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Theory of microwave superradiance from a Bose-Einstein condensate of magnons

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We show that the nearly uniform mode generated by the confluence of Bose-Einstein condensate (BEC) magnon pairs produced by microwave radiation in a film of yttrium iron garnet (YIG) is a coherent magnon state. This state corresponds to a macroscopic precessing magnetization which emits a superradiant microwave signal as a result of the cooperative action of the spins. The theory explains quantitatively recent experimental observations of microwave emission from a BEC of magnons in a YIG film when the driving power exceeds a critical value.

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In a recent series of papers Demokritov *et al.*[1–3] reported the experimental observation of quasiequilibrium Bose-Einstein condensation (BEC) of magnons at room temperature in films of yttrium iron garnet (YIG) driven by microwave radiation. When magnetized in the plane these films have a dispersion relation (frequency ω_k versus wave vector k) for magnons propagating along the field that exhibits a minimum frequency $\omega_{k0}=2\pi f_{\min}$ at $k_0 \sim 10^5$ cm^{-1}. Magnon pairs with opposite wave vectors pumped parametrically in large numbers by the microwave field are redistributed over the spectrum creating a hot magnon gas that remains decoupled from the lattice for some time due to the relatively slow spin-lattice relaxation. The experiments show that if the microwave power exceeds a certain threshold value the intensity of the light scattered by magnon with frequency ω_{k0} increases by orders of magnitude indicating the formation of a Bose-Einstein condensate.[1–3]

We have shown[4] that the cooperative action in an interacting magnon gas in a YIG film creates a coherent magnon state at ω_{k0} when the magnon population exceeds a critical density. The coherent state corresponds to a macroscopic precessing magnetization providing theoretical support for the experimentally observed Bose-Einstein condensate of magnons. Since the wavelength of the associated spin wave is on the order of 10^{-5} cm, the direct microwave radiation from the BEC is negligible. However, as observed by Dzyapko *et al.*,[5] if the applied in-plane static field has a value such that the frequency of the $k\approx 0$ magnon is $\omega_0=2\omega_{k0}$, a microwave radiation is generated by $k\approx 0$ magnons created by pairs of BEC magnons $k_0,-k_0$ through a three-magnon confluent process. The $k\approx 0$ value is necessary for emission because the wave number of electromagnetic radiation with frequency of 1.5 GHz, as in the experiments,[5] is $k=2\pi/f \approx 0.3$ cm^{-1}. In this Rapid Communication we show that the $k\approx 0$ magnons created by the BEC are coherent magnons states, which correspond to a nearly uniform magnetization precessing with frequency ω_0 and generating a microwave signal. The microwave emitted by the collective action of the spins is identified with superradiance. We show that this process occurs when the driving microwave power exceeds a critical value and that the theoretical results for the emitted signal power agree very well with the experimental data.[5]

The term superradiance was introduced by Dicke[6] in 1954 to designate the type of spontaneous emission of radiation from an assembly of N atoms that has as an intensity proportional to N^2 instead of N as in usual situations. This emission required some kind of quantum coherence in the atomic states, a topic which only many years later was treated formally. Shortly after Dicke's paper appeared it was shown that the magnetic dipole emission from a collection of spins naturally leads to superradiance if the spin system is treated semiclassically.[7] The observation of macroscopic superradiance of microwaves in ferromagnetic resonance in YIG was achieved only in the 1970s.[8] The recent paper by Dzyapko *et al.*[5] reports on the observation of superradiance originating from a Bose-Einstein condensate. The total power radiated by a magnetization precessing about the static field with frequency ω_0 is given by[9]

