Generation of turbulence by vibrating forks and other structures in superfluid ^{4}He

M. Blažková,[1] D. Schmoranzer,[2] L. Skrbek,[2] and W. F. Vinen[3]

[1]*Institute of Physics, ASCR, v.v.i., Na Slovance 2, 182 21 Prague, Czech Republic*

[2]*Faculty of Mathematics and Physics, Charles University, Ke Karlovu 3, 121 16 Prague, Czech Republic*

[3]*School of Physics and Astronomy, University of Birmingham, Birmingham B15 2TT, United Kingdom*

(Received 28 October 2008; revised manuscript received 6 January 2009; published 20 February 2009)

A study of the drag on the prongs of a number of quartz forks vibrating in the superfluid phase of ^{4}He is reported, and particular attention is paid to the transitions from laminar to turbulent flow over a wide range of temperature. Behavior in the normal phase is consistent with that for a classical fluid, as has already been reported [Phys. Rev. E **75**, 025302 (2007)]. Behavior in the superfluid phase is compared to that of other structures vibrating in superfluid ^{4}He, and similarities and differences are noted. We focus on the observed behavior of the drag coefficient as a function of velocity, and the problems posed by this behavior are explored. There is evidence for a sharp critical velocity at which significant turbulence starts to be generated in the superfluid component. At high velocities the drag coefficient tends to that observed in a classical fluid, suggesting that the two fluids, strongly coupled by mutual friction, are then behaving like a single classical viscous fluid. Behavior in the intermediate region seems to vary from one case to another. Evidence is presented that in the case of some structures the transition to single-fluid behavior takes place rather abruptly at a velocity that is only slightly greater than the sharp superfluid critical velocity, but that in other structures the transition is more gradual. Observed values of both the superfluid critical velocity and the effective viscosity of the fully coupled fluids are presented and discussed. It is suggested that the critical superfluid velocity is always closely similar to that at which the coupled fluids would be expected to undergo a classical transition between a flow that is strictly laminar and one that displays the first instability, and a possible reason is discussed.

DOI: 10.1103/PhysRevB.79.054522 PACS number(s): 67.25.dk, 67.25.dm, 47.27.Cn, 47.37.+q

I. INTRODUCTION

Recently many studies of the behavior of vibrating wires,[1,2] spheres,[3] and grids[4–6] in the superfluid phases of both ^{4}He and ^{3}He were reported. Attention has been focused especially on the change in damping associated with the transition to turbulent flow. Critical velocities have been reported, together with the dependence of the damping on velocity in the supercritical regime; sometimes this transition is accompanied by hysteresis or switching. Important questions in classical turbulence relate to the forms of turbulence generated by flow past various types of obstacle and to the processes by which these forms are produced. The experiments on vibrating structures in superfluids allow us to address analogous questions relating to quantum turbulence. Turbulence in a superfluid, or quantum turbulence,[7] differs from that in a classical fluid for three reasons: except at the lowest temperatures, the superfluid exhibits two-fluid behavior, a normal-fluid component coexisting with a superfluid component—the two fluids being able to support independent velocity fields; the superfluid component can flow without dissipation; and flow of the superfluid is subject to severe quantum restrictions. These restrictions mean that the only form of rotational motion allowed in the superfluid component is a quantized vortex line, in which there is an irrotational circulation equal to $2\pi\hbar/m$ round a thin vortex core, where m is the mass of a ^{4}He atom or two ^{3}He atoms. Turbulent flow of the superfluid component must therefore take the form of some irregular tangle of vortex line. On length scales greater than the spacing between the vortex lines, flow of the superfluid component can mimic that of a classical fluid. However, on smaller length scales, including typically those at which dissipation occurs, the flow must be very different. At a finite temperature any vortex line moving relative to the normal fluid suffers a drag (mutual friction), which can cause motion in the two fluids to become strongly coupled. The study of quantum turbulence combines the challenges we meet in the study of classical turbulence with those associated with quantum phenomena.

The first aim of this paper is to describe observations made with a number of vibrating forks[8] in superfluid ^{4}He at temperatures above 1 K, where there is a significant fraction of normal fluid. The observations are compared to those reported very recently for vibrating forks in ^{4}He at lower temperatures and with those relating to other forms of oscillating structure in ^{4}He. Experimental results are presented in the form of plots of the drag coefficient against velocity. We demonstrate that all these experimental results exhibit certain common features: a sharp critical velocity, with or without hysteresis or switching, associated with the onset of turbulence in the superfluid component; a tendency for the drag coefficient to approach a constant value, of order unity, at high velocities, similar to the behavior observed with classical fluids; and an intermediate region in which the drag coefficient varies with velocity in a way that differs to some extent from one type of oscillating structure to another. We go on to set out the nature of the problems raised by these observations, emphasizing both the similarities to, and the differences from, those relevant to classical fluids. We argue that the observed behavior at high velocities indicates "quasiclassical behavior," in which, even at high temperatures, there is a single turbulent velocity field (arising from coupled motion of the two fluids) that mimics that occurring in a classical fluid. We consider the intermediate region, arguing

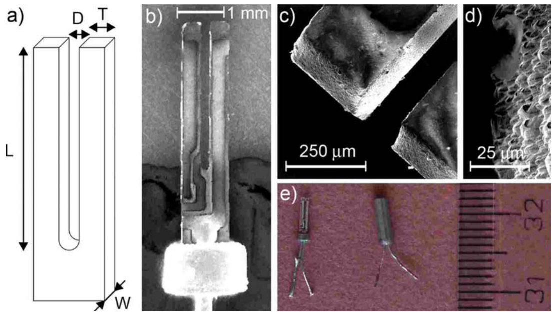

FIG. 1. (Color online) A commercially available quartz tuning fork (Ref. 12). (a) A schematic drawing of the fork; (b) micrograph of the entire fork; (c) micrograph of the ends of its prongs; (d) detail of its surface roughness; (e) fork in its original can and with the can removed.

that for some structures there is a rather sudden transition to quasiclassical behavior, with a more gradual transition for others. We observe that the superfluid critical velocity seems to coincide with the velocity at which flow of the quasiclassical coupled fluids would make a transition between a strictly laminar form and one in which the first instability of that flow is evident, and we suggest a tentative reason. We shall wish to emphasize that the way in which the drag coefficient is observed to vary with velocity, over the whole range of velocities, contains much information about the processes accompanying the vibration of the structure over and above that contained simply in the observed superfluid critical velocity. Our discussion relates only to superfluid ^{4}He; superfluid ^{3}He, even in the B phase, appears to behave differently.[9,10]

There are two different theoretical (or computational) approaches to the generation of quantum turbulence by vibrating structures. In the case of very small structures, when only small numbers of vortex lines may be involved, the evolution of the turbulence might well be examined by computer simulation.[11] However, for larger structures, when large numbers of vortex lines are likely to be involved, especially in fully developed turbulence, computer simulations of the type so far available are inadequate since they cannot cope with large vortex densities.[10] In this latter case, we must rely on general physical arguments, essentially statistical in nature, such as those associated with the concept of an eddy viscosity. Discussion of our experimental results in this paper will be based on such general arguments, so that they may not be relevant to the behavior of the smallest structures. We note that the characteristic dimensions of the forks with which this paper is primarily concerned are significantly larger than those of most other vibrating structures that have been studied in superfluid helium.

