Cloaking bending waves propagating in thin elastic plates

Mohamed Farhat,[1] Sebastien Guenneau,[2] Stefan Enoch,[1] and Alexander B. Movchan[2]
[1]*Institut Fresnel, CNRS, Aix-Marseille Université, Campus Universitaire de Saint-Jérôme, 13013 Marseille, France*
[2]*Department of Mathematical Sciences, Liverpool University, Peach Street, Liverpool L69 3BX, United Kingdom*
(Received 3 December 2008; published 12 January 2009)

We introduce a cylindrical cloak to control the bending waves propagating in thin plates. This is achieved through radially dependent isotropic mass density and radially dependent and orthotropic flexural rigidity deduced from a coordinate transformation for the biharmonic propagation equation in the spirit of the paper of Pendry *et al.* [Science **312**, 1780 (2006)]. We analyze the response of the cloak surrounding a clamped obstacle in the presence of a cylindrical excitation. We note that whereas the studied bending waves are of different physical and mathematical nature, they are cloaked in many ways as their electromagnetic and acoustic counterparts; e.g., when the source lies inside the coating, it seems to radiate from a shifted location (mirage effect).

DOI: 10.1103/PhysRevB.79.033102 PACS number(s): 46.40.−f, 43.40.+s, 62.30.+d

The theory describing the flexural motion of thin homogeneous plates is well established and can be found in many classical books (see, e.g., Refs. 1 and 2). In contrast, it only gave rise to a handful of research papers on scattering of flexural waves,[3] owing to the fourth-order derivatives involved in the governing equations (requiring a special numerical treatment to take into account additional limit conditions compared with acoustic wave equations). Moreover, analysis of flexural waves in structured plates is even less well studied. In 2007, Evans and Porter[4] presented the first analysis of the propagation of bending waves in a doubly periodic array of rigid pins in an infinite thin plane. The same year, Movchan *et al.*[5] gave a detailed analysis of the band structure (Floquet-Bloch spectral problem) for the biharmonic equation in a doubly periodic domain of circular holes with either clamped or stress free boundary conditions.

In 2006, Pendry *et al.*[6] and Leonhardt[7] independently showed the possibility of designing a cloak that renders any object inside it invisible to electromagnetic radiation. This coating consists of a metamaterial whose physical properties (permittivity and permeability) are deduced from a coordinate transformation in the Maxwell system. The anisotropy and the heterogeneity of the parameters of the coat work as a deformation of the space around the object we want to hide by bending the wavefront around it and enabling waves to emerge on the other side in the original propagation direction without any perturbation. The experimental validation[8] of these theoretical considerations was given, a few months later, by an international team involving the former authors who used a cylindrical cloak consisting of concentric arrays of split ring resonators. This cloak makes a copper cylinder invisible to an incident plane wave at 8.5 GHz as predicted by the numerical simulations.

In the same vein, Cummer and Schurig[9] analyzed the two-dimensional (2D) acoustic cloaking for pressure waves in a transversely anisotropic fluid by exploiting the analogy with TE electromagnetic waves. Torrent and Sanchez-Dehesa[10] subsequently investigated this cloaking for concentric layers of solid lattices behaving as anisotropic fluids in the homogenization limit. Using a similar approach, Farhat *et al.*[11] independently demonstrated cloaking of surface liquid waves using a microstructured metallic cloak which was experimentally validated at 10 Hz. Quite remarkably, Chen and Wu[12] and Cummer *et al.*[13] noticed that a three-dimensional (3D) acoustic cloaking for pressure waves in a fluid can be envisaged since the wave equation retains its form under geometric changes.

However, Milton *et al.*[14] showed back in 2006 that the elasticity equations are not invariant under coordinate transformations, and consequently that if cloaking exists for such classes of waves, it would be of a different nature from its acoustic and electromagnetic counterparts. Indeed, the equations governing the propagation of elastodynamic waves with a time harmonic dependence are written (in weak sense) as

$$\nabla \cdot \mathbf{C}{:}\nabla \mathbf{u} + \rho_0 \omega^2 \mathbf{u} = 0, \tag{1}$$

where ρ_0 is the (scalar) density of an isotropic heterogeneous elastic medium, $\mathbf{C}$ is the rank-4 elasticity tensor, ω is the wave angular frequency, and $\mathbf{u}(x_1,x_2,x_3,t) = \mathbf{u}(x_1,x_2,x_3)e^{-i\omega t}$, where t is the time, the associated three-component vector displacement field.

