators in the barrier region). Our purpose is to determine how the ground-state energy depends on the difference in phase between the superconducting order parameter on the left and right sides of the barrier, $\theta_R-\theta_L$. Specifically, we will consider the limit in which the matrix elements in H' tend to zero, so that the leading θ-dependent term in the ground-state energy can be computed in fourth-order perturbation theory,

$$\Delta E = -2N(t_L t_R)^2 [J \cos(\theta_L - \theta_R) + \cdots], \tag{4.5}$$

where $\cdots \to 0$ as $|t_\alpha| \to 0$ and $2|t_R t_L|^2 J$ is the Josephson-coupling density across the barrier.

An explicit expression can be obtained for J in terms of imaginary time-ordered correlation functions, as was shown in Ref. 44. This can be derived either by making a Laplace transform of the ordinary perturbative expression, or directly from a Euclidean path integral

$$J = \frac{1}{N\beta} \int d1d2d3d4 F_L(1,2) F^\star_R(4,3) \tilde{\Gamma}(1,2;3,4), \tag{4.6}$$

where $1 \equiv (\tau_1, \vec{r}_1)$,

$$\int d1 \equiv \sum_{\vec{r}_1} \int_0^\beta d\tau_1 \tag{4.7}$$

(in the limit $\beta \to \infty$), and

$$F_\alpha(1,2) \equiv \langle T_\tau [c^\dagger_{\alpha,\vec{r}_1,\uparrow}(\tau_1) c^\dagger_{\alpha,\vec{r}_2,\downarrow}(\tau_2)] \rangle,$$

$$\tilde{\Gamma}(1,2;4,3) \equiv \langle T_\tau[c^\dagger_{\vec{r}_1,\uparrow}(\tau_1)c^\dagger_{\vec{r}_2,\downarrow}(\tau_2)c_{\vec{r}_3,\downarrow}(\tau_3)c_{\vec{r}_4,\uparrow}(\tau_4)]\rangle. \tag{4.8}$$

It is convenient to express $\tilde{\Gamma}$ as the sum of a noninteracting piece, corresponding to processes in which two electrons tunnel through the barrier independently of each other, plus a correction term Γ, which expresses the effects of interactions *within the barrier region* between the tunneling electrons,

$$\tilde{\Gamma}(1,2;3,4) = G_\uparrow(1,4)G_\downarrow(2,3) + \Gamma(1,2;3,4), \tag{4.9}$$

where

$$G_\sigma(1,2) \equiv \langle T_\tau[c^\dagger_{\vec{r}_1,\sigma}(\tau_1)c_{\vec{r}_2,\sigma}(\tau_2)]\rangle. \tag{4.10}$$

Finally, we define two contributions to J:

$$J = J_1 + J_2, \tag{4.11}$$

where J_1 is the noninteracting portion and J_2 is the portion proportional to Γ.

B. General considerations

Under most circumstances, $J_1 > 0$ since it is proportional to F^2G^2 and hence does not depend on the sign of G (see Ref. 45). In Ref. 44, the situation in which there are attractive interactions between electrons in the barrier was investigated. In this case, J_2 is also, generally, positive. This earlier study dealt with the conditions under which $J_2 > J_1$, leading to anomalously large values of the I_cR product of the junction. In a sense, we are investigating the converse problem, in which repulsive interactions lead to $J_2 < 0$, while at the same time $|J_2| > J_1$.

In most cases with conventional superconductors, in which $|\Delta|$ is the smallest energy scale and ξ_0 (the superconducting coherence length) is the largest length scale in the problem, it is typically the case that $J_1 \gg |J_2|$. The dominant processes that contribute to J_1 involve pairs tunneling within an (imaginary) time interval of each other $\sim 1/|\Delta|$ and within a radius $\sim \xi_0$ of each other. In contrast, the processes that dominate J_2 involve electrons tunneling within an "interaction time" $\tau_{\rm int}$ and an interaction range $r_{\rm int}$ of each other. Thus, one generally expects $|J_2/J_1|$ is small in proportion to a positive power of $|\Delta|\tau_{\rm int}$ and $r_{\rm int}/\xi_0$. In conventional superconductors, where these factors are very small, this is the deciding factor, so that except under very special circumstances, one expects positive ("ferromagnetic") coupling across any junction. Indeed, as discussed in Ref. 44, in the limit that Δ is small, Eq. (4.6) reduces to the familiar form for a tunneling Hamiltonian with an effective hopping $t_{\rm eff} = t_L g t_R$ across the barrier, where

$$g \equiv \int \frac{d1d2}{N\beta} G_\sigma(1,2). \tag{4.12}$$

Conversely, in "high-temperature superconductors," in which Δ is not all that small, it is reasonable to expect, at least under some circumstances, that the coupling is negative—the long-sought π junction.

There is one other factor that can suppress J_1 relative to J_2: when the two electrons tunnel across the barrier independently of each other, their individual crystal momenta are conserved in the process. Thus, if we remove a pair of electrons at momenta $\vec{k}$ and $-\vec{k}$ from the right superconductor, they must be injected into the left superconductor at the same momenta. If the right and left superconductors are the same, this is not a problem—electrons can be removed from near the Fermi energy on the left and can be added on the right at low energy. However, if the left and right superconductors have different values of k_F, then no such low-energy process is possible. Conversely, for correlated tunneling, only the center of mass momentum of the tunneling electrons is conserved, so a pair of electrons removed from the left with momenta $\vec{k}$ and $-\vec{k}$ can be injected into the right superconductor with different momenta $\vec{q}$ and $-\vec{q}$.

C. Explicit model barriers

To make the considerations explicit, we will compute J for a model of the barrier in several limits. Specifically, we will consider the case in which the barrier consists of a Hubbard chain,

$$H_B = \sum_{\vec{r}} \epsilon \hat{n}(\vec{r}) - t \sum_{\langle \vec{r},\vec{r}'\rangle,\sigma} [c^\dagger_{\vec{r},\sigma}c_{\vec{r},\sigma} + \text{H.c.}] + U\sum_{\vec{r}} c^\dagger_{\vec{r},\uparrow}c^\dagger_{\vec{r},\downarrow}c_{\vec{r},\downarrow}c_{\vec{r},\uparrow} \tag{4.13}$$

in the strong-coupling limit, where the hopping matrix element $t \to 0$, with site energy ϵ and on-site repulsion between two electrons on the same site $U > 0$. We will consider two cases: the empty chain case in which $\epsilon > 0$ and the half-filled case, in which $\epsilon < 0 < U - |\epsilon|$. In the half-filled case, the ground state of the barrier is 2^N-fold degenerate; we resolve this degeneracy by applying a staggered magnetic field $h\Sigma_{\vec{r},\sigma} e^{i\vec{Q}\cdot\vec{r}} \sigma c^\dagger_{\vec{r},\sigma}c_{\vec{r},\sigma}$, where $\vec{Q}$ is the Néel ordering vector, and we take the limit $h \to 0$ at the end of the calculation.

For simplicity, we will also assume that Δ_L and Δ_R are independent of $\vec{k}$, and that over the relevant range of energies, the density of states in both superconductors can be approximated by constants ρ_L and ρ_R, respectively.

1. Empty barrier

For the case of an empty barrier, $\epsilon > 0$, a straightforward (but tedious) calculation reveals that

$$J_1 = \frac{2}{N}\sum_{\vec{k}} \left(\frac{\Delta_{L,\vec{k}}}{2E_{L,\vec{k}}}\right)\left(\frac{\Delta_{R,\vec{k}}}{2E_{R,\vec{k}}}\right)\left[\frac{1}{E_{L,\vec{k}}+\epsilon}\right]\left[\frac{1}{E_{R,\vec{k}}+\epsilon}\right] \times \left[\frac{1}{E_{L,\vec{k}}+E_{R,\vec{k}}} + \frac{1}{\epsilon}\right]. \tag{4.14}$$

If the left and right superconductors are the same,

$$J_1 = \frac{\rho}{2\epsilon\Delta}\left[\frac{f(y) - y}{1 - y^2}\right], \tag{4.15}$$

where

$$f(y_\alpha) = \frac{\cos^{-1}(y_\alpha)}{\sqrt{1-y_\alpha^2}},$$

$y_\alpha \equiv \epsilon/|\Delta_\alpha|$, and in this case $y=y_L=y_R$. Here $f(y)$ is a smooth function with $f(0)=\pi/2$, $f(1)=1$, and $f(y)\sim \ln(2y)/y$ as $y\to\infty$.

