

Family of non-Abelian Kitaev models on a lattice: Topological condensation and confinement

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We study a family of non-Abelian topological models in a lattice that arise by modifying the Kitaev model through the introduction of single-qudit terms. The effect of these terms amounts to a reduction in the discrete gauge symmetry with respect to the original systems, which corresponds to a generalized mechanism of explicit symmetry breaking. The topological order is either partially lost or completely destroyed throughout the various models. The systems display condensation and confinement of the topological charges present in the standard non-Abelian Kitaev models, which we study in terms of ribbon operator algebras.

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I. INTRODUCTION

The subject of topological orders poses new challenges in the understanding of new phases of matter due to novel effects in quantum many-body physics.¹ There is by now a good deal of examples in condensed matter, such as in fractional Hall-effect systems,^{2–6} in short-range resonating valence bond (RVB) models,^{7–10} or in quantum spin liquids.^{5,11–17} There exist also exactly solvable models,^{18–20,39,40} which are paradigmatic examples for exhibiting topological properties that can be addressed in full detail since the whole spectrum of those models is known. Although topological orders typically arise in the quantum physics of two spatial dimensions, it is possible to construct exactly solvable models in three spatial dimensions and beyond.²¹ There is yet another field in which topological orders appear naturally. It corresponds to discrete gauge theories that arise as a consequence of a spontaneous symmetry-breaking mechanism from a continuous gauge group down to a discrete gauge group.^{22–28} In these two-dimensional topological quantum field theories, the standard algebraic language to describe the residual gauge-invariant properties of the excitations is that of quasitriangular Hopf algebras (quantum groups).²⁹

At the same time, quantum systems with topological order provide new expectations for finding alternative ways of robust quantum computation.^{18,30,31} In fact, there are several forms to set up schemes for topological quantum computation, some of them based on the braiding of quasiparticles,^{18,32–38} some of them based solely on the topological entangled properties of the degenerate ground states, without selective addressing of the physical qubits and without resorting to braiding of excitations,^{39,40} and others based on cluster states.⁴¹

Topological orders can be thought of as new forms of long-range entanglement and they are at the crossroads of condensed matter and quantum information.^{42–50} Some forms of hidden topological orders in quantum spin chains can be detected with string order parameters, which in turn can be interpreted in the light of quantum information techniques, and their long-range entanglement detected with them⁴⁵ using matrix product states from condensed matter.

There are experimental proposals based on optical lattices^{51,52} to implement models with Abelian topological

orders,⁵³ and in particular, the study of the string order parameter mentioned above can also be proposed by means of these techniques.⁵⁴ There are also proposals for non-Abelian models based on Josephson-junction arrays,^{55–57} in addition to the largely studied case of the fractional quantum Hall effect.³⁸

One of the emblematic examples of exactly solvable models to study topological orders on a lattice is the Kitaev model,¹⁸ both in its Abelian and non-Abelian versions. It captures the algebraic properties exhibited by the discrete gauge theories mentioned above. In addition, it provides us with an explicit realization of a Hamiltonian on a lattice, with the bonus that it allows for a model of topological quantum computation.

Comparatively, there are much less works on the non-Abelian Kitaev model than in the Abelian case (toric code). This is due, to some extent, to the additional mathematical technical difficulties presented by the non-Abelian case which is traditionally introduced with the language of quasitriangular Hopf algebras and their representations.^{58,59} Here we have made an effort to explain its contents in full detail and clarity with simpler algebraic tools based on group theory and their representations. Our goal is twofold: to make the model more accessible to a broader audience with a previous knowledge on the Abelian toric code and to use that simpler presentation as a starting point for considering more general models.

In this paper we introduce a family of non-Abelian topological models on a lattice, such that the standard Kitaev model corresponds to a particular case. More specifically, we study a two-parameter family labeled by a pair of subgroups $N \subset M \subset G$, N normal in G , where G is a discrete non-Abelian gauge group. In the particular case $N=1$, $M=G$ correspond to the original Kitaev models. The Hamiltonians of the family, denoted $H_G^{N,M}$, are explicitly constructed in Eq. (35). The standard vertex (“electric”) operators are modified according to the subgroup M , while the face (“magnetic”) operators change in accordance with N . In addition, there are new terms entering in the Hamiltonians which act on the edges of the lattice. Since there is a qudit attached to each edge these are single-qudit terms. Depending on the choice of the pair of subgroups (M, N) with respect to G , the non-Abelian discrete gauge group of the whole Hamiltonian $H_G^{N,M}$ may range from G down to the trivial group when $M=N$.

This is so because the gauge group for these models turns out to be given by $G' = M/N$. Therefore, the family of non-Abelian models provides us with a mechanism of explicit symmetry breaking of an original Hamiltonian with large discrete gauge symmetry group. In other words, this mechanism can also be seen as a symmetry-reduction mechanism, since we may have still a smaller gauge symmetry present in the Hamiltonian.

The new edge terms do not commute with the vertex and face terms of the original Hamiltonian, but this can be compensated by slightly changing these vertex and face terms. This change corresponds to studying the regimen in which the single-qudit terms have a higher coupling constant. Choosing the models this way, we can study their ground state and also the charge condensation phenomena. At least in some cases, single-qudit terms can be understood as a mechanism for introducing string tension, or more appropriately “ribbon tension,” to some of the quasiparticle excitations which thus get confined. In those cases, a complete characterization of the charge types and domain-wall fluxes will be given.

In order to facilitate both the exposition of the results and the readability of the paper, throughout the main text we will be giving the main constructions and results omitting many auxiliary details or proofs. However, all of these can be found in a well-ordered form in Appendixes A–F.

We hereby summarize briefly some of our main results. (i) We introduce a family of Hamiltonians defined on two-dimensional spatial lattices of arbitrary topology which exhibit a variety of discrete non-Abelian gauge group symmetry and topological orders. (ii) The ground state of the models can be exactly given and characterized in terms of opening a boundary ribbon operators. In many interesting cases the spectrum of excitations can be characterized accordingly. (iii) The new models show condensation and confinement of the charges in the original models with Hamiltonian H_G . (iv) In order to facilitate and complement the study of the family of models, we have carried out a thorough clarification of the main properties of the standard non-Abelian Kitaev model. In particular, (a) the ribbon operator algebra is introduced in an intrinsic way, with the motivation to find operators that describe excitations. (b) We study in detail and generalize the concept of ribbon. In particular, closed ribbons and a related algebra are defined and their transformation properties are described. (c) The vertex and face operators that appear in the Hamiltonian are related to elementary closed ribbon operators, showing that everything in the models can be translated to the language of ribbons. (d) We give a detailed account of two-particle states, giving explicitly a basis for the states that clarifies the meaning of the labels for topological charge. (v) A description of the ground state in terms of boundary ribbon operators is given.

This paper is organized as follows. In Sec. II we treat the standard non-Abelian Kitaev model. We start explaining the terms appearing in the Hamiltonian and go on characterizing the ground state and quasiparticle excitations by means of closed ribbon operators. We also present an explicit characterization of the topological charges of the model and study when single-quasiparticle states are possible. In Sec. III we motivate the family of non-Abelian model Hamiltonians and

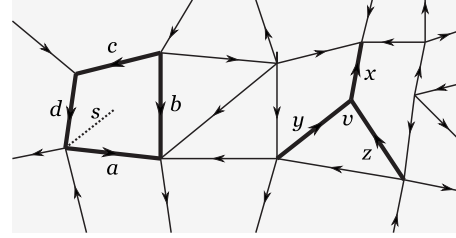


FIG. 1. The two-dimensional lattices that we consider are arbitrary in shape and have oriented edges. Thick lines display the support of a face operator (left) and a vertex operator (right).

present their generic properties. Then, we show how these models exhibit topological condensation and confinement described by domain walls. To this end we make use of closed and open ribbon operator algebras. Section IV is devoted to conclusions.

Appendixes A–F deserve special attention since they contain the detailed and basic explanations of all the constructions used throughout the text. Specifically, Appendix A contains a brief summary of representation theory for group algebras; their centers, and induced characters. In Appendix B we perform an extensive treatment of ribbon operators, which are necessary to describe the whole spectrum of the models. We define ribbons as geometrical objects and then construct and characterize a series of ribbon operator algebras. In Appendix C we study the relationship between certain ribbon transformations and the action of ribbon operator algebras on suitable subspaces, which is a key ingredient in describing the topological properties of the models. In Appendix D we give some details about the local degrees of freedom that appear in the Hilbert space of two-particle excitations. In Appendix E we explain why single-quasiparticle states exist in non-Abelian models on surfaces of nontrivial topology. Finally, in Appendix F we show several results needed for condensation and ground-state characterization.

II. NON-ABELIAN KITAEV MODEL

A. Hamiltonian

The data necessary for building up the model, as introduced by Kitaev,¹⁸ are any given finite group G and a lattice embedded in an orientable surface. The edges of the lattice must be oriented, as shown in Fig. 1. At every edge of the lattice we place a qudit, that is, a $|G|$ -dimensional quantum system, with Hilbert space \mathcal{H}'_G with orthonormal basis $\{|g\rangle | g \in G\}$. This way, we identify \mathcal{H}'_G with the group algebra $\mathbf{C}[G]$. The Hilbert space for the whole system is then $\mathcal{H}_G := \mathcal{H}'_G^{\otimes n}$, with n as the number of edges in the lattice. For notational convenience, we will denote the inverse of elements of G as \bar{g} instead of the usual g^{-1} . For completeness, we give a recollection of some basic properties of the group algebra $\mathbf{C}[G]$ in Appendix A.

Usually, when we talk about sites in a lattice we mean its vertices. However, here we will say that a site s is a pair $s = (v, f)$ with f as a face and v as one of its vertices.¹⁸ The need to consider sites will be clarified later when we discuss the excitations of the model in terms of strips associated with

ribbon operators. This is in contrast with the Abelian case where one only needs to consider strings both in the direct and dual lattices. As it happens, to obtain a non-Abelian generalization we need to consider vertices and faces (plaquettes) in a unified manner through the concepts of sites and strings and dual strings in a unified manner through the concept of ribbons.

The Hamiltonian of interest, as introduced in Ref. 18 is

$$H_G = - \sum_v A_v - \sum_f B_f, \tag{1}$$

where the sums run over vertices v and faces f . The terms A_v and B_f are projectors, called, respectively, vertex and face operators, or electric and magnetic operators. They commute with each other [Eq. (B37)]. In what follows, we give their explicit form.

First, we need a group of local operators at each vertex. We label its elements as A_v^g , $g \in G$, with $A_v^g A_v^{g'} = A_v^{gg'}$ so that they form a representation of G on \mathcal{H}_G . The operators A_v act only on those edges that meet at v , and this action depends on the orientation of the edge, inward or outward v . For example, for the vertex v of Fig. 1 we have

$$A_v^g |x, y, z, \dots\rangle := |g x, y \bar{g}, z \bar{g}, \dots\rangle, \tag{2}$$

where the dots represent other qudits, which do not change. These are the “local gauge transformation”¹⁸ operators. The vertex operators A_v that appear in the Hamiltonian are projectors onto the trivial sector of the representation of G at v , that is

$$A_v := \frac{1}{|G|} \sum_{h \in G} A_v^h. \tag{3}$$

Now let $s=(v, f)$ be a site and p_s denote the closed path with its endpoints in v and running once and counterclockwise through the border of f . That is, p_s is related to an elementary plaquette. We can then consider operators B_s^g , $g \in G$, which project onto those states with value g for the “product along p_s .” For example, for the site s of Fig. 1 we have

$$B_s^g |a, b, c, d, \dots\rangle := \delta_{g, a\bar{b}c\bar{d}} |a, b, c, d, \dots\rangle. \tag{4}$$

These are the “magnetic charge”¹⁸ operators. Note that the orientation of the edges with respect to the path is relevant. The face operators B_f that appear in the Hamiltonian are projectors onto the trivial flux, that is,

$$B_f := B_s^1, \tag{5}$$

where s is any site with $s=(v, f)$ and 1 is the unit of G . The operator B_f can be labeled just with the face, not with the particular site, because if the flux is trivial for a site then it is so for any other in the same face.

Since the Hamiltonian is a sum of projector operators, the ground-state subspace contains those states $|\xi\rangle$ which are left invariant by the action of the vertex and face operators, namely,

$$A_v |\xi\rangle = B_f |\xi\rangle = |\xi\rangle, \tag{6}$$

for every v and f . That is, the projector onto the ground state is

$$P_{GS} = \prod_v A_v \prod_f B_f. \tag{7}$$

In the sphere or in the plane, there is no ground-state degeneracy.¹⁸ In particular, the ground state can be obtained easily,

$$|\psi_G\rangle = P_{GS} |\mathbf{1}\rangle = \prod_v A_v |\mathbf{1}\rangle, \tag{8}$$

where $|\mathbf{1}\rangle$ is the state with all the qudits in the state $|1\rangle$.

If an eigenstate violates some of the conditions [Eq. (6)] it is an excited state. Note that there is an energy gap from the ground state to excited states and that excitations are localized. If $A_v |\xi\rangle = 0$, then we say that there is an electric quasiparticle at vertex v . If $B_f |\xi\rangle = 0$, then we say that there is a magnetic quasiparticle at face f . In general electric and magnetic charges are interrelated, as we will see, and one says that quasiparticles are dyons that live at sites.