Milton *et al.* noticed[14] that under a change in coordinates $\mathbf{x}\rightarrow\mathbf{x}'$ such that $\mathbf{u}'(\mathbf{x}')=\mathbf{A}^{-T}\mathbf{u}(\mathbf{x})$ with $A_{ij}=\partial x_i'/\partial x_j$, Eq. (1) takes the form

$$\nabla' \cdot (\mathbf{C}' + \mathbf{S}'){:}\nabla' \mathbf{u}' + \underline{\underline{\rho'}}\omega^2 \mathbf{u}' = \mathbf{D}'{:}\nabla' \mathbf{u}', \tag{2}$$

which importantly preserves the symmetry of the new elasticity tensor $\mathbf{C}'+\mathbf{S}'$. However, this transformed equation contains $\mathbf{S}'$ and $\mathbf{D}'$, which are two rank-3 (symmetric) tensors such that $D'_{pqr}=S'_{qrp}$ and ρ'_{pq} is a rank-2 tensor whose expressions can be found in Ref. 14.

In this Brief Report, we show that a heterogeneous orthotropic cloak can be performed via an anisotropic Young modulus $\underline{\underline{E}}$ and a radially dependent isotropic mass density ρ to make an object surrounded by such a coat neutral for flexural waves in thin elastic plates. Thus, although Eq. (2) suggests that cloaking by coordinate transformation cannot be applied, in general, to elastic waves,[14] we demonstrate that the specific case of thin plates leads to an opposite conclusion.

The equations for bending of plates are well known and can be found in many textbooks, such as those of Timoshenko[1] or Graff.[2] The wavelength λ is supposed to be large enough compared to the thickness of the plate h and small compared to its in-plane dimension L, i.e., $h \ll \lambda \ll L$. In this case we can adopt the hypothesis of the theory of von Kármán.[1,2] The displacement field $\mathbf{u}$ solution of Eq. (1) is $U(x_1,x_2)$ in the x_3 direction (along the vertical axis) and we further choose to work in cylindrical coordinates.

Following the proposal of Pendry *et al.*[6] to build an electromagnetic cloak of interior and exterior radii a and b, we now consider the change in coordinates $\mathbf{x}=(r,\theta,x_3)\rightarrow\mathbf{x}'=[a+(b-a)r/b,\theta,x_3]$. With all the above assumptions, we find that the out-of-plane displacement $\mathbf{u}'=[0,0,U'(r,\theta)]$ solution of Eq. (2) is such that

$$\nabla\cdot\{\underline{\underline{\zeta}}^{-1}\nabla[\lambda\nabla\cdot(\underline{\underline{\zeta}}^{-1}\nabla U')]\}-\lambda^{-1}\beta_0^4U'=0, \tag{3}$$

with $\beta_0^4=\omega^2\rho_0 h/D_0$, where D_0 is the flexural rigidity of the plate, ρ_0 its density, and h its thickness. $\underline{\underline{\zeta}}$ is a diagonal rank-2 tensor and λ is a varying coefficient of the material, a case encompassed in Refs. 1 and 2.

Physical considerations based on dimensional analysis lead to the following choice:

$$\underline{\underline{\zeta}}=\underline{\underline{E}}^{-1/2} \quad\text{and}\quad \lambda=\rho^{-1/2}. \tag{4}$$