II. EXPERIMENTAL RESULTS WITH VIBRATING FORKS IN SUPERFLUID ^{4}He

The forks are made of quartz and are available commercially as frequency standards,[12] typically for 2^{15} Hz ($\sim$33 kHz) at room temperature; those we have used in most of our work have the shapes shown in Fig. 1 and their dimensions are given in Table I. The forks are excited at their resonant frequencies, and we measure the amplitude of the resonant response as a function of the drive force F.[13] The way in which the amplitude of the response and the drive force are derived from the electrical measurements is described in Ref. 8. It is illuminating to plot the results in terms of a drag coefficient, C_D, which is defined by the equation

TABLE I. Dimensions of forks.

Fork	Freq. (kHz)	L (mm)	T (mm)	W (mm)	D (mm)
A1	32	3.71	0.42	0.35	0.21
B1	32	3.65	0.68	0.46	0.18
C3	32	2.51	0.25	0.10	0.13
U1	4	19.70	2.20	0.80	
U2	8	9.50	0.45	0.90	0.50
L2	32	2.17	0.21	0.10	0.12
L1	32	2.17	0.21	0.10	0.12
K1	32	3.9	0.39	0.28	

$$F = \frac{1}{2} C_D \rho A U^2, \qquad (1)$$

where ρ is the appropriate fluid density, $A = LW$ is the area of a prong projected on a plane normal to the motion, and U is the amplitude of the velocity response at the ends of the prongs of the fork. (Strictly speaking we should take into account the fact that the velocity with which any particular part of a prong moves varies along the length of the prong. However, in the overall damping there is a heavy weighting in favor of the region of a prong near its tip.) Figures 2–4 show typical results for three forks (A1, L2, and U1) in the form of plots of C_D against U for different temperatures.

III. DISCUSSION OF EXPERIMENTAL RESULTS

A. Normal phase

Consider first the drag exerted on the prongs of the fork A1 at a temperature of 2.16 K. The fraction of superfluid is then very small, and we can assume that the observed form of dependence of C_D on U is that for a classical fluid with kinematic viscosity, ν, equal to the viscosity of the normal fluid divided by the total fluid density. (We attempted to take data above the lambda transition, but we found that at the highest velocities they were affected by cavitation.[14]) At the angular frequency, ω, appropriate to the fork, the classical viscous penetration depth, $\delta = (2\nu/\omega)^{1/2}$, is small compared to all dimensions of a prong. We can then expect the drag coefficient at low velocities (laminar flow) to have the form[15]

$$C_D = 2\alpha \frac{S}{A} (\omega \nu)^{1/2} \frac{1}{U}, \qquad (2)$$

where $S \approx 2L(T+W)$ is the total surface area of a prong and α is a constant of order unity that depends on the shape of the fork. We ignore streaming effects.[15] In the limit of high

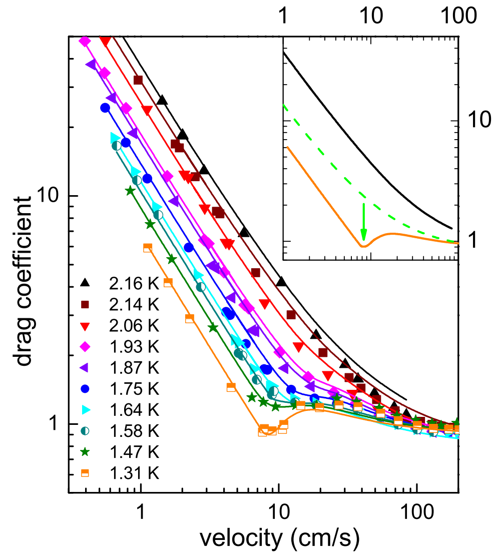

FIG. 2. (Color online) Plots of the observed drag coefficient at various temperatures and at the saturated vapor pressure for vibrating tuning fork A1 (Ref. 12). The lines are fits to a theoretical expression that is described in the text. In the inset we show the experimental curves for temperatures of 1.31 and 2.16 K, together with the broken (green) line, which shows how the drag coefficient would behave for a classical fluid with the kinematic viscosity ν_c evaluated for fork A1 at a temperature of 1.31 K (ν_c is defined in the text).

velocities we expect to have fully developed turbulent flow, for which the drag coefficient is expected to be a constant, β, of order unity. These expectations are consistent with our experimental results at 2.16 K, and they are consistent also with results already published.[16] A drag coefficient given by the simple interpolation formula,

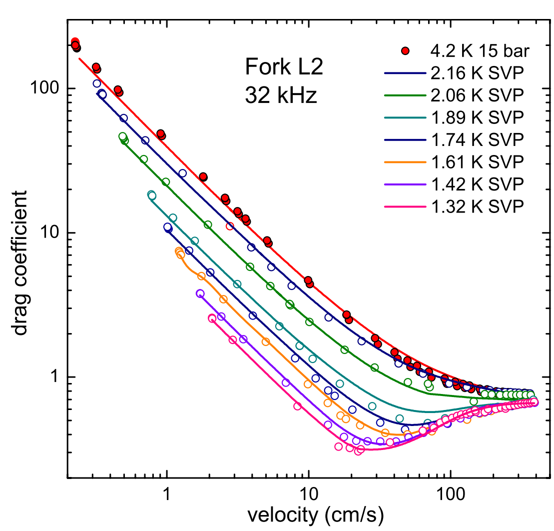

FIG. 3. (Color online) Plots of the observed drag coefficient at various temperatures and at the saturated vapor pressure for vibrating tuning fork L2. Behavior of the fork L1 is very similar. The lines are fits to a theoretical expression that is described in the text.

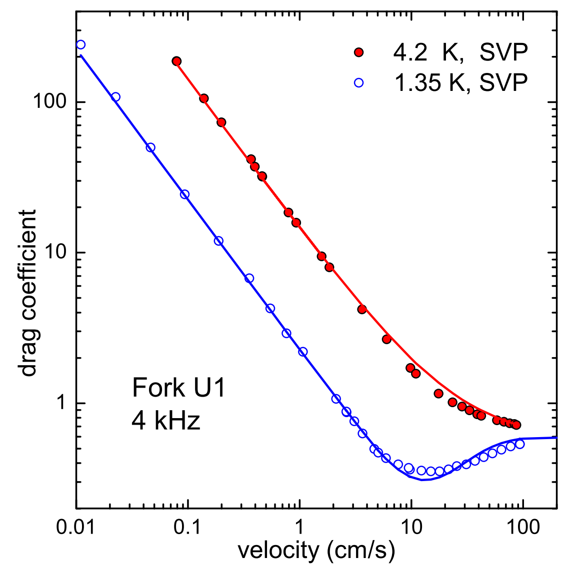

FIG. 4. (Color online) Plots of the observed drag coefficient at two temperatures and at the saturated vapor pressure for vibrating tuning fork U1. The lines are fits to a theoretical expression that is described in the text.