The excitations of these models carry topological charge. Let us explain what this means. First, consider a configuration with several excitations, far apart from each other. Each of these excitations has a type, a property that can be measured locally and does not change.¹⁸ It is this type what we refer as a topological charge. The point is that there exist certain degrees of freedom with a global topological nature. In particular, there exists a subsystem which depends on the value of the charges and such that no local measurement is able to distinguish its states.¹⁸ This subsystem is thus protected and a good place to store quantum information. When two quasiparticles get close, some degrees of freedom of the protected subsystem become local. This operation, called fusion, allows us to perform measurements. Finally, one can perform unitary operations on the protected subsystem by suitably “braiding” the excitations.

We will not be concerned with the particular rules that govern the processes of fusion and braiding. Instead, we only want to be able to label the topological charges. But for this, as we shall see, it is enough to study certain ribbon operator algebras, which are introduced next.

B. Ribbon operators

This section is devoted to ribbon operators,¹⁸ which will be extensively employed throughout the paper. The main motivation is that ribbon operators describe quasiparticle excitations above the ground state in the non-Abelian Kitaev model, much like string operators describe the corresponding excitations in the Abelian case. A full account of the properties and definitions for ribbon operators used in this section is presented in Appendix B, especially in Appendix B 8 where a basic characterization theorem for ribbon operators is proven.

The basic idea behind ribbon operators is the following. First, ribbons are certain “paths” that connect sites (not vertices), as shown in Fig. 2. Suppose that for every pair of sites

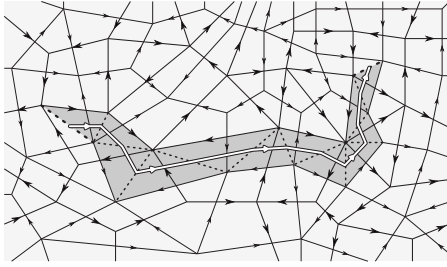


FIG. 2. Thick lines correspond to the lattice and thin lines to the dual lattice. Arrows show the orientation of edges and dual edges. Note that dual edges are oriented in agreement with edges (see explanation in main text). The shaded area is a ribbon. All the sites that form the ribbon are displayed as dashed lines, thicker in the case of the two sites in the ends. The arrowed thick white line shows the orientation of the ribbon.

s and s' and for every ribbon ρ connecting them we have at our disposal certain family of operators $\{O_\rho^i\}_i$ with support in the ribbon ρ . In particular, suppose that any state $|\psi\rangle$ with no excitations along ρ except possibly at s and s' can be written as

$$|\psi\rangle = \sum_i O_\rho^i |\psi_i\rangle \quad (9)$$

in terms of certain states $|\psi_i\rangle$ which have no excitations along ρ except possibly at s' but *not* at s . Then, any state can be obtained from states with one excitation less by application of such ribbon operators. In the sphere, where as we will see there are no states with one excitation, this means that any configuration of excited sites can be obtained from the GS by application of ribbon operators connecting these sites. Thus, we are addressing a situation for quasiparticle excitations which clearly resembles that of the Abelian Kitaev model, where strings in the dual and direct lattice have operators attached to them that create excitations at their endpoints.

Before ribbons can be further considered, we need to give more structure to our lattice. In particular, we will have to deal with a “merged” lattice in which the lattice and its dual play a simultaneous role. The reason to consider this merged lattice is that the excitations, as commented above, are related to sites, i.e., pairs $s=(v, f)$ of a vertex and a face. Since the dual of a face is a vertex in the dual lattice, we could equally well say that a site is a pair of a vertex v and a neighboring dual vertex $v'=f^*$. Thus, a site is best visualized as a line connecting these two vertices, as the dashed lines show in Fig. 2.

In order to have an oriented merged lattice, we orient the edges of the dual lattice in such a way that a dual edge e^* crosses the edge e “from right to left,” as in Fig. 2. This can be done because we are considering orientable surfaces only. Just as edges connect vertices in a normal lattice, we need something that connects sites in the merged lattice. These connectors turn out to be certain oriented triangles that come into two types: direct and dual triangles. A direct triangle τ is formed with two sites and an edge, as shown in Figs. 3(a) and 3(b). The idea is that τ points from a site $\partial_0\tau$ (dashed side to the left) to a site $\partial_1\tau$ (dashed side to the right) through

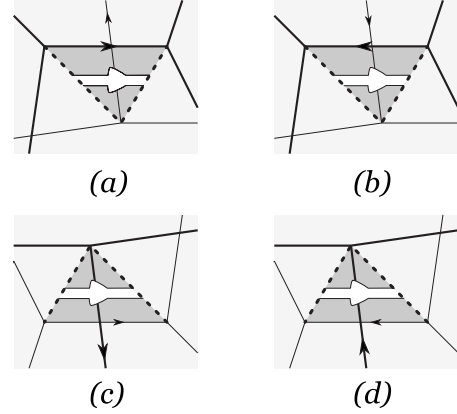


FIG. 3. Each figure represents a triangle τ (shaded area) that connects two sites (dashed lines): $\partial_0\tau$ to the left and $\partial_1\tau$ to the right. Thick lines correspond to the lattice and thin lines to the dual lattice. Arrows show the orientation of edges and dual edges. (a) A direct triangle with an edge which matches its direction. (b) A direct triangle with an edge which does not match its direction. (c) A dual triangle with a dual edge which matches its direction. (d) A dual triangle with a dual edge which does not match its direction.

an edge e_τ in the direct lattice. Note that the directions of τ and e_τ can either match or not, as the figure shows. A dual triangle τ' is formed with two sites and a dual edge [see Figs. 3(c) and 3(d)]. Again, it points from a site $\partial_0\tau'$ to a site $\partial_1\tau'$ through an edge $e_{\tau'}^*$, which now belongs to the dual lattice. Again, the directions of τ' and $e_{\tau'}^*$ can either match or not, as the figure shows.

Just as in a usual lattice a list of composable edges forms a path: in the merged lattice a list of composable triangles forms a triangle strip. So a strip is a sequence of triangles $\rho=(\tau_1, \dots, \tau_n)$ with the end of a triangle being the beginning of the next one, $\partial_1\tau_i=\partial_0\tau_{i+1}$. The ends of a strip are $\partial_0\rho=\partial_0\tau_1$ and $\partial_1\rho=\partial_1\tau_n$. A triangle strip is called a ribbon when it does not self-overlap, except possibly on its ends. A generic example of ribbon is shown in Fig. 2. For a detailed description of triangles, strips, and ribbons on a lattice, we refer to Appendix B 1.

Our next task is to attach to each triangle an algebra of operators which is enough to move quasiparticles between its two ends in the sense of Eq. (9). With this aim in mind, we first define triangle operators, which are single qudit operators acting on the edge e_τ of a triangle τ . These operators depend on whether the triangle is direct or dual and on the relative orientation of e_τ . The four possibilities are depicted in Fig. 3. The corresponding operators are

$$(a) T_\tau^g|k\rangle = \delta_{g,k}|k\rangle, \quad (b) T_\tau^g|k\rangle = \delta_{g,k}|k\rangle, \quad (10)$$

$$(c) L_\tau^g|k\rangle = |gk\rangle, \quad (d) L_\tau^g|k\rangle = |k\bar{g}\rangle, \quad (11)$$

where $|k\rangle$ is the state of the qudit at the edge e_τ . Thus, the triangle operators T_τ^g of direct triangles are projectors, like the B_s^g , and the triangle operators L_τ^g of dual triangles form a representation of G , like the A_v^g .

We start considering a direct triangle τ . Since direct triangles connect sites with the same face but different vertices,

triangle operators for direct triangles must be able to move electric or vertex excitations. Let v, v' be the two vertices of τ . Then, as a special case of Eq. (B46),

$$|G| \sum_{g \in G} T_\tau^g A_v T_\tau^g = 1. \quad (12)$$

Thus, any state $|\psi\rangle$ can be expressed as

$$|\psi\rangle = \sum_{g \in G} T_\tau^g |\psi_g\rangle, \quad (13)$$

with $|\psi_g\rangle = |G| A_v T_\tau^g |\psi\rangle$ as a state with no excitation at v because A_v projects out electric excitations. Moreover, T_τ^g commutes with all face operators and all vertex operators apart from those in the ends of τ , so that $|\psi_g\rangle$ has no excited spots which are not already in $|\psi\rangle$, except possibly at v' . These are the properties we were looking for and thus we define the algebra \mathcal{A}_τ as that with basis $\{T_\tau^g\}_{g \in G}$.

Next we consider a dual triangle τ . Since dual triangles connect sites with the same vertex but different face, triangle operators for dual triangles must be able to move magnetic or face excitations. Let f, f' be the two faces of τ . Then, as a special case of Eq. (B47),

$$\sum_{g \in G} L_\tau^g B_f L_\tau^g = 1. \quad (14)$$

Thus, any state $|\psi\rangle$ can be expressed as

$$|\psi\rangle = \sum_{g \in G} L_\tau^g |\psi_g\rangle, \quad (15)$$

with $|\psi_g\rangle = B_f L_\tau^g |\psi\rangle$ as a state with no excitation at f because B_f projects out magnetic excitations. Moreover, L_τ^g commutes with all vertex operators apart from those in the only vertex of τ and all face operators except those from the two faces connected by τ , so that $|\psi_g\rangle$ has no excited sites which are not already in $|\psi\rangle$, except possibly at f' . These are the properties we were looking for and thus we define the algebra \mathcal{A}_τ as that with basis $\{L_\tau^g\}_{g \in G}$.

Now that we have triangle operators at our disposal, we can move quasiparticles at will in the sense of Eq. (9). In particular, if we want to move an excitation from one end of a ribbon $\rho = (\tau_1, \dots, \tau_n)$ to the other end, we just proceed triangle by triangle. In other words, we can introduce an algebra $\mathcal{A}_\rho := \otimes_i \mathcal{F}_{\tau_i}$ which contains a family of operators $\{O_\rho^i\}$ with the properties related to Eq. (9). \mathcal{A}_ρ can be thought of as the algebra of all quasiparticle processes along ρ . Note that it is closed under the adjoint operator, $\mathcal{A}_\rho^\dagger = \mathcal{A}_\rho$.

However, if we are just interested in processes where no quasiparticles are created or destroyed but in the ends of ρ , as is the case for Eq. (9), then \mathcal{A}_ρ is just too general. Instead, we consider the ribbon operator algebra $\mathcal{F}_\rho \subset \mathcal{A}_\rho$, which contains those operators that do not create or destroy excitations along ρ . In other words $F \in \mathcal{F}_\rho$ if $[F, A_v] = [F, B_f] = 0$ for any vertex v and face f which do not lie in the ends of ρ . Note that \mathcal{F}_ρ is closed under the adjoint operator because $A_v = A_v^\dagger$, $B_f = B_f^\dagger$. These are the operators we were searching for in Eq. (9): a basis of \mathcal{F}_ρ gives the desired operators O_ρ^i [see Eqs. (B46) and (B47)]. \mathcal{F}_ρ can be thought of as the algebra of processes in which a pair of quasiparticles is created in one

end of the ribbon and then one of them is moved to the other end. In these terms, it is clear why excited states are expressible by means of ribbon operators acting on ground states.

A particularly meaningful basis for \mathcal{F}_ρ , explicitly given in Eq. (B66), consists of certain operators $F_\rho^{RC;uv}$, labeled by C , a conjugacy class of the group G , R , an irreducible representation of certain group \mathbf{N}_C defined below, and the indices $\mathbf{u} = (i, j)$, $\mathbf{v} = (i', j')$ with $i, i' = 1, \dots, |C|$, $j, j' = 1, \dots, n_R$. Here $|C|$ is the cardinality of C and n_R is the degree of the representation R . The group \mathbf{N}_C is defined as that with elements $g \in G$ with $gr_C = r_C g$ for some chosen representative $r_C \in C$. In order to construct the operators $F_\rho^{RC;uv}$, one also has to choose a particular unitary matrix representation Γ_R for R and enumerate the elements of the conjugacy class as $C = \{c_i\}$, together with a suitable subset $\{q_i\}_{i=1}^{|C|} \subset G$ such that $c_i = q_i r_C q_i^{-1}$. Later we will relate the labels R, C to the topological charges of the model and show how the indices \mathbf{u}, \mathbf{v} are related to local degrees of freedom at both ends of the ribbon. We will use the following notation to denote linear combinations of ribbon operators with the same topological charge labels R, C :

$$F_\rho^{RC}(\boldsymbol{\alpha}) := \sum_{\mathbf{u}, \mathbf{v}} \alpha^{\mathbf{u}, \mathbf{v}} F_\rho^{RC; \mathbf{u}, \mathbf{v}}, \quad (16)$$

where $\alpha^{\mathbf{u}, \mathbf{v}} \in \mathbb{C}$.

