Interaction of an electromagnetic wave with a cone-shaped invisibility cloak and polarization rotator

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A cone-shaped cloak whose cross section gradually increases along the axial direction (z direction) is proposed in this paper. We present full wave analysis of this cloak in response to electromagnetic waves, showing that a perfect conical cloak can support the propagation of any kind of fields. In addition, the reduced set of cloaking parameters is derived for azimuthally invariant ($\partial/\partial\varphi=0$) incident fields. The advantage of this simplified cloaking structure is that all the components of material parameters are spatially invariant with relative magnitude larger than one. Hence, conical cloak with small scattering is physically realizable within a wide band of frequency for this specific type of incident fields. Finally, we apply similar transformation to achieve a polarization rotator which can arbitrarily control the polarization of the electromagnetic wave getting through. The proposed design provides a practical way to realize invisible cloak and some other electromagnetic devices, especially in the conditions that the source distribution is rotationally symmetric.

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I. INTRODUCTION

Creating electromagnetic (EM) cloak of invisibility has attracted much attention in the recent years.[1–22] One important approach to achieve invisibility is based on the form-invariant coordinate transformation of Maxwell's equations.[3] This concept which generalized a similar idea on the transformation of conductivity equation[1] has triggered enormous interests since the proposed cloaking device can effectively exclude EM wave from the interior region and rend object with arbitrary shape and size invisible. A similar conformal mapping method was applied to two-dimensional (2D) space to produce similar effect in geometric limit.[4] Afterward, the first experimental demonstration of such a cloak with simplified parameters was reported.[5] Further full wave simulations and theoretical treatments of the problems were also presented.[6–23]

Most of the cloak designs considered in the literatures have been restricted to 2D geometrics, where the cross sections of the cloaks remain unchanged and an obstacle is wrapped completely with the cloak shell. In this paper, we propose a cone-shaped cloak with varying cross section along the axial direction. Full wave scattering model is established to analyze the electromagnetic characteristics of this type of cloak. It is demonstrated that a perfect conical cloak can guide any kind of EM waves, since the transcendental condition is naturally satisfied for arbitrary separation parameter ν, which is quite different from classical cone scattering problems.[24–26] Therefore, the conical cloak is quite useful in many practical cases. Furthermore, it is shown that for azimuthally invariant ($\partial/\partial\varphi=0$) incoming field, where only H_φ, E_r, and E_θ (or E_φ, H_r, and H_θ) enter into Maxwell's equations, it is possible to construct a simplified conical cloak whose relative permittivity and permeability elements are all spatially invariant and larger than one. Consequently, this simplified cloaking structure can be created with natural material, and unlimited bandwidth can be realized for this specific incident waves. Apart from cloak of invisibility, some other interesting EM devices can be constructed by extending the same idea, such as conical polarization rotator, which can arbitrarily control the polarization of EM wave as it transmits from one side of half space to another. All these interesting phenomena are evaluated by full wave analysis and further potential applications are also discussed.

II. MATERIAL PARAMETERS OF A CONICAL CLOAK

We use transformation approach to obtain the material parameters of a conical cloak. The associated spatial distortion is described as a mapping between the original spherical coordinate (r',θ',ϕ') and the new physical one (r,θ,ϕ). The compressing of the space from a solid cone $(0\leq\theta'\leq\alpha)$ into a conical cover $(\beta\leq\theta\leq\alpha)$ can therefore be given by the following equations:

$$r'=r,\quad \theta'=g(\theta),\quad \varphi'=\varphi, \tag{1}$$

where $g(\theta)$ is an arbitrary monotonic function with $g(\alpha)=\alpha$ and $g(\beta)=0$. α and β represent coordinate parameters of the outer and inner surfaces of the conical cloak. As shown in Fig. 1, these two interfaces share the same vertex at the original point. In our study, the original space is assumed to be free space (characterized by $\bar{\bar{\varepsilon}}'=\varepsilon_0\bar{\bar{I}}$ and $\bar{\bar{\mu}}'=\mu_0\bar{\bar{I}}$). Hence the

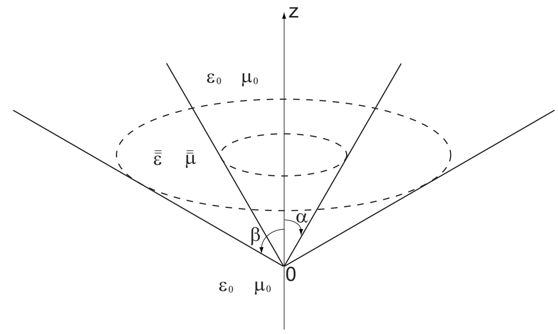

FIG. 1. Geometry of a conical cloak.

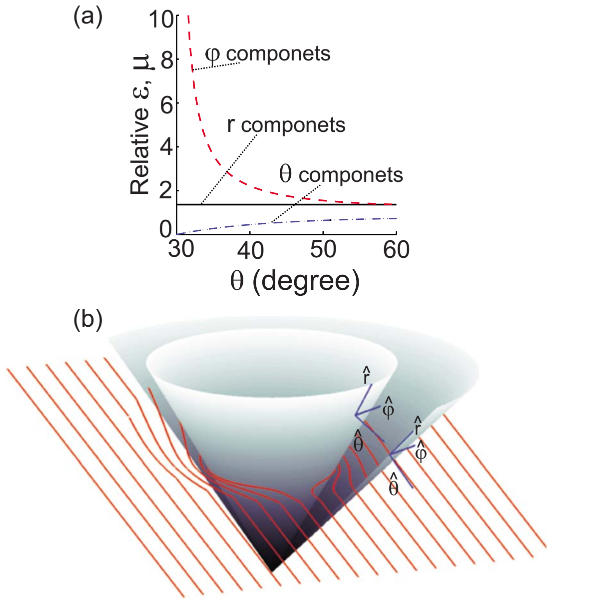

FIG. 2. (Color online) (a) The material parameters of the conical cloak as a function of the longitudinal angle θ. (b) Configuration of rays traversing a conical cloak at an oblique angle. The transformation medium that composes the cloak lies between the two cones (α and β are 60° and 30°, respectively).