$$C_D = 2\alpha \frac{S}{A}(\omega\nu)^{1/2}\frac{1}{U} + \beta, \tag{3}$$

describes the experimental results rather well over the whole range of velocities. In earlier publications we have defined the critical velocity for the classical transition from laminar to turbulent flow as the velocity at which the two terms on the right-hand side of Eq. (3) are equal, so that

$$U_c = 2\frac{\alpha}{\beta}\frac{S}{A}(\omega\nu)^{1/2}. \tag{4}$$

In fact this definition is misleading because, almost certainly, it does not correspond to any significant change in flow pattern. It is probable that in reality the gradual change in the form of the drag coefficient, from that corresponding to strictly laminar flow to that corresponding to a fully turbulent regime with eddy motion on scales up to width of a prong, reflects a sequence of different flow patterns; the first one displaying the primary instability of the laminar flow. For the case of a vibrating fork the details of this sequence are not known. The details are known for the case of a cylinder with circular cross section, undergoing transverse oscillations in water, with visualization of the flow,[17–20] and in this case the initial instability is a Taylor (Taylor-Görtler) instability in the viscous penetration depth arising from the curvature of the surface of the cylinder.[18,21,22] Attempts to carry out similar experiments with a rod of rectangular cross section in water (i.e., with a suitably scaled version of the prong of a fork) are in progress in our own laboratory in Prague and by Donnelly and Hershberger[23] at the University of Oregon, but they are proving hard to interpret (in setting up these classical experiments we have assumed that the two prongs of a fork do not lead to flows that interfere with each other); it might be expected that the first instability would arise at the corners of the rod, but, if this is indeed the case, the instability seems to spread rapidly over the whole surface of the rod. In the present context we shall find later that the physically significant classical critical velocity seems to be that at which the first instability occurs, which we identify with the first departure of the drag coefficient from that corresponding to strictly laminar flow and which we denote by U_{c1}. For the purposes of rough quantitative analysis we take U_{c1} as the velocity at which the first term on the right-hand side of Eq. (3) is four times β; i.e., we take

$$U_{c1} = \frac{\alpha}{2\beta}\frac{S}{A}(\omega\nu)^{1/2}. \tag{5}$$

B. Superfluid phase: Forks A1, B1, and C3

Looking at Figs. 2–4, we see that the curves for temperatures well below the lambda transition are quite different in shape from those for the normal phase. Especially at the lowest temperatures, there is a minimum in the drag coefficient; in the case of fork A1 this minimum is rather sharp, although it is less sharp in the case of forks L1/L2 and U1. A sharp minimum suggests the existence of a sharp critical velocity. Since this feature becomes well marked at only the lowest temperatures we make the reasonable assumption that it is associated with the superfluid component; i.e., that it marks the onset of significant vorticity, in the form of a significant density of vortex lines, in the superfluid component. We denote this critical velocity by U_{cs}. We emphasize that no hysteretic behavior is seen; the form of the drag versus velocity is the same for increasing velocities as it is for decreasing velocities.

Let us now focus our attention on the results obtained with fork A1; those obtained with forks B1 and C3 are similar. We see that, at velocities greater than that which we identify as the superfluid critical velocity, the drag coefficient passes through a broad maximum and then follows a curve that is very similar to that for a classical fluid, tending to a constant value of order unity at the highest velocity. Thus there appears to be a critical velocity analogous to that given by Eq. (4) in addition to the critical velocity U_{cs}. At first sight we were tempted to associate this second transition with the normal fluid,[24] but in fact this is not reasonable because of the strong coupling between the two fluids that must result from the vortex lines produced by the transition in the superfluid component. Instead we start by suggesting the following tentative scenario. The transition in the superfluid component leads to the formation of a random tangle of vortex line in the neighborhood of the fork. The resulting mutual friction couples the two fluids together, so that they behave as a single quasiclassical fluid. We assume that this single coupled fluid can be characterized by an effective kinematic viscosity, ν_e, which describes the combined effect of the viscosity of the normal fluid and an eddy viscosity characterizing small-scale momentum transfer in the superfluid due to small-scale vortex motion in the random tangle. The length scale associated with this small-scale motion is the vortex spacing, ℓ, and the velocity scale is of order κ/ℓ, where κ is the quantum of circulation. The eddy kinematic viscosity must therefore be of order κ, which has the same order of magnitude as the kinematic viscosity, η_n/ρ, of the normal fluid (ρ is the total density of the helium). Thus the effective kinematic viscosity of the coupled fluids is likely to be of order η_n/ρ; the precise value may be either larger or smaller by a factor of order unity. It is natural now to assume that, provided that the vortex tangle extends far enough from the surface of the fork, this coupled fluid system can undergo a gradual transition to large-scale turbulence (turbulent eddies on a scale of order the width of the prongs of the fork), in a way similar to that occurring in a classical fluid. Thus we might assume tentatively that there are two transitions: the first establishes a more or less random tangle in the superfluid component, allowing that component to undergo rotational motion and to couple its motion to the normal component; the second being a quasiclassical transition to large-scale turbulence in the coupled components. At velocities greater than the superfluid critical velocity but less than that associated with the coupled fluids, the "random" vortex tangle in the superfluid component might allow that component to flow in a quasiclassical laminar mode, matching the laminar flow in the normal fluid; the vortex tangle is then not strictly random to the extent that it allows the superfluid to flow with the large-scale vorticity field characteristic of laminar viscous flow. We remark that in this laminar viscous flow the parallel component of the velocity field, relative to the velocity of the oscillating fork, would need to vanish at the boundaries of the fork; this would be achieved by partial pinning of the vortices on the rough surface of the fork. We emphasize that these are tentative ideas. We shall find later, following a more detailed consideration of the experimental results, that they need some modifications; in particular coupled laminar viscous flow does not seem to be observed, the transition in the superfluid at the velocity U_{cs} leading immediately to turbulent flow of the coupled fluids.

Our ideas can be expressed in terms of a model, according to which the total drag coefficient can be represented mathematically by the following equations:

$$C_D = 2\alpha\frac{S}{A}(\omega x_e \nu_e)^{1/2}\frac{1}{U} + x_e\beta, \tag{6}$$

where

$$x_e = x + (1-x)\Phi(U-U_{\mathrm{cs}})\frac{(U-U_{\mathrm{cs}})^2}{\varepsilon + (U-U_{\mathrm{cs}})^2}, \tag{7}$$

$$\nu_e = \nu + (\nu_c - \nu)\Phi(U-U_{\mathrm{cs}})\frac{(U-U_{\mathrm{cs}})^2}{\varepsilon + (U-U_{\mathrm{cs}})^2}, \tag{8}$$

where U is the velocity amplitude of a fork, x is the normal-fluid fraction, ρ_n/ρ, $\nu=\eta_n/\rho$ is the kinematic viscosity of the normal fluid referred to the total fluid density, $\Phi(y)$ is the Heaviside step function, U_{cs} is the critical velocity of the superfluid component, ν_c is the effective kinematic viscosity of the fully coupled fluids, and α, β, and ε are constants. Equation (6) has the form that applies to a classical fluid with an effective density $x_e\rho$ and effective kinematic viscosity ν_e [(cf. Eq. (3)]. We see that, according to these equations, the effective normal-fluid fraction starts to rise from x to unity

TABLE II. Values of the adjustable parameters α, β, and ϵ.