In the case of Abelian groups there are no local degrees of freedom and the elements of the basis are $F_\rho^{RC} = F_\rho^{\chi, g}$, with $g \in G$ and χ as an element of the character group of G . These operators are unitary and form a group,

$$F_\rho^{\chi, g} F_\rho^{\chi', g'} = F_\rho^{\chi\chi', gg'}, \quad F_\rho^{\chi, g}{}^\dagger = F_\rho^{\bar{\chi}, \bar{g}}. \quad (17)$$

Indeed, $T_\rho^\chi := F_\rho^{\chi, 1}$ are the string operators of Abelian models and $L_\rho^g := F_\rho^{e, g}$ the costring operators, with e as the identity character.

An essential property of ribbon operators, which reflects the topological nature of the model, is that in the absence of excitations the particular shape of the ribbon is unimportant; we can deform the ribbon while keeping the action of the ribbon operator invariant. More exactly, if the state $|\psi\rangle$ is such that the ribbon ρ can be deformed, with its ends fixed, to obtain another ribbon ρ' without crossing any excitation, then

$$F_\rho^{RC}(\boldsymbol{\alpha})|\psi\rangle = F_{\rho'}^{RC}(\boldsymbol{\alpha})|\psi\rangle. \quad (18)$$

This is illustrated in Fig. 4.

C. Closed ribbons

For a closed ribbon σ we mean one for which both ends coincide, so that we can set $\partial\sigma := \partial_0\sigma = \partial_1\sigma$. In view of the definition of \mathcal{F}_ρ , in the case of closed ribbons it is natural to consider a subalgebra $\mathcal{K}_\sigma \subset \mathcal{A}_\sigma$ such that it forgets the single end $\partial\sigma$. With this goal in mind, we let $\mathcal{K}_\sigma \subset \mathcal{A}_\sigma$ contain those operators in \mathcal{A}_σ that commute with all vertex and face operators A_v, B_f . In terms of quasiparticle processes, such closed ribbon operators are related to processes in which a pair of quasiparticles is created and one end of them is moved along the ribbon until they meet again to fuse into

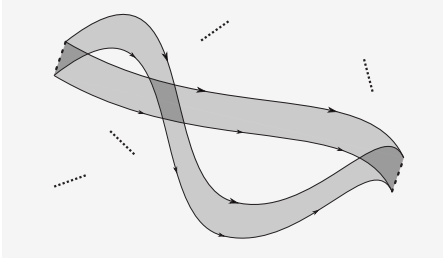


FIG. 4. An example of a deformation of a ribbon. The endpoints are fixed, and the area in between the two ribbons does not contain any excited site, which are represented with dotted lines.

vacuum. Closed ribbon operators play a fundamental role in characterizing the ground state of the model in a similar fashion as how closed strings are the building blocks for the ground state in the Abelian case (toric code). A detailed analysis of closed ribbon operators is performed in Appendix B 9.

We first consider the smallest examples of closed ribbons, i.e., dual and direct closed ribbons. We say that a ribbon is direct (dual) if it consists only of direct (dual) triangles. A dual ribbon like α in Fig. 5 encloses a single vertex v , and \mathcal{K}_α has as basis the operators A_v^h , $h \in G$. A direct ribbon like β in Fig. 5 encloses a single face f , and \mathcal{K}_β has as basis the operators B_f^C . These are labeled by the conjugacy classes C of G and take the form $B_s^C = \sum_{g \in C} B_s^g$ for any $s = (v, f)$. Thus, after defining ribbon operators by means of vertex and face operators, we now see that vertex and face operators are themselves ribbon operators.

As for the rest of closed ribbons σ , which we call proper closed ribbons, it turns out that \mathcal{K}_σ has as basis certain orthogonal projectors K_σ^{RC} that form a resolution of the identity, as shown in proposition (9). The labels R, C of these projectors are the same appearing in the basis for \mathcal{F}_ρ . In fact, in Sec. II D we will characterize excitations in terms of closed ribbon operators.

The algebra \mathcal{K}_σ does not see the ends of σ . Because of this, unlike \mathcal{F}_σ , it can stand deformations in which the end $\partial\sigma$ is not fixed or, for that matter, rotations of the ribbon. More exactly, if the state $|\psi\rangle$ is such that the closed ribbon σ can be deformed to obtain another ribbon σ' without crossing any excitation then

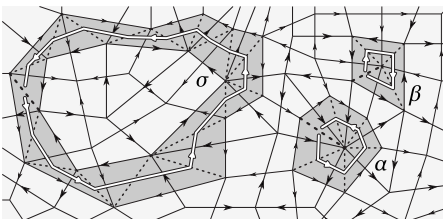


FIG. 5. Three examples of closed ribbons. σ is a proper closed ribbon, containing both dual and direct triangles. It is also a boundary ribbon, as it encloses an area with the topology of a disk. α is a dual closed ribbon and thus encloses a single vertex. β is a direct closed ribbon and thus encloses a single face.

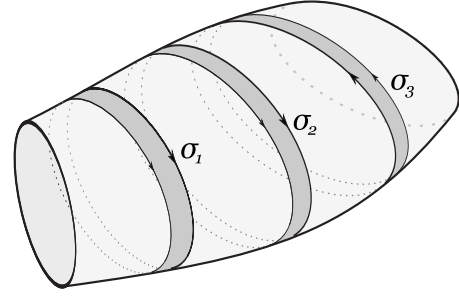


FIG. 6. Examples of closed ribbon transformations. A tubular piece of surface is displayed. The closed ribbon σ_1 is a deformation of σ_2 as long as there are no excitations between them. The ribbon σ_3 has an inverse orientation, and thus to obtain it from σ_2 we have to consider a deformation plus an inversion.

$$K_\sigma^{RC}|\psi\rangle = K_{\sigma'}^{RC}|\psi\rangle \tag{19}$$

(see Appendix C 2). This is illustrated in Fig. 6. Another kind of transformation is possible for closed ribbons. In particular, we can consider deformations plus inversions of the orientation of the ribbon, as shown in Fig. 6. When σ' is a transformation of σ which includes an inversion we have

$$K_\sigma^{\bar{R}\bar{C}}|\psi\rangle = K_{\sigma'}^{RC}|\psi\rangle, \tag{20}$$

where \bar{C} is the inverse conjugacy class of C , \bar{R}^C is the conjugate representation of R^C , and R^C is an irreducible representation of $\mathbf{N}_{\bar{C}}$ defined by $R^C(\cdot) := R(g \cdot \bar{g})$ if $\bar{r}_C = gr\bar{c}\bar{g}$ for some $g \in G$. In Sec. II D we relate this to inversion of topological charge.

D. Topological charges

Let s_0, s_1 be two nonadjacent sites in a lattice embedded in the sphere. From the discussion on ribbon operators it follows that the states

$$|RC; \mathbf{u}\mathbf{v}\rangle := F_\rho^{RC; \mathbf{u}\mathbf{v}}|\psi_G\rangle \tag{21}$$

form a basis for the subspace with excitations only at s_0 and s_1 . Here $|\psi_G\rangle$ is ground state (8) and ρ is any ribbon with $\partial_i\rho = s_i$.

For each site $s = (v, f)$, we introduce the algebra \mathcal{D}_s with basis $\{D_s^{hg} := A_v^h B_s^g\}_{h, g \in G}$. The reason to introduce it is that its action on an excitation at s gives all possible local action on the excitation.¹⁸ In other words, \mathcal{D}_s is useful to show why \mathbf{u}, \mathbf{v} are just local degrees of freedom. The action of the algebras \mathcal{D}_{s_i} on the states [Eq. (21)] is

$$D_{s_0}^{h,g}|RC; \mathbf{u}\mathbf{v}\rangle = \delta_{g, c_i} \sum_{s=1}^{n_R} \Gamma_R^{s,j}[n(hq_i)]|RC; \mathbf{u}(s)\mathbf{v}\rangle, \\ D_{s_1}^{h,g}|RC; \mathbf{u}\mathbf{v}\rangle = \delta_{g, \bar{c}_i'} \sum_{s=1}^{n_R} \bar{\Gamma}_R^{s,j'}[n(hq_i')]|RC; \mathbf{u}\mathbf{v}(s)\rangle, \tag{22}$$

where $\mathbf{u} = (i, j)$, $\mathbf{v} = (i', j')$, $\mathbf{u}(s) = [i(hq_i), s]$, and $\mathbf{v}(s) = [i(hq_i'), s]$, and we set for any $g \in G$ $g =: q_{i(g)}n(g)$ with $n(g) \in \mathbf{N}_{\bar{C}}$. Equation (22) is a consequence of Eqs. (6) and (B69).

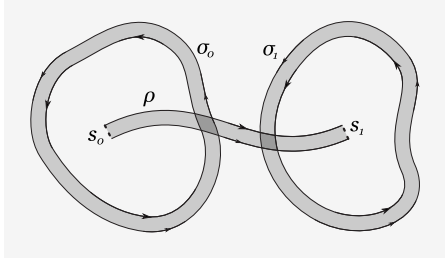


FIG. 7. An open ribbon ρ that connects two sites s_0 and s_1 and two closed ribbons σ_0 and σ_1 that surround counterclockwise s_0 and s_1 , respectively. The ribbon operators $F_\rho^{h,s}$ of the open ribbon change the excitations at s_0, s_1 . The ribbon operators $K_{\sigma_0}^{RC}, K_{\sigma_1}^{RC}$ of the closed ribbons project the system onto states with a given topological charge at s_0, s_1 .

As shown in detail in AppendixD, it is possible to find operators $d_{\mathbf{u}''}^{\mathbf{u}'} \in \mathcal{D}_{s_0}$ and $d_{\mathbf{v}''}^{\mathbf{v}'} \in \mathcal{D}_{s_1}$ with

$$d_{\mathbf{u}''}^{\mathbf{u}'} d_{\mathbf{v}''}^{\mathbf{v}'} |RC; \mathbf{u}\mathbf{v}\rangle = \delta_{\mathbf{u}, \mathbf{u}''} \delta_{\mathbf{v}, \mathbf{v}''} |RC; \mathbf{u}' \mathbf{v}'\rangle. \quad (23)$$

Thus we see that a state with particular labels \mathbf{u}, \mathbf{v} can be transformed with local operators into one with any other labels \mathbf{u}', \mathbf{v}' . Roughly speaking, for local operators we mean operators which act on a neighborhood of the excitations. More exactly, local operators should have a support which does not connect excitations.

What about the degrees of freedom related to the labels R and C ? They can certainly be measured locally because there exists a set of projectors $D_{s_0}^{RC} \in \mathcal{D}_{s_0}$ with

$$D_{s_0}^{RC} |R' C'; \mathbf{u}\mathbf{v}\rangle = \delta_{R, R'} \delta_{C, C'} |RC; \mathbf{u}\mathbf{v}\rangle. \quad (24)$$

However, R and C cannot be changed locally in the sense that an operator with a support not connecting both sites and which creates no additional excitations will not change their values. To see this, consider two closed ribbons σ_0 and σ_1 that enclose, respectively, the sites s_0 and s_1 counterclockwise, as in Fig. 7. From the discussion in AppendixC 3 it follows that

$$K_{\sigma_0}^{RC} |R' C'; \mathbf{u}\mathbf{v}\rangle = K_{\sigma_1}^{\bar{R}\bar{C}} |R' C'; \mathbf{u}\mathbf{v}\rangle = \delta_{R, R'} \delta_{C, C'} |RC; \mathbf{u}\mathbf{v}\rangle. \quad (25)$$

Any operator with no common support with σ will commute with the projectors K_σ^{RC} and thus cannot change the value of R and C . In particular, any operator which changes R and C must have a support that connects the sites s_0 and s_1 .

Indeed, the preceding discussion shows that R and C are the labels of the topological charges of the model. Thus the charge of an excitation is the pair (R, C) , with C as a conjugacy class of G and R as an irreducible representation of \mathbf{N}_C . If a closed ribbon σ encloses certain amount of excitations, as in Fig. 8, the projectors K_σ^{RC} correspond to sectors with different total topological charges in the region surrounded. If $|\xi\rangle$ is a state with no excitations in the area enclosed by σ , we have

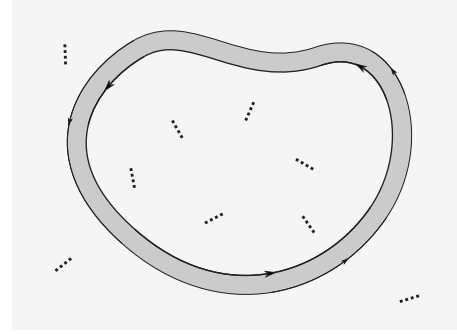


FIG. 8. A boundary ribbon σ that encloses several excitations counterclockwise. The corresponding operators K_σ^{RC} are projectors onto the sector with total topological charge (R, C) inside the ribbon.

$$K_\sigma^{e1} |\xi\rangle = |\xi\rangle, \quad (26)$$

with e as the identity representation (see AppendixF). Thus, $(e, 1)$ is the trivial charge. This offers a way to describe the ground state of Eq. (1) as the space of states for which Eq. (26) holds for any boundary ribbon, that is, any closed ribbon enclosing a disk or simply connected region.