$$P=\frac{2N^2\Omega^2\omega_0^4}{3c^3}(m_x^2+m_y^2), \tag{1}$$

where N is the number of spins in the region of emission, Ω is the volume of the spin unit cell, c is the speed of light, and m_x and m_y are the small-signal components of the transverse magnetization. In Eq. (1) we have written the volume of the sample as $V=N\Omega$ to stress the dependence of the radiated power on the square of the number of spins. In this Rapid Communication we employ a quantum treatment of the magnetic excitations considering that they are spin waves with frequency ω_k and wave vector k described by magnon creation and annihilation operators c_k^+ and c_k with an unperturbed Hamiltonian $H_0=\hbar\Sigma_k\omega_k c_k^+ c_k$. As is well known the eigenstates $|n_k\rangle$ of the Hamiltonian H_0 and of the number operator $n_k=c_k^+c_k$ can be obtained by applying integral powers of the creation operator to the vacuum. These stationary states have precisely defined number of magnons n_k and uncertain phase. They are used in nearly all quantum treatments of thermodynamic properties, relaxation mechanisms, and other phenomena involving magnons. However, they have zero expectation value for the small-signal transverse magnetization operators m_x and m_y and consequently cannot generate microwave radiation. The states that correspond to classical spin waves are the coherent magnon states,[10,11] defined in analogy to the coherent photon states introduced by Glauber.[12] A coherent magnon state is the eigenket of the circularly polarized magnetization operator $m^+=m_x+im_y$. It can be written as the direct product of single-mode coherent states, defined as the eigenstates of the annihilation operator $c_k|\alpha_k\rangle=\alpha_k|\alpha_k\rangle$, where the eigenvalue α_k is a complex number. Although the coherent states are not eigenstates of the unperturbed Hamiltonian and as such do not have a well-defined number of magnons, they have nonzero expectation values for the magnetization m^+ with a well-defined phase. The coherent state $|\alpha_k\rangle$ can be expanded in terms of the eigenstates $|n_k\rangle$ and has an expectation value for the number operator given by $\langle n_k\rangle = |\alpha_k|^2$. The coherent states are not orthogonal to one another, but they form a complete set, so they can be used as a basis for the expansion of an arbitrary state. Another important property is that a coherent state can be generated by the application of a displacement operator to the vacuum,[10–12]

$$|\alpha_k\rangle = D(\alpha_k)|0\rangle, \tag{2}$$

where

$$D(\alpha_k) = \exp(\alpha_k c_k^+ - \alpha_k^* c_k). \tag{3}$$

The theory presented here considers a Hamiltonian that contains contributions from the exchange and dipolar interactions between the spins as well as Zeeman interactions with the static field H applied in the plane of the film and with the parallel-pumping microwave magnetic field. Expressing the spin operators in terms of the magnon operators, one can write the Hamiltonian in the form[9,11,13–15]

$$H = H_0 + H^{(3)} + H^{(4)} + H'(t), \tag{4}$$

where H_0 is the unperturbed Hamiltonian, $H^{(3)}$ and $H^{(4)}$ represent the three- and four-magnon interactions arising from the nonlinear magnetic interaction, and $H'(t)$ describes the microwave driving. In another paper[4] we show that the BEC magnons created by the cooperative action of the magnon hot gas through the four-magnon interaction are represented by pairs of coherent magnon states with wave vectors in a narrow range around $k_0, -k_0$. Here we initially consider only the process by which the $k \approx 0$ mode is generated by the confluence of the BEC magnon pair $k_0, -k_0$. Later we will calculate the number of magnons as a function of the driving microwave power to compare the theory with experiments. The Hamiltonian for the three-magnon confluence process can be written as[15]

$$H^{(3)} = \hbar V_{(3)} c_0^+ c_{k0} c_{-k0} + \text{H.c.}, \tag{5}$$

where the vertex of the interaction for small wave vectors is dominated by the dipolar interaction between the spins S and is given by $V_{(3)} = \omega_M/(2SN)^{1/2}$, where $\omega_M = \gamma 4\pi M$, γ is the gyromagnetic ratio, and M is the magnetization. In Eq. (5) we can take quasiaverages of pairs of destruction operators for BEC magnons to form a correlation function $\sigma_{k0} = \langle c_{k0} c_{-k0}\rangle = n_{k0} e^{i\varphi_{k0}} e^{-i2\omega_{k0}t}$, where n_{k0} is the magnon number and φ_{k0} is the phase between the states of the pair. We assume that interaction (5) is turned on at $t=0$ when the BEC magnons are excited, so that the Hamiltonian for the system is $H = H_0 + H^{(3)}(t)$, where