Fork	α	β	ϵ
A1	0.72	0.85	0.015
B1	0.65	0.43	0.045
C3	0.36	0.42	0.075
U1	0.26	0.52	0.14
U2	0.27	0.5	
L2	0.4	0.63	0.85
L1	0.38	0.63	0.95
K1	0.55	0.85	0.003

when the superfluid velocity exceeds the critical velocity U_{cs}, the change taking place over range of velocity determined by the parameter ε, and that the effective kinematic viscosity changes (increases or decreases) from ν to ν_c in a similar way. The precise description of the transition region of width ϵ implied by these equations should not be taken too seriously. The equations are intended to provide simply an interpolation between the regime in which the two fluids are uncoupled, for $U < U_{cs}$, and that in which the fluids are fully coupled, for $U \gg U_{cs}$. (It can be added, however, that other simple forms of interpolation seem less successful when attempting to fit all available experimental data. Better agreement between the fits and the experimental data might be achieved with additional free fitting parameters, but it is hardly justified in view of the limited experimental accuracy.)

The solid lines in Fig. 2 are obtained with these fitting functions, α, U_{cs}, β, ν_e, and ε being adjustable parameters (normal-fluid parameters are taken from Donnelly and Barenghi[25]). We see that the fits are rather good, providing evidence in favor of our model. We note in particular that the model reproduces the maximum in C_D at a velocity of order 200 mm s^{-1}. As expected, the fitting parameters α, β, and ε are independent of temperature, and they are displayed in Table II. Values of the superfluid critical velocity, U_{cs}, and the effective kinematic viscosity, ν_c, will be discussed later. As expected, the parameters α and β are of order unity, while ν_c is not very different from ν. The parameter ε is small, showing that the initial transition that generates a vortex tangle in the superfluid leads to a sharp rise in the coupling between the two fluids.

C. Superfluid phase: Forks L1/L2 and U1

Looking at Figs. 3 and 4 we see what appears to be a rather different behavior: the minimum that we associate with the superfluid critical velocity is less sharp and at velocities greater than the superfluid critical velocity there is no maximum before the drag coefficient levels off at a constant value. Nevertheless, as we see from Figs. 3 and 4, our trial functions 6, 7, and 8 can still be fitted, but only with a value of ϵ that is significantly increased. This difference implies that vortex line starts to be produced at velocities above the superfluid critical value at a rate that is significantly smaller, so that full coupling between the two fluids sets in more gradually. The reason for this difference is not known. The difference does not seem to be associated with a different size or frequency of fork. A possible reason is that forks L1/L2 and U1 have prongs with surfaces that are smoother than is the case with forks A1, B1, and C3.

We remark at this point that not only does ϵ vary from one fork to another, but also α and β, although the variation of these latter parameters is much smaller. The variation of α reflects an expected dependence of the laminar drag on the exact shape of the fork. The variation of β reflects some not-unexpected dependence of the limiting value of the drag coefficient at large velocity on the detailed geometry. We can add, however, that the values of β set out in Table II agree within experimental error ($\sim$20%) in cases where we have measured them with those obtained when the fork oscillates in a classical fluid (either gaseous helium at 77 K or the normal phase of liquid helium at an elevated pressure, the elevated pressure ensuring that the cavitation observed at the vapor pressure in the normal phase is suppressed[14]); this observation is consistent with our assertion that at high velocities the superfluid in the neighborhood of the fork is behaving like a classical fluid.

D. Comparison to vibrating forks at very low temperatures

The experimental results that we have been presenting relate to vibrating forks in superfluid ^{4}He at temperatures above 1 K, where there is a significant fraction of normal fluid. Very recently measurements at lower temperatures were performed in Kharkov[26] and we now compare them to our own.

Details of the fork (K1) used in Kharkov are included in Table I. For reasons connected with the way the experiments were carried out, the response of the fork and the driving force could not be calibrated in absolute units. Absolute values have therefore been obtained by direct comparison to the measurements on our fork A1, which displays almost identical behavior in an overlapping range of temperature. Typical results so obtained for the fork K1 are shown in Fig. 5, along with those for our fork A1 for comparison. We see a very similar behavior, except that the drag on fork K1 at subcritical velocities is much smaller. There is no hysteresis.

Various forks suffer some damping due to internal friction even *in vacuo* (the Q factor *in vacuo* is typically 5×10^5). At temperatures above 1 K this damping can be neglected at all velocities in comparison to that due to the helium. In the case of the Kharkov results, however, relating to temperatures below 1 K, the damping cannot be neglected. It leads to an apparently temperature-independent contribution to the drag at low velocities, a contribution that is dominant at the lowest temperatures. Subtraction of this contribution leads to a linear drag that is proportional to T^4 at the lowest temperatures, which is to be expected from the scattering of ballistic phonons. In Fig. 5 we show both the uncorrected and the corrected values of the drag.

If our ideas were correct, we can expect that our Eqs. (6)–(8) will still describe the corrected results, provided that we modify our model to take account of this ballistic scattering, which replaces laminar drag from the normal fluid.

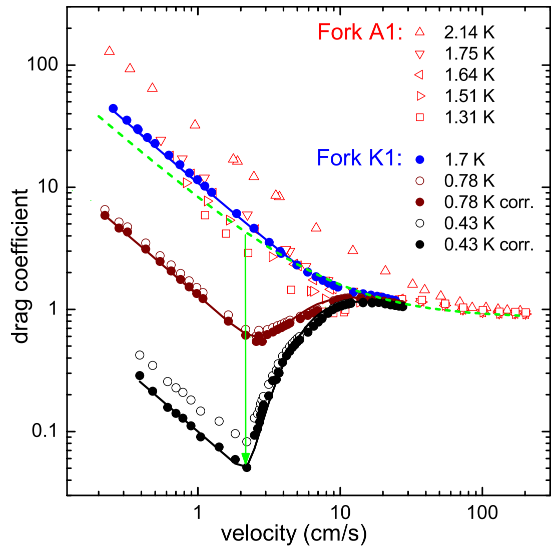

FIG. 5. (Color online) Plots of the observed drag coefficient at several low temperatures and at the saturated vapor pressure for the vibrating tuning fork K1. Results for our fork A1 are included for comparison. The lines are fits to a theoretical expression that is described in the text. The data for K1 do not extend to high enough velocities to allow an accurate determination of the parameter β and therefore β is taken to have the same value as that for A1. The uncorrected data for K1 are shown as open circles; data corrected for the internal damping of the fork (see text) are shown as filled circles. The broken (green) line is derived from Eq. (3) for a classical fluid of kinematic viscosity ν_c.

The phonons form what is in effect a fluid with high viscosity, which cannot become turbulent. We shall suppose that the drag due to the phonons is unaffected by the transition to turbulence in the superfluid component. This may not be strictly correct because the ballistic scattering may be affected by interaction between the phonons and the vortex lines, but the drag due to the phonons is small in comparison to that due to the superfluid turbulence, so that corrections to it are unlikely to be important. We can therefore modify Eqs. (6)–(8) by putting $x=0$ and adding to Eq. (6) a term $\gamma(T)/U$ to represent the effect of the ballistic scattering. The fits seen in Fig. 5 are obtained by taking $\gamma(0.78\ \text{K})=0.013\ \text{ms}^{-1}$ and $\gamma(0.43\ \text{K})=0.001\ \text{ms}^{-1}$. The good quality of these fits provides evidence in favor of our model. Values of the other adjustable parameters are given in Table II. Judging from the value of ε we see that the transition to turbulence in the superfluid is such that the vortex line is produced at a rate that increases rather sharply with increasing velocity, as was seen with forks A1, B1, and C3, but not with the other forks.