In a region with no excitations, quasiparticles can only be locally created in pairs so that the two excitations have opposite charges and the total charge in the region remains trivial. From Eq. (20) and (25) it follows that the opposite of the charge (R, C) is (\bar{R}^C, \bar{C})

E. Single-quasiparticle states

In a sphere there do not exist states with a single excitation. The reason, as shown in Fig. 9, is that any closed ribbon σ divides the sphere into two regions, both of them simply connected. The ribbon σ surrounds one of this region counterclockwise, call it R_1 , and the other one clockwise, call it R_2 . Then not only the operator K_σ^{RC} is a projector onto the subspace with total charge (R, C) in R_1 but also a projector onto the subspace with total charge (\bar{R}^C, \bar{C}) in R_2 . Thus, if

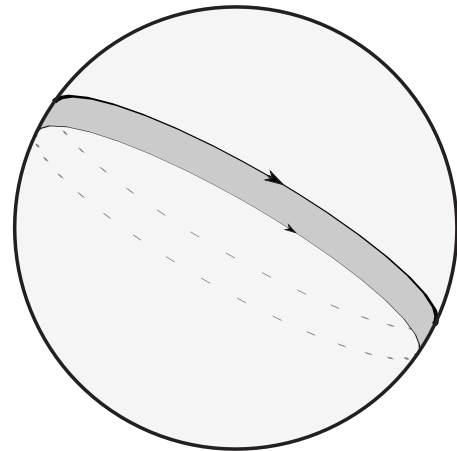


FIG. 9. A closed ribbon in a sphere. Its ribbon operators K_σ^{RC} project onto states with topological charge (R, C) in the upper side of the sphere and (\bar{R}^C, \bar{C}) in the lower side.

there are no excitations in R_1 , we have a total charge $(e, 1)$ in R_1 and also a total charge $(e, 1)$ in R_2 . But a single excited site cannot have trivial charge, and thus R_2 contains either zero or more than one excitation

What about surfaces with nontrivial topology, such as a torus? In the case of Abelian groups, the situation is the same as in the sphere: there are no states with a single excitation. In the case of vertex excitations, that is, electric charges, this follows from the fact that

$$\prod_{v \in V} A_v^g = 1. \quad (27)$$

For face excitations, that is, magnetic charges, an analogous result holds. For any character χ of G , let $B_f^\chi := \sum_{g \in G} \chi(g) B_s^g$ for $s=(v, f)$. Then

$$\prod_{f \in F} B_f^\chi = 1. \quad (28)$$

For non-Abelian groups, the situation is very different. In fact, examples of single-quasiparticle states can be constructed (see Appendix E).

III. CONDENSATION AND CONFINEMENT

A. Models

We want to modify the Hamiltonian H_G by introducing single qudit terms. In particular, we propose to consider projectors of the form

$$L_\tau^N := \frac{1}{|N|} \sum_{n \in N} L_\tau^n, \quad T_\tau^M := \sum_{m \in M} T_\tau^m, \quad (29)$$

where τ is a dual or direct triangle and N, M are subgroups of G . Thus L_τ^N projects out the trivial representation of N and T_τ^M selects those states within M . We want to have single qudit operators that do not depend on the orientation of the edge $e=e_\tau$. This is automatic for $T_e^M := T_\tau^M$, but in the case of dual triangles this is true if and only if N is normal, so that we can set $L_e^N := L_\tau^N$. That is, if $n \in N$ and $g \in G$, then $gn\bar{g} \in N$. Moreover, we want these two kinds of single-qudit terms to commute,

$$[L_e^N, T_e^M] = 0, \quad (30)$$

which is true if and only if $N \subset M$.

Now consider a Hamiltonian of the form

$$H = H_G - \mu \sum_e (L_e^N + T_e^M), \quad (31)$$

where μ is a positive coupling constant and the sum runs over edges e . The problem with this Hamiltonian is that the new terms do not commute with H_G . However, as we show now, we can still consider the limit of large μ . In this limit, the low energy sector is projected out by

$$P := \otimes_e T_e^M L_e^N. \quad (32)$$

Let us define the following vertex and face projectors:

$$A_v^M := \frac{1}{|M|} \sum_{m \in M} A_v^m, \quad B_f^N := B_s^N := \sum_{n \in N} B_s^n, \quad (33)$$

where $s=(v, f)$ is a site. Note that B_s^N only depends on f because N is normal. We now make the following observation:

$$\begin{aligned} |M| P A_v^M P &= |G| P A_v P, \\ |N| P B_f^N P &= |N| P B_f P. \end{aligned} \quad (34)$$

Thus, studying the low energy sector of Eq. (31) for large μ amounts to study the sector with no edge excitations of the Hamiltonian,

$$H_G^{N,M} := - \sum_v A_v^M - \sum_f B_f^N - \sum_e (T_e^M + L_e^N). \quad (35)$$

The point of these Hamiltonians is that all its vertex, face, and edge terms commute, and thus the ground state of the system can be exactly given. It turns out that it is related to that of Eq. (1) but for the group $G' := M/N$, as we will see in Sec. III B. Note that $H_G^{1,G}$ is just the original Hamiltonian (1) up to a constant. Although we have motivated the introduction of Eq. (35) through Eq. (31), our aim is to study the models $H_G^{N,M}$ in their own right for arbitrary subgroups $N \subset M \subset G$ with N normal.

B. Ground state

The ground state of Hamiltonian (35) is described by the conditions

$$A_v^M |\psi\rangle = B_f^N |\psi\rangle = L_e^N |\psi\rangle = T_e^M |\psi\rangle = |\psi\rangle, \quad (36)$$

where v is any vertex, f any face, and e any edge. Violations of these conditions amount to vertex, face, or edge excitations. Let V be the subspace of states with no edge excitations, which is projected out by the projector P of Eq. (32). V is a tensor product of single qudit subspaces $V := \otimes_e V_{M/N}$, with $V_{M/N} \subset \mathcal{H}_G'$ as the subspace with orthonormal basis,

$$|\tilde{m}\rangle := |N|^{-1/2} \sum_{n \in N} |mn\rangle, \quad \tilde{m} \in M/N. \quad (37)$$

Thus $V \simeq \mathcal{H}_{M/N}$; that is, within the subspace V we are effectively dealing with qudits of dimension $|M/N|$ which are naturally labeled through the group quotient. We denote the corresponding isomorphism by

$$p: \mathcal{H}_{M/N} \rightarrow V. \quad (38)$$

Let us write

$$H'_{M/N} := p H_{M/N} p^{-1}, \quad (39)$$

that is, $H'_{M/N}$ is Hamiltonian (1), for the group M/N , applied to the subspace $V_{M/N}$. We have

$$H'_{M/N} P = (H_G^{N,M} + 2|E|) P. \quad (40)$$

Thus, within the sector with no edge excitations we are effectively dealing with Hamiltonian $H_{M/N}$ (1). Moreover, the ground state of $H_G^{N,M}$ in \mathcal{H}_G is that of $H'_{M/N}$ in V . The projector onto the ground state is

$$P_{\text{GS}}^{N,M} := P \prod_v A_v^M \prod_f B_f^N = P \prod_v A_v \prod_f B_f, \quad (41)$$

where $A_v := pA_v p^{-1}$, $B_f := pB_f p^{-1}$, with A_v and B_f acting in $\mathcal{H}_{M/N}$. In the sphere, the normalized ground state is

$$|\psi_G^{N,M}\rangle \propto P_{\text{GS}}^{N,M} |\mathbf{1}\rangle \propto \prod_v A_v' |\tilde{\mathbf{1}}\rangle. \quad (42)$$

Thus the new edge terms in the Hamiltonian, which can be thought of as a sort of generalized ‘‘Zeeman terms,’’⁶⁰ have the role of selecting a particular sector of the Hilbert space in which a new non-Abelian discrete gauge symmetry appears, namely, $G' = M/N$. Thus, edge terms amount to an explicit symmetry-breaking mechanism, since in general the gauge symmetry is reduced, even to a trivial one if $M=N$. Alternatively, we can say that they provide a symmetry-reduction mechanism. Thus, the sector with no edge excitations is completely understood. In Secs. III C–III G we study the meaning of edge excitations.

C. Example

Before we go on with the general case and its details, let us first give a flavor of what is going on by considering a family of examples. We take $N=1$ and M normal in G , so that the new gauge group is $G' = M$. Note that in this case we can forget about the L_e^N terms because $L_e^1 = 1$. Our aim is to study the result of applying quasiparticle creation operators (16) on a ground state of Hamiltonian (35),

$$|\psi\rangle := F_\rho^{RC}(\boldsymbol{\alpha}) |\psi_G^{NM}\rangle. \quad (43)$$

We first consider purely magnetic quasiparticle creation operators, fixing R as the identity representation. For simplicity we set $\alpha^{uv} = c \in \mathbb{C}$. Then from Eq. (B69) it follows that $[A_v^M, F_\rho^{RC}(\boldsymbol{\alpha})] = 0$ for every vertex v and from Eq. (B50) it follows that $[T_e^M, F_\rho^{RC}(\boldsymbol{\alpha})] = 0$ for any edge e not in a dual triangle of ρ . Then due to Eq. (36) the state $|\psi\rangle$ can have, at most, face excitations on the ends of ρ and edge excitations on dual triangles of ρ . In particular, from Eqs. (36), (B69), and (B52) it follows that if f is an end face of ρ and e is any dual edge of ρ we have

$$B_f^1 |\psi\rangle = u |\psi\rangle, \quad T_e^M |\psi\rangle = u' |\psi\rangle, \quad (44)$$

with $u, u' = 0, 1$. As long as $C \neq 1$, we have $u=0$ and thus $|\psi\rangle$ contains a pair of face excitations. On the other hand, $u' = 1$ if $C \subset M$, which means that $|\psi\rangle$ contains a chain of edge excitations along ρ if we try to create magnetic charges which do not belong to the new gauge group G' . Therefore, we find out that some face excitations are confined, in particular those created with $C \not\subset M$. By this, we mean that the energy of $|\psi\rangle$ increases linearly with the length of ρ in terms of dual triangles.

Next, we consider purely electric quasiparticle creation operators, that is, we set $C=1$. Reasoning in the same way as in the previous case, one finds out that the state $|\psi\rangle$ can have, at most, vertex excitations on the ends of ρ but no face or edge excitations. In particular, if v is an end vertex of ρ we have

$$A_v^M |\psi\rangle = u |\psi\rangle, \quad (45)$$

with $u=1$ if the restriction of R to M is an identity representation and $u=0$ otherwise. That is, in some cases $|\psi\rangle$ is a ground state although R is not trivial. Since there is no local degeneracy in the ground state we know that $|\psi\rangle = c |\psi_G^{NM}\rangle$ for some $c \in \mathbb{C}$. Moreover, c can be nonzero, because, as we will see below, for $\alpha^{uv} = \delta_{u,v}$ and R trivial in M ,

$$\langle F_\rho^{R1}(\boldsymbol{\alpha}) \rangle_{\psi_G^{NM}} = 1. \quad (46)$$

Moreover, if σ is any boundary ribbon we have for R trivial in M

$$\langle K_\sigma^{R1} \rangle_{\psi_G^{NM}} = \frac{n_R |M|}{|G|}. \quad (47)$$

Thus, those electric charges with trivial restriction of R to M are condensed; they are part of the ground state.

D. Condensation

Let σ be a boundary ribbon, that is, a ribbon that encloses some region r . Motivated by the previous example, we want to study the expectation value in the ground state of H_G^{NM} of the operators K_σ^{RC} . Recall that these operators project onto the space with total topological charge (R, C) in systems with Hamiltonian H_G . Then if for a particular charge type we have

$$\langle K_\sigma^{RC} \rangle := \langle \psi_G^{NM} | K_\sigma^{RC} | \psi_G^{NM} \rangle > 0, \quad (48)$$

we say that the charges (R, C) of the original Hamiltonian H_G get condensed in the system with Hamiltonian H_G^{NM} ; if one measures the charge of the region r in the ground state of H_G^{NM} there exists some probability of finding the charge (R, C) .

As we will show in Appendix F,

$$\langle K_\sigma^{RC} \rangle = \frac{n_R |M|}{|G| |N|} (\chi_R, \chi_{e_M \uparrow})_{N_C} |C \cap N|, \quad (49)$$

where the product $(\cdot, \cdot)_M$ is defined in Eq. (A1) and $e_M \uparrow$ is the induced representation in G of the identity representation of M (see Appendix A 2). Another way to write the product is

$$(\chi_R, \chi_{e_M \uparrow})_{N_C} = \frac{1}{|N_C| |M|} \sum_{g \in G} |M_C^g| (\chi_R, 1)_{M_C^g}, \quad (50)$$

where $M_C^g := N_C \cap \bar{g} M g$. Note in particular that for M normal the sum has a single term, simplifying the form of Eq. (49). Result (49) not only shows that some of the charges are condensed but also that the expectation value is independent of the shape or size of the ribbon, a feature that underlines the topological nature of the condensation. Such behavior for a perimeter expectation is called a zero law.⁶¹

Let us consider several examples. First, under Kitaev's original Hamiltonian $H_G^{1,G}$ we have

$$\langle K_\sigma^{RC} \rangle = \delta_{C,1} \delta_{R,e_G}, \quad (51)$$

where e_G is the identity representation of G . Thus none of the nontrivial charges is condensed, as expected. In the case $N = M = G$ we have

$$\langle K_\sigma^{RC} \rangle = \frac{|C|}{|G|} \delta_{R, \epsilon_{N_C}}, \quad (52)$$

which means that the purely magnetic charges are condensed. On the contrary, in the case $N=M=1$ we have

$$\langle K_\sigma^{RC} \rangle = \frac{n_R^2}{|G|} \delta_{C,1}, \quad (53)$$

which means that the purely electric charges are condensed. Another illustrative case is that of an Abelian group G . In that case we can label the projectors as $K_\sigma^{\chi, g}$ with $g \in G$ and χ is an element of the character group of G . Then

$$\langle K_\sigma^{\chi, g} \rangle = \frac{|M|}{|G||N|} \delta_{g, N, N} \delta_{\chi_M, e_M}, \quad (54)$$

where χ_M is the restriction of χ to M .