permittivity ($\bar{\bar{\varepsilon}}$) and permeability ($\bar{\bar{\mu}}$) tensors of the conical cloak can be expressed in the transformed physical coordinate system (r,θ,ϕ) as follows:

$$\bar{\bar{\varepsilon}}=\varepsilon_0\begin{bmatrix}\varepsilon_{rr} & 0 & 0\\ 0 & \varepsilon_{\theta\theta} & 0\\ 0 & 0 & \varepsilon_{\varphi\varphi}\end{bmatrix},\quad \bar{\bar{\mu}}=\mu_0\begin{bmatrix}\mu_{rr} & 0 & 0\\ 0 & \mu_{\theta\theta} & 0\\ 0 & 0 & \mu_{\varphi\varphi}\end{bmatrix},\quad (2)$$

where $\varepsilon_{rr}=\mu_{rr}=g'(\theta)\sin g(\theta)/\sin\theta$, $\varepsilon_{\theta\theta}=\mu_{\theta\theta}=\sin g(\theta)/[g'(\theta)\sin\theta]$, and $\varepsilon_{\varphi\varphi}=\mu_{\varphi\varphi}=g'(\theta)\sin\theta/\sin g(\theta)$. ε_{rr}, $\varepsilon_{\theta\theta}$, and $\varepsilon_{\varphi\varphi}$ represent the relative permittivity elements along r, θ, and φ directions, while μ_{rr}, $\mu_{\theta\theta}$, and $\mu_{\varphi\varphi}$ are the relative permeability elements along r, θ, and φ directions, respectively. In order to simplify the material parameters we select the transformation function $g(\theta)$ as

$$g(\theta)=\arccos[A(\cos\theta-\cos\beta)+1], \quad (3)$$

where $A=(1-\cos\alpha)/(\cos\beta-\cos\alpha)>1$. Substitute Eq. (3) into Eq. (2), the corresponding relative material parameters can be cast in the following forms:

$$\varepsilon_{rr}=\mu_{rr}=A,\quad \varepsilon_{\theta\theta}=\mu_{\theta\theta}=\frac{\sin^2 g(\theta)}{A\sin^2\theta},$$

$$\varepsilon_{\varphi\varphi}=\mu_{\varphi\varphi}=\frac{A\sin^2\theta}{\sin^2 g(\theta)}. \quad (4)$$

Equation (4) shows that the radial components of the effective constitutive parameters become a constant larger than 1. To create such a cloak, we need to carefully control the other two components with artificially structured metamaterial.[27,28] Figure 2(a) depicts the three components of material parameters as functions of the longitudinal angle θ. It is worth noticing that under the transformation, the permittivity and permeability tensors of the conical cloak are strongly anisotropic and severely inhomogeneous. In addition, at the cloak's interior surface ($\theta=\beta$), $\varepsilon_{\theta\theta}$ and $\mu_{\theta\theta}$ are zero while $\varepsilon_{\varphi\varphi}$ and $\mu_{\varphi\varphi}$ tend to become infinite. To verify the validity of the derived parameters, the ray tracing exercise[3,7] is performed, shown in Fig. 2(b) when a bulk of light is obliquely incident upon the conical cloak. This result indicates that the configuration specified by Eq. (4) can exclude rays from the internal region, and the light emerged from the cloak is the same as if it had just passed through free space.

III. INTERACTIONS BETWEEN ELECTROMAGNETIC WAVE AND A PERFECT CONICAL CLOAK

The ray tracing exercise in Sec. II shows that the conical cloak can bend and guide the incoming light smoothly around the internal region. In this section, we use full wave scattering method to study the response of a perfect conical cloak to electromagnetic waves. We assume all the sources are located beyond the region of interest. Similar to the process in Refs. 18 and 20, we utilize Mie scattering theory and express the fields through two scalar potentials $U(r,\theta,\phi)$ and $V(r,\theta,\phi)$ as follows:[29]

$$E_r=i\omega\left(\frac{1}{k_0^2\mu_{\theta\theta}\varepsilon_{\varphi\varphi}}\frac{\partial^2}{\partial r^2}U+U\right), \quad (5a)$$

$$E_\theta=\frac{i\omega}{k_0^2\mu_{\varphi\varphi}\varepsilon_{\theta\theta}}\frac{1}{r}\frac{\partial^2}{\partial r\,\partial\theta}U-\frac{1}{\varepsilon_0\varepsilon_{\theta\theta}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}V, \quad (5b)$$

$$E_\varphi=\frac{i\omega}{k_0^2\mu_{\theta\theta}\varepsilon_{\varphi\varphi}}\frac{1}{r\sin\theta}\frac{\partial^2}{\partial r\,\partial\varphi}U+\frac{1}{\varepsilon_0\varepsilon_{\varphi\varphi}}\frac{1}{r}\frac{\partial}{\partial\theta}V, \quad (5c)$$

$$H_r=i\omega\left(\frac{1}{k_0^2\mu_{\varphi\varphi}\varepsilon_{\theta\theta}}\frac{\partial^2}{\partial r^2}V+V\right), \quad (5d)$$

$$H_\theta=\frac{i\omega}{k_0^2\mu_{\theta\theta}\varepsilon_{\varphi\varphi}}\frac{1}{r}\frac{\partial^2}{\partial r\,\partial\theta}V+\frac{1}{\mu_0\mu_{\theta\theta}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}U, \quad (5e)$$

$$H_\varphi=\frac{i\omega}{k_0^2\mu_{\varphi\varphi}\varepsilon_{\theta\theta}}\frac{1}{r\sin\theta}\frac{\partial^2}{\partial r\,\partial\varphi}V-\frac{1}{\mu_0\mu_{\varphi\varphi}}\frac{1}{r}\frac{\partial}{\partial\theta}U, \quad (5f)$$

where $k_0=\omega\sqrt{\mu_0\varepsilon_0}$ is the wave number in free space. Substituting Eq. (4) into Eqs. (5a)–(5f), we can obtain the wave equations for U and V:

$$\left[r^2\frac{\partial^2}{\partial r^2}+\frac{1}{\sin g(\theta)g'(\theta)}\frac{\partial}{\partial\theta}\frac{\sin g(\theta)}{g'(\theta)}\frac{\partial}{\partial\theta}+\frac{1}{\sin^2 g(\theta)}\frac{\partial^2}{\partial\varphi^2}+k_0^2\right]\begin{pmatrix}U\\V\end{pmatrix}=0. \tag{6}$$

In what follows, we develop some general properties of the U function, which also concern V.