Cloaking further requires that

$$E_r=\left(\frac{r-a}{r}\right)^2,\quad E_\theta=\left(\frac{r}{r-a}\right)^2,\quad\text{and}\quad \rho=\alpha^4\left(\frac{r-a}{r}\right)^2, \tag{5}$$

where $\alpha=b/(b-a)$; a and b are the interior and the exterior radii of the elastic coat of thickness h. Then, in cylindrical coordinates and assuming that $U'=\sum_{n=-\infty}^{\infty}U_n'(r)e^{in\theta}$, the equation satisfied by U_n' is written as

$$\begin{aligned}&\partial_r\left\{(r-a)\partial_r\left[\frac{1}{r-a}\partial_r[(r-a)\partial_rU_n']-\frac{n^2}{(r-a)^2}U_n'\right]\right\}\\&\quad-\frac{n^2}{r-a}\left\{\frac{1}{r-a}\partial_r[(r-a)\partial_rU_n']-\frac{n^2}{(r-a)^2}U_n'\right\}\\&\quad-(r-a)\beta_1^4U_n'=0,\end{aligned} \tag{6}$$

which is the cylindrical expression of the corresponding fourth-order equation in anisotropic media with the variable $r-a$ and where $\beta_1=\alpha\beta_0$. This equation presents some similarities with the wave equations governing the acoustic and electromagnetic cloaks in Refs. 13 and 15, which were derived from the coordinate transformation introduced for electromagnetic waves in Ref. 6: the cornerstone is always the radial contraction $r'=a+r(b-a)/b$. The radial shift from r to $r-a$ in Eq. (6) can actually be seen as the bottom neck of the cloaking mechanism. Whereas Chen *et al.*[15] performed a spherical harmonic scattering for the 3D electromagnetic cloak which is governed by a Riccati-Bessel equation, and Cummer *et al.*[13] obtained an acoustic solution in terms of spherical Bessel and Hankel functions J_n and $H_n^{(1)}$, our flexural wave solution is a combination of cylindrical Bessel and Hankel functions J_n and $H_n^{(1)}$, as well as their modified counterparts I_n and K_n, owing to the fourth-order nature of the biharmonic equation, which we can split into two coupled Helmholtz and modified Helmholtz equations.[2] The arguments in the special functions are either $\beta_0 r$ or $\beta_1(r-a)$ depending upon whether we are outside or inside the coating, and taking into account their singular behavior at the origin as well as outgoing wave conditions leads to a linear algebraic system. We note that in the derivation of the linear system we used not only the continuity of the displacement U_n and its radial derivative $\partial_r U_n'$ at $r=b$ (as required for a cylindrical acoustic cloak), but also the conservation of the bending moment and the generalized Kirchoff stress[2,3] in the radial direction through this interface. The usefulness of such a multipole analysis has been already exemplified in Ref. 13 as being the cornerstone of 3D acoustic cloaking. Here, we shall only emphasize that the limiting analysis (not required in the 3D electromagnetic case since it involves Riccati-Bessel functions that do all vanish at the origin, unlike Bessel and Hankel functions[15]), showing that the field vanishes inside the acoustic cloak, carries through in our case with additional technicalities resulting from the inversion of a larger algebraic system involving four linearly independent equations. It turns out that a prerequisite criterion for a vanishing displacement (acoustic neutrality) in the disk $r<a$ is that the number of wavelengths in $a<r<b$ (inside the shell that is over a distance $b-a$) must be exactly the number of wavelengths for $r>b$ over a distance b (outside the cloak), which could be expected from $\beta_1=\alpha\beta_0$, upon inspection of Eq. (6). The other criterion for cloaking being that the wave impedances must be appropriately matched at the outer boundary $r=b$ of the cloak, which is an additional constraint met by the wave speed associated with material parameters (4) and (5).

We now turn to the numerical analysis of the field radiated by a point source vibrating harmonically in the x_3 direction and generating a harmonic vibration on the plate. Note that all the lengths are in arbitrary units (e.g., mm). This point source is located at the point (0.5,0.5) in the vicinity of a clamped obstacle shaped as the letter E and surrounded by the heterogeneous orthotropic cloak, which consists of a circular coating of radii $a=0.2$ and $b=0.4$ which is centered about the origin. Its elastic parameters are characterized by a spatially varying scalar density ρ and a spatially varying rank-2 tensor $\underline{\underline{E}}$ given by Eq. (5). Figure 1 represents the real part of the vertical displacement U' in the presence of the point source that is nothing but the bending waves at the instant $t=0$. As expected, the vertical displacement outside the cloak is identical to the one we obtain when the plate is homogeneous (without any obstacle and cloak). This shows the efficiency of the cloaking by coordinate transformation for this type of waves. Moreover, one can observe that the field inside the cloaked region (central disk) is null, which is another confirmation of the relevance of the approach.