E. Values of the effective kinematic viscosity

Values of the effective kinematic viscosity ν_c deduced by fitting Eqs. (6)–(8) (modified in the case of K1) to the experimental data for the various forks are shown in Fig. 6. Although the values have the expected orders of magnitude, they fall into two distinct groups: those with $\nu_c \sim 10^{-8}\ \text{m}^2\ \text{s}^{-1}$ and those with $\nu_c \sim 2\times10^{-9}\ \text{m}^2\ \text{s}^{-1}$. There is no obvious explanation of this fact; the forks belonging to one group do not seem to have any obvious characteristic that is different from those in the other group.

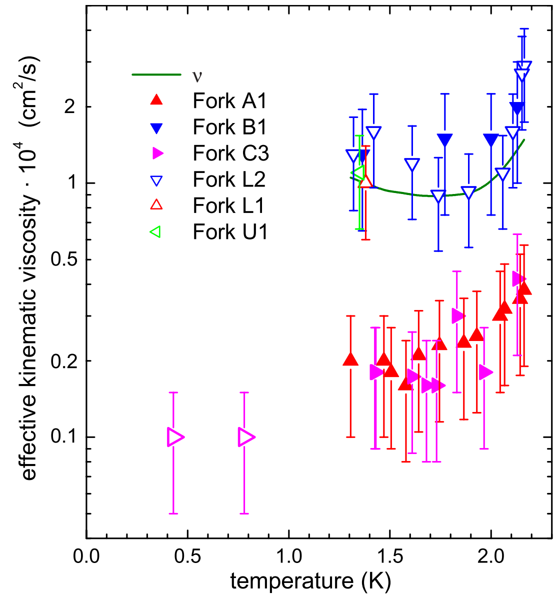

FIG. 6. (Color online) Plots of the effective kinematic ν_c against temperature for the various forks as indicated. The solid (green) line is a plot of kinematic viscosity of the normal fluid (referred to the total fluid density; i.e., η_n/ρ), based on values tabulated by Donnelly and Barenghi (Ref. 25).

IV. COMPARISON TO OTHER VIBRATING STRUCTURES

In Figs. 7 and 8 we show plots of the drag coefficient versus velocity for an oscillating grid[4,5] and an oscillating sphere.[3] The results for the grid, which show little or no hysteresis, have a form very similar to that for the forks. However, the data do not extend to sufficiently large velocities for us to determine the limiting value of the drag coefficient (i.e., the value of β). The sphere exhibits hysteretic

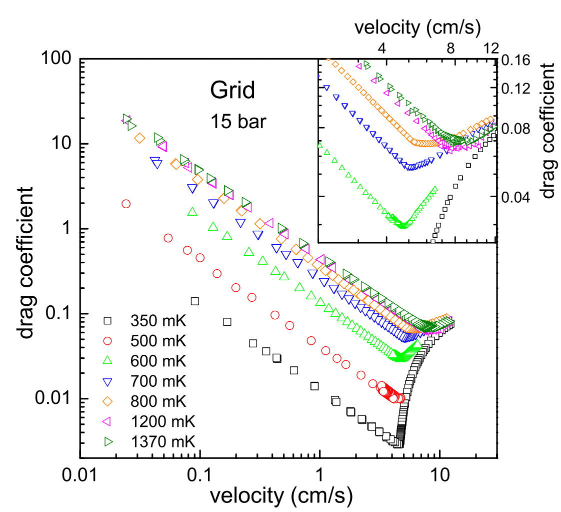

FIG. 7. (Color online) Plots of the observed drag coefficient at various temperatures and at the saturated vapor pressure for a vibrating grid.

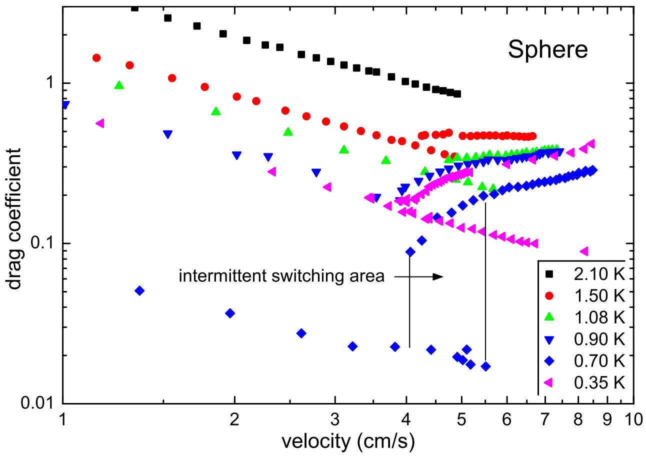

FIG. 8. (Color online) Plots of the observed drag coefficient at various temperatures and at the saturated vapor pressure for vibrating sphere.

and switching effects,[3] which we shall discuss later, but again the data seem not to extend to high enough velocities (the apparent rise in drag coefficient at high velocities suggests that the data might then be affected by heating). We do not show data for vibrating wires:[1,2] again they do not extend to large enough velocities and, more seriously, it seems difficult from the available data to correct for significant nonlinear internal damping. Furthermore, the wires have very small diameters, so that our type of analysis may not be applicable. We conclude that the available data for oscillating grids, spheres, and wires are not sufficiently extensive to permit the type of detailed analysis that underlies our Eqs. (6)–(8) and that they can be used only to extract values of the superfluid critical velocity. Further experiments are clearly required.

Values of the observed critical velocities for the various oscillating structures are displayed in Fig. 9 (the data labeled "Fork Lanc" were provided by Haley[27]). In cases where there is hysteresis we have taken the critical velocity that is observed on reducing the velocity. We see that there is a clear indication that the critical velocities increase with increasing temperature. Structures of different size seem to have critical velocities that are very similar in magnitude, but not exactly the same. The suggestion has been made (see Sec. V) that the critical velocities increase with increasing angular frequency, ω, as $\omega^{1/2}$. To test this suggestion we have plotted in Fig. 10 the dimensionless ratio $U_{cs}/(\omega\kappa)^{1/2}$ against temperature. There is little evidence that the data then collapse onto a single line, as would be required by this suggestion. We return to this question in Sec. V.