Utilizing separation of variables method, the U function can be expressed for all values of $r\in(0,\infty)$ as a linear combination of the form $U=R_\nu(r)\Theta_\nu^m(\theta)\Phi_m(\varphi)$, where the explicit expressions of $R_\nu(r)$, $\Theta_\nu^m(\theta)$, and $\Phi_m(\varphi)$ as well as the separation parameters ν and m can be determined through the boundary condition, radiation condition, and sources. For problems with rotational symmetry, the azimuthal function Φ is of the trivial form $\Phi_m(\varphi)=e^{im\varphi}$ and the parameter m is an integer ($m\in\mathbb{Z}$). The radial function $R_\nu(r)$ and the angular function $\Theta_\nu^m(\theta)$ are the solutions of the following equations, respectively:

$$r^2\frac{\partial^2}{\partial r^2}R+\left[k_0^2r^2-\left(\nu^2-\frac{1}{4}\right)\right]R=0, \tag{7}$$

$$\frac{1}{\sin g(\theta)g'(\theta)}\frac{\partial}{\partial\theta}\frac{\sin g(\theta)}{g'(\theta)}\frac{\partial}{\partial\theta}\Theta+\left[\left(\nu^2-\frac{1}{4}\right)-\frac{m^2}{\sin^2 g(\theta)}\right]\Theta=0. \tag{8}$$

Equation (7) shows that the radial function $R_\nu(r)$ is a solution of Bessel's function and can be written in terms of appropriate combination of the following form:[24]

$$R_\nu(r)=\sqrt{k_0r}J_\nu(k_0r)+C_\nu\sqrt{k_0r}J_{-\nu}(k_0r),$$

$$\nu\notin(0,\pm1,\pm2,\pm3,\ldots) \tag{9a}$$

or

$$R_n(r)=\sqrt{k_0r}J_n(k_0r)+C_n\sqrt{k_0r}Y_n(k_0r),$$

$$\nu=n\in(0,\pm1,\pm2,\pm3,\ldots), \tag{9b}$$

where C_n (or C_ν) is an arbitrary constant while J_n and Y_n represent the Bessel functions of the first and second kinds, respectively. As to the angular function $\Theta_\nu^m(\theta)$, with the substitution $\eta=\cos[g(\theta)]$ it can be reduced to a solution of Legendre's equation and is of the general form

$$\Theta_\nu^m(\theta)=P_{\nu-1/2}^m[\cos g(\theta)]+D_{\nu,m}Q_{\nu-1/2}^m[\cos g(\theta)], \tag{10}$$

where P and Q are associated Legendre polynomials of the first and second kinds, respectively. Therefore, the general expression of U can be expressed as

$$U(r,\theta,\varphi)=\sum_{\nu\in\Im}\sum_{m\in\mathbb{Z}}a_{\nu,m}R_\nu(r)\Theta_\nu^m(\theta)\Phi_m(\varphi) \tag{11a}$$

or

$$U(r,\theta,\varphi)=\int_\Gamma d\nu\sum_{m\in\mathbb{Z}}b_{\nu,m}R_\nu(r)\Theta_\nu^m(\theta)\Phi_m(\varphi) \tag{11b}$$

depending on whether the separation parameter ν are discrete or continuous.

The admissible values of ν can be determined by Eq. (16) in the following part. It should be pointed out that any function $U(r,\theta,\phi)$ satisfying certain integrability and continuity conditions can always be represented through an expression similar to Eqs. (11a) and (11b) by appropriately choosing $\Im$ (or Γ).[24] Thus, the solution of a scattering problem under given boundary condition consists of the determination of the set $\Im$ or the contour Γ appearing in Eqs. (11a) and (11b) along with the expansion constant $a_{\nu,m}$ (or $b_{\nu,m}$).

Consider the case where the parameter ν is continuous (discrete ν can be derived in similar manner), we can express the scalar potentials, respectively, for the exterior fields ($\alpha<\theta\le\pi$), internal fields ($0\le\theta\le\beta$), and the fields of the cloak layer ($\beta<\theta<\alpha$) as follows:

$$U^{\text{ext}}=\sqrt{k_0r}\int_\Gamma d\nu\sum_{m\in\mathbb{Z}}a_{\nu,m}^{(U)}J_\nu(k_0r)P_{\nu-1/2}^m(\cos\theta)e^{im\varphi}, \tag{12a}$$

$$V^{\text{ext}}=\sqrt{k_0r}\int_\Gamma d\nu\sum_{m\in\mathbb{Z}}a_{\nu,m}^{(V)}J_\nu(k_0r)P_{\nu-1/2}^m(\cos\theta)e^{im\varphi}, \tag{12b}$$

$$U^{\text{int}}=\sqrt{k_0r}\int_\Gamma d\nu\sum_{m\in\mathbb{Z}}c_{\nu,m}^{(U)}J_\nu(k_0r)P_{\nu-1/2}^m(\cos\theta)e^{im\varphi}, \tag{13a}$$

$$V^{\text{int}}=\sqrt{k_0r}\int_\Gamma d\nu\sum_{m\in\mathbb{Z}}c_{\nu,m}^{(V)}J_\nu(k_0r)P_{\nu-1/2}^m(\cos\theta)e^{im\varphi}, \tag{13b}$$

$$U^{\text{cloak}}=\sqrt{k_0r}\int_\Gamma d\nu\sum_{m\in\mathbb{Z}}d_{\nu,m}^{(U)}J_\nu(k_0r)\{P_{\nu-1/2}^m[\cos g(\theta)]+D_{\nu,m}^{(U)}Q_{\nu-1/2}^m[\cos g(\theta)]\}e^{im\varphi}, \tag{14a}$$

$$V^{\text{cloak}}=\sqrt{k_0r}\int_\Gamma d\nu\sum_{m\in\mathbb{Z}}d_{\nu,m}^{(V)}J_\nu(k_0r)\{P_{\nu-1/2}^m[\cos g(\theta)]+D_{\nu,m}^{(V)}Q_{\nu-1/2}^m[\cos g(\theta)]\}e^{im\varphi}, \tag{14b}$$

where $a_{\nu,m}^{(U)}$, $a_{\nu,m}^{(V)}$, $c_{\nu,m}^{(U)}$, $c_{\nu,m}^{(V)}$, $d_{\nu,m}^{(U)}$, $d_{\nu,m}^{(V)}$, $D_{\nu,m}^{(U)}$, and $D_{\nu,m}^{(V)}$ are all undetermined expansion coefficients. With the help of Eqs.