If the two superconductors have quite different values of k_F, then J_1 gets substantially suppressed. Let E_L and E_R be, respectively, the energy of the left quasiparticle at $k_{F,R}$ and the energy of the right quasiparticle at $k_{F,L}$. So long as $E_L, E_R \ll \Delta$, the magnitude of J_1 is hardly changed. However, if the two superconductors are sufficiently different that $E_L, E_R \gg \Delta$, then

$$J_1 = \rho_L f(y_L)\left(\frac{\Delta_R}{E_R^2\epsilon}\right) + \rho_R f(y_R)\left(\frac{\Delta_L}{E_L^2\epsilon}\right). \tag{4.16}$$

The interaction correction derives from terms that would have contributed to J_1, but are suppressed due to the repulsive interaction, and so has the opposite sign,

$$J_2 = -2f(y_L)f(y_R)\rho_L\rho_R\left[\frac{U}{\epsilon(U+2\epsilon)}\right]. \tag{4.17}$$

In conventional superconductors, typically the bandwidth W is the largest energy in the problem, and hence $\rho\sim W^{-1}$ is "small." Since $J_1\sim\rho$ and $J_2\sim\rho^2$, this means that $J_1 \gg |J_2|$. The exception to this rule occurs in cases in which $\Delta E_F \equiv \min[E_R, E_L]$ is a substantial fraction of the bandwidth.

However, in high-temperature superconductors, we expect to find that Δ is not too much smaller than W. In this case, it will depend on details whether or not J_1 or J_2 dominate. Clearly, this situation is more likely the larger U. In the case of tunneling between two identical superconductors, and in the limit $U\to\infty$, $|J_2| > J_1$ so long as

$$\rho\Delta > x_c \equiv \frac{[f(y)-y]}{4f^2(y)(1-y^2)}. \tag{4.18}$$

For $y \gg 1$, $x_c \sim y/\ln^2 y$, i.e., condition (4.18) requires impossibly large gap scales. However, $x_c = 19/96$ for $y=1$ and $x_c \to 1/2\pi$ as $y\to 0$.[46] Manifestly, J_1 is reduced in magnitude and J_2 is unaffected by a substantial value of ΔE_F. For instance, if we consider the case in which the left and right superconductors differ only in the position of the Fermi surface, J_2 dominates so long as

$$[\rho\Delta E_F]^2 f(y) > \rho\Delta. \tag{4.19}$$

2. Antiferromagnetic barrier

We now turn to discuss the case $\epsilon < 0 < U - |\epsilon|$, in which the barrier is half filled. As mentioned previously, we introduce a staggered Zeeman field h to lift the ground-state degeneracy and we take $h\to 0$ at the end of the calculation. The lowest-order perturbative term in the Josephson coupling consists of all processes in which a pair of electrons tunnels through the barrier, subjected to the constraint that the barrier has to return to its Néel ordered ground state at the end of the process.

For simplicity, we consider the $U\to\infty$ case, in which no doubly occupied sites are allowed in the barrier. (We have obtained similar expressions for finite U but they contain no qualitatively different physics.) In this case, processes in which electrons tunnel though different sites contribute to J_1 and those in which they tunnel through the same site contribute to J_2. As discussed in Ref. 32, the single-site process necessarily involves an exchange of two fermions and so makes a negative contribution to J. In the present problem, there is an antiferromagnetic "umklapp" contribution to J_1, in which the tunneling electron exchanges momentum π/a with the antiferromagnet; this makes a negative contribution to J_1 which, under appropriate circumstances, can largely cancel the usual positive contribution.

It is a straightforward exercise to obtain the following resulting expressions:

$$J_1 = \frac{1}{2N}\sum_{\vec{k},\vec{k}'}\left(\frac{\Delta_{L,\vec{k}}}{2E_{L,\vec{k}}}\right)\left(\frac{\Delta_{R,\vec{k}'}}{2E_{R,\vec{k}'}}\right)(\delta_{\vec{k}-\vec{k}'} - \delta_{\vec{k}-\vec{k}'+\pi\hat{y}}) \times\left[\frac{1}{E_{L,\vec{k}}+E_{R,\vec{k}'}} + \frac{1}{|\epsilon|}\right]\left[\frac{1}{E_{L,\vec{k}}+|\epsilon|}\right]\left[\frac{1}{E_{R,\vec{k}'}+|\epsilon|}\right], \tag{4.20}$$

where $\delta_{\vec{k}}$ is the Kronecker delta function and

$$J_2 = -\frac{1}{N^2}\sum_{\vec{k},\vec{k}'}\left(\frac{\Delta_{L,\vec{k}}}{2E_{L,\vec{k}}}\right)\left(\frac{\Delta_{R,\vec{k}'}}{2E_{R,\vec{k}'}}\right) \times\left[\frac{1}{E_{L,\vec{k}}+|\epsilon|}\right]\left[\frac{1}{E_{R,\vec{k}'}+|\epsilon|}\right]\left[\frac{1}{E_{L,\vec{k}}+E_{R,\vec{k}'}}\right]. \tag{4.21}$$

Generically, the magnitude of the umklapp term (proportional to $\delta_{\vec{k}-\vec{k}'+\pi\hat{y}}$) is considerably smaller than the momentum conserving term since it is typically not possible for both $\vec{k}$ or $\vec{k}+\pi\hat{y}$ to be close to the Fermi surface *unless* a special nesting condition is satisfied. For example, if the two superconductors are one-dimensional wires close to half filling, then the separation between the two Fermi points is indeed close to π. In that case, the umklapp term tends to cancels the momentum-conserving term. (If particle-hole symmetry is present, this cancellation is exact.)

We have evaluated the expressions in Eqs. (4.20) and (4.21) numerically for the case in which the two superconducting strips are identical chains with a k-independent gap function, $\Delta_{\alpha,k}=\Delta$ and the band structure is the simplest tight-binding result, $\epsilon_{\alpha,k} = -2t\cos(k) - \mu$. The results are summarized in Fig. 3, where the x axis is the average density of electrons $\langle n\rangle$ in the superconductors and the y axis is the magnitude of Δ; the solid lines separate the region of net negative from the regions of net positive Josephson coupling. The general trends alluded to in the above discussion are clear from this figure.

D. Model with a striped superconducting ground state

The basic ingredients necessary for finding a model with a striped superconducting ground state are present in the above discussed examples. However, in the interest of making everything explicit, we consider the following problem. First,

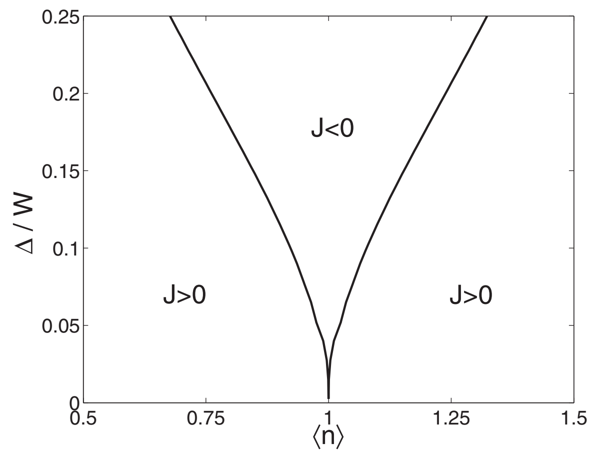

FIG. 3. Phase diagram for the antiferromagnetic barrier problem as a function of the pairing gap Δ (in units of the bandwidth $W=4t$) and the average density $\langle n\rangle$ in the superconductors on either side of the barrier. When Δ is large or $\langle n\rangle$ is close to 1 (half filling), then a negative Josephson coupling ($J<0$) is favored.

we generalize the problem to $d+1$ dimensions. We consider an array of parallel d-dimensional hyperplanes with weak attractive interactions and a particle-hole symmetric dispersion. Each hyperplane thus has a uniform superconducting order parameter. For weak attractions between electrons and large enough d, the superconducting correlations in each hyperplane can, presumably, be accurately treated in the context of BCS mean-field theory. The particle-hole symmetry implies that $E_{\vec{k}}=E_{\vec{k}+\vec{\pi}}$, where $\vec{\pi}$ is the d-dimensional antiferromagnetic Néel ordering vector. Sandwiched between each superconducting hyperplane there is a d-dimensional insulting hyperplane, which is the d-dimensional generalization of the antiferromagnetic barrier discussed above. Finally, we couple neighboring hyperplanes with an arbitrarily weak hopping matrix t. The Josephson coupling per unit hyperarea is then computed as above.

The particle-hole symmetry of the model insures that $J_1=0$, and that therefore the Josephson coupling per unit hyperarea is proportional to J_2 and is hence negative. Thus, the ground state of this system is a state with coexisting SDW and striped superconducting orders. Note that the phase of the antiferromagnetic order is determined by the sign of h. If we take $h>0$ in all cases, then the antiferromagnetic order is "in phase." In this case, the Hamiltonian is invariant under translations in the direction perpendicular to the hyperplanes by the distance between insulating hyperplanes, and hence the striped superconductor spontaneously breaks translational symmetry. However, we can also imagine staggering the sign of h from one hyperplane to the next so as to mimic the antiphase spin-stripe order seen in the cuprates. In this case, the underlying unit cell is twice as large, and the striped superconductor does not spontaneously break translational symmetry. However, in both cases, the spatial average of any component of the superconducting order parameter is zero, so this state conforms to our definition of a PDW.