$$H^{(3)}(t) = \theta(t)[g_0(t)c_0 + g_0^*(t)c_0^+], \tag{6a}$$

$$g_0(t) = \hbar V_{(3)} n_{k0} e^{i\varphi_{k0}} e^{-i2\omega_{k0}t}, \tag{6b}$$

and $\theta(t)$ is the Heaviside step function. In the Schrödinger picture the state $|t\rangle$ at an instant t is related to its value at $t=0$ through the time-evolution operator $|t\rangle = U(t,0)|0\rangle$. To find the evolution operator $U(t,0)$ we use the interaction picture and write $U = U^0 U^I$, where $U^0(t,0) = \exp(-iH_0 t/\hbar)$. The intermediate evolution operator U^I is governed by $i\hbar dU^I/dt = H^I U^I$, where

$$H^I(t) = \theta(t)[g_0(t)e^{-i\omega_0 t}c_0 + \text{H.c.}]. \tag{7}$$

As the commutator of H^I with itself at different times is a c-number function, an explicit solution for U^I can be found leading to[16]

$$U(t,0) = \exp(-iH_0 t/\hbar + i\beta)\exp(\gamma_0 c_0^+ - \gamma_0^* c_0), \tag{8}$$

where β is an arbitrary phase resulting from an integration constant and the steady-state form of $\gamma_k(t)$ is obtained with

$$\gamma_0(t) = \frac{1}{i\hbar}\int_0^t g_0^*(t')e^{-i\omega_0 t'}dt' \Rightarrow \frac{-g_0^*(t=0)}{\hbar(\omega_0 - 2\omega_{k0})} e^{i(\omega_0 - 2\omega_{k0})t}. \tag{9}$$

The second exponential function in Eq. (8) is the displacement operator (3), so that at time t the state is a coherent magnon with wave vector $k=0$, $|t\rangle = |\alpha_0\rangle \exp(-i2\omega_{k0}t)$, where the eigenvalue is given by $\alpha_0 = \gamma_0^*(t=0)$. Using Eq. (9) and introducing the relaxation rate η_0 for the magnon $k=0$ by means of $\omega_0 \to \omega_0 + i\eta_0$, one can write the eigenvalue of the coherent state as

$$\alpha_0 = \frac{-\omega_M e^{-i\varphi_{k0}}}{(2SN)^{1/2}(2\omega_{k0} - \omega_0 - i\eta_0)} n_{k0}, \tag{10}$$

where we have used $\beta = \pi$ appropriate for steady state as will be shown later. This result has several implications, namely, the $k=0$ mode generated by the confluence of a BEC coherent magnon pair $k_0, -k_0$ is a quantum coherent magnon state. Thus it corresponds to a macroscopic magnetization precessing with frequency $2\omega_{k0}$ and produces superradiance with this frequency. Since the amplitude of the magnetizations m_x and m_y is proportional to the eigenvalue (10), the emitted microwave power given by Eq. (1) is proportional to $n_0 = |\alpha_0|^2$. Thus, according to Eq. (10) the emitted power is proportional to n_{k0}^2. As shown in Ref. 4 the number of BEC magnons n_{k0} is above thermal value only when the microwave power exceeds a critical value p_{c2}. However, the confluent process itself has no threshold condition; the BEC magnon pairs $k_0, -k_0$ with any value of n_{k0} will create $k=0$ magnons. This is in contrast to the so-called subsidiary resonance instability process[15,17] in which the three-magnon splitting process occurs only if the microwave field exceeds a critical value. The amplitude of the $k=0$ mode is maximum at the resonance condition $\omega_0 = 2\omega_{k0}$, which in the experiments of Ref. 5 is attained for an external field $H=520$ Oe.