(5a)–(5f), the expressions of EM fields in the three regions can be obtained. By applying the continuity of E_r, E_φ, H_r, and H_φ at the two boundaries $\theta=\alpha$ and $\theta=\beta$, we can get eight boundary equations, four for U function and the other four for V function. Here we only consider the case of U function (the V function case can be similarly deduced) and list the four boundary equations as follows:

$$d_{\nu,m}^{(U)}\{P_{\nu-1/2}^{m}[\cos g(\beta)]+D_{\nu,m}^{(U)}Q_{\nu-1/2}^{m}[\cos g(\beta)]\} = c_{\nu,m}^{(U)}P_{\nu-1/2}^{m}(\cos\beta), \tag{15a}$$

$$d_{\nu,m}^{(U)}\sin^2 g(\beta)\{P_{\nu-1/2}^{m}{}'[\cos g(\beta)]+D_{\nu,m}^{(U)}Q_{\nu-1/2}^{m}{}'\,[\cos g(\beta)]\} = c_{\nu,m}^{(U)}\sin^2\beta P_{\nu-1/2}^{m}{}'(\cos\beta), \tag{15b}$$

$$d_{\nu,m}^{(U)}\{P_{\nu-1/2}^{m}[\cos g(\alpha)]+D_{\nu,m}^{(U)}Q_{\nu-1/2}^{m}[\cos g(\alpha)]\} = a_{\nu,m}^{(U)}P_{\nu-1/2}^{m}(\cos\alpha), \tag{15c}$$

$$d_{\nu,m}^{(U)}\sin^2 g(\alpha)\{P_{\nu-1/2}^{m}{}'[\cos g(\alpha)]+D_{\nu,m}^{(U)}Q_{\nu-1/2}^{m}{}'\,[\cos g(\alpha)]\} = a_{\nu,m}^{(U)}\sin^2\alpha P_{\nu-1/2}^{m}{}'(\cos\alpha). \tag{15d}$$

Since $g(\beta)$ is zero and $Q_{\nu-1/2}^{m}[\cos g(\beta)]$ is an infinite term, it can be seen from Eqs. (15a) and (15b) that $c_{\nu,m}^{(U)}=D_{\nu,m}^{(U)}=0$, indicating that the internal fields are exactly zero. It is worth noticing that for the case where $m=0$, the product $D_{\nu,m}^{(U)}Q_{\nu-1/2}^{m}[\cos g(\beta)]$ is equal to -1, although $D_{\nu,m}^{(U)}$ is zero. Obviously, the value of this product is nonzero only at the inner boundary ($\theta=\beta$) of the cloak. With the help of Eq. (5f), it can be demonstrated that the φ-directed magnetic-flux density B_φ blows up[11] at the cloak's interior surface, which can be characterized as a magnetic surface current induced along φ direction.[19] However, when m is a nonzero term, $D_{\nu,m}^{(U)}Q_{\nu-1/2}^{m}[\cos g(\beta)]$ is exactly equal to zero. Thus, surface current will not be induced in this case. Afterwards, by substituting $D_{\nu,m}^{(U)}=0$ into Eqs. (15c) and (15d), we obtain the transcendental equation for the separation parameter ν:

$$\frac{\sin^2 g(\alpha)P_{\nu-1/2}^{m}{}'[\cos g(\alpha)]}{\sin^2\alpha P_{\nu-1/2}^{m}[\cos g(\alpha)]}=\frac{P_{\nu-1/2}^{m}{}'(\cos\alpha)}{P_{\nu-1/2}^{m}(\cos\alpha)}. \tag{16}$$

Since $g(\alpha)$ is equal to α, Eq. (16) holds true for arbitrary real or complex ν, indicating that the whole system behaves as free space which supports the propagation of any kind of EM fields. This result provides another confirmation that the cloak achieved with the parameters given by Eq. (4) is perfect. It is worthwhile to point out that in some nonideal cases where the outer boundary is not matched to free space $[g(\alpha)\neq\alpha]$, the solution set of ν can be still determined from Eq. (16). Thereby, for any given source distribution, all the expansion coefficients can be determined by matching the potential functions U and V to corresponding incident field.[25] Here we show an example where a perfect conical cloak with $\alpha=60°$ and $\beta=30°$ is subject to an illumination at 2 GHz from a linear polarized plane wave propagating in the $+x$ direction with electric field $\bar{E}^i=\hat{y}e^{ik_0x}$. Figures 3(a) and 3(b) depict the calculated E_y field distribution in two orthogonal planes. The results indicate that the fields have been excluded from the interior region while no scattering is induced outside. There is no near-field enhancement[24,26] at the apex of the perfect conical cloak even when the internal cone is very sharp. However, it can be demonstrated that the performance of the cloak is rather sensitive to tiny perturbations at the inner boundary. Even a minor change will cause a noticeable scattered field outside, which has something in common with the cylindrical cloak case.[19,22]

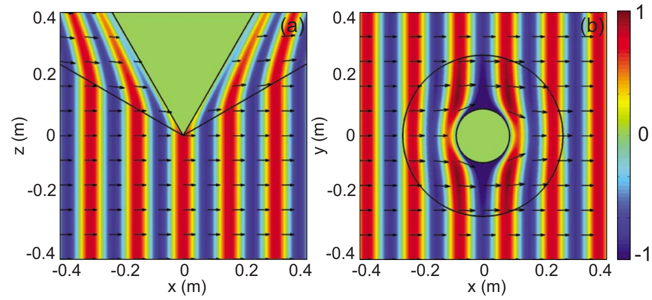

FIG. 3. (Color online) (a) The y-directed total electric-field distribution in the x-z plane for a plane-wave scattering problem. (b) E_y field distribution in the same case observed at the plane $z=0.15$ m. The incident plane wave propagates in the $+\hat{x}$ direction with normalized E_y polarized electric field. The arrows represent the Poynting vectors. The outer longitudinal angle (α) is set to be 60°, while the inner one (β) is 30°.