This model is, admittedly, somewhat contrived especially in the assumed particle-hole symmetry. However, since at this point J_2 has a finite magnitude, the state is robust with respect to small deformations of the model. The construction of solvable microscopic models with a striped superconducting ground state and the elucidation of the ingredients necessary to get antiphase superconducting order across a correlated insulating barrier are the principle results of the present paper.

E. Quasiparticle spectrum of a striped superconductor

The spectrum of an s-wave superconductor is typically fully gapped, while a d-wave superconductor has a gap that vanishes only on a few isolated points (lines in $3d$) on the Fermi surface. These features persist even in the presence of coexisting CDW or SDW order of various sorts.[17] In particular, the density of quasiparticle states is either zero for energies less than a nonzero minimum value (in the s-wave case) or vanishes (linearly) as the energy approaches the Fermi surface. This effect of superconducting order is in contrast to most other orders, including SDW and CDW orders, in which, so long as the effective ordering potential is weak compared to the Fermi energy, at most a portion of the Fermi surface is gapped leaving the system with a (smaller) reconstructed Fermi surface but a Fermi surface none the less.

The gap character in a PDW is more similar to that of a CDW than to a uniform superconductor. Specifically, so long as the order parameter is not too large, an ungapped reconstructed Fermi surface remains.[17,18] Thus, a PDW typically has a finite density of states in the superconducting phase.

For band-structure parameters characteristic of a typical underdoped or optimally doped curpate, the quasiparticle spectrum of a striped superconductor has a large (d-wave-like) gap near the "antinodal points" [i.e., near $(\pi,0)$ and $(0,\pi)$] but has a closed Fermi-surface pocket which closely follows the contours of the portion of the normal-state Fermi surface near the "nodal points" [i.e., where the Fermi surface crosses the (0,0) to (π,π) cord]. In short, it closely resembles certain phenomenological descriptions[47] of the "pseudogap." The low-energy spectral weight is substantial only on a portion of the bare Fermi surface near the nodal (π,π) direction,[31] forming a "Fermi arc."[48,49]

V. COUPLED ORDER PARAMETERS

In this section, we explore the aspects of the theory of a PDW that can be analyzed without reference to microscopic mechanisms. We focus on the properties of ordered states at $T=0$, far from the point of any quantum phase transition, where for the most part fluctuation effects can be neglected. (The one exception to the general rule is that, where we discuss effects of disorder, we will encounter various spin-glass-related phases where fluctuation effects, even at $T=0$, can qualitatively alter the phases.) For simplicity, most of our discussion is couched in terms of a Landau theory, in which the effective free energy is expanded in powers of the order parameters; this is formally not justified deep in an ordered phase, but it is a convenient way to exhibit the consequences of the order-parameter symmetries.

Specifically, the emphasis in this section is on the interrelation between striped superconducting order and other orders. There is a necessary relation between this order and CDW and nematic (or orthorhombic) orders since the striped superconductor breaks both translational and rotational symmetries of the crystal. From the microscopic considerations above and from the phenomenology of the cuprates, we are also interested in the relation of superconducting and SDW orders. We assume that the host crystal is tetragonal, and that there are therefore two potential symmetry-related ordering wave vectors Q and $\bar{Q}$, which are mutually orthogonal. Spin-orbit coupling is assumed to be negligible. We introduce the following order-parameter fields: the nematic N, the PDW Δ_Q, the SDW $\vec{S}_Q$, and the CDW ρ_K (where $K=2Q$), and of course the corresponding orders at the symmetry-related wave vector $\bar{Q}$. The Landau effective free energy density can then be expanded in powers of these fields as follows:

$$\mathcal{F} = \mathcal{F}_2 + \mathcal{F}_3 + \mathcal{F}_4 + \cdots, \tag{5.1}$$

where $\mathcal{F}_2$, the quadratic term, is simply a sum of decoupled terms for each order parameter,

$$\begin{aligned}\mathcal{F}_3 = {} & \gamma_s[\rho_K^\star \vec{S}_Q \cdot \vec{S}_Q + \rho_{\bar{K}}^\star \vec{S}_{\bar{Q}} \cdot \vec{S}_{\bar{Q}} + \text{c.c.}] \\ & + \gamma_\Delta[\rho_K^\star \Delta_{-Q}^\star \Delta_Q + \rho_{\bar{K}}^\star \Delta_{-\bar{Q}}^\star \Delta_{\bar{Q}} + \text{c.c.}] \\ & + g_\Delta N[\Delta_Q^\star \Delta_Q + \Delta_{-Q}^\star \Delta_{-Q} - \Delta_{\bar{Q}}^\star \Delta_{\bar{Q}} - \Delta_{-\bar{Q}}^\star \Delta_{-\bar{Q}}] \\ & + g_s N[\vec{S}_Q^\star \cdot \vec{S}_Q - \vec{S}_{\bar{Q}}^\star \cdot \vec{S}_{\bar{Q}}] + g_c N[\rho_K^\star \rho_K - \rho_{\bar{K}}^\star \rho_{\bar{K}}],\end{aligned} \tag{5.2}$$

and the fourth-order term, which is more or less standard, is shown explicitly in Appendix A. (Detailed discussions of the microscopic definition of the PDW order parameter is also discussed in this appendix.)

The effect of the cubic term proportional to γ_s on the interplay between the spin and charge components of stripe order has been analyzed in depth in Ref. 50. Similar analysis can be applied to the other terms. In particular, it follows that the existence of superconducting stripe order ($\Delta_Q \neq 0$ and $\Delta_{\bar{Q}}=0$), implies the existence of nematic order ($N \neq 0$) and charge-stripe order with half the period ($\rho_{2Q} \neq 0$). However, the converse statement is not true: while CDW order with ordering wave vector $2Q$ or nematic order tend to promote PDW order, depending on the magnitude of the quadratic term in $\mathcal{F}_2$, PDW order may or may not occur.

Aother feature of the coupling between the PDW and CDW orders is that it produces sensitivity to disorder which is not normally a feature of the superconducting state. In the presence of quenched disorder, there is always some amount of spatial variation in the charge density $\rho(r)$ of which the important portion for our purposes can be thought of as being a pinned CDW that is a CDW with a phase which is a pinned slowly varying function of position, $\rho(r) = |\rho_K| \cos[K \cdot r + \phi(r)]$. Below the nominal striped superconducting ordering temperature, we can similarly express the PDW order in terms of a slowly varying superconducting phase, $\Delta(r) = |\Delta_Q| \exp[iQ \cdot r + i\theta_Q(r)] + |\Delta_{-Q}| \exp[-iQ \cdot r + i\theta_{-Q}(r)]$. The resulting contribution to $\mathcal{F}_3$ is

$$\mathcal{F}_{3,\gamma} = 2\gamma_\Delta |\rho_K \Delta_Q \Delta_{-Q}| \cos[2\theta_-(r) - \phi(r)], \tag{5.3}$$

where

$$\theta_\pm(r) \equiv [\theta_Q(r) \pm \theta_{-Q}(r)]/2,$$

$$\theta_{\pm Q}(r) = [\theta_+(r) \pm \theta_-(r)]. \tag{5.4}$$

The aspect of this equation that is notable is that the disorder couples directly to a piece of the superconducting phase θ_-. No such coupling occurs in usual zero-momentum superconductors.

It is important to note that the condition that $\Delta(r)$ be single valued imposes a nontrivial topological constraint on possible vortices. Specifically, an isolated half vortex (about which the phase winds by π) is forbidden in either θ_+ or θ_-; vortices must occur either as full 2π vortices in one or the other phase field or as a bound pair of a θ_+ and a θ_- half vortex. An important consequence of this phase coupling is that the effect of quenched disorder, as in the case of the CDW itself, destroys long-range superconducting stripe order. (This statement is true,[51] even for weak disorder, in dimensions $d < 4$.) Naturally, the way in which this plays out depends on the way in which the CDW state is disordered.