As shown in Ref. 4 when the confluent process is not effective, as is the case when ω_0 is far apart from $2\omega_{k0}$, the number of BEC magnons generated in the microwave driven YIG film varies with microwave pumping power p above a critical value p_{c2} as $n_{k0} \propto (p - p_{c2})^{1/2}$. If n_{k0} had this same dependence here, Eqs. (1) and (10) would predict a microwave signal power p_s continuously increasing with p as $p_s \propto (p - p_{c2})$. This is very different from the experimental data of Ref. 5 which exhibit a fast initial increase in p_s with p

above the critical power, followed by a rapid saturationlike behavior. This indicates that other effects must be introduced in the model to account for the experimental observations. Two effects have been found to be very important in the theory, namely, the counter-reaction of the $k=0$ mode on the BEC magnons and, most importantly, the detuning from resonance caused by the frequency shift of the $k=0$ magnon due to the effect of the large population created by the BEC modes on the nonlinear interaction.

The complete theory presented here to explain the experiments of Ref. 5 is based on the full Hamiltonian given by Eq. (4) with the nonlinear four-magnon interaction represented by[13–15]

$$H^{(4)} = \hbar \sum_{k,k'} \left(\frac{1}{2} S_{kk'} c_k^+ c_{-k}^+ c_{k'} c_{-k'} + T_{kk'} c_k^+ c_{k'}^+ c_k c_{k'} \right), \quad (11)$$

where the interaction coefficients for small k values are dominated by the dipolar energy. In films the coefficients contain several terms multiplied by a factor F_k involving the film thickness d and the wave numbers.[18] For the values relevant to the experiments[1–3] the product kd is on the order of 10, so that F_k is negligible and the coefficients are given approximately by the bulk values $S_{kk'}=2T_{kk'}=2\omega_M/NS$. The following sequence of events for the formation of the quasi-equilibrium BEC has been observed experimentally[1–3] and confirmed theoretically.[4] The microwave parallel-pumping magnetic field with frequency f_p (8.1 GHz in the experiments of Refs. 1–3) excites parametrically magnon pairs with frequencies in a narrow range around $f_p/2$ and wave vectors $\vec{k},-\vec{k}$ with directions determined by the coupling parameter and magnitudes given by the dispersion relation. Since the critical power for this process is low ($\sim 10^{-4}-10^{-3}$ W), the primary magnons are excited in large numbers at the power levels used in the experiments (a few watts). The population of the primary magnons quickly spreads over a broad frequency range leading to the formation of a hot magnon gas as a result of four-magnon scattering events that conserve the number of magnons. Since the spin-lattice relaxation time is relatively long the magnon gas remains decoupled from the lattice for about 500 ns. The theory assumes that the population of the primary magnons is distributed among the k_R modes in the magnon reservoir with frequency below $f_p/2$ according to the Bose-Einstein statistics with a chemical potential μ appropriate for a system with constant number of particles.[1] The theory also assumes that the correlation between the phases of the magnon pairs lasts for a time $t\sim 4/\eta_m$, where η_m is the magnetic relaxation rate. Note that if all the reservoir states had the same magnon number the sum in k_R in Eq. (11) would reproduce the density of states with a peak at the frequency ω_{k0}. As the microwave pumping power is increased, the number of magnons in the gas increases and so does the chemical potential. When the power is close to p_{c2}, μ is close to $\hbar\omega_{k0}$, so that the Bose-Einstein distribution has a peak at ω_{k0} which combined with the peak in the density of states results in a coefficient in the Hamiltonian (11) strongly dominated by the modes with $\omega_k \sim \omega_{k0}$. Thus if the microwave power exceeds a critical value the modes in the hot magnon gas with frequency $\omega_k \sim \omega_{k0}$ drive BEC magnon pairs $\vec{k}_0$, $-\vec{k}_0$ by a cooperative action through the four-magnon interaction. In Ref. 4 it has been shown that the effective Hamiltonian for driving $\vec{k}_0,-\vec{k}_0$ magnon pairs can be written as