Notice that apart from a sharp cone, the surface of the conical cloak can be a plane. As depicted in Fig. 4(a) where the outer boundary of cloak is flat ($\alpha=90°$), the incident plane wave is compressed and spread along the inner surface ($\beta=60°$) of the concave conical shell. Figure 4(b) shows the case where a plane wave is obliquely incident upon a conical cloak with $\alpha=90°$ and $\beta=85°$. It can be seen from these two examples that if the conical cover is fabricated to a thin layer ($\beta\approx\alpha=90°$), the incident wave will be compressed from any direction into a very thin quasiplane. In Figs. 4(c) and 4(d), the outer longitudinal angle α is larger than 180° and the EM wave can be trapped in a small cone.

IV. CONICAL CLOAK WITH SIMPLIFIED MATERIAL PARAMETERS

The current fabrication of electromagnetic cloak requires anisotropic media with each component of permittivity and permeability independently controlled.[1,3,5–9,12–15,21,22] Although this is within the realm of metamaterial technique,[27,28] it still requires the radial permittivity and permeability elements with relative magnitudes to be less than one. Even the cylindrical cloak with simplified parameters[5,6,9,12] cannot overcome this drawback. Therefore, the bandwidth of a passive cloaking structure will be always limited. However, we find that the conical cloak based on the reduced set of material parameters offers a feasible solution to this problem.

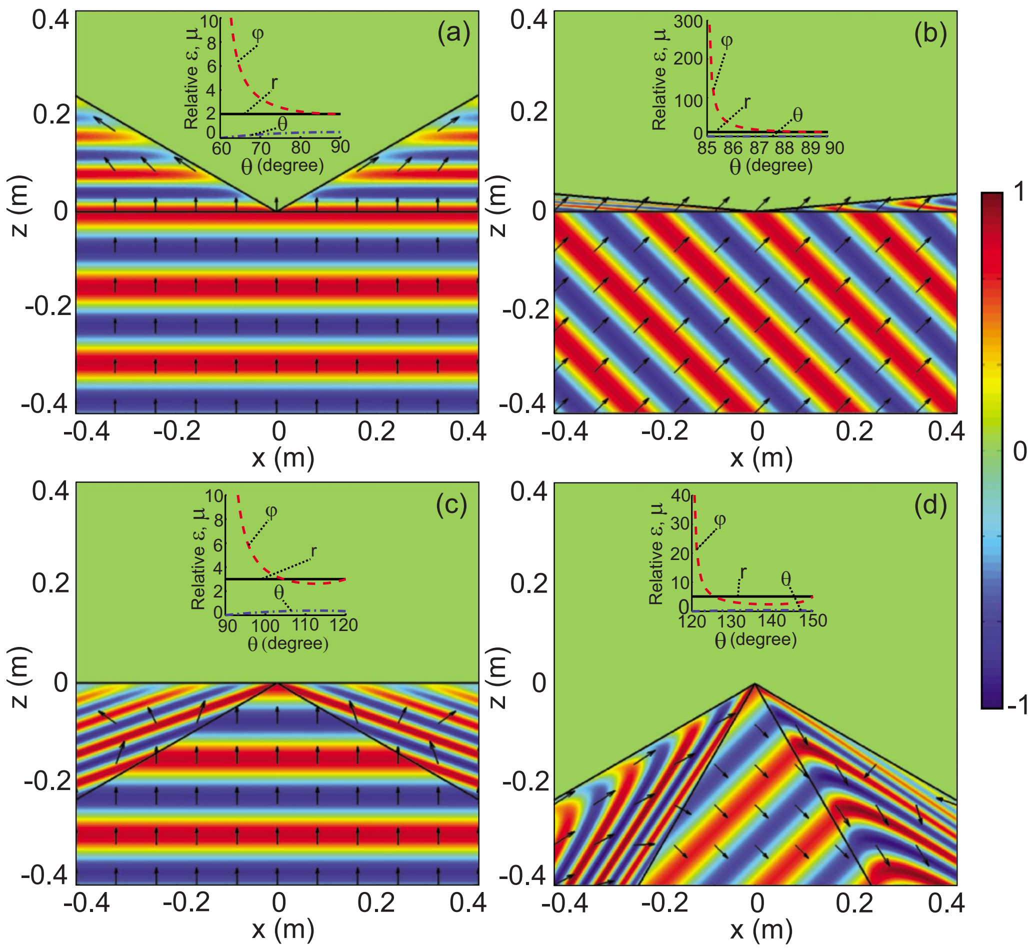

FIG. 4. (Color online) The y-directed total electric-field distribution and Poynting vectors (arrows) in the x-z plane for a plane-wave scattering problem by a perfect conical cloak with different inner and outer longitudinal angles. Here, four cases are considered: (a) $\alpha=90°$ and $\beta=60°$. (b) $\alpha=90°$ and $\beta=85°$. (c) $\alpha=120°$ and $\beta=90°$. (d) $\alpha=150°$ and $\beta=120°$. The insets in these figures depict the r, θ, and φ components of the material parameters of the conical cover.