In Fig. 2 the vertical displacement U' outside the cloak seems to originate from a location $r_s'=\alpha^{-1}r_s+a$, which is slightly shifted with respect to the real position r_s of the source (located inside the cloak itself). This effect is similar to the mirage effect already observed in electromagnetic cloaks.[16] In Fig. 3 the vertical displacement U' on the line

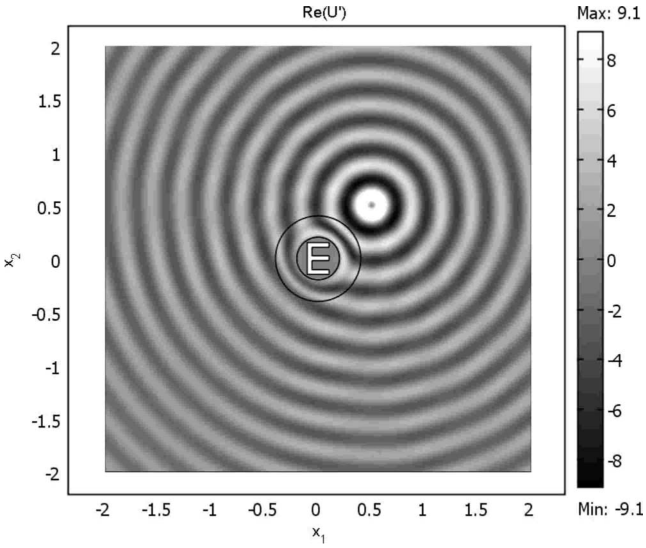

FIG. 1. Real part of the displacement field U' distribution in the vicinity of the cloaked E-shaped rigid clamped obstacle. The source is located at point (0.5,0.5) and its wavelength is $\lambda_0=2\pi/\beta_0=0.28$, which is of the same order as the inner and outer radii of the cloak: $a=0.2$ and $b=0.4$.

$x_2=0$ is shown for both a source located at r_s' inside the coating and a source shifted laterally at r_s in a homogeneous plate (i.e., without any cloak and inclusion). The respective positions are given in the figure caption.

In order to better illustrate the efficiency of the proposed invisibility cloak we have computed the diffracted vibration amplitude that is the difference between the total vibration and the incident one. Figures 4 and 5 show the squared modulus of the vibration amplitude in the case of the object alone and the cloaked object. Thus one can immediately see that the scattered field vanishes outside the cloak itself,

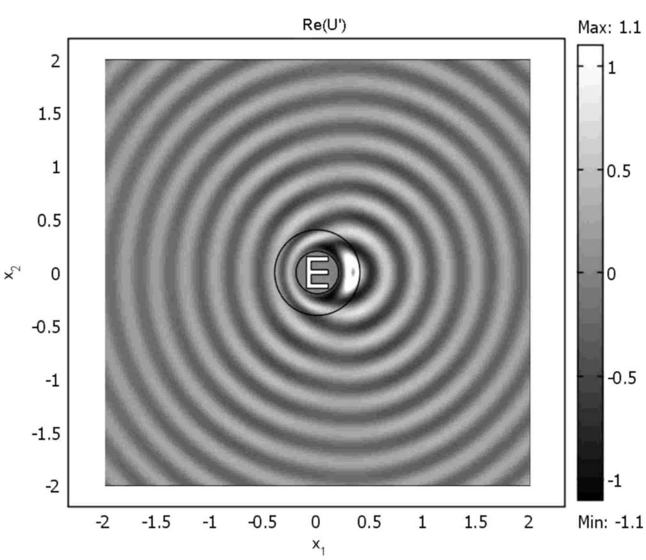

FIG. 2. Real part of the displacement field U' when the source is located inside the coating at the point (0.335,0). The field seems to be emitted by a shifted source located at point (0.27,0); see also Fig. 3 (mirage effect).