In the most straightforward case, the CDW order is punctuated by random pinned dislocations, i.e., 2π vortices of the ϕ field. The existence of the coupling in Eq. (5.3) implies that there must be an accompanying π vortex in θ_-. The condition of single-valued-ness implies that there must also be an associated half vortex or antivortex in θ_+.[52] If these latter vortices are fluctuating, they destroy the superconducting state entirely, leading to a resistive state with short-ranged striped superconducting correlations. If they are frozen, the resulting state is analogous to the ordered phase of an *XY* spin glass. Such a state has a nonvanishing Edwards-Anderson order parameter, spontaneously breaks time-reversal symmetry, and, presumably, has vanishing resistance but no Meissner effect and a vanishing critical current. In 2D, according to conventional wisdom, a spin-glass phase can only occur at $T=0$, but in 3D there can be a finite-temperature glass transition.[53]

In 3D there is also the exotic possibility that, for weak enough quenched disorder, the CDW forms a Bragg-glass phase, in which long-range order is destroyed but no free dislocations occur.[54–56] In this case, ϕ can be treated as a random but single-valued function—correspondingly, so is θ_-. The result is a superconducting Bragg-glass phase which preserves time-reversal symmetry and, presumably, acts more or less the same as a usual superconducting phase. It is believed that a Bragg-glass phase is not possible in 2D.[55] A summary of the characterization of the PDW, SC, and CDW phases in terms of their order parameters, their interrelations, and their sensitivity to quenched disorder appears in Table I.

Another perspective on the nature of the superconducting state can be obtained by considering a composite order parameter which is proportional to $\Delta_Q \Delta_{-Q}$. There is a cubic term which couples a uniform charge $4e$ superconducting order parameter Δ_4 to the PDW order,

$$\mathcal{F}_3' = g_4\{\Delta_4^\star[\Delta_Q\Delta_{-Q} + \Delta_{\bar{Q}}\Delta_{-\bar{Q}}] + \text{c.c.}\}. \quad (5.5)$$

This term implies that whenever there is PDW order, there is also necessarily charge $4e$ uniform superconducting order. However, since this term is independent of θ_-, it would be totally unaffected by Bragg-glass formation of the CDW. The half vortices in θ_+ discussed above can simply be viewed as the fundamental $(hc/4e)$ vortices of a charge $4e$ superconductor.

Turning now to the quartic terms in $\mathcal{F}_4$, several features of the ordered phases depend qualitatively on the sign of various couplings. Again, this is very similar to what happens in the case of CDW order—see, for example, Ref. 57. For instance, depending on the sign of a certain biquadratic term, either unidirectional (superconducting stripe) or bidirectional (superconducting checkerboard) order is favored.

Finally, we comment on the case of coexisting uniform and striped superconducting order parameters. Such a state is not thermodynamically distinct from a regular (uniform) superconductor coexisting with a charge-density wave, even if the uniform superconducting component is in fact weaker than the striped component. Therefore, we expect many of the special features of the striped superconductor (such as its sensitivity to potential disorder) to be lost. In Appendix B, we extend the Landau free energy to include a uniform superconducting component and show that this is indeed the case.

VI. MODEL FOR A LAYERED STRIPED SUPERCONDUCTOR

We now discuss a low-energy effective theory for a layered disordered striped superconductor. As we saw in Sec. V, disorder inevitably nucleates half vortices which are pinned to dislocations in the CDW. The system therefore has Ising-type degrees of freedom which are the "charges" (or vorticities) of these π vortices. We assume that their positions are quenched random variables. As we discussed in Ref. 20, a PDW in a layered superconductor can give rise to a frustration of the interlayer Josephson coupling, given a structure in which the CDW is rotated by 90° between layers [as occurs in the LTT phase of $La_{2-x}Ba_xCuO_4$]. To make the model relevant to this system, we consider the case where the interlayer Josephson coupling vanishes identically.

Under these assumptions, the interaction energy between half vortices is composed of two parts: the magnetic energy and the kinetic energy associated with the screening currents that surround the vortices. The total interaction energy for a given configuration of half vortices is[58]

$$U = \frac{1}{2}\sum_{ij\alpha\beta} u(\vec{R}_{i\alpha} - \vec{R}_{j\beta})q_{i\alpha}q_{j\beta} + \frac{1}{2}\sum_{\alpha\beta} v(z_\alpha - z_\beta)Q_\alpha Q_\beta. \quad (6.1)$$

Here, $q_{i\alpha}=\pm 1$ and $\vec{R}_{i\alpha}$ are the charge (or vorticity) and the position of the ith vortex in layer α, respectively, z_α is the z coordinate of layer α, and $Q_\alpha=\Sigma_i q_{i\alpha}$. The interaction potentials $u(\vec{R})$ and $v(z)$ are given by

$$u(\vec{R}) = \frac{B_0^2\lambda^2 d}{\mu_0}\int \frac{d^2k}{(2\pi)^2}\left[\frac{\delta_{\alpha\beta}}{k^2} - \frac{d}{2\lambda}\frac{e^{-\alpha(k)|z|/\lambda}}{\alpha(k)k^2}\right][e^{i\vec{k}\cdot\vec{\rho}} - 1], \quad (6.2)$$

where $\vec{\rho}$ and z are the radial and the z axis separations between the two vortices, respectively, $\alpha(k)\equiv\sqrt{1+(\lambda k)^2}$, $B_0 \equiv \frac{h}{4e\lambda^2}$, $\delta_{\alpha\beta}$ is a Kronecker delta function of the layer indices α,β of the two vortices, λ is the in-plane penetration depth, d is the interlayer distance, and

$$v(z) = \frac{B_0^2\lambda^2 d}{2\pi\mu_0}\left\{\delta_{\alpha\beta}\ln\frac{L}{\xi} - \frac{d}{2\lambda}e^{-|z|/\lambda}\ln\frac{L}{\lambda} + \frac{d}{4\lambda}e^{-2|z|/\lambda}\left[\ln\frac{|z|}{2\lambda} + \gamma + e^{2|z|/\lambda}E_1\left(\frac{2|z|}{\lambda}\right)\right]\right\}, \quad (6.3)$$

where L is a long-wavelength cutoff of the order of the linear dimension of the system, ξ is the coherence length, $E_1(x) = \int_x^\infty \frac{e^{-t}}{t}dt$ is the exponential integral function, and γ is Euler's constant. The second term in Eq. (6.1) diverges as $\ln L$ unless $Q_\alpha=Q=$const for all α. In the absence of an external magnetic field, this term constrains $Q_\alpha=0$. This is the usual "charge neutrality" condition which comes from the infrared divergence of the vortex self-energy.

For $\sqrt{\rho^2+z^2}\gg\lambda$, Eq. (6.2) reduces to

$$u(\vec{R}) \approx \frac{B_0^2\lambda^2 d}{2\pi\mu_0}\left[\delta_{\alpha\beta} - \frac{d}{2\lambda}e^{-|z|/\lambda}\right]\ln\frac{\xi}{\rho}. \quad (6.4)$$

The statistical-mechanics problem of a finite density of half vortices with quenched random positions, and whose interaction is given by Eq. (6.1) is an interesting unsolved statistical-mechanics problem.[59,60]

As we mentioned, the putative superconducting glass phase (in which the half vortices are frozen) necessarily involves time-reversal symmetry breaking. It should therefore be detectable by measuring the magnetic fields associated with the spontaneous half vortices which occur at dislocations in the charge order. For example, consider a half vortex at the surface of the sample (taken to be at $z=0$). The asymptotic form of the magnetic field above the surface in the limit $0<z\ll\lambda$ is

$$B_z(z\ll\lambda) \approx \frac{\hbar}{8\lambda^2 e}\left(\frac{d}{z}\right), \quad (6.5)$$

where d is the interlayer distance. [For $z\gtrsim\lambda$, screening by diamagnetic currents in the other planes becomes considerable and B_z crosses over to $\sim\frac{\hbar d}{8\lambda e}(\frac{1}{z^2})$.] Assuming $\lambda\approx 2000$ Å and $d\approx 15$ Å, Eq. (6.5) gives $B_z\approx 300(\frac{\text{Å}}{z})$ G. At a distance of about 1000 Å from surface, the resulting characteristic fields on the order of 0.3 G (assuming a half vortex right at the surface) are well within the resolution of current local magnetic field measurement techniques. The onset of spontaneous random magnetic fields of this order at the glass transition temperature (which is presumably also signaled by the vanishing of the linear resistivity) would be a dramatic confirmation of the PDW scenario in $\frac{1}{8}$-doped LBCO.

VII. FINAL THOUGHTS

The strong suppression of the three-dimensional superconducting T_c in $La_{2-x}Ba_xCuO_4$ at $x=1/8$, and other "1/8 anomalies" in the $La_{2-x}Sr_xCuO_4$ family of cuprate superconductors have long been interpreted in the literature as evidence that charge order competes with high-temperature superconductivity. (More recently,[61–63] clear evidence of a 1/8 anomaly has been adduced in $YBa_2Cu_3O_{7-\delta}$ as well.) However, the remarkable properties of $La_{2-x}Ba_xCuO_4$, particularly the fact that the antinodal gap is largest at $x=1/8$, where the dynamical layer decoupling is observed, strongly suggests that charge-stripe order *can be* part of the mechanism of superconductivity as argued in Ref. 64.

The superconducting state presented here represents another face of the interplay between superconductivity and spin and charge orders. If the SC and SDW orders simply competed, the PDW would be a rather unnatural state; it is natural if self-organized inhomogeneities are an essential feature of the mechanism. In this sense, the PDW state should also be regarded as an electronic liquid crystal phase.[65] Numerous and important implications follow from the properties of this state.