Here we consider transverse magnetic case, where only H_φ, E_r, and E_θ enter into Maxwell's equations (transverse electric case can be derived in the same way). Similar to the realization of simplified cylindrical cloak,[5,6] by keeping the products $\mu_{\varphi\varphi}\varepsilon_{rr}$ and $\mu_{\varphi\varphi}\varepsilon_{\theta\theta}$ unchanged, the material parameters of a reduced conical cloak can be obtained as

$$\varepsilon_{rr}=[g'(\theta)]^2, \quad \varepsilon_{\theta\theta}=1, \quad \mu_{\varphi\varphi}=1. \tag{17}$$

If $g(\theta)$ is selected as a linear transformation function $g(\theta)=\alpha(\theta-\beta)/(\alpha-\beta)$, Eq. (17) is reduced to

$$\varepsilon_{rr}=\left(\frac{\alpha}{\alpha-\beta}\right)^2, \quad \varepsilon_{\theta\theta}=1, \quad \mu_{\varphi\varphi}=1. \tag{18}$$

It is interesting to see that all the permittivity and permeability elements are spatially uniform with relative magnitude larger than one. Consequently, this reduced conical cloak can be realized with natural materials. Furthermore, it is possible to be designed within a wide band of frequency. However, undesired scattering is inevitable since the impedance is mismatched at the outer boundary of the cloak. Therefore, it is necessary to use high-order transformation[12,16] to create smooth impedance at the outer interface. Here, we propose another transformation function $g(\theta)=\alpha[(\theta-\beta)/(\alpha-\beta)]^{(\alpha-\beta)/\alpha}$, which is different from the quadratic transformation used in Refs. 12 and 16. As a result, the effective constitutive parameters of the cloak become

$$\varepsilon_{rr}=\left(\frac{\alpha-\beta}{\theta-\beta}\right)^{\frac{2\beta}{\alpha}}, \quad \varepsilon_{\theta\theta}=1, \quad \mu_{\varphi\varphi}=1. \tag{19}$$

Although the radial relative permittivity element ε_{rr} is dependent of θ, it is always larger than 1. Hence, this reduced conical cloak working for azimuthally invariant incident fields is still physically realizable within a wide band of frequency.

It should be pointed out that the simplified cloaking structure specified by Eq. (17) requires that the magnetic field should be polarized along φ direction. Thus, it is only applicable to the case where the fields of the incoming wave are independent of φ. From calculations, we find that the magnetic field of this azimuthally invariant incident wave in free space must take the following general form:

$$\bar{H} = \hat{\varphi} J_1(k_\rho \rho) e^{k_z z} \quad \text{for} \quad k_z \neq k_0, \tag{20a}$$

$$\bar{H} = \hat{\varphi} \rho e^{k_0 z} \quad \text{for} \quad k_z = k_0, \tag{20b}$$

where $k_\rho^2 + k_z^2 = k_0^2$ and $k_0 = \omega\sqrt{\mu_0 \varepsilon_0}$ is the wave number in free space.

In what follows, we will perform full wave analysis to study the electromagnetic characteristics of the reduced conical cloak based on linear transformation. In the transverse magnetic (TM) case, only U function contributes to the field distribution and the scalar wave equation is described as

$$\left(\frac{r^2}{\mu_{\varphi\varphi}\varepsilon_{\theta\theta}}\frac{\partial^2}{\partial r^2} + \frac{1}{\varepsilon_{rr}\sin\theta}\frac{\partial}{\partial\theta}\frac{\sin\theta}{\mu_{\varphi\varphi}}\frac{\partial}{\partial\theta} + k_0^2\right)U = 0. \tag{21}$$

Since $\partial/\partial\varphi = 0$, the U function may be written in terms of the separation solutions of Eq. (22) as $U = \sum_\nu R_\nu(r)\Theta_\nu(\theta)$. After some algebraic manipulations, we find that the radial function $R_\nu(r)$ still takes exactly the same form as that of perfect conical cloak, namely, Eqs. (9a) and (9b). However, the angular function Θ is different. It is the solution of the following equation:

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}\Theta + \varepsilon_{rr}\left(\nu^2 - \frac{1}{4}\right)\Theta = 0. \tag{22}$$

In the case of linear transformation where the permittivity and permeability elements of the cloak are given by Eq. (19), the solution of Eq. (22) is

$$\Theta_\nu(\theta) = P_\xi(\cos\theta) + D_\nu Q_\xi(\cos\theta), \tag{23}$$

where $\xi = \sqrt{[\alpha/(\alpha-\beta)]^2[\nu^2-(1/4)]+1/4} - 1/2$ and D_ν is an arbitrary constant. Following the same process in Sec. III, the scalar potentials in the three regions ($\alpha < \theta \leq \pi$), ($0 \leq \theta < \beta$), and ($\beta < \theta < \alpha$) can be listed. By applying the boundary conditions (continuity of E_r and H_φ) at the two interfaces $\theta = \alpha$ and $\theta = \beta$, we find that the internal fields are no longer zero and transcendental equation for the parameter ν can be expressed as

$$\begin{vmatrix} P_{\nu-1/2}(\cos\alpha) & -P_\xi(\cos\alpha) & -Q_\xi(\cos\alpha) & 0 \\ P'_{\nu-1/2}(\cos\alpha) & -P'_\xi(\cos\alpha) & -Q'_\xi(\cos\alpha) & 0 \\ 0 & -P_\xi(\cos\beta) & -Q_\xi(\cos\beta) & P_{\nu-1/2}(\cos\beta) \\ 0 & -P'_\xi(\cos\beta) & -Q'_\xi(\cos\beta) & P'_{\nu-1/2}(\cos\beta) \end{vmatrix} = 0. \tag{24}$$

Generally, Eq. (24) has a lot of solutions, which can be determined through numerical method. With the solution set of Eq. (24), we can expand the potential for any azimuthally invariant incident field in eigenfunctions. Finally, all the expansion coefficients can be determined by matching the incoming part of the total potential to the incoming part of the incident potential.[25]