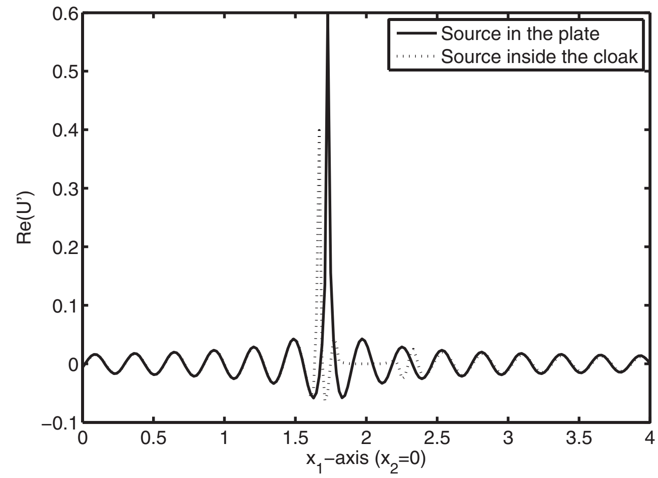

FIG. 3. Real part of the displacement U' along the line $y=0$ for a source located inside the coating at point (0.335.0) as shown in Fig. 2 (dashed line) and for a source located in a homogeneous plate at point (0.27,0) (solid line).

hence showing that the total field is identical to the incident one as expected. The small remaining variation is inherent to the numerical computation of the fields. Moreover, the comparison with the object alone allows illustrating that the scattering due to the chosen object is important, and confirming the effect of the invisibility cloak.

Note that we consider a linear model, and that the behavior of the cloak does not depend on the frequency; thus, the structure works whatever the incident field including nonharmonic excitation such as pulses or any superposition of vibrating frequencies. Of course this supposes that the amplitude of vibration is such that the linear regime is ensured. Moreover, manufacturing such an invisibility cloak would require structuring the material to design a metamaterial that would mimic the required properties (mass density and Young modulus). Actual realizations of such metamaterials

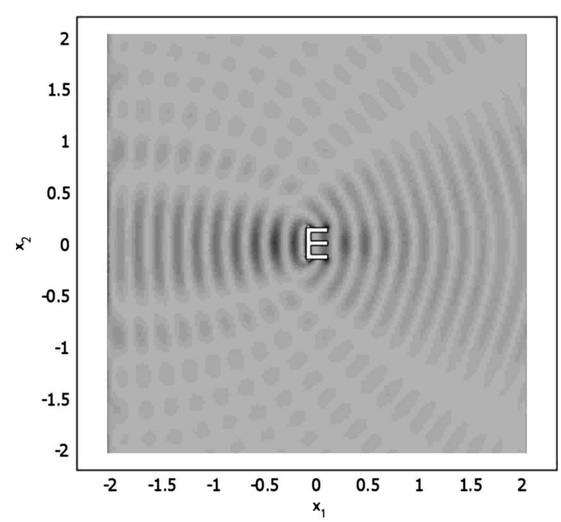

FIG. 4. Snapshot of the square modulus of the diffracted vibration amplitude around the uncloaked object. The light gray level corresponds to small values of the field (vanishing) while the black corresponds to its maximum (2).

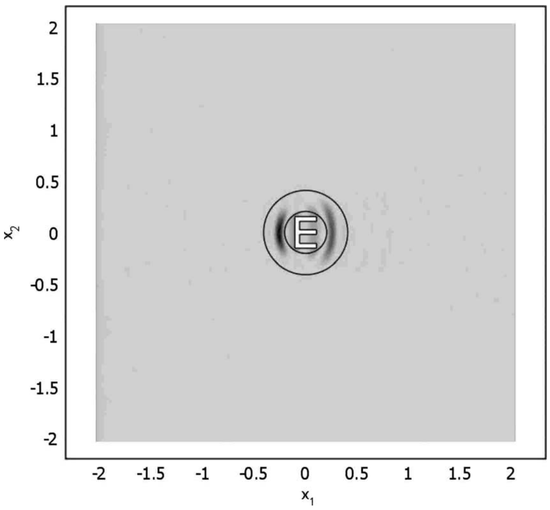

FIG. 5. Snapshot of the square modulus of the diffracted vibration amplitude around the cloak described in Fig. 1. The scale of colors is the same as in Fig. 4.