A. Implications

1. Glassy superconductors

The most dramatic consequence of the PDW physics is that, in the presence of weak quenched disorder, the superconducting phase gives way to a regime of glassy behavior, where strong local superconducting correlations extend up to a finite correlation length set by the strength of the disorder. In the regime of temperature below the onset of substantial local superconducting coherence but above the transition to a fully superconducting state (if one occurs), the system can be characterized as a "failed superconductor." [30] Clearly, in this regime, the longitudinal resistivity will be small compared to normal-state values, as will the quasiparticle contributions to the thermopower and linear Hall resistance. Moreover, one expects to see strong local indications of superconductivity, especially the formation of a superconducting pseudogap in the single-particle spectrum with a form similar to that of the ordered phase.

Taking into account that the materials are ultimately 3D, there will generally be a glass transition at $T=T_g$ in this regime. In the absence of an applied magnetic field, the glass phase can be characterized by its spontaneously broken time-reversal symmetry. Presumably, the glass phase has zero resistance, zero critical current, and fails to exhibit a full Meissner effect. More generally, for a range of temperatures, including temperatures above T_g, the magnetic response of the PDW will be characterized by slow dynamics, a broad distribution of relaxation rates, and probably a certain degree of history dependence, which are all the dramatic and confusing features of spin glasses but amplified by the coherent orbital organization of a superconducting state.

2. Interlayer decoupling

When a PDW state occurs in a quasi-2D (layered) material, it will frequently be the case (depending on the interlayer geometry) that the usual Josephson coupling between neighboring layers vanishes. This feature was explored at length in our earlier paper.[20] In the case of the LTT structure of $La_{2-x}Ba_xCuO_4$, the fact that the stripes in neighboring planes run at right angles to each other insures that (in the absence of disorder), the Josephson coupling between neighboring planes vanishes. Strictly speaking, this does not mean that there is no coupling between plains, as would occur in a putative "floating phase." [66] There are always couplings between farther neighbor layers, and higher-order Josephson couplings between neighboring planes (which couple Δ_4 in 3D), and these interactions are relevant in the renormalization group sense at any temperature below the putative Kosterlitz-Thouless (KT) transition temperature. However, at the very least, it means that the 3D superconducting state is enormously more anisotropic than would be expected on the basis of the bare electronic anisotropy. Moreover, as long as superconducting coherence in a given plane is limited due to quenched disorder, if these higher-order couplings are weak enough, they can be essentially ignored.

3. Striped superconductors in strongly correlated models

While we have established, as a point of principle, that well-defined models exist that support a PDW phase, it still makes sense to make predictions that can be tested in "numerical experiments" on *t-J* and Hubbard models. The fact that spontaneously occurring π junctions have not yet been reported in extensive previous DMRG studies[40] is worrisome in this regard.

Based on the insights gained above, we propose DMRG studies of microscopic models in which we expect indications of PDW formation can be observed. Two examples are:

(i) A three-leg Hubbard ladder in which the outer two legs have a negative U and the inner leg has a positive U. The chemical potential and on-site energies should be chosen, so that the inner leg is near half filling and so strongly antiferromagnetically correlated. The density of electrons on the outer ("superconducting") legs can be varied and need not be identical. This model is thus a close relative of the model that we treated in Sec. IV C 2, except that we do not treat the superconducting legs in mean-field theory and we are not restricted to considering parametrically small coupling between the legs. The tendency to a striped SC phase can be tested by studying the sign of the pair-field correlations between the upper and lower legs. When the outer legs are near half filling, we expect to see negative correlations indicative of PDW formation. Indeed, we have already seen such behavior in preliminary DMRG studies of this model.[67]

(ii) A five-leg Hubbard ladder with all repulsive interaction, constructed to consist of two outer two-leg ladders and an inner Hubbard chain. Again, the chemical potential and on-site energies should be chosen, so that the inner chain is near half filling, making it strongly antiferromagnetically correlated, while the density of electrons on the outer ladders can be varied and need not be identical. By making the densities on the outer legs different enough, we are confident that a PDW state can be induced. However, we are very curious to learn how robust this state is under less contrived conditions. We are now undertaking such an investigation.[67]

B. Speculations

1. Striped SC phases in $La_{2-x}Ba_xCuO_4$ *at* $x=1/8$

We were originally motivated in this study by experiments in $La_{2-x}Ba_xCuO_4$, and we remain optimistic that they are, indeed, evidence of the existence of a PDW phase. In order to make more than an impressionistic comparison with experiment, the nature of the superconducting glass phase will need to be understood theoretically, much better than we do at present. However, in the absence of such theoretical control, we can still make some speculative statements concerning the relation between experiment and theory.

Tentatively, we would like to identify the point at which resistivity vanishes in the c direction as the true point of the 3D glass transition $T_g \approx 10$ K. If this is the case, the linear-response resistivity at lower temperatures is truly zero, although various nonlinear and hysteretic processes may complicate the measurements. We expect very long time scales and a degree of history dependence of macroscopic measurements to begin being significant at considerably higher temperatures $T > T_g$.

It is natural to associate the point at which the in-plane resistivity apparently vanishes, $T_{KT} \approx 16$ K, with what would have been the KT transition of a single plane in the absence of quenched disorder. We do not think that this is a true transition and would expect that if experiments could be carried out with higher precision than has currently been possible, the in-plane resistivity would be found to have a finite value for $T_g < T < T_{KT}$. In relatively clean samples, simple scaling arguments suggest that the residual resistivity should be proportional to ξ^{-2}, where ξ is the coherence length, which in turn is roughly the distance between dislocations.

The sharp drop in the resistivity at the spin-ordering temperature $T_{SDW} \approx 42$ K is probably not a true phase transition either but rather marks the sudden onset of significant intermediate scale superconducting coherence. However, we suspect that local pairing correlations persist to higher temperatures; as long as the spin order is strongly fluctuating at $T > T_{SDW}$, we imagine that phase coherence between neighboring superconducting stripes is prevented, while for $T < T_{SDW}$, antiphase superconducting correlations extend over multiple stripes. We favor this viewpoint for several reasons, most importantly because the gap features seen in ARPES persist to higher temperatures.[30]

2. ARPES spectrum of $La_{2-x}Ba_xCuO_4$

A recent study[30] of the ARPES spectrum of $La_{2-x}Ba_xCuO_4$ confirmed the conclusion of an earlier[29] ARPES/scanning tunneling microscopy (STM) study concerning the existence of a gap consistent with a generally d-wave angle dependence. However, the higher-resolution study revealed that this gap has what appears to be a two-component structure, in that the gap near the antinodal point at $(\pi,0)$, is considerably larger (by a factor of 2–3) than a simple extrapolation from the nodal direction would suggest. Since there are several forms of density wave order known to be present in $La_{2-x}Ba_xCuO_4$, it is not straightforward to unambiguously identify particular spectral features with particular types of order. This is particularly problematic since fluctuating order can also lead to a pseudogap with many similarities to the gap that would be produced in the corresponding ordered phase.

That having been said, it is striking how much the observed spectrum in $La_{2-x}Ba_xCuO_4$ resembles the mean-field spectrum found under the assumption that there is large amplitude PDW (striped) order coexisting with small amplitude uniform superconducting order. The PDW has a large gap along the antinodal $[(\pi,0)$ or $(0,\pi)]$ direction, and a Fermi pocket in the nodal $[(\pi,\pi)]$ direction, whose spectral weight is considerable only along an open Fermi arc region which nearly coincides with the bare Fermi surface. If, in addition, there is a uniform d-wave component to the order parameter, this completes the gapping of the Fermi surface. A spectrum that resembles the measured ARPES spectrum in $La_{2-x}Ba_xCuO_4$ can be obtained[17] under the assumption that the PDW has a gap magnitude $\Delta_{PDW}=40$ meV and uniform gap magnitude $\Delta_d=8$ meV. The ARPES spectrum is always measured at temperatures well above the bulk superconducting $T_c=4$ K, so to the extent that this identification is correct, what is being observed is a pseudogap. Moreover, as mentioned above, the gap structure persists to temperatures above T_{SDW}, although at the higher temperatures the gap in the nodal regions decreases appreciably. Much more detailed analysis of the energy, momentum, and temperature dependences of the ARPES spectrum will be necessary to test this interpretation.

3. Relevance to other cuprates

We have summarized the evidence for a state with the symmetries of a PDW state (or the glassy version of it) in $\frac{1}{8}$-doped $La_{2-x}Ba_xCuO_4$. There is also evidence for dynamical layer decoupling effects in $La_{1.6-x}Nd_{0.4}Sr_xCuO_4$ (Ref. 68) and also in $La_{2-x}Sr_xCuO_4$ in a magnetic field.[26,69] (For the possible relevance of these ideas to certain heavy fermion and organic superconductors, see Refs. 70–73.)