It is possible to show which charges condense using another kind of expectation values, namely those for operators (16), which create a particle-antiparticle pair in original model (1). From the discussion in Appendix F it follows that if $|C \cap N| = \emptyset$ or $(\chi_R, 1)_{M_C^g} = 0$ for all g , we have

$$\langle F_\rho^{RC}(\alpha) \rangle = 0. \quad (55)$$

In the case of M normal, for a charge (R, C) that is not condensed according to Eq. (49) the operator $F_\rho^{RC}(\alpha)$ always has expectation value of zero. Even if M is not normal, if we trace out the local degrees of freedom and set $F_\rho^{RC} := F_\rho^{RC}(\alpha)$ with $\alpha^{uv} = \delta_{u,v}$ we get

$$\langle F_\rho^{RC} \rangle = \frac{1}{|C|} (\chi_R, \chi_{e_M})_{N_C} |C \cap N|. \quad (56)$$

Therefore, for that particular choice we get an expectation value which vanishes if and only if Eq. (49) does, showing that both approaches agree. Again topology makes its appearance in the fact that the length or shape of the ribbon ρ are not relevant. For Abelian groups Eq. (56) reads

$$\langle F_\rho^{\chi, g} \rangle = \delta_{g, N, N} \delta_{\chi_M, e_M}. \quad (57)$$

E. Confinement

The example studied in Sec. III C suggests that the edge terms L_e^N and T_e^M could be interpreted as string tension terms, which in turn would confine some of the charges of the original model H_G . However, one has to be a bit cautious with such a viewpoint in general. Certainly, in those cases in which N is central in M and M is normal (from now on, case I) such a viewpoint makes sense. Only in those cases do certain properties hold (see Appendix B 6). In case I we can write the relations [see Eq. (B52)]

$$P_{e, e'}^{NM} F_\rho^{RC; uv} P_{e, e'}^{NM} = d_{RC}^{NM} F_\rho^{RC; uv} P_{e, e'}^{NM}, \quad (58)$$

where d_{RC}^{NM} equals one (zero) if $C \subset M$ and the restriction of R to $N \subset N_C$ is trivial (in other case), $P_{e, e'}^{NM} = L_e^N T_{e'}^M$ and $e = e_\tau, e' = e_{\tau'}$ with τ, τ' as direct and dual triangles in an open ribbon ρ , respectively. Relation (58) shows that edge opera-

tors project out certain states among those which were created by applying a string operator to a ground state. Note that the projection only takes into account the quasiparticle labels R, C of the string operators. Moreover, it is not important which the particular edges are. Outside of case I such nice properties, reasonable for string tension terms, do not hold. As a consequence of Eq. (58), a state of form (43) will have a chain of edge excitations along ρ unless $C \subset M$ and R is trivial in N : all other charges get confined when moving from H_G to H_G^{NM} , which means that they exist at the end of chains of edge excitations. We shall refer to these chains of excitations as domain walls. They can be labeled just as we labeled topological charges in H_G , something that we will do in Sec. III F.

Unfortunately, as soon as any of the mentioned conditions for case I fails many nice properties of the models are lost. Indeed, only for those systems that fall in that class will we be able to classify domain walls and confined charges in terms of open and closed ribbon operator algebras in the fashion of what we already did for topological charges in H_G . On the other hand, in certain more general cases it is still possible to classify domain-wall types. In particular, we will show that this can be done in all models with N Abelian (from now on, case II). Interestingly enough, the domain-wall fluxes in case II are qualitatively richer than those in case I. Such fluxes have in general a non-Abelian character for case II systems, whereas for case I they have always an Abelian nature.

F. Case I systems

This section is devoted to those models $H_G^{N, M}$ with N central in M and M normal in G . We will show that edge excitations appear in the form of domain walls that terminate in certain site excitations which are therefore confined. With this goal in mind, we start introducing the excitations which will turn out to be confined. Consider the projectors

$$A_v^N := \frac{1}{|N|} \sum_{n \in N} A_v^n, \quad B_f^M := B_s^M = \sum_{m \in M} B_s^m. \quad (59)$$

They commute among each other and with the terms of Hamiltonian (35), so that we could choose the energy eigenstates to be eigenstates of projectors (59). We say that the state $|\psi\rangle$ has a confined excitation at a site $s=(v, f)$ whenever $A_v^N B_f^M |\psi\rangle = 0$. These are really excitations that follow from

$$A_v^N B_f^M |\psi\rangle = 0 \Rightarrow A_v^M B_f^N |\psi\rangle = 0, \quad (60)$$

where we have used $A_v^N A_v^M = A_v^M$ and $B_f^M B_f^N = B_f^N$. That they are really confined will be revealed later, but we can already give a clue: a state with a confined excitation at the site s must have an edge excitation at least at one of the edges e meeting at v or in the border of f . Thus, confined excitations cannot appear isolated: there must be edge excitations around them. Conversely, it is also true that a chain of edge excitations cannot terminate without a confined excitation in its end. However, this is not enough to demonstrate confinement, but we can do it better by introducing suitable ribbon operator algebras.

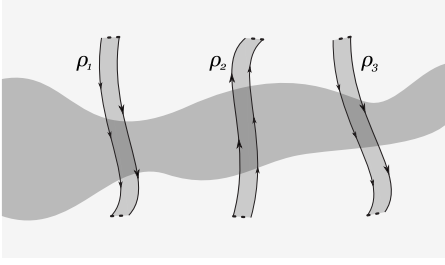


FIG. 10. Examples of open ribbon transformations for \mathcal{I}_ρ . Edge excitations are present only in the shaded area. No confined quasiparticle excitations are present. The ribbon ρ_1 is a deformation of ρ_3 , and ρ_2 has an inverse orientation with respect to them.

1. Open ribbon operators

Domain walls have a type, and this type behaves as a flux in the absence of confined excitations. We thus need a family of projectors that distinguish between the domain-wall fluxes that cross a particular line, analogous to the projectors K_σ^{RC} that distinguished the topological charge in a the area enclosed by σ . So we consider as our starting point the ribbon operator algebra \mathcal{F}_ρ for an open ribbon ρ . Since the flux should be the same if we move the ends of our flux-measuring ribbon in an area with no edge excitations, we need a ribbon algebra which to some extent forgets the ends of the ribbon. In particular, we should choose ribbon operators that do not create or destroy excitations. Thus, we define $\mathcal{I}_\rho \subset \mathcal{F}_\rho$ as the subalgebra containing those operators $F \in \mathcal{F}_\rho$ which commute with all vertex operators A_v^M and face operators B_f^N . Such operators also commute with all edge terms L_e^N and T_e^M (see colollary 12).

As we show in proposition 11, \mathcal{I}_ρ is linearly generated by certain orthogonal projectors $J_\rho^{\chi t}$ that form a resolution of the identity. The labels (χ, t) of these projectors are χ , an element of the character group of N , and t , an element of the quotient group G/M . If $|\psi\rangle$ has no edge excitations along ρ then

$$J_\rho^{e1}|\psi\rangle = |\psi\rangle, \quad (61)$$

where e and 1 are both identity elements [see Eqs. (B85) and (F4)].

The algebra \mathcal{I}_ρ can stand deformations in which the ends of ρ are not fixed. More exactly, if the state $|\psi\rangle$ is such that the open ribbon ρ can be deformed, without fixing its ends, to obtain another ribbon ρ' in such a way that no confined excitations are crossed and the ends do not touch edge excitations, then

$$J_\rho^{\chi t}|\psi\rangle = J_{\rho'}^{\chi t}|\psi\rangle \quad (62)$$

(see Appendix C 2). This is illustrated in Fig. 10. As in the case of closed ribbons in H_G , here we can consider inversions in the orientation of the ribbon, as shown in Fig. 10. When ρ' is a transformation of ρ which includes an inversion we have

$$J_\rho^{\chi t}|\psi\rangle = J_{\rho'}^{\bar{\chi}t}|\psi\rangle, \quad (63)$$

where for any $n \in N$ we set $\chi'(n) := \chi(\bar{n})$ (see Appendix C 2).

When a ribbon ρ_2 crosses two domain walls which are, respectively, crossed by two other ribbons ρ_3, ρ_4 with the same orientation as ρ , as in Fig. 12, we have

$$J_{\rho_2}^{\chi t} = \sum_{\xi} \sum_{k \in G/M} J_{\rho_3}^{\bar{\xi}k} J_{\rho_4}^{\xi \chi k t}, \quad (64)$$

where ξ runs over the group of characters (see Appendix B 10).

2. Closed ribbon operators

Before we can further analyze the consequences of the properties of \mathcal{I}_ρ , as we shall do in Sec. III F 3, we have to introduce a family of projectors that distinguishes the charges in our models. The situation is different to the one we found in the models H_G because now some charges are confined, and thus the rules for deforming closed ribbons will change qualitatively to reflect this fact. Our starting point is the ribbon operator algebra \mathcal{F}_σ for a closed ribbon σ , from which we want to select certain suitable operators, just as we did for other projector algebras. In the case of open ribbons just considered, we made this selection requiring that the operators commuted with all vertex and face terms. Here that will not be enough because since σ is closed we would be considering operators that create a closed domain wall, with no ends and no confined excitations. Therefore, we define $\mathcal{K}'_\sigma \subset \mathcal{F}_\sigma$ as the subalgebra containing those operators $K \in \mathcal{F}_\sigma$ which commute with all vertex operators A_v^M , face operators B_f^N , and edge operators L_e^N, T_e^M .

As we show in proposition 14, \mathcal{K}'_σ is linearly generated by certain orthogonal projectors K_σ^{RC} that form a resolution of the identity. The labels (R, C) of these projectors are C , a set of the form $\{mg\bar{m} | m \in M\}$ for some $g \in G$, and R , an irreducible representation of the group $\mathbf{N}_C := \{m \in M | mr_C \bar{m} r_C \in N\}$ for some fixed $r_C \in C$. If σ is a boundary ribbon surrounding an area with no vertex or face excitations in the state $|\psi\rangle$, then

$$K_\sigma^{e1}|\psi\rangle = |\psi\rangle, \quad (65)$$

where e and 1 are both identity elements [see Eqs. (B75), (B86), and (F6)].

The algebra \mathcal{K}'_σ can stand deformations in which the end $\partial\sigma$ is not fixed, as long as it crosses no domain walls. If the state $|\psi\rangle$ is such that the open ribbon σ can be deformed to obtain another ribbon σ' in such a way that no vertex or face excitations are crossed and the end of σ touches no edge excitations, then

$$K_\sigma^{RC}|\psi\rangle = K_{\sigma'}^{RC}|\psi\rangle \quad (66)$$

(see Appendix C 2). This is illustrated in Fig. 11. As in the case of closed ribbons in H_G , here we can consider inversions in the orientation of the ribbon, as shown in Fig. 11. When σ' is a transformation of σ which includes an inversion we have

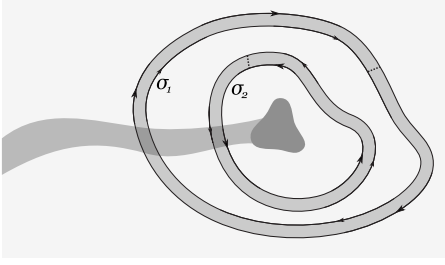


FIG. 11. Examples of closed ribbon transformations for \mathcal{K}'_ρ . The light shaded area represents edge or domain-wall excitations and the dark shaded area confined quasiparticle excitations. The ribbon σ_1 is a deformation of σ_2 which includes an inversion.

$$K_\sigma^{RC}|\psi\rangle = K_{\sigma'}^{\bar{R}\bar{C}\bar{C}}|\psi\rangle, \quad (67)$$

where R^C is an irreducible representation of \mathbf{N}'_C defined by $R^C(\cdot) := R(m \cdot \bar{m})$ if $\bar{r}_C = mr\bar{m}$ for some $m \in M$ (see Appendix C 2).