in electromagnetism could be found in Ref. 8 and in acoustics, with an example of possible structure given in Ref. 10. A consequence of structuring the material is that the metamaterial itself is usually dispersive, thus it becomes much more difficult to ensure a broadband cloaking. However, we have proposed in a recent paper a way to design cloaks that allows us to get a significantly larger working frequency domain.[11]

Another consequence of structuring the metamaterial is that the object hidden cannot be as large or as small as one wishes with respect to the wavelength. Indeed, one has to play with the following constraints: the characteristic size of the structure of the material should be small with respect to the wavelength, whereas the size of the coating itself should not be too small with respect to the wavelength. Moreover the manufacturing technology used adds itself its own constraints that should be taken into account.

In conclusion, we have studied analytically and numerically the extension of the cloaking mechanism described in Refs. 6, 9, 13, and 14 to the domain of flexural waves propagating in thin infinite elastic plates. These waves obey the biharmonic equation so that the problem we have treated above is neither isomorphic to the Maxwell system nor to the acoustic case. Nevertheless, our numerical computations based on the finite elements method proved that a rigid clamped obstacle surrounded by a coating with anisotropic Young modulus and a radially dependent mass density is neutral for the bending waves generated by a point source located in its vicinity. Moreover, we have shown that a counterpart of the classical mirage effect in optics could be observed for bending waves when a source is located inside the coating.

Lastly, the required elastic parameters for this bending cloak could be achieved through homogenization of a perforated plate displaying concentric structured layers (design similar to the structured cloak for liquid surface waves[11]) since its density need not be anisotropic. An alternative design could be based upon a generalization of the work by Torrent and Sanchez-Dehesa[10] to plate theory. We thus believe our study could have important medical applications in isolating structures from vibrations of metallic (or nonmetallic) plates. Obviously, aeronautic, ship, and car industries could benefit from such acoustic insulations. On a larger scale, we hope that our study will foster efforts in the control of flexural waves in geophysics, potentially opening new vistas in the design of passive antiearthquake systems.

1 S. Timoshenko, *Theory of Plates and Shells* (McGraw-Hill, New York, 1940).
2 K. F. Graff, *Wave Motion in Elastic Solids* (Dover, New York, 1975).
3 A. N. Norris and C. Vemula, J. Sound Vib. **181**, 115 (1995).
4 D. V. Evans and R. Porter, J. Eng. Math. **58**, 317 (2007).
5 A. B. Movchan, N. V. Movchan, and R. C. McPhedran, Proc. R. Soc. London, Ser. A **463**, 2505 (2007).
6 J. B. Pendry, D. Schurig, and D. R. Smith, Science **312**, 1780 (2006).
7 U. Leonhardt, Science **312**, 1777 (2006).
8 D. Schurig, J. J. Mock, B. J. Justice, S. A. Cummer, J. B. Pendry, A. F. Starr, and D. R. Smith, Science **314**, 977 (2006).
9 S. A. Cummer and D. Schurig, New J. Phys. **9**, 45 (2007).
10 D. Torrent and J. Sanchez-Dehesa, New J. Phys. **10**, 063015 (2008).
11 M. Farhat, S. Enoch, S. Guenneau, and A. B. Movchan, Phys. Rev. Lett. **101**, 134501 (2008).
12 H. Chen and C. T. Chan, Appl. Phys. Lett. **91**, 183518 (2007).
13 S. A. Cummer, B. I. Popa, D. Schurig, D. R. Smith, J. Pendry, M. Rahm, and A. Starr, Phys. Rev. Lett. **100**, 024301 (2008).
14 G. W. Milton, M. Briane, and J. R. Willis, New J. Phys. **8**, 248 (2006).
15 H. Chen, B. I. Wu, B. Zhang, and J. A. Kong, Phys. Rev. Lett. **99**, 063903 (2007).
16 F. Zolla, S. Guenneau, A. Nicolet, and J. B. Pendry, Opt. Lett. **32**, 1069 (2007).