V. ORIGIN OF THE CRITICAL SUPERFLUID VELOCITY

It is generally accepted that when the superfluid critical velocity is exceeded a tangle of vortex lines forms in the superfluid component in the neighborhood of the oscillating structure. We argued in earlier sections that at high temperatures this leads to a strong coupling between the two fluids, and that the coupled fluids, behaving like a single fluid with

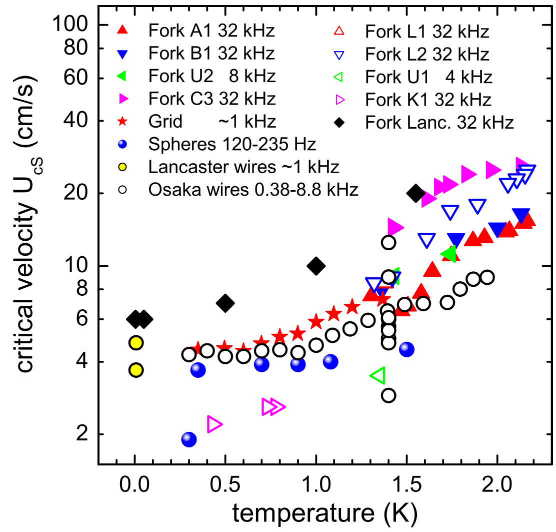

FIG. 9. (Color online) Values of the critical superfluid velocity, U_{cs}, observed for various oscillating structures, plotted against temperature. The different structures oscillate at different frequencies as shown.

a kinematic viscosity determined by a combination of the viscosity of the normal fluid and the eddy viscosity of the superfluid, undergoes a gradual transition to a fully turbulent state in a way closely similar to that observed when the structure oscillates in a classical fluid. At very low temperatures, when the normal fluid is effectively absent, there is a similar gradual transition, but one in which the superfluid component acts alone as a single quasiclassical fluid with a kinematic viscosity equal to the eddy viscosity associated with a tangle of vortex lines. These pictures leave us with an important open question: can we understand the value of the velocity at which the superfluid makes the initial transition to a tangled vortex state?

The first point that must be made is that the nucleation of vortex line at the critical velocity must be "extrinsic"; i.e., it must arise from the multiplication or stretching of existing remanent vortex line. Intrinsic nucleation of vortex line is

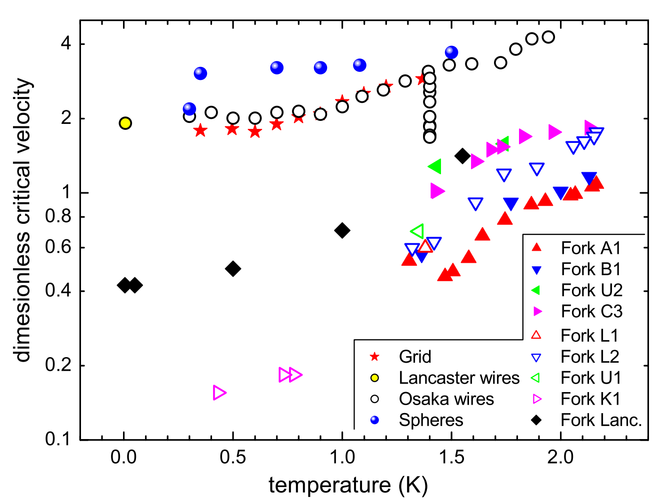

FIG. 10. (Color online) Values of the dimensionless ratio $U_{cs}/(\omega\kappa)^{1/2}$ plotted against temperature.

possible in superfluid ^{4}He only at velocities that are much larger than those encountered here. Very strong experimental evidence comes from the recent work of the Osaka group with vibrating wires.[1]

As we have already mentioned, the superfluid critical velocity seems not to depend very strongly on the size of the oscillating structure. If it were accurately independent of size and shape and if it were then to depend on only the angular frequency, ω, and the quantum of circulation, κ, a dimensional argument leads to

$$U_{cs} = \mathrm{const}(\omega\kappa)^{1/2}. \tag{9}$$

This is the predicted dependence on frequency that we mentioned earlier. However, as we also mentioned earlier, there is as yet little convincing experimental evidence for this dependence. The wide scatter on the experimental points in Figs. 9 and 10 leads us to believe that the critical velocity depends to a significant extent on the detailed geometry of the structure and perhaps also on the state of roughness of the surface. Moreover, it may depend also on the form of the remanent vortex or vortices.

We shall now focus largely on the behavior of the forks because, as we have explained, it is for this case that we have the most complete sets of experimental data. We recall from Sec. III A that the transition to turbulence in a classical fluid is rather gradual and described by Eq. (3). Let us look especially at the superfluid data in Fig. 2 and consider the way in which the drag coefficient varies as the velocity is reduced from a large value. The drag coefficient is observed to rise with decreasing velocity, just as it does in a classical fluid. As it approaches proportionality to $1/U$, the drag coefficient drops rather suddenly to the value appropriate to the normal fluid acting alone. In other words, the superfluid critical velocity appears at a velocity closely equal to that at which flow of a classical fluid with density equal to the total helium density and kinematic viscosity equal to the parameter ν_c makes a transition to laminar flow; i.e., at a velocity analogous to the classical critical velocity U_{c1} introduced in Sec. III A (see especially the broken green lines in Figs. 2 and 5). Although data for other forks are less clear cut, owing to a larger value of the smoothing parameter, ϵ, they nevertheless show very similar behavior. Furthermore, this behavior seems to be common to all oscillating structures, although more detailed experimental data will be needed to provide really convincing evidence. Evidence that the superfluid critical velocity is generally very similar to the classical critical velocity U_{c1} was discussed in more detail in a recent review by Skrbek and Vinen.[9]

Of course this similarity may result from a numerical accident. The classical critical velocity [Eq. (4)] is of the order $(\omega\nu)^{1/2}$. The effective kinematic viscosity of the fully coupled fluids is of the order 0.1κ and therefore the critical velocity given by Eq. (9) is similar in order of magnitude to that given by Eq. (5). But the observations suggest that the two critical velocities are not merely similar in order of magnitude but are actually closely equal.

At first sight this equality seems very strange. The two transitions are apparently quite different in character: the classical transition is from a state of laminar *viscous* flow with zero slip at a solid boundary; the quantum transition is from a state of laminar *potential* flow with complete slip at a solid boundary. However, this picture of the quantum transition is not quite correct because of the need for one or more nucleating vortices. Superfluid transitions arise from the multiplication and stretching of these nucleating vortices. The way in which this multiplication and stretching occurs is not generally known. Appropriate simulations would be helpful, but only those of Hänninen *et al.*[10] are available and they relate to an oscillating structure (a sphere) the surface of which is smooth, so that the nucleating vortices are not pinned to particular points on the sphere. However, the results of these simulations ought to be considered. The temperature is assumed to be zero. The nucleating vortex is in the form of a vortex stretching across the helium, to which the sphere is attached. Oscillation of the sphere at angular frequency ω leads to the generation of Kelvin waves on the nucleating vortex with wave number equal to approximately $k=(\omega/\kappa)^{1/2}$, and as the amplitude of these waves increases to values of order k^{-1} self-reconnections result in the production of vortex rings with radius of order k^{-1}. These rings are produced even at quite low velocities. However, above a certain critical velocity, which we denote by U_{cH}, the density of these rings in the neighborhood of the sphere suddenly increases and leads to the formation of what appears to be a random vortex tangle, which is usually localized in the form of a wake that oscillates from one side of the sphere to the other as the sphere itself oscillates. It should be noted that the critical velocity obtained in the simulations is significantly larger than the observed value of U_{cs} for a sphere.