3. Domain walls and charges

The deformation properties of the projectors in \mathcal{J}_ρ that we have introduced indicate that edge excitations appear in the form of domain walls to which a flux can be attached. Branching points are possible in these walls. Each value of the flux corresponds to a projector $J_\rho^{X^t}$, and we know that it is preserved along a domain wall due to deformation property (62). The trivial flux is given by Eq. (61) and the inverse flux by Eq. (63). Since deformations of $J_\rho^{X^t}$ only require that no confined excitations are crossed, a domain wall can only end in the presence of confined excitations. Domain-wall fluxes have an Abelian nature: as indicated by Eq. (64), the addition of two given fluxes always produces the same combined flux. All these ideas are reflected in Fig. 12.

Regarding charge labeling, the properties of the projectors in \mathcal{K}'_σ very much resemble those already found in the study of \mathcal{K}_σ in systems with Hamiltonian H_G . The trivial charge is

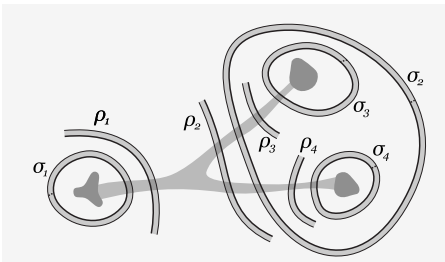


FIG. 12. An illustration of the relationship between domain-wall fluxes and quasiparticle charges. The light shaded area represents edge or domain-wall excitations and the dark shaded area quasiparticle excitations. Each of the open ribbons ρ_i can be deformed to the boundary ribbon σ_i , so that the domain-wall flux measured by ρ_i corresponds to the confined charge measured by σ_i . Since ρ_1 and ρ_2 are equivalent up to an inversion, the confined charges measured by σ_1 and σ_2 are inverses. The total flux in ρ_2 is the combination of that in ρ_3 and ρ_4 and thus the confined charge measured by σ_2 is the combination of that measured by σ_3 and σ_4 .

given by Eq. (65) and the inverse charge by Eq. (67). Indeed, the new element that appears is that now deformation properties (66) take into account that the charge could be attached to a domain wall.

But if we want to neatly describe confinement, we have to establish the relationship between domain walls and charges. To this end, consider Fig. 12. We can deform each ribbon ρ_i , without changing the flux it measures, until ρ_i is equal to σ_i , the boundary ribbon enclosing the charge at the end of the domain wall. At that point we can compare both projector algebras, with the following result: there exists a function f onto the group character of N such that

$$J_\sigma^{X^t} = \sum_{C \subset tM} \sum_R \delta_{X, f(R)} K_\sigma^{RC}, \quad (68)$$

where the sum on R runs over irreducible representations in \mathbf{N}'_C [see Eq. (B90)]. Equation (68) tells us at the end of which domain walls can each charge exist. In other words, the projectors in \mathcal{J}_σ classify charge types from \mathcal{K}'_σ in different compartments or sectors: each of these sectors gives a confined charge type. The different charge labels within a sector give us the topological part of the charge. In particular, for the trivial confined charge we recover the topological charge types for a system with Hamiltonian $H_{M/N}$, in accordance with the study of the ground state of Sec. III B. Finally, all charges which do not belong to the trivial confined charge sector are indeed confined because if we take any circle surrounding them we must always have a domain wall crossing it. When excitations are localized in a single site, it turns out that confined excitations are exactly described, as expected, by projectors (59).

As we already did in the particular case of H_G models, the ground state can be described in terms of ribbon operators of arbitrary size. In this case, we have to impose that no region should contain a nontrivial charge *and* no line should be crossed by a nontrivial domain-wall flux. That is, a state ξ is a ground state if and only if

$$K_\sigma^{e1}|\xi\rangle = |\xi\rangle, \quad J_\rho^{e1}|\xi\rangle = |\xi\rangle \quad (69)$$

for all boundary ribbons σ and proper ribbons ρ . This conditions generalize to arbitrary N and M (see Appendix F).

G. Case II systems

This section is devoted to those models $H_G^{N,M}$ with N Abelian. As we have already commented, in this case we will only be able to describe, using ribbon projector algebras, domain-wall fluxes but not charge types, except those in the sector with no edge excitations which are already classified through the mapping to $H_{M/N}$. Thus we proceed to describe the algebra \mathcal{J}_ρ , which is defined exactly as in case I. As discussed in Appendix B 10, in this case the projectors are J_ρ^{RT} , with T as an element of the double coset $M \backslash G / M$ and R an induced representation in \mathbf{N}_T of an irreducible representation in N , with \mathbf{N}_T as the group $\{m \in M | mr_T M = r_T M\}$ for some fixed $r_T \in T$. We recall that each element of the double coset takes the form $T = \{mr_T m' | m, m' \in M\}$. From the double coset structure, we can obtain a subalgebra of the group algebra $\mathbf{C}(G)$. Indeed, for $T, T' \in M \backslash G / M$ we have

$$TT' = \sum_{T'' \in MG/M} C_{TT'}^{T''} T'', \quad (70)$$

where $C_{TT'}^{T''}$ are positive integers. The rules for deformations of J_ρ^{RT} , discussed in Appendix C 2, state that the ribbon ρ can move across regions in which $B_f^N|\psi\rangle=|\psi\rangle$ and $A_v^N|\psi\rangle=|\psi\rangle$ are satisfied, and the ends can be displaced as long as they do not touch edge excitations, with the result that the flux measured by J_ρ^{RT} does not change. The inverse flux of (R, T) is (\bar{R}^T, \bar{T}) , where \bar{M} contains the inverses of the elements of M and R^T is defined by $R^T(\cdot) := R(r_T m \cdot \bar{m} \bar{r}_T)$ if $m \in M$ is such that $M = r_T m \bar{r}_T M$. Finally, there is no analog of Eq. (64) in the sense that the knowledge of two fluxes is not enough to determine the total combined flux. This can be seen, for example, in Eq. (70). When M is normal (case I), the sum is reduced to a single term, so that combining a flux (e, T) with a flux (e, T') will give certain determinate total flux (e, T') . This is no longer true when M is not normal (case II), giving a non-Abelian nature to the domain-wall fluxes, which disappears in case I.

IV. CONCLUSIONS

In this paper we have introduced a family of quantum lattice Hamiltonians with a discrete non-Abelian gauge symmetry such that the standard Kitaev model for topological quantum computation is a particular case of this class. The ground state of the models can be exactly given and, in many cases, quasiparticles or at least domain-wall excitations can be classified. They can be characterized by operator algebras corresponding to closed and open ribbon operators. This is done in full generality for arbitrary topologies. The models can be understood in terms of topological charge condensation and confinement with respect to the standard Kitaev models.

We have given a detailed account of the quasiparticle excitations in the standard non-Abelian Kitaev model. In particular, we have seen that for orientable closed surfaces other than the sphere, such as the torus, excitations may show up in the form of single quasiparticles.

One of the features exhibited by the family of non-Abelian models considered in this paper is the existence of a string tension for the motion of quasiparticle excitations. It would be interesting to study the role of such tensions when the action of external source of decoherence such as local external fields or thermal effects are studied.^{30,62} Another aspect that deserves further study is the properties of these models for topological quantum computation, since here we have only focused on their properties as far as topological order is concerned.

ACKNOWLEDGMENTS

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APPENDIX A: GROUP ALGEBRAS

We present some basic properties of the algebras of finite groups. We will find them useful for establishing several results on ribbon operator algebras in Appendix A 1. In particular, we need them to label the charges and domain walls of our models.

1. Representations and classes

Given a finite group G , let $(G)_{\text{cj}}$ be the set of conjugacy classes of G and $(G)_{\text{ir}}$ be the set of irreducible representations of G up to isomorphisms.⁶³ For $R \in (G)_{\text{ir}}$, $g \in G$, we denote by R_g the image of g under R and by $\Gamma_R(g)$ the unitary matrix of R_g in a particular basis. The character of a representation R is $\chi_R(g) := \text{Tr} \Gamma_R^{ii}(g)$. Characters are examples of class functions $\phi, \psi, (G)_{\text{cj}} \rightarrow \mathbb{C}$, for which we introduce the product⁶³

$$(\phi, \psi)_G := \frac{1}{|G|} \sum_{C \in (G)_{\text{cj}}} |C| \phi(C) \bar{\psi}(C), \quad (A1)$$

where the bar denotes complex conjugation.

The following are the well-known orthogonality relations for irreducible representations, characters, and conjugacy classes,⁶³

$$\sum_{g \in G} \Gamma_R^{ij}(g) \bar{\Gamma}_{R'}^{i'j'}(g) = \frac{|G|}{n_R} \delta_{R,R'} \delta_{i,i'} \delta_{j,j'}, \quad (A2)$$

$$\sum_{C \in (G)_{\text{cj}}} |C| \chi_R(C) \bar{\chi}_{R'}(C) = |G| \delta_{R,R'}, \quad (A3)$$

$$\sum_{R \in (G)_{\text{ir}}} \chi_R(C) \bar{\chi}_R(C') = \frac{|G|}{|C|} \delta_{C,C'}, \quad (A4)$$

where $R, R' \in (G)_{\text{ir}}$, $C, C' \in (G)_{\text{cj}}$, and $n_R = \chi_R(1)$ is the degree of R . Equations (A3) and (A4) imply that $|(G)_{\text{cj}}| = |(G)_{\text{ir}}|$. Identities (A3), which are just a particular case of Eq. (A2), can be written more concisely as $(\chi_R, \chi_{R'})_G = \delta_{R,R'}$.

2. Induced representations

Given a finite group G and a normal subgroup $H \subset G$, let $(H, G)_{\text{cj}}$ be the set of conjugacy classes of G contained in H and $(H, G)_{\text{ir}}$ be the set of induced representations in G of irreducible representations in H up to isomorphisms.⁶³ Recall that for each representation R of H there exists a representation R_{ind} , called the induced representation of R in G , such that for $g \in G$,

$$\chi_{R_{\text{ind}}}(g) := \begin{cases} \sum_{r \in G/H} \chi_R(r g \bar{r}), & g \in H \\ 0, & g \notin H. \end{cases} \quad (A5)$$

The Frobenius reciprocity formula asserts that for $\phi, (G)_{\text{cj}} \rightarrow \mathbb{C}$ and $R \in (H)_{\text{ir}}$,

$$(\phi|_H, \chi_R)_H = (\phi, \chi_{R_{\text{ind}}})_G, \quad (A6)$$

where $\phi|_H$ is the restriction of ϕ to H .

Let us introduce an equivalence relation in $(H)_{\text{cj}}$. For $D, D' \in (H)_{\text{cj}}$, we set $D \sim D'$ if $D' = gD\bar{g}$ for some $g \in G/H$. Each $C \in (H, G)_{\text{cj}}$ is related to a unique equivalence class \tilde{C} in the following way:

$$C = \bigcup_{D \in \tilde{C}} D. \quad (\text{A7})$$

Given $S \in (H)_{\text{ir}}$ and $r \in G/H$, define a representation S' in $(H)_{\text{ir}}$ by setting $S'_h := S_{rhr}$. We introduce an equivalence relation in $(H)_{\text{ir}}$. For $S, S' \in (H)_{\text{ir}}$, we set $S \sim S'$ if $S = S^g$ for some $g \in G/H$. Each $R \in (H, G)_{\text{ir}}$ is related to a unique equivalence class \tilde{R} in the following way:

$$\chi_R(g) = \begin{cases} \frac{|G|}{|H||\tilde{R}|} \sum_{S \in \tilde{R}} \chi_S(g), & g \in H \\ 0, & g \notin H, \end{cases} \quad (\text{A8})$$

and $S \in \tilde{R}$ if $S_{\text{ind}} = R$.

As a generalization of Eqs. (A3) and (A4) we have the following orthogonality relations:

$$\sum_{C \in (H, G)_{\text{cj}}} |C| \chi_R(C) \bar{\chi}_{R'}(C) = \frac{|G|^2}{|H||\tilde{R}|} \delta_{R, R'}, \quad (\text{A9})$$

$$\sum_{R \in (H, G)_{\text{ir}}} |\tilde{R}| \chi_R(C) \bar{\chi}_R(C') = \frac{|G|^2}{|H||C|} \delta_{C, C'}, \quad (\text{A10})$$

where $R, R' \in (H, G)_{\text{ir}}$ and $C, C' \in (H, G)_{\text{cj}}$. Equations (A9) and (A10) imply that $|(H, G)_{\text{cj}}| = |(H, G)_{\text{ir}}|$. Note that Eq. (A9) can be rewritten in terms of the product $(\cdot, \cdot)_G$ and derived from Eq. (A6) as follows:

$$(\chi_R, \chi_{R'})_G = (\chi_{S_0}, \chi_{R'})_H = \frac{|G|}{|H||\tilde{R}|} \delta_{R, R'}, \quad (\text{A11})$$

where $S_0 \in \tilde{R}$. As for Eq. (A10), it follows from Eqs. (A4) and (A8)

$$\begin{aligned} & \sum_{R \in (H, G)_{\text{ir}}} |\tilde{R}| \chi_R(C) \bar{\chi}_R(C') \\ &= \sum_{R \in (H, G)_{\text{ir}}} \frac{|G|^2}{|H|^2 |\tilde{R}|} \sum_{S, S' \in \tilde{R}} \chi_S(D_0) \bar{\chi}_{S'}(D_0) \\ &= \sum_{g \in G/H} \sum_{R \in (H, G)_{\text{ir}}} \frac{|G|}{|H|} \sum_{S \in \tilde{R}} \chi_S(D_0) \bar{\chi}_S(gD_0\bar{g}) \\ &= \frac{|G|}{|D_0|} \sum_{g \in G/H} \delta_{D_0, gD_0\bar{g}} = \frac{|G|^2}{|H||C|} \delta_{C, C'}, \end{aligned} \quad (\text{A12})$$

where $D_0 \in \tilde{C}$, $D_0 \in \tilde{C}'$.