The quasiparticle spectrum of a PDW state has striking qualitative resemblance to the spectra seen in ARPES in other cuprates, especially $Bi_2Sr_2CaCu_2O_{8+\delta}$ and $Bi_2Sr_2CuO_{6+\delta}$. This possibly sheds light on the issue of the "nodal-antinodal dichotomy": According to this interpretation, both the nodal and the antinodal gaps are superconducting gaps, with the first being uniform and the other modulated. There are two distinct types of order ("two gaps"), but they are both superconducting, and so they can smoothly evolve into one another. Perhaps, as the temperature is decreased, the PDW gradually decreases and the uniform order parameter increases, while their sum (which is determined by relatively high-energy microscopic physics) remains approximately constant. [Note that an early study[74] of modulated structures seen in STM (Refs. 75 and 76) concluded that they could be understood in terms of just such a two-superconducting-gap state.]

More generally, one of the most remarkable features of the pseudogap phenomena is the existence of effects of superconducting fluctuations, detectable[77–79] for instance in the Nernst signal, over a surprisingly broad range of temperatures and doping concentrations. At a broad-brush level,[80] these phenomena are a consequence of a phase stiffness scale that is small compared to the pairing scale. However, it is generally difficult to understand the existence of such a broad fluctuational regime on the basis of any sensible microscopic considerations. The glassy nature of the ordering phenomena in a PDW may hold the key to this central paradox of HTC phenomenology, as it gives rise to an intrinsically broad regime in which superconducting correlations extend over large but not infinite distances.

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APPENDIX A: ORDER PARAMETERS

We will now define the various order parameters introduced in Sec. V and discuss their symmetry properties. The striped superconducting order parameter Δ_Q is a charge $2e$ complex scalar field, carrying momentum Q. To define it microscopically, we write the superconducting order parameter as

$$\Delta(\vec{r},\vec{r}\,') \equiv \langle \psi_\uparrow^\dagger(\vec{r})\psi_\downarrow^\dagger(\vec{r}\,')\rangle = F(\vec{r}-\vec{r}\,')[\Delta_0 + \Delta_Q e^{i\vec{Q}\cdot\vec{R}} + \Delta_{-Q}e^{-i\vec{Q}\cdot\vec{R}}], \quad (A1)$$

where $R=(\vec{r}+\vec{r}\,')/2$, $F(\vec{r}-\vec{r}\,')$ is some short-range function [for a "d-wave-like" striped superconductor, $F(\vec{r})$ changes sign under 90° rotation], and Δ_0 is a uniform order parameter. In the remaining of this appendix, we set $\Delta_0=0$. The effect of Δ_0 is discussed in Appendix B. In cases where a $\Delta_{\bar{Q}}$ is also nonzero (where $\bar{Q}$ is related to Q by a 90° rotation) analogous terms with Δ_Q replaced with $\Delta_{\bar{Q}}$ have to be added.

The order parameters that may couple to Δ_Q and their symmetry properties are as follows. The nematic order parameter N is a real pseudoscalar field; the CDW ρ_K with $K=2Q$ is a complex scalar field; $\vec{S}_Q$ is a neutral spin-vector complex field. All these order parameters are electrically neutral. Under spatial rotation by $\pi/2$, $N\to -N$, $\rho_K \to \rho_{\bar{K}}$, $\vec{S}_Q \to \vec{S}_{\bar{K}}$, and $\Delta_Q \to \pm\Delta_{\bar{Q}}$, where $\pm$ refers to a d-wave or s-wave version of the striped superconductor. Under spatial translation by r, $N\to N$, $\rho_K \to e^{iK\cdot r}\rho_K$, $\vec{S}_Q \to e^{iQ\cdot r}\vec{S}_K$, and $\Delta_Q \to e^{iq\cdot r}\Delta_Q$.

With these considerations, we write all the possible fourth-order terms consistent with symmetry as follows:

$$\begin{aligned}\mathcal{F}_4 = &+ u(\vec{S}_Q\cdot\vec{S}_Q\Delta_Q^\star\Delta_{-Q} + \vec{S}_{\bar{Q}}\cdot\vec{S}_{\bar{Q}}\Delta_{\bar{Q}}^\star\Delta_{-\bar{Q}} + \text{c.c.})\\ &+ [v_+(\vec{S}_Q^\star\cdot\vec{S}_Q + \vec{S}_{\bar{Q}}^\star\cdot\vec{S}_{\bar{Q}}) + \tilde{v}_+(|\rho_K|^2 + |\rho_{\bar{K}}|^2)]\\ &\times(|\Delta_Q|^2 + |\Delta_{-Q}|^2 + |\Delta_{\bar{Q}}|^2 + |\Delta_{-\bar{Q}}|^2)\\ &+ [v_-(\vec{S}_Q^\star\cdot\vec{S}_Q - \vec{S}_{\bar{Q}}^\star\cdot\vec{S}_{\bar{Q}}) + \tilde{v}_-(|\rho_K|^2 - |\rho_{\bar{K}}|^2)]\\ &\times(|\Delta_Q|^2 + |\Delta_{-Q}|^2 - |\Delta_{\bar{Q}}|^2 - |\Delta_{-\bar{Q}}|^2)\\ &+ vN^2[(|\Delta_Q|^2 + |\Delta_{-Q}|^2) + (|\Delta_{\bar{Q}}|^2 + |\Delta_{-\bar{Q}}|^2)]\\ &+ \lambda_+[(|\Delta_Q|^2 + |\Delta_{-Q}|^2)^2 + (|\Delta_{\bar{Q}}|^2 + |\Delta_{-\bar{Q}}|^2)^2]\\ &+ \lambda_-[(|\Delta_Q|^2 - |\Delta_{-Q}|^2)^2 + (|\Delta_{\bar{Q}}|^2 - |\Delta_{-\bar{Q}}|^2)^2] + \cdots,\end{aligned} \quad (A2)$$

where we have explicitly shown all the terms involving Δ_Q, while the terms $\cdots$ represent the remaining quartic terms all of which, with the exception of those involving N, are exhibited explicitly in Ref. 50.

On physical grounds, we have some information concerning the sign of various terms in $\mathcal{F}_4$. The term proportional to u determines the relative phase of the spin and superconducting stripe order—we believe $u>0$ which thus favors a $\pi/2$ phase shift between the SDW and the striped superconducting orders; i.e., the peak of the superconducting order occurs where the spin order passes through zero. The other interesting thing about this term is that it implies an effective cooperativity between spin and striped superconducting orders. The net effect, i.e., whether spin and striped superconducting orders cooperate or fight, is determined by the sign of $|u|-v_+-v_-$, such that they cooperate if $|u|-v_+-v_->0$ and oppose each other if $|u|-v_+-v_-<0$. It is an interesting possibility that spin order and superconducting stripe order can actually favor each other even with all "repulsive" interactions. The term proportional to λ_- determines whether the superconducting stripe order tends to be real ($\lambda_->0$), with a superconducting order that simply changes sign as a function of position, or a complex spiral, which supports ground-state currents ($\lambda_-<0$).

APPENDIX B: COEXISTING UNIFORM AND STRIPED ORDER PARAMETERS

In this appendix, we analyze the coupling of a striped superconducting order parameter Δ_Q to a uniform order parameter Δ_0. In this case, we have to consider in addition to the order parameters introduced in Sec. V a CDW order parameter with wave vector Q, denoted by ρ_Q. The additional cubic terms in the Ginzburg-Landau free energy are

$$\begin{aligned}\mathcal{F}_{3,u} = &\gamma_Q\Delta_0^\star[\rho_Q\Delta_{-Q} + \rho_Q^\star\Delta_Q + \rho_{\bar{Q}}\Delta_{-\bar{Q}} + \rho_{\bar{Q}}^\star\Delta_{\bar{Q}}] + \text{c.c.}\\ &+ g_\rho[\rho_{2Q}^\star\rho_Q^2 + \rho_{2\bar{Q}}^\star\rho_{\bar{Q}}^2 + \text{c.c.}].\end{aligned} \quad (B1)$$

Equation (B1) shows that if both Δ_0 and Δ_Q are nonzero, there is necessarily a coexisting nonzero ρ_Q through the γ_Q term. The additional quartic terms involving Δ_0 are

$$\mathcal{F}_{4,u} = u_\Delta(\Delta_0^{\star 2}\Delta_Q\Delta_{-Q} + \Delta_0^{\star 2}\Delta_{\bar{Q}}\Delta_{-\bar{Q}} + \text{c.c.}) + \delta|\Delta_0|^2[|\Delta_Q|^2 + |\Delta_{\bar{Q}}|^2] + |\Delta_0|^2[u_\rho(|\rho_Q|^2 + |\rho_{\bar{Q}}|^2) + u'_\rho(|\rho_{2Q}|^2 + |\rho_{2\bar{Q}}|^2)] + v'|\Delta_0|^2[\vec{S}^\star_Q \cdot \vec{S}_Q + \vec{S}^\star_{\bar{Q}} \cdot \vec{S}_{\bar{Q}}]. \tag{B2}$$