$$H'_{\text{eff}}(t) \cong \hbar(h\rho)_{\text{eff}} e^{-i2\omega_{k0}t} c_{k_0}^+ c_{-k_0}^+ + \text{H.c.}, \quad (12)$$

where $(h\rho)_{\text{eff}} \cong -i\eta_{k0}(p/p_{c2})^{1/2}$ is an effective field determined by the ratio of the microwave pumping power p to the critical value p_{c2} and the relaxation rate of the BEC magnons η_{k0}.[4]

In order to study the process by which the pairs of BEC coherent magnons $\vec{k}_0,-\vec{k}_0$, are produced and then generate $k\sim 0$ modes by three-magnon coalescence, we employ the total Hamiltonian (4) using Eq. (12) as the driving term. The Heisenberg equation is then used to obtain the equations of motion for the operators c_{k0}, c_{-k0}, and c_0. We consider that all states involved are coherent magnon states as demonstrated earlier and work with the corresponding eigenvalues α_k. In addition we assume that there are p_{k0} pair modes with wave vectors equal or close to $\vec{k}_0,-\vec{k}_0$ to drive the $k\sim 0$ modes and that the resonance condition is satisfied, $\omega_0 = 2\omega_{k0}$, as determined by the value of the applied field H. The equations of motion for the eigenvalue α_0 and the correlation function $\sigma_{k0}=\alpha_{k0}\alpha_{-k0}$ in a frame rotating with frequency ω_0 become

$$\frac{d\alpha_0}{dt} = -(\eta_0 + iV_{(4)}n_0)\alpha_0 - ip_{k0}V_{(3)}\sigma_{k0}, \quad (13)$$

$$\frac{d\sigma_{k0}}{dt} = -2(\eta_{k0} + i2V_{(4)}n_{k0})\sigma_{k0} - i2[V_{(3)}\alpha_0/p_{k0} + (h\rho)_{\text{eff}}]n_{k0}, \quad (14)$$

where $V_{(4)}=S_{kk'}+2T_{kk'}=4\omega_M/SN$. Note that in Eq. (14) the term representing the coupling with the $k=0$ mode is divided by the number of modes p_{k0} assumed in the driving because σ_{k0} represents only one pair mode k_0. The coupling term in Eq. (14) represents a reaction of the $k=0$ mode that influences the behavior of the BEC modes. In steady state $d/dt=0$ Eq. (13) leads to

$$\alpha_0 = \frac{-ip_{k0}V_{(3)}\sigma_{k0}}{\eta_0 + iV_{(4)}n_0}. \quad (15)$$

This result is consistent with Eq. (10) considering that here $\omega_0=2\omega_{k0}$ and we have assumed that there are p_{k0} modes to drive the $k=0$ mode. Note that the factor i in Eq. (15) that corresponds to a phase $\beta=\pi$ in Eq. (8) is determined by the form of the interaction between the modes. The presence of the term $iV_{(4)}n_0$ in the denominator of Eq. (15) is a result of the four-magnon interaction and represents a detuning from the resonance condition due to the renormalization of the $k=0$ mode frequency. In fact, this term is responsible for the saturation of the emitted signal with microwave pumping power observed experimentally.[5] As the pumping power increases and the population of the $k=0$ mode becomes very large the detuning from resonance increases causing a saturation in the growth of the $k=0$ mode amplitude. In order to

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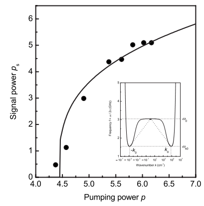

FIG. 1. Microwave emission signal power vs pumping power. Symbols represent the experimental data of Dzyapko *et al.* (Ref. 5) and the solid line is the fit with theory with parameters as in the text. The inset shows the dispersion relation for magnons propagating along the field $H=520$ Oe applied in the plane of a YIG film with thickness of 5 μm.