For the high-order transformation case where ε_{rr} is not spatially uniform, Eq. (21) can no longer be solved with analytic Frobenius method.[30] Hence, we deal with the problem trough the finite-element method (FEM). Suppose the incident wave is polarized along φ direction with magnetic field specified by Eq. (20a) (here k_ρ is equal to $0.1k_0$). Figure 5(a) displays the H_φ field distribution when the incoming wave is propagating in free space. Figure 5(b) shows the field distribution of a perfect conductive cone exposed directly to the incoming wave, where distinct scattering can be observed. Figure 5(c) depicts the scattering of a perfect conical cloak. It can be seen that the incident wave outside is almost unchanged and the internal fields are equal to zero. In Fig. 5(d), we plot the numerical magnetic-field distribution when the incoming wave is incident upon a reduced conical cloak obtained from linear transformation. Compared with the perfect conductive cone shown in Fig. 5(b), this simplified cloak has the effect of guiding waves around. However, due to the impedance mismatching at the outer boundary, noticeable field scattering is still induced and some waves can even penetrate the core. Figure 5(e) depicts the numerical total magnetic field for the case where a simplified conical cloak based on high-order transformation is exposed to the incident wave. The result shows that although the field penetrating the inner region is still nonzero in this case, the scattering outside is dramatically reduced.

V. CONICAL POLARIZATION ROTATOR

Other than the cloak, in this section we will propose a conical device which can be used to arbitrarily control the polarization of EM waves. Consider the following transformation:

$$r' = r \quad \theta' = \theta \quad \varphi' = \varphi, \quad (\text{for } \theta > \alpha), \tag{25a}$$

$$r' = r \quad \theta' = \theta \quad \varphi' = \varphi + \varphi_0, \quad (\text{for } \theta < \beta), \tag{25b}$$

$$r' = r \quad \theta' = \theta \quad \varphi' = \varphi + h(\theta), \quad (\text{for } \beta < \theta < \alpha), \tag{25c}$$

where $h(\theta)$ is a continuous function with $h(\alpha)=0$ and $h(\beta) = \varphi_0$. It can be seen that the above mapping rotates an angle φ_0 for region $\theta < \beta$. Assume the background is free space. We find the permittivity and permeability tensors of the conical cover (in the region $\beta < \theta < \alpha$) can be expressed in the spherical coordinate (r, θ, ϕ) as follows:

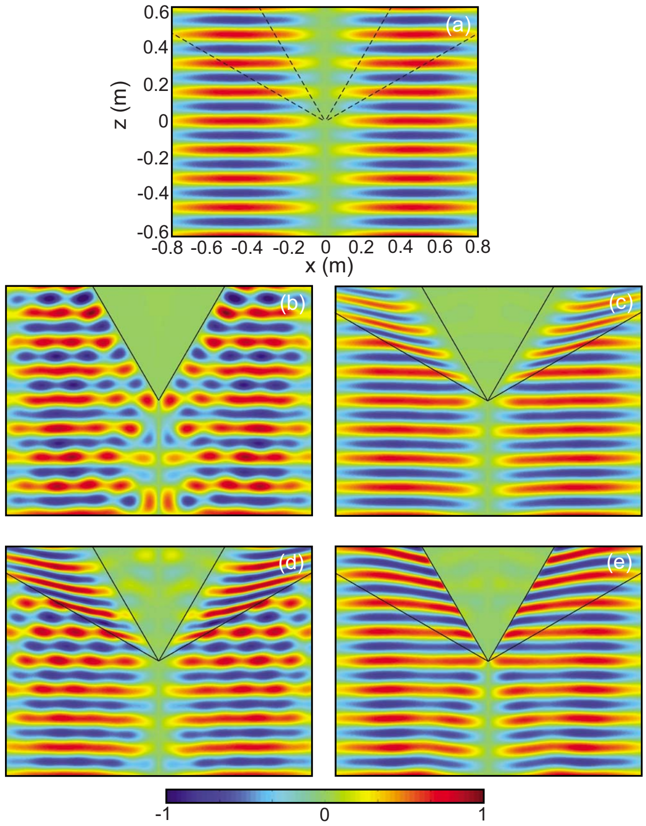

FIG. 5. (Color online) φ-directed total magnetic-field distribution for TM mode scattering problem. (a) The incoming wave with magnetic field $\bar{H}=\hat{\varphi}J_1(k_\rho\rho)e^{i\sqrt{k_0^2-k_\rho^2}z}$ (where $k_\rho=0.1k_0$) propagates in free space. (b) A perfect electric conductor cone with half cone angle equal to 30° is exposed directly to this incoming wave. (c) The incoming wave is incident upon a perfect conical cloak. (d) The incoming wave is incident upon a nonideal conical cloak obtained from linear transformation. (e) The wave is incident upon a simplified cloak achieved with high-order transformation. In cases (c), (d), and (e) the outer longitudinal angle α and the inner longitudinal angle β are set to be 60° and 30°, respectively.

$$\bar{\bar{\varepsilon}}=\varepsilon_0\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -A(\theta)\\ 0 & -A(\theta) & 1+A^2(\theta)\end{bmatrix},$$

$$\bar{\bar{\mu}}=\mu_0\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -A(\theta)\\ 0 & -A(\theta) & 1+A^2(\theta)\end{bmatrix}, \tag{26}$$

where $A(\theta)=h(\theta)\sin\theta$. If the function $h(\theta)$ is selected as

$$h(\theta)=B\left(\ln\tan\frac{\alpha}{2}-\ln\tan\frac{\theta}{2}\right), \tag{27}$$

where $B=\varphi_0/[\ln\tan(\alpha/2)-\ln\tan(\beta/2)]$, $A(\theta)$ can therefore be simplified to a constant $[A(\theta)=-B]$. As a result, all the material parameters become spatially invariant. However, the relative permittivity and permeability tensors are still nondiagonal. For the purpose of simplifying the fabrication process, we let $\sin^2\tau=1/2[1+(B/\sqrt{1+B^2})]$ and define a new orthogonal coordinate system (u,ν,w) with the unit vectors $\hat{u}=\hat{r}$, $\hat{\nu}=\hat{\theta}\cos\tau+\hat{\varphi}\sin\tau$, and $\hat{w}=-\hat{\theta}\sin\tau+\hat{\varphi}\cos\tau$. As a result, the relative permittivity and permeability tensors in the new coordinate are diagonal

$$\bar{\bar{\varepsilon}}=\varepsilon_0\begin{bmatrix}\varepsilon_{uu} & 0 & 0\\ 0 & \varepsilon_{\nu\nu} & 0\\ 0 & 0 & \varepsilon_{ww}\end{bmatrix},\quad \bar{\bar{\mu}}=\mu_0\begin{bmatrix}\mu_{uu} & 0 & 0\\ 0 & \mu_{\nu\nu} & 0\\ 0 & 0 & \mu_{ww}\end{bmatrix}, \tag{28}$$

with $\varepsilon_{uu}=\mu_{uu}=1$, $\varepsilon_{\nu\nu}=\mu_{\nu\nu}=(B^2+2+B\sqrt{B^2+4})/2$, and $\varepsilon_{ww}=\mu_{ww}=(B^2+2-B\sqrt{B^2+4})/2$.