Let us now consider the effect of quenched disorder. Following the discussion preceding Eq. (5.4), we write the order parameters in real space as $\Delta(r)=|\Delta_0|e^{i\theta_0} + |\Delta_Q|e^{i(\theta_Q+Q\cdot r)} + |\Delta_{-Q}|e^{i(\theta_{-Q}-Q\cdot r)}$ and $\rho(r)=|\rho_Q|\cos(Q\cdot r+\phi_Q) + |\rho_{2Q}|\cos(2Q\cdot r+\phi)$. Let us assume that the disorder nucleates a point defect in the CDW, which in this case corresponds to a 2π vortex in the phase ϕ_Q. By the g_ρ term in Eq. (B1), this induces a 4π vortex in ϕ. (Note that a in the presence of ρ_Q, a 2π vortex in ϕ is not possible.) The γ_Δ term in Eq. (5.3) then dictates a 2π vortex in the phase $\theta_- = (\theta_Q - \theta_{-Q})/2$. However, unlike before, this vortex does not couple to the global superconducting phase $\theta_+ = (\theta_Q + \theta_{-Q})/2$. Therefore, an arbitrarily small uniform superconducting component is sufficient to remove the sensitivity of a striped superconductor to disorder, and the system is expected to behave more or less like a regular (uniform) superconductor.

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[1] J. R. Schrieffer, *Theory of Superconductivity* (Addison-Wesley, Redwood City, CA, 1964).

[2] H.-D. Chen, O. Vafek, A. Yazdani, and S.-C. Zhang, Phys. Rev. Lett. **93**, 187002 (2004); L. Balents, Lorenz Bartosch, Anton Burkov, Subir Sachdev, and K. Sengupta, Phys. Rev. B **71**, 144508 (2005); A. Melikyan and Z. Tesanovic, *ibid.* **71**, 214511 (2005).

[3] The term pair-density-wave has been used in the literature to describe two different states. In Ref. 2 it was used to describe a state with coexisting superconducting and CDW orders. While there may be limits where this term conveys a useful intuitive picture of the microscopic nature of the state, it is not a truly distinct state since it has a nonvanishing uniform component of the SC order parameter and the period of the modulations of the SC order is the same as that of the CDW. The PDW state described here is the same as that of Ref. 22, namely, a state in which the superconducting order parameter is modulated (such that its average vanishes exactly).

[4] P. Fulde and R. A. Ferrell, Phys. Rev. **135**, A550 (1964); A. I. Larkin and Y. N. Ovchinnikov, Zh. Eksp. Teor. Fiz. **47**, 1136 (1964) [Sov. Phys. JETP **20**, 762 (1965)].

[5] Close in spirit to the present work is the staggered superconducting state found in a study of the one-dimensional (1D) Kondo lattice (Ref. 6), and in various studies of the superconducting state with odd-frequency pairing (Ref. 7). There have also been some interesting papers on a somewhat different but very attractive mechanism of PDW formation in systems with multiple Fermi surfaces and an appropriate nesting condition (Refs. 8 and 9). Recent work found evidence for coexistence of superconducting correlations and incommensurate charge order in the 1D Hubbard model with bond-charge interaction (Ref. 10).

[6] O. Zachar, Phys. Rev. B **63**, 205104 (2001).

[7] A. V. Balatsky and E. Abrahams, Phys. Rev. B **45**, 13125 (1992); E. Abrahams, A. Balatsky, J. R. Schrieffer, and P. B. Allen, *ibid.* **47**, 513 (1993); P. Coleman, E. Miranda, and A. Tsvelik, *ibid.* **49**, 8955 (1994).

[8] T. Datta, arXiv:0805.4028 (unpublished).

[9] K. Kubo, J. Phys. Soc. Jpn. **77**, 043702 (2008).

[10] A. A. Aligia, A. Anfossi, L. Arrachea, C. Degli Esposti Boschi, A. O. Dobry, C. Gazza, A. Montorsi, F. Ortolani, and M. E. Torio, Phys. Rev. Lett. **99**, 206401 (2007).

[11] S.-C. Zhang, Science **275**, 1089 (1997).

[12] S.-C. Zhang, J. Phys. Chem. Solids **59**, 1774 (1998).

[13] The SO(5) spiral generalizes the notion of an antiferromagnetic [SO(3)] spiral and is for the same reasons (Ref. 14) relatively difficult to realize in any microscopic model in the absence of spin-orbit coupling.

[14] L. P. Pryadko, S. A. Kivelson, V. J. Emery, Y. B. Bazaliy, and E. A. Demler, Phys. Rev. B **60**, 7541 (1999).

[15] One can also imagine similar interplay between PDW order and more exotic states, such as DDW (Ref. 16), which follows similar lines as the interplay with SDW order.

[16] S. Chakravarty, R. B. Laughlin, D. K. Morr, and C. Nayak, Phys. Rev. B **63**, 094503 (2001).

[17] E. Berg, C.-C. Chen, and S. A. Kivelson, Phys. Rev. Lett. **100**, 027003 (2008).

[18] S. Baruch and D. Orgad, Phys. Rev. B **77**, 174502 (2008).

[19] B. Spivak, P. Oreto, and S. A. Kivelson, Phys. Rev. B **77**, 214523 (2008).

[20] E. Berg, E. Fradkin, E.-A. Kim, S. A. Kivelson, V. Oganesyan, J. M. Tranquada, and S.-C. Zhang, Phys. Rev. Lett. **99**, 127003 (2007).

[21] Q. Li, M. Hücker, G. D. Gu, A. M. Tsvelik, and J. M. Tranquada, Phys. Rev. Lett. **99**, 067001 (2007).

[22] D. F. Agterberg and H. Tsunetsugu, Nat. Phys. **4**, 639 (2008).

[23] J. M. Tranquada, G. D. Gu, M. Hücker, Q. Jie, H.-J. Kang, R. Klingeler, Q. Li, N. Tristan, J. S. Wen, G. Y. Xu, Z. J. Xu, J. Zhou, and M. v. Zimmermann, Phys. Rev. B **78**, 174529 (2008).

[24] J. M. Tranquada, B. J. Sternlieb, J. D. Axe, Y. Nakamura, and S. Uchida, Nature (London) **375**, 561 (1995).

[25] For a review, see S. A. Kivelson, I. P. Bindloss, E. Fradkin, V. Oganesyan, J. Tranquada, A. Kapitulnik, and C. Howald, Rev. Mod. Phys. **75**, 1201 (2003).

[26] A. A. Schafgans A. D. LaForge, S. V. Dordevic, M. M. Qazilbash, Seiki Komiya, Yoichi Ando, and D. N. Basov (unpublished).

[27] Static charge-stripe order onsets at 54 K at the low-temperature tetragonal/low-temperature orthorhombic (LTT)/(LTO) transition (Ref. 28).

[28] P. Abbamonte, A. Rusydi, S. Smadici, G. D. Gu, G. A. Sawatzky, and D. L. Feng, Nat. Phys. **1**, 155 (2005).