compare theory with data we have solved the coupled equations (13) and (14) numerically to find the steady-state values of the magnon populations n_0 and n_{k0}. The calculations were done considering that the relaxation of all modes involved is dominated by the magnetic interactions, $\eta_0=\eta_{k0}=\eta_m$. We also use the following normalized variables and parameters: $n'_k=n_k/SN$, $t'=\eta_m t$, $V'_{(3)}=V_{(3)}(SN)^{1/2}/2\eta_m$, $V'_{(4)}=V_{(4)}(SN)/2\eta_m$, and $(h\rho)'_{\text{eff}}=(h\rho)_{\text{eff}}/\eta_m$. With $4\pi M=1.76$ kG and $\eta_m=5\times10^7$ s^{-1} we have $V'_{(3)}=219.0$ and $V'_{(4)}=1440.0$. Since the microwave signal power is a fraction of the total radiated power given by Eq. (1), we use the expression $p_s=Cn'_0$ to fit the data of Dzyapko *et al.*[5] In Fig. 1 the symbols represent the data of Ref. 5 and the solid line represents the theoretical fit with using $C=13.2$ μW, $p_{k0}=5\times10^3$, and $p_{c2}=4.45$ W. The fit is quite good but it is important to check if the values of the fitting parameters bear connection to reality. A good estimate for the number of BEC modes that drive the $k=0$ magnon is obtained by counting the modes with frequency in the range $\omega_{k0}-\omega_{k0}+\eta_m/2$ and with $\vec{k}$ in the z direction of the static field, $k_z=n\pi/L_z$, where n is an integer and L_z is the sample length. The result calculated numerically with the dispersion relation for the YIG film obtained from Ref. 19 is 20×10^3. The value of p_{k0} obtained from the fitting is somewhat smaller than this, which is expected since it represents the number of modes weighted by the number of magnons of the mode k relative to the maximum number at k_0. To calculate the emitted microwave signal we use in Eq. (1) the known relation between the magnetization components and the coherent state eigenvalue,[4] obtaining

$$\langle P\rangle \cong \frac{V^2\omega_0^4M^2}{c^3}n'_0. \tag{16}$$

Using in Eq. (16) $\omega_0=2\pi\times3.0$ GHz, $M=300$ G, $c=3\times10^{10}$ cm/s, and an estimated emission volume $V=1$ mm $\times0.5$ mm$\times5$ μm$=2.5\times10^{-6}$ cm^3, we obtain for the factor of n'_0 in Eq. (16) approximately 400 μW. This is 2 orders of magnitude larger than the value of C obtained from the fit of theory to experiment, which is quite reasonable considering that the measured signal power is only a very small fraction of the total radiated power given by Eq. (16). It is important to note that if Eqs. (13) and (14) are solved considering $p_{k0}=1$, the calculated n'_0 is smaller than the value obtained with $p_{k0}=5\times10^3$ by a factor of 10^7. This means that with $p_{k0}=1$ the total emitted power calculated with Eq. (16) is smaller than the measured signal power by a factor of 10^5, which is completely unrealistic.

In conclusion we have shown that the nearly uniform mode generated by BEC magnon pairs produced by microwave radiation in a film of YIG is a coherent magnon state. This state corresponds to a macroscopic precessing magnetization which emits superradiance as a result of the cooperative action of the spins. The theory explains quantitatively recent experimental observations of Dzyapko *et al.*[5] of microwave emission when the driving power exceeds a critical value. The theoretical results fit very well the data for the emitted signal power versus microwave pumping power with realistic parameters. The theoretical model is also consistent with the 6 MHz linewidth of the microwave emission spectrum observed in Ref. 5. This value was considered too large by the authors of Ref. 5 who expected a linewidth 1 order of magnitude smaller corresponding to the spin-lattice relaxation rate. In fact the linewidth is close to the value determined by the magnetic relaxation rate, $\eta_m/2\pi=8$ MHz, which in our theory dominates decay process.

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