The formation of this tangle would undoubtedly be modified if the oscillating structure were rough, but let us suppose that something similar still occurs, only at a lower velocity. We suppose then that when the critical velocity U_{cs} is exceeded a dense tangle of vortex line envelopes the oscillating structure. At very low temperatures this tangle causes the superfluid in the region of the tangle to behave like a classical viscous fluid, with viscosity equal to the eddy viscosity associated with a random tangle of vortex line; at a higher temperature, mutual friction gives rise to a strong coupling between the two fluids, so that they behave as a single fluid with an effective viscosity, ν_c, that depends on both the normal-fluid viscosity and the superfluid eddy viscosity. This is of course the idea that we used in developing the model on which Eqs. (6)–(8) are based. We assumed also that the enveloping tangle of vortex line extends away from the oscillating structure to a distance of order that at which flow is induced by the oscillation of the structure concerned.

However, this quasiclassical behavior requires presumably that the density of vortex line be sufficiently large. In other words the line spacing, ℓ, must be small compared with some characteristic lengths in the quasiclassical flow. One obvious characteristic length in the quasiclassical flow is the classical viscous penetration depth, given by

$$\delta = \left(\frac{2\nu_c}{\omega}\right)^{1/2}. \tag{10}$$

This penetration depth is clearly relevant to quasiclassical *laminar* flow; such laminar flow is possible only if $\ell < \delta$. The characteristic length that is relevant to flow at velocities greater than those for which laminar flow is stable is less obvious. We can, however, assert with fair certainty that the *initial* instability occurs on a scale that is equal to the viscous penetration depth (this is known to be true for an oscillating cylinder of circular cross section[21,22]). As long as there is no hysteresis, which is the case with our tuning forks, the scale of the turbulent motion at velocities close to critical must also be confined to a region of size comparable with the viscous penetration depth, even when the velocity has been reduced from a large value. Thus we conclude that turbulent motion also requires that $\ell < \delta$, at least at velocities close to critical.

We return to the way in which the drag coefficient changes as the velocity is reduced from a large value. Experiment tells us that, as the velocity is reduced toward a value at which quasiclassical turbulence might be expected to give way to quasiclassical purely laminar flow (the analog of U_{c1}), the vortex density must decrease rather suddenly to a value that is too small to maintain quasiclassical behavior; i.e., to a density such that $\ell > \delta$. A sensible conclusion is that the high density of vortex lines at velocities where there is quasiclassical turbulence is being maintained by the quasiclassical turbulence itself.

This view is not unreasonable. The large-scale quasiclassical turbulence is maintained by the large-scale flow round the structure (combined with the no-slip boundary condition). There is an injection of energy into the large-scale turbulent motion in much the same way as occurs in flow through a grid (the large superfluid eddies being associated with a partial polarization of the underlying vortex tangle). The energy in the large eddies tends to flow through nonlinear coupling into smaller-scale turbulence until it reaches a scale of order the vortex spacing, where it serves to generate extra length of vortex line. The rate of decay of the large eddies is governed by their lifetimes, which are equal to their turnover times. Thus an important contribution to the generation of vortex line could come from the large-scale turbulence, so that failure to maintain this turbulence, for quasiclassical reasons, could result in a rather sudden reduction in the vortex line density. The observed link between the critical velocity U_{cs} and the velocity at which quasiclassical turbulence is extinguished then becomes natural. Similarly, during an increase in velocity, the vortex line density cannot increase to a value large enough to produce an observable drag until there is a development of large-scale turbulence. This situation implies an instability, so that the development of large-scale turbulence linked to the generation of a large vortex density may take place in practice only at a velocity larger than that at which the large-scale turbulence disappears on reducing the velocity. As we have noted, such hysteresis is indeed observed with some, but not all, structures, especially at lower temperatures.

We have noted that a critical velocity, U_{cH}, is observed in the simulations of Hänninen *et al.*,[10] although its magnitude is larger than the observed U_{cs}. This critical velocity seems to be unrelated to any quasiclassical critical velocity. There are various possibilities: the existence of U_{cH} may be peculiar to a structure with a smooth surface; U_{cH} may always be larger than U_{cs} and therefore unobservable; or the line density produced in a steady state above U_{cH} may remain quite small and too small to allow a quasiclassical transition to turbulent flow.

It is instructive to estimate the rate of production of vortex line resulting from the decay of large-scale turbulence. In the case of homogeneous isotropic turbulence the largest eddies, characterized by a velocity U and size D, decay on a timescale D/U, so that they lead to a flux of energy to smaller scales equal to roughly U^3/D per unit mass of helium. If we suppose that this decay rate applies also to eddies of size of order W in a volume of order $WTL/2$ around each prong of a tuning fork (W, L, and T are the fork dimensions shown in Fig. 1), then the rate of production of vortex line must be given in order of magnitude by

$$\rho_s \kappa^2 \frac{d\mathcal{L}}{dt} = \frac{U_s^3}{W} \rho \frac{WTL}{2}, \tag{11}$$

where $\mathcal{L}$ is the total length of vortex line, U_s is the superfluid velocity relative to a prong of the fork, and where we have taken the energy per unit length of vortex line, in a random tangle, to be $\rho_s \kappa^2$. For typical values of the various parameters, we find that $d\mathcal{L}/dt \sim 7 \times 10^6 \ \mathrm{ms}^{-1}$. This is much larger than the values found in the simulations of Ref. 10, tending therefore to confirm that vortex production by decay of large-scale eddy motion is more effective than by direct stretching of remanent vortices (although the proviso must be added that the simulations of Ref. 10 were not extended to very large times).

Our model of the physics underlying the critical velocity U_{cs} implies that there is a small density of vortex lines in the neighborhood of the oscillating structure at velocities less than U_{cs}. The model does not tell us the magnitude of this small density. Experiments reported so far indicate a density that is too small to have an observable effect on the drag, although we note that the drag is likely to be determined not only by the density but also by the configuration of vortex line. Experiments at lower temperatures than have so far been studied, where the drag due to the normal fluid has become very small, ought to throw light on this question.

We return to the actual value of the critical superfluid velocity U_{cs}. We have suggested that it is equal to U_{c1}, evaluated for a classical fluid with kinematic viscosity equal to the effective kinematic viscosity ν_c; i.e.,

$$U_{cs} = \frac{\alpha}{2\beta} \frac{S}{A} (\omega \nu_c)^{1/2}. \tag{12}$$

Thus we predict that U_{cs} is proportional to the square root of the frequency only if the parameters α, β, S/A, and ν_c are constant. This prediction is consistent with our earlier suggestion that the critical velocity seems to depend to a significant extent on the detailed geometry of the structure.

The views that we have been expressing are of course speculative. Significant features of the experimental results, especially the variation of both the parameter ϵ and the effective kinematic viscosity ν_c from one structure to another, remain puzzling. But we hope that our views will serve to stimulate further work, both in the acquisition of more extensive experimental data and in the development of the theory. If confirmed, a link between a superfluid critical velocity and an instability of quasiclassical flow would be an interesting feature of quantum turbulence.

We have noted that we have as yet little detailed knowledge or understanding of the transition to turbulence in the flow round our oscillating forks in a classical fluid. We emphasize, however, that this fact does not seriously affect our argument. In essence we have based this argument on the idea that quasiclassical flow of the superfluid around our forks mimics the purely classical flow, whatever that classical flow might be. This is not to say that a detailed knowledge of this classical flow would not lead ultimately to a better understanding of the quantum case.