[29] T. Valla, A. V. Fedorov, J. Lee, J. C. Davis, and G. D. Gu,

Science **314**, 1914 (2006).
[30] R.-H. He, K. Tanaka, S.-K. Mo, T. Sasagawa, M. Fujita, T. Adachi, N. Mannella, K. Yamada, Y. Koike, Z. Hussain, and Z.-X. Shen, Nat. Phys. **5**, 119 (2009).
[31] S. Chakravarty (private communication).
[32] B. I. Spivak and S. A. Kivelson, Phys. Rev. B **43**, 3740 (1991).
[33] A. V. Rozhkov and D. P. Arovas, Phys. Rev. Lett. **82**, 2788 (1999).
[34] J. A. van Dam, Y. V. Nazarov, E. P. A. M. Bakkers, S. De Franceschi, and L. P. Kouwenhoven, Nature (London) **442**, 667 (2006).
[35] A. V. Andreev, A. I. Buzdin, and R. M. Osgood III, Phys. Rev. B **43**, 10124 (1991).
[36] B. M. Andersen, I. V. Bobkova, P. J. Hirschfeld, and Yu. S. Barash, Phys. Rev. Lett. **96**, 117005 (2006); B. M. Andersen, Y. S. Barash, S. Graser, and P. J. Hirschfeld, Phys. Rev. B **77**, 054501 (2008).
[37] D. A. Wollman, D. J. Van Harlingen, W. C. Lee, D. M. Ginsberg, and A. J. Leggett, Phys. Rev. Lett. **71**, 2134 (1993).
[38] C. C. Tsuei, J. R. Kirtley, C. C. Chi, Lock See Yu-Jahnes, A. Gupta, T. Shaw, J. Z. Sun, and M. B. Ketchen, Phys. Rev. Lett. **73**, 593 (1994).
[39] M. Beasley (private communication).
[40] S. R. White and D. J. Scalapino, Phys. Rev. Lett. **80**, 1272 (1998).
[41] A. Himeda, T. Kato, and M. Ogata, Phys. Rev. Lett. **88**, 117001 (2002).
[42] M. Raczkowski, M. Capello, D. Poilblanc, R. Frésard, and A. M. Oleś, Phys. Rev. B **76**, 140505(R) (2007).
[43] M. Capello, M. Raczkowski, and D. Poilblanc, Phys. Rev. B **77**, 224502 (2008).
[44] V. Oganesyan, S. A. Kivelson, T. H. Geballe, and B. Moyzhes, Phys. Rev. B **65**, 172504 (2002).
[45] The one sort of counterexample to this general statement that we have encountered is the case in which F_L and F_R are sufficiently different from each other, *and* that in at least one of the superconducting regions Δ changes sign as a function of $\vec{k}$. (See, e.g., the system described in Sec. III B.) Clearly, even if Δ changes sign, in cases in which the left and right superconductors are identical, $J_1 > 0$.
[46] When $\rho\Delta \sim 1$, the basis for the various expressions for J_n are violated, as it is then no longer well justified to treat the density of states as a constant. However, this should not change the qualitative conclusions.
[47] W. S. Lee, I. M. Vishik, K. Tanaka, D. H. Lu, T. Sasagawa, N. Nagaosa, T. P. Devereaux, Z. Hussain, and Z. X. Shen, Nature (London) **450**, 81 (2007).
[48] M. R. Norman, H. Ding, M. Randeria, J. C. Campuzano, T. Yokoya, T. Takeuchi, T. Takahashi, T. Mochiku, K. Kadowaki, P. Guptasarma, and D. G. Hinks, Nature (London) **392**, 157 (1998).
[49] A. Kanigel, M. R. Norman, M. Randeria, U. Chatterjee, S. Suoma, A. Kaminski, H. M. Fretwell, S. Rosenkranz, M. Shi, T. Sato, T. Takahashi, Z. Z. Li, H. Raffy, K. Kadowaki, D. Hinks, L. Ozyuzer, and J. C. Campuzano, Nat. Phys. **2**, 447 (2006).
[50] O. Zachar, S. A. Kivelson, and V. J. Emery, Phys. Rev. B **57**, 1422 (1998).
[51] A. Larkin, Zh. Eksp. Teor. Fiz. **58**, 1466 (1970) [Sov. Phys. JETP **31**, 784 (1970)].
[52] The possibility of half vortices in a striped superconductor and their effect on the phase diagram in the clean case was discussed by Agterberg and Tsunetsugu (Ref. 22).
[53] E. S. C. Ching, P. T. Leung, A. Maassen van den Brink, W. M. Suen, S. S. Tong, and K. Young, Rev. Mod. Phys. **70**, 1545 (1998).
[54] T. Giamarchi and P. LeDoussal, Phys. Rev. B **52**, 1242 (1995).
[55] D. S. Fisher, Phys. Rev. Lett. **78**, 1964 (1997); Chen Zeng, P. L. Leath, and Daniel S. Fisher, *ibid.* **82**, 1935 (1999).
[56] M. J. P. Gingras and D. A. Huse, Phys. Rev. B **53**, 15193 (1996).
[57] J. A. Robertson, S. A. Kivelson, E. Fradkin, A. C. Fang, and A. Kapitulnik, Phys. Rev. B **74**, 134507 (2006); A. Del Maestro, B. Rosenow, and S. Sachdev, *ibid.* **74**, 024520 (2006).
[58] See for instance, M. Tinkham, *Introduction to Superconductivity* (Dover, New York, 1996).
[59] K. S. Raman, V. Oganesyan, and S. L. Sondhi, (unpublished) considered the statistical mechanics of a magnetically coupled layered superconductor. The problem they were interested in differs from the one discussed here in that the vortices can appear anywhere, provided the condition of charge neutrality is satisfied on each plane. Instead, in the problem discussed here, the half vortices are quenched at the impurity coordinates and only the sign of their vorticity is allowed to fluctuate. Although the interactions are similar, the case at hand is driven by the effects of disorder leading to a massively frustrated state.
[60] Related models have been considered in the context of granular *d*-wave superconductors. See S. V. Panyukov and A. D. Zaikin, Physica B **203**, 527 (1994).
[61] Ruixing Liang, D. A. Bonn, and W. N. Hardy, Phys. Rev. B **73**, 180505(R) (2006).
[62] Y. Ando, K. Segawa, S. Komiya, and A. N. Lavrov, Phys. Rev. Lett. **88**, 137005 (2002).
[63] D. LeBoeuf, N. Doiron-Leyraud, J. Levallois, R. Daou, J.-B. Bonnemaison, N. E. Hussey, L. Balicas, B. J. Ramshaw, R. Liang, D. A. Bonn, W. N. Hardy, S. Adachi, Cyril Proust, and L. Taillefer, Nature (London) **450**, 533 (2007); L. Taillefer (private communication).
[64] E. Arrigoni, E. Fradkin, and S. A. Kivelson, Phys. Rev. B **69**, 214519 (2004).
[65] S. A. Kivelson, E. Fradkin, and V. J. Emery, Nature (London) **393**, 550 (1998).
[66] C. S. O'Hern, T. C. Lubensky, and J. Toner, Phys. Rev. Lett. **83**, 2745 (1999).
[67] E. Berg, S. A. Kivelson, E. Fradkin, and J. M. Tranquada (unpublished).
[68] J. F. Ding, X. Q. Xiang, Y. Q. Zhang, H. Liu, and X. G. Li, Phys. Rev. B **77**, 214524 (2008); K. Fujita, T. Noda, K. M. Kojima, H. Eisaki, and S. Uchida, Phys. Rev. Lett. **95**, 097006 (2005).
[69] B. Lake, H. M. Rønnow, N. B. Christensen, G. Aeppli, K. Lefmann, D. F. McMorrow, P. Vorderwisch, P. Smeibidl, N. Mangkorntong, T. Sasagawa, M. Nohara, H. Takagi, and T. E. Mason, Nature (London) **415**, 299 (2002).
[70] Recent experiments suggest that an FFLO state, a close relative of the PDW, may be present in the $CeCoIn_5$ heavy-fermion superconductor (Refs. 71 and 72) and also in κ-$(BEDT\text{-}TTF)_2Cu(NCS)_2$ organic superconductors (Ref. 73). The phenomenology of the PDW discussed here, particularly the sensitivity to disorder, should then apply to these systems as well.
[71] A. Bianchi, R. Movshovich, C. Capan, P. G. Pagliuso, and J. L. Sarrao, Phys. Rev. Lett. **91**, 187004 (2003).
[72] M. Kenzelmann, Th. Strässle, C. Niedermayer, M. Sigrist, B. Padmanabhan, M. Zolliker, A. D. Bianchi, R. Movshovich,

E. D. Bauer, J. L. Sarrao, and J. D. Thompson, Science **321**, 1652 (2008).
[73] R. Lortz, Y. Wang, A. Demuer, P. H. M. Böttger, B. Bergk, G. Zwicknagl, Y. Nakazawa, and J. Wosnitza, Phys. Rev. Lett. **99**, 187002 (2007).
[74] D. Podolsky, E. Demler, K. Damle, and B. I. Halperin, Phys. Rev. B **67**, 094514 (2003).
[75] C. Howald, H. Eisaki, N. Kaneko, and A. Kapitulnik, Proc. Natl. Acad. Sci. U.S.A. **100**, 9705 (2003); C. Howald, H. Eisaki, N. Kaneko, M. Greven, and A. Kapitulnik, Phys. Rev. B **67**, 014533 (2003).
[76] J. E. Hoffman, E. W. Hudson, K. M. Lang, V. Madhavan, H. Eisaki, S. Uchida, and J. C. Davis, Science **295**, 466 (2002); J. E. Hoffman, K. McElroy, D.-H. Lee, K. M. Lang, H. Eisaki, S. Uchida, and J. C. Davis, *ibid.* **297**, 1148 (2002).
[77] Yayu Wang, Z. A. Xu, T. Kakeshita, S. Uchida, S. Ono, Yoichi Ando, and N. P. Ong, Phys. Rev. B **64**, 224519 (2001).
[78] D. Podolsky, S. Raghu, and A. Vishwanath, Phys. Rev. Lett. **99**, 117004 (2007).
[79] V. Oganesyan, D. A. Huse, and S. L. Sondhi, Phys. Rev. B **73**, 094503 (2006).
[80] V. J. Emery and S. A. Kivelson, Nature (London) **374**, 434 (1995).