3. Group algebra $\mathbf{C}[G]$

Given a finite group G , the group algebra $\mathbf{C}[G]$ consists of formal sums $\sum_{g \in G} c_g g$, $c_g \in \mathbf{C}$. We are interested in certain representation $\mathcal{R}: G \times G \rightarrow \mathbf{GL}(\mathbf{C}[G])$. Let us denote by \mathcal{R}_{g_1, g_2} the image of $(g_1, g_2) \in G \times G$. Then \mathcal{R} is defined by

$$\mathcal{R}_{g_1, g_2}(g) := g_1 g \bar{g}_2, \quad g \in G, \quad (\text{A13})$$

where \bar{g} denotes the inverse of g . It turns out that the following isomorphism holds, as we shall check below,

$$\mathbf{C}[G] \simeq \sum_{R \in (G)_{\text{ir}}} V_R \otimes V_{\bar{R}}, \quad (\text{A14})$$

where V_R is the representation space of R .

With the aim of checking Eq. (A14) explicitly, let us consider the following elements of $\mathbf{C}[G]$:

$$e_R^{ij} := \frac{n_R}{|G|} \sum_{g \in G} \bar{\Gamma}_R^{ij}(g) g, \quad (\text{A15})$$

where R is a representation of G and $i, j = 1, \dots, n_R$. For irreducible R , there are $|G|$ in such elements because $\sum_{R \in (G)_{\text{ir}}} n_R^2 = |G|$ due to Eq. (A14). In fact, they give a new basis for $\mathbf{C}[G]$,

$$g = \sum_{R \in (G)_{\text{ir}}} \sum_{i, j=1}^{n_R} \Gamma_R^{ij}(g) e_R^{ij}, \quad (\text{A16})$$

which can be checked using Eq. (A4). In this basis,

$$\mathcal{R}_{g_1, g_2} e_R^{ij} = \sum_{k, l=1}^{n_R} \Gamma_R^{ki}(g_1) \bar{\Gamma}_R^{lj}(g_2) e_R^{kl}, \quad (\text{A17})$$

which gives explicitly isomorphism (A14) as desired. Let us define

$$\left(\sum_g c_g g \right) := \sum_g \bar{c}_g \bar{g}. \quad (\text{A18})$$

Then, we have

$$e_R^{ij} e_{R'}^{i'j'} = \delta_{R, R'} \delta_{j, i'} e_R^{ij'}, \quad \bar{e}_R^{ij} = e_R^{ji}. \quad (\text{A19})$$

which follow from Eq. (A2).

4. Algebra \mathcal{Z}_G

The center of a group algebra $\mathbf{C}[G]$, denoted \mathcal{Z}_G , is the subalgebra of elements that commute with all the elements of $\mathbf{C}[G]$. A basis for the center of $\mathbf{C}[G]$ is the following:

$$e_C := \sum_{g \in C} g, \quad C \in (G)_{\text{cj}}. \quad (\text{A20})$$

We have

$$e_C e_{C'} := \sum_{C'' \in (G)_{\text{cj}}} N_{C, C'}^{C''} e_{C''}, \quad \bar{e}_C = e_{\bar{C}}, \quad (\text{A21})$$

where \bar{C} denotes the inverse class of C and $N_{C, C'}^{C''} \geq 0$ are integers.

With the aim of finding an alternative basis for \mathcal{Z}_G , we define

$$e_R := \sum_i e_R^{ii}, \quad R \in (G)_{\text{ir}}, \quad (\text{A22})$$

which are a nice set of projectors,

$$\begin{aligned}
 e_R e_{R'} &= \delta_{R,R'} e_R, \\
 \bar{e}_R &= e_R, \\
 \sum_{R \in (G)_{\text{ir}}} e_R &= 1,
 \end{aligned} \tag{A23}$$

as follows from Eq. (A4). They provide us with a new basis for \mathcal{Z}_G since

$$e_R = \frac{n_R}{|G|} \sum_{C \in (G)_{\text{cj}}} \bar{\chi}_R(C) e_C, \tag{A24}$$

$$e_C = \sum_{R \in (G)_{\text{ir}}} \frac{|C|}{n_R} \chi_R(C) e_R, \tag{A25}$$

as can be checked using Eq. (A4).

5. Algebra $\mathcal{Z}_{H,G}$

Let H be a normal subgroup of G . There is a natural inclusion $\mathbf{C}[H] \subset \mathbf{C}[G]$. We are interested in the intersection of their centers $\mathcal{Z}_{H,G} := \mathcal{Z}_H \cap \mathcal{Z}_G$. A basis for the algebra $\mathcal{Z}_{H,G}$ is

$$e_C^G := \sum_{g \in C} g, \quad C \in (H,G)_{\text{cj}}. \tag{A26}$$

Its elements can be rewritten in terms of elements of \mathcal{Z}_H as follows:

$$e_C^G = \sum_{D \in \tilde{C}} e_D^H = \frac{|H| |\tilde{C}|}{|G|} \sum_{g \in G/H} g e_{D\bar{g}}, \tag{A27}$$

where we are using the notation of Eq. (A7).

We have the aim of finding a basis of projectors for $\mathcal{Z}_{H,G}$, analogous to Eq. (A23). Let

$$e'_R := \frac{|H| |\tilde{R}|}{|G|} e_R^G = \sum_{S \in \tilde{R}} e_S^H, \quad R \in (H,G)_{\text{ir}}, \tag{A28}$$

so that

$$\begin{aligned}
 e'_R e'_{R'} &= \delta_{R,R'} e'_R, \\
 e'_R{}^\dagger &= e'_R, \\
 \sum_{R \in (H,G)_{\text{ir}}} e'_R &= 1,
 \end{aligned} \tag{A29}$$

showing also that the e_R are linearly independent. Indeed, $\{e'_R | R \in (H,G)_{\text{ir}}\}$ is the desired projector basis for $\mathcal{Z}_{H,G}$ because we have

$$e'_R = \frac{n_R |\tilde{R}| |H|}{|G|^2} \sum_{C \in (H,G)_{\text{cj}}} \bar{\chi}_R(C) e_C^G, \tag{A30}$$

$$e_C^G = \sum_{R \in (H,G)_{\text{ir}}} \frac{|C|}{n_R} \chi_R(C) e'_R, \tag{A31}$$

where $R \in (H,G)_{\text{ir}}$ and $C \in (H,G)_{\text{cj}}$. In order to check Eq. (A31), insert Eq. (A30) and apply Eq. (A10). Finally, note that if H belongs to the center of G we have $e'_R = e_R^H$.

APPENDIX B: RIBBON OPERATORS

In this appendix we discuss ribbon operator algebras. We will first introduce the geometric aspects of ribbons, then we will attach operators to ribbons, and we will finish describing and characterizing certain projector ribbon subalgebras. These projectors are directly related to the topological charges and domain-wall fluxes in the models under study.

1. Sites, triangles, and strips

Our starting point is a lattice embedded in an orientable two-dimensional manifold. Let V , E , and F be the sets of its vertices, edges, and faces, respectively. The edges of the direct and dual lattices must be oriented accordingly, as explained in the main text in Sec. II and Fig. 1. A direct edge e points from the vertex $\partial_0 e$ to the vertex $\partial_1 e$ and a dual edge e^* points from the dual vertex $\partial_0 e^*$ to the dual vertex $\partial_1 e^*$. The shape of the lattice is arbitrary but with certain conditions. Namely, (i) if e is an edge, then $\partial_0 e \neq \partial_1 e$ and (ii) a face with s edges must have s different vertices. The same conditions must hold for the dual lattice.

For each edge $e \in E$, we introduce an inverse edge \bar{e} which is an edge with the direction reversed, i.e., with $\partial_0 \bar{e} = \partial_1 e$ and $\partial_1 \bar{e} = \partial_0 e$. We let $\bar{\bar{e}} = e$ and denote by $E_{\text{ext}} := E \cup \bar{E}$ the disjoint union of the original and inverse edges. For dual edges, we set $(e^*) = (\bar{e})^*$ so that $E_{\text{ext}}^* = E^* \cup \bar{E}^*$.

A (direct) path p is a list $(v_0, e_1, v_1, \dots, e_n, v_n)$ such that $v_i \in V$, $e_i \in E_{\text{ext}}$, $\partial_0 e_i = v_{i-1}$, and $\partial_1 e_i = v_i$. A dual path p^* is a list $(f_0^*, e_1^*, f_1^*, \dots, f_r^*)$ such that $f_i \in F$, $e_i \in E_{\text{ext}}$, $\partial_0 e_i^* = f_{i-1}^*$, and $\partial_1 e_i^* = f_i^*$.

Sites. A site is a pair $s = (v, f)$ with f as a face and v as one of its vertex. We visualize sites as dashed lines connecting the vertex v and the dual vertex f^* , as shown in Fig. 13 and use the notation $s := (v_s, f_s)$.

Triangles. A direct triangle $\tau = (s_0, s_1, e)$ consists of two sites s_i and a direct edge $e \in E_{\text{ext}}$ such that (i) $f_{s_0} = f_{s_1}$, (ii) $\partial_i e = v_{s_i}$, and (iii) s_0, s_1 and e form a triangle with sides listed in *counterclockwise* order. We use the notation $\tau := (\partial_0 \tau, \partial_1 \tau, e_\tau)$ and say that τ points from $\partial_0 \tau$ to $\partial_1 \tau$ through e_τ .

A dual triangle $\tau = (s_0, s_1, e^*)$ consists of two sites s_i and a dual edge $e^* \in E_{\text{ext}}^*$ such that (i) $v_{s_0} = v_{s_1}$, (ii) $\partial_i e^* = f_{s_i}^*$, and (iii) s_0, s_1 and e^* form a triangle with sides listed in *clockwise* order. We use the notation $\tau := (\partial_0 \tau, \partial_1 \tau, e_\tau^*)$ and say that τ points from $\partial_0 \tau$ to $\partial_1 \tau$ through e_τ^* . Each direct (dual) triangle τ has a complementary triangle $\bar{\tau}$, the unique direct (dual) triangle with $e_{\bar{\tau}} = \bar{e}_\tau$.

Two triangles overlap if they “share part of their area.” Specifically, a dual triangle τ and a direct triangle τ' overlap if $\partial_i \tau = \partial_i \tau'$ either for $i=0$ or $i=1$, and two triangles of the

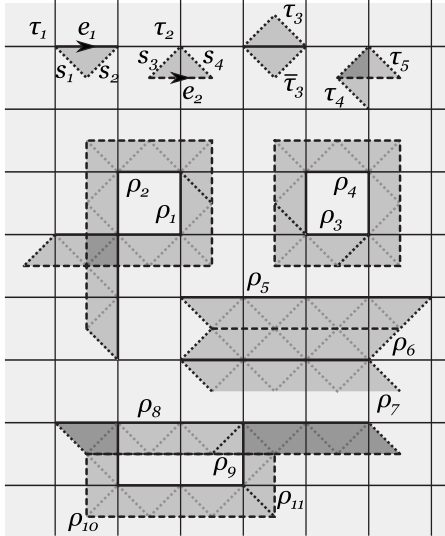


FIG. 13. Several examples of sites, triangles, and strips. Sites are displayed as black dotted lines when they are at end of some ribbon. Triangles and strips are displayed as gray bands. The direct path of each strip is a thick black line and the dual path a dashed line. The s_i are sites, the τ_i are triangles, and the ρ_i strips (indeed ribbons). $\tau_1=(s_1, s_2, e_1)$ is a direct triangle and $\tau_2=(s_3, s_4, e_2^*)$ is a dual triangle. $\bar{\tau}_3$ is the complementary triangle of τ_3 . τ_4 and τ_5 overlap. $\rho=\rho_1\rho_2$ is a strip but not a ribbon. $\sigma=\rho_3\rho_4$ and $\sigma'=\rho_4\rho_3$ are closed strips (indeed ribbons), with σ' as a rotation of σ and $\sigma \triangleright \sigma'=\rho_3$. ρ_5 is a dual complementary ribbon or ρ_6 , $(\rho_5, \rho_6)_\Delta$, and ρ_7 are direct complementary ribbons of ρ_6 , $(\rho_6, \rho_7)_\nabla$. The ribbons ρ_8 , ρ_9 , ρ_{10} , and ρ_{11} illustrate various types of joints, so that $(\rho_8, \rho_{10})_<$, $(\rho_9, \rho_{11})_>$, and $(\rho_8\rho_9, \rho_{10}\rho_{11})_<>$.

same type overlap if they are the same triangle. Triangles and their properties are illustrated in Fig. 13.