VI. SUMMARY AND CONCLUSIONS

We have reported the results of experiments in which we have measured the drag on the prongs of a number of small tuning forks oscillating in superfluid ^{4}He, over a range of temperatures above 1 K, and we have compared our results to those obtained with forks at lower temperatures and with other forms of oscillating structure. We have presented our results in the form of plots of the drag coefficient against velocity; we have argued that the detailed form of these plots contains valuable information that extends beyond a knowledge of a critical superfuid velocity above which there is an increased drag; and we have noted similarities with the behavior of oscillating structures in classical fluids. In the light of this information we have discussed the nature of the critical superfluid velocity. We have observed that the velocity seems often to be associated with a transition to turbulence that is essentially classical in its characteristics, and we have tentatively suggested that this quasiclassical behavior has its origin in an instability in which the generation of vortex line at the rate required to produce large-scale turbulence is a by-product of the large-scale turbulence itself.

ACKNOWLEDGMENTS

We are grateful to many colleagues for stimulating discussions, especially R. Hänninen, M. Krusius, M. Tsubota, W. Schoepe, P. Skyba, and H. Yano. We thank O. Kolosov for supplying some of the quartz forks. This research is supported by research Plans No. MS 0021620834, No. AVOZ 10100520, No. GAUK 7953/2007, and No. GAČR 202/08/0276.

[1] H. Yano, A. Handa, H. Nakagawa, K. Obara, O. Ishikawa, T. Hata, and M. Nakagawa, J. Low Temp. Phys. **138**, 561 (2005); H. Yano, A. Handa, H. Nakagawa, M. Nakagawa, K. Obara, O. Ishikawa, and T. Hata, J. Phys. Chem. Solids **66**, 1501 (2005); H. Yano, N. Hashimoto, A. Handa, M. Nakagawa, K. Obara, O. Ishikawa, and T. Hata, Phys. Rev. B **75**, 012502 (2007); N. Hashimoto, R. Goto, H. Yano, K. Obara, O. Ishikawa, and T. Hata, *ibid.* **76**, 020504(R) (2007).

[2] D. I. Bradley, D. O. Clubb, S. N. Fisher, A. M. Guenault, R. P. Haley, C. J. Matthews, G. R. Pickett, and K. Zaki, J. Low Temp. Phys. **138**, 493 (2005); D. I. Bradley, Phys. Rev. Lett. **84**, 1252 (2000); D. I. Bradley, S. N. Fisher, A. M. Guenault, M. R. Lowe, G. R. Pickett, A. Rahm, and R. C. V. Whitehead, *ibid.* **93**, 235302 (2004).

[3] J. Jäger, B. Schuderer, and W. Schoepe, Phys. Rev. Lett. **74**, 566 (1995); J. Jäger, B. Schuderer, and W. Schoepe, Physica B **210**, 201 (1995); J. Luzuriaga, J. Low Temp. Phys. **108**, 561 (1997); M. Niemetz, H. Kerscher, and W. Schoepe, *ibid.* **126**, 287 (2002).

[4] H. A. Nichol, L. Skrbek, P. C. Hendry, and P. V. E. McClintock, Phys. Rev. Lett. **92**, 244501 (2004); Phys. Rev. E **70**, 056307 (2004).

[5] D. Charalambous, L. Skrbek, P. C. Hendry, P. V. E. McClintock, and W. F. Vinen, Phys. Rev. E **74**, 036307 (2006).

[6] D. I. Bradley, D. O. Clubb, S. N. Fisher, A. M. Guenault, C. J. Matthews, and G. R. Pickett, J. Low Temp. Phys. **134**, 381 (2004); D. I. Bradley, D. O. Clubb, S. N. Fisher, A. M. Guenault, R. P. Haley, C. J. Matthews, G. R. Pickett, V. Tsepelin, and K. Zaki, Phys. Rev. Lett. **95**, 035302 (2005).

[7] W. F. Vinen and J. J. Niemela, J. Low Temp. Phys. **128**, 167 (2002).

[8] R. Blaauwgeers, M. Blazkova, M. Clovecko, V. B. Eltsov, R. de Graaf, J. Hosio, M. Krusius, D. Schmoranzer, W. Schoepe, L. Skrbek, P. Skyba, R. E. Solntsev, and D. E. Zmeev, J. Low Temp. Phys. **146**, 537 (2007).

[9] L. Skrbek and W. F. Vinen, *Progress in Low Temperature Physics* (Elsevier, Amsterdam, 2009), Vol. XVI, Chap. 4.

[10] R. Hänninen, M. Tsubota, and W. F. Vinen, Phys. Rev. B **75**, 064502 (2007).

[11] R. Goto, S. Fujiyama, H. Yano, Y. Nago, N. Hashimoto, K. Obara, O. Ishikawa, M. Tsubota, and T. Hata, Phys. Rev. Lett. **100**, 045301 (2008).

[12] Forks specified as type DT26 (A1) and DT38 (B1) used in this work have been produced by Fronter Electronics, China, www.chinafronter.com. Forks C, U1, U2, L1, and L2 were provided by O. Kolosov, Lancaster University.

[13] The measurements have been performed in a brass pressure cell, which was filled by helium through a thin capillary via a nitrogen trap and copper sinter inside the cell to prevent the deposit of frozen air or water on the fork. To keep the highest possible Q value, forks A and B have only the tops of their original metal cans removed. The small surrounding volume of liquid helium of extremely low viscosity also results in a shorter characteristic time required to obtain steady-state flow.

[14] M. Blažková, T. V. Chagovets, M. Rotter, D. Schmoranzer, and L. Skrbek, J. Low Temp. Phys. **150**, 194 (2008); M. Blažková, D. Schmoranzer, and L. Skrbek, Low Temp. Phys. **34**, 298 (2008).

[15] G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge University Press, Cambridge, 1967).

[16] M. Blažková, D. Schmoranzer, and L. Skrbek, Phys. Rev. E **75**, 025302(R), (2007).

[17] C. H. K. Williamson, J. Fluid Mech. **155**, 141 (1985).

[18] T. Sarpkaya, J. Fluid Mech. **165**, 61 (1986).

[19] E. D. Obasaju, P. W. Bearman, and J. M. R. Graham, J. Fluid Mech. **196**, 467 (1988).
[20] M. Tatsuno and P. W. Bearman, J. Fluid Mech. **211**, 157 (1990).
[21] H. Honji, J. Fluid Mech. **107**, 509 (1981).
[22] P. Hall, J. Fluid Mech. **146**, 347 (1984).
[23] R. J. Donnelly and R. Hershberger (private communication).
[24] M. Blažková, M. Človečko, V. B. Eltsov, E. Gažo, R. de Graaf, J. J. Hosio, M. Krusius, D. Schmoranzer, W. Schoepe, L. Skrbek, P. Skyba, R. E. Solntsev, and W. F. Vinen, J. Low Temp. Phys. **150**, 525 (2008).
[25] R. J. Donnelly and C. F. Barenghi, J. Phys. Chem. Ref. Data **27**, 1217 (1998).
[26] G. A. Sheshin, A. A. Zadorozhko, E. Y. Rudavskii, V. K. Chagovets, L. Skrbek, and M. Blazkova, Low Temp. Phys. **34**, 875 (2008).
[27] R. P. Haley (private communication).