Full wave scattering method is employed to verify the validity of derived parameters [Eq. (28)]. The transformation conical cover with $\alpha=120°$ and $\beta=90°$ is exposed to a E_y polarized plane wave propagating in the $+\hat{z}$ direction at 2 GHz. The rotation angle was chosen as $\varphi_0=90°$. From calculations (similar to that of Sec. III), we find the expressions of the scalar potentials for the exterior fields $(\alpha<\theta\leq\pi)$, internal fields $(0\leq\theta<\beta)$, and the fields of the conical cover $(\beta<\theta<\alpha)$, respectively, as follows:

$$U^{\text{ext}}=\sqrt{k_0r}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}a_{n,m}^{(U)}J_{n+1/2}(k_0r)P_n^m(\cos\theta)e^{im\varphi}, \tag{29a}$$

$$V^{\text{ext}}=\sqrt{k_0r}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}a_{n,m}^{(V)}J_{n+1/2}(k_0r)P_n^m(\cos\theta)e^{im\varphi}, \tag{29b}$$

$$U^{\text{int}}=i\sqrt{k_0r}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}a_{n,m}^{(U)}J_{n+1/2}(k_0r)P_n^m(\cos\theta)e^{im\varphi}, \tag{30a}$$

$$V^{\text{int}}=i\sqrt{k_0r}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}a_{n,m}^{(V)}J_{n+1/2}(k_0r)P_n^m(\cos\theta)e^{im\varphi}, \tag{30b}$$

$$U^{\text{con}}=\sqrt{k_0r}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}a_{n,m}^{(U)}J_{n+1/2}(k_0r)P_n^m(\cos\theta)e^{imh(\theta)}e^{im\varphi}, \tag{31a}$$

$$V^{\text{con}}=\sqrt{k_0r}\sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}a_{n,m}^{(V)}J_{n+1/2}(k_0r)P_n^m(\cos\theta)e^{imh(\theta)}e^{im\varphi}, \tag{31b}$$

where $a_{n,m}^{(U)}$ and $a_{n,m}^{(V)}$ are the expansion coefficients of the incident wave which can be determined by appropriate projections with the spherical harmonic.[25] Finally, by substituting Eqs. (29a), (29b), (30a), (30b), (31a), and (31b) into Maxwell's equations, all the components of the EM fields can be deduced. Figures 6(a) and 6(b) shows the calculated E_y and E_x field distributions in the x-z plane, respectively. It is interesting to see that the wave is totally transmitted from an open region to another without introducing any scattering. Nothing has been changed except that the polarization of the

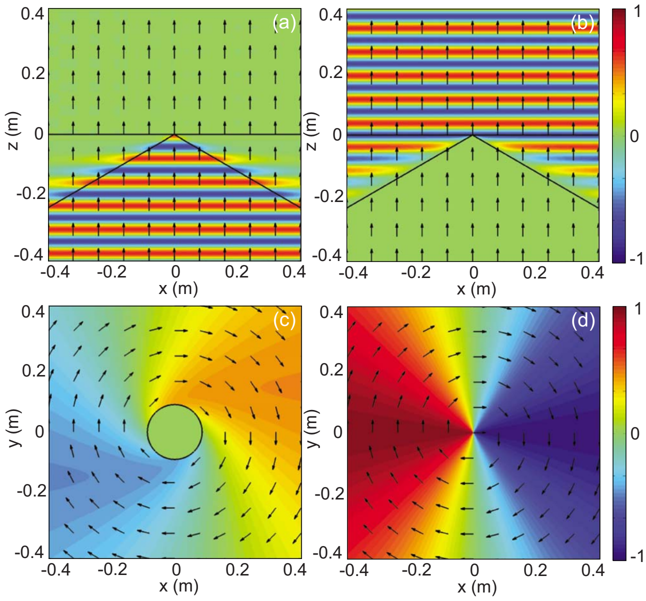

FIG. 6. (Color online) (a) E_y fields distribution and Poynting vectors (arrows) in the x-z plane when an E_y polarized plane wave is incident upon the system along z direction. (b) E_x fields distribution in the x-z plane of this case. (c) E_z field distribution and Poynting vectors observed in the plane $z=-0.05$ m. (d) E_z field distribution and Poynting vectors observed in x-y plane ($z=0$ m).

field has been rotated by an angle of 90°. In order to observe it more clearly, we have also computed the E_z field distribution in two planes vertical to the z axis. As depicted in Figs. 6(c) and 6(d), the field is rotated and the wave inside the conical cover transmits in a spiral trajectory.

VI. CONCLUSION

A peculiar cloaking structure—conical cloak—is proposed in this paper. We have analyzed the EM behavior of this cone-shaped cloak by rigorously solving Maxwell's equations in spherical coordinate system. For the scattering problems of a perfect conical cloak, we show that the transcendental equation is naturally satisfied for the separation parameter ν with arbitrary real or complex values. Under the condition that the source is rotationally symmetric, with the resulting fields azimuthally invariant, it is possible to construct a simplified conical cloak with wide working frequency. A conical polarization rotator whose effective constitute parameters are spatially uniform is also studied. This device can arbitrarily control the polarization of EM waves getting through without introducing any scattering. The conical devices proposed here provide good candidates for lots of applications such as absorbing materials, cylindrical waveguide devices, field rotators, etc.

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