Strips. A (triangle) strip of length $n \geq 0$ is an alternating sequence of sites and triangles $\rho=(s_0, \tau_1, s_1, \tau_2, \dots, s_{n-1}, \tau_n, s_n)$ with $\partial_0 \tau_i = s_{i-1}$ and $\partial_1 \tau_i = s_i$. We define $\partial_0 \rho := s_0$ and $\partial_1 \rho := s_n$. Strips can be given just as a list of sites $\rho=(s_0, s_1, \dots, s_n)$ or, if they have nonzero length, as a list of triangles $\rho=(\tau_1, \dots, \tau_n)$. They could also be given as a pair $\rho=(s, \mathbf{x})$, with $s=s_0$ as the initial site and $\mathbf{x} \in 2^n$ as a binary vector, because for any given site there exists exactly a dual and a direct triangle pointing from it. Examples of strips are given in Fig. 13.

Let ρ be a strip with $s_i = \partial_i \rho$. We say that ρ is (i) trivial if it has length zero, (ii) direct (dual) if it consists only of direct (dual) triangles, (iii) proper if it is neither direct nor dual, (iv) open if $v_{s_0} \neq v_{s_1}$ and $f_{s_0} \neq f_{s_1}$, and (v) closed if $s_0 = s_1$. When σ is a closed strip we write $\partial \sigma$ instead of $\partial_i \sigma$.

Two strips are composable if $\partial_1 \rho_1 = \partial_0 \rho_2$. In that case the composed strip is $\rho = \rho_1 \rho_2 := (s_0, \dots, s_m)$ with $\rho_1 = (s_0, \dots, s_n)$ and $\rho_2 = (s_n, \dots, s_m)$. This composition operation is clearly associative. The cyclic nature of the list of triangles of a closed strip allows us to rotate it (see Fig. 13). We say that σ' is a rotation of σ , denoted $(\sigma, \sigma')_\circ$, if $\sigma \neq \sigma'$, $\sigma = \rho_1 \rho_2$, and $\sigma' = \rho_2 \rho_1$. In that case we set $\sigma \triangleright \sigma' := \rho_1$. Note that if $(\sigma, \sigma')_\circ$ then both σ and σ' are closed.

We can attach both a direct and a dual path to a strip ρ (see Fig. 13). If (τ_1, \dots, τ_q) is the ordered list of direct tri-

angles in ρ , we set $p_\rho = (v_{\partial_0 \tau_1}, e_{\tau_1}, v_{\partial_1 \tau_1}, \dots, e_{\tau_q}, v_{\partial_1 \tau_q})$. Similarly, if $(\tau'_1, \dots, \tau'_r)$ is the ordered list of dual triangles in ρ , we set $p_\rho^* = (f_{\partial_0 \tau'_1}, e_{\tau'_1}^*, f_{\partial_1 \tau'_1}, \dots, e_{\tau'_r}^*, f_{\partial_1 \tau'_r})$.

Consider two strips ρ_1 and ρ_2 . We say that ρ_1 and ρ_2 (i) do not overlap, denoted $(\rho_1, \rho_2)_\circ$, if no triangle of ρ_1 overlaps with a triangle of ρ_2 , (ii) form a left joint, denoted $(\rho_1, \rho_2)_<$, if $\rho_i = \rho \tau_i \rho'_i$ with τ_1 as a dual triangle, τ_2 as a direct triangle, and $(\rho'_1, \rho'_2)_\circ$, (iii) form a right joint, denoted $(\rho_1, \rho_2)_>$, if $\rho_i = \rho'_i \tau_i \rho$ with τ_1 as a dual triangle, τ_2 as a direct triangle, and $(\rho'_1, \rho'_2)_\circ$, (iv) form a left-right joint, denoted $(\rho_1, \rho_2)_<>$, if $\rho_i = \rho \tau_i \rho'_i \tau'_i \rho'$ with τ_1, τ'_1 as dual triangles, τ_2, τ'_2 direct triangles, and $(\rho'_1, \rho'_2)_\circ$, and (v) form a crossed joint, denoted $(\rho_1, \rho_2)_+$, if $\rho_i = \rho'_i \rho'_i$ with $(\rho'_1, \rho'_2)_<$, $(\rho'_2, \rho'_1)_>$, $(\rho'_1, \rho'_2)_\circ$, and $(\rho''_1, \rho''_2)_\circ$. Closed crossed joints are only possible in surfaces of nontrivial topology (see Fig. 15). Indeed, their name was chosen with those cases in which ρ_1, ρ_2 are closed in mind. The other joint types are illustrated in Fig. 13.

We denote by $V_\rho(F_\rho)$ the set of vertices (faces) in a strip ρ , by $E_\rho^\Delta(E_\rho^\nabla)$ the set of edges $e \in E_{\text{ext}}$ with $e = e_\tau$ for some dual (direct) triangle τ in ρ , and by $\bar{E}_\rho^\Delta(\bar{E}_\rho^\nabla)$ the set of their inverses. If ρ and ρ' are nondirect (non dual) closed strips with $E_\rho^\Delta = \bar{E}_{\rho'}^\Delta$ ($E_\rho^\nabla = \bar{E}_{\rho'}^\nabla$), we say that ρ' is a dual (direct) complementary ribbon of ρ , denoted $(\rho, \rho')_\Delta$ [$(\rho, \rho')_\nabla$] (see Fig. 13).

2. Ribbons

Ribbons are strips such that its direct and dual path do not self-cross.

Definition 1. Let ρ be a triangle strip, $p_\rho = (v_0, \dots, e_q, v_q)$ and $p_\rho^* = (f_0^*, \dots, e_r^*, f_r^*)$. We say that ρ is a ribbon if for any triangle τ in ρ , $\bar{\tau}$ is not in ρ , for $0 \leq i \neq j \leq q$, with $i \neq 0$ or $j \neq q$, $v_i \neq v_j$, and for $0 \leq i \neq j \leq r$, with $i \neq 0$ or $j \neq r$, $f_i \neq f_j$.

A rotation of a ribbon is a ribbon. Two ribbons ρ_1 and ρ_2 are composable if they are composable as strips and $\rho = \rho_1 \rho_2$ is a ribbon. If ρ is a ribbon and $\rho = \rho_1 \rho_2$ as strips, then ρ_1 and ρ_2 are ribbons also and $(\rho_1, \rho_2)_\circ$. Complementary ribbons do not overlap.

Consider any two sites s and s' . If $v_s = v_{s'}$, we denote by $\alpha_{s,s'}$ the unique nontrivial dual ribbon ρ with $\partial_0 \rho = s$ and $\partial_1 \rho = s'$. If $f_s = f_{s'}$, we denote by $\beta_{s,s'}$ the unique nontrivial direct ribbon ρ with $\partial_0 \rho = s$ and $\partial_1 \rho = s'$. All nontrivial dual (direct) ribbons take the form $\alpha_{s,s'}(\beta_{s,s'})$ for some s, s' . We also write $\alpha_s := \alpha_{s,s}$ and $\beta_s := \beta_{s,s}$. Then if $(\alpha_s, \alpha_{s'})_\circ$ we have $\alpha_s \triangleright \alpha_{s'} = \alpha_{s,s'}$ and if $(\beta_s, \beta_{s'})_\circ$ we have $\beta_s \triangleright \beta_{s'} = \beta_{s,s'}$.

3. Triangle operators

From this point on we are working with a fixed finite group G . To each edge $e \in E$ we attach a Hilbert space \mathcal{H}'_G with orthonormal basis $\{|g\rangle\}_{g \in G}$. The total Hilbert space of our system is then $\mathcal{H}_G := \mathcal{H}'_G^{\otimes |E|}$. If O is a single-qudit operator, O_e with $e \in E_{\text{ext}}$ denotes that operator acting on the qudit attached to the edge $e(\bar{e})$ if $e \in E(e \in \bar{E})$.

Before we can define operators for arbitrary ribbons we must consider their elementary components, triangles. To this end, let

$$L^h := \sum_{g \in G} |hg\rangle\langle g|, \quad T^g := |g\rangle\langle g|, \quad I := \sum_{g \in G} |\bar{g}\rangle\langle g|. \quad (\text{B1})$$

If τ is a dual triangle, we set

$$L_\tau^h := F L_{e_\tau}^h F^x, \quad (\text{B2})$$

with $x=0$ ($x=1$) if $e_\tau \in E(e_\tau \in \bar{E})$. If τ' is a direct triangle, we set

$$T_{\tau'}^g := F^x T_{e_{\tau'}}^g F^x, \quad (\text{B3})$$

with $x=0$ ($x=1$) if $e_\tau \in E(e_\tau \in \bar{E})$. With these definitions we have

$$L_{\tau'}^h L_\tau^{h'} = L_{\tau'}^{hh'}, \quad T_{\tau'}^g T_{\tau'}^{g'} = \delta_{g,g'} T_{\tau'}^g, \quad (\text{B4})$$

$$L_\tau^{h^\dagger} = L_{\tau'}^{\bar{h}}, \quad T_{\tau'}^{g^\dagger} = T_{\tau'}^g,$$

$$L_\tau^1 = 1, \quad \sum_{g \in G} T_{\tau'}^g = 1.$$

As for the commutation rules, we have

$$L_\tau^h T_{\tau'}^g = \begin{cases} T_{\tau'}^g L_\tau^h & \text{if } \partial_0 \tau = \partial_0 \tau' \\ T_{\tau'}^g L_\tau^h & \text{if } \partial_1 \tau = \partial_1 \tau' \\ T_{\tau'}^g L_\tau^h & \text{otherwise.} \end{cases} \quad (\text{B5})$$

If $\tau_1 \neq \tau_2$ are dual triangles and τ'_1, τ'_2 are direct triangles then

$$[L_{\tau_1}^h, L_{\tau_2}^{h'}] = [T_{\tau'_1}^g, T_{\tau'_2}^{g'}] = 0. \quad (\text{B6})$$

Thus, nonoverlapping triangles have commuting triangle operators.

4. Ribbon operators

For each ribbon ρ we introduce a set of operators $\{F_\rho^{h,g}\}$ with $h, g \in G$. We call them ribbon operators. First, if ε is a trivial ribbon we define

$$F_\varepsilon^{h,g} := \delta_{1,g}. \quad (\text{B7})$$

If τ is a dual triangle and τ' a direct triangle we set¹⁸

$$F_\tau^{h,g} := \delta_{1,g} L_\tau^h, \quad F_{\tau'}^{h,g} := T_{\tau'}^g. \quad (\text{B8})$$

If ρ is an arbitrary ribbon of length $l > 1$, we let $\rho = \rho_1 \rho_2$ and recursively define a gluing or composition procedure by means of the following relations:

$$F_\rho^{h,g} := \sum_{k \in G} F_{\rho_1}^{h,k} F_{\rho_2}^{\bar{k}h, \bar{k}g}. \quad (\text{B9})$$

We must of course ensure that this definition of $F_\rho^{h,g}$ is independent of the particular choice of ρ_1 and ρ_2 . But this amounts to check that if $\rho = \rho_1 \rho_2 \rho_3$ then

$$\sum_{k \in G} F_{\rho_1}^{h,k} F_{\rho_2 \rho_3}^{\bar{k}h, \bar{k}g} = \sum_{k \in G} F_{\rho_1 \rho_2}^{h,k} F_{\rho_3}^{\bar{k}h, \bar{k}g}, \quad (\text{B10})$$

which follows by expanding $F_{\rho_2 \rho_3}^{\bar{k}h, \bar{k}g}$ and $F_{\rho_1 \rho_2}^{h,k}$ with Eq. (B9). We also have to check that if $\rho = \varepsilon \rho = \rho \varepsilon'$ then

$$F_\rho^{h,g} = \sum_{k \in G} F_\varepsilon^{h,k} F_\rho^{\bar{k}h, \bar{k}g} = \sum_{k \in G} F_\rho^{h,k} F_{\varepsilon'}^{\bar{k}h, \bar{k}g}, \quad (\text{B11})$$

which indeed holds true.

We will find useful the notation

$$T_\rho^g := F_\rho^{1,g}, \quad L_\rho^h := \sum_{g \in G} F_\rho^{h,g} \quad (\text{B12})$$

conceived so that

$$F_\rho^{h,g} = L_\rho^h T_\rho^{g'} = T_\rho^g L_\rho^h. \quad (\text{B13})$$

Also, for any $S \subset G$, $g \in G$ we set

$$F^{S,g} := \frac{1}{|S|} \sum_{s \in S} F^{s,g}, \quad F^{g,S} := \sum_{s \in S} F^{g,s}. \quad (\text{B14})$$

We now list several properties of ribbon operators. They follow from the properties of triangle operators and Eq. (B9). For any ribbon ρ

$$F_\rho^{h,g} F_\rho^{h',g'} = \delta_{g,g'} F_\rho^{hh',g}, \quad F_\rho^{h,g^\dagger} = F_\rho^{\bar{h},g}, \quad (\text{B15})$$

$$L_\rho^1 = \sum_{g \in G} T_\rho^g = 1. \quad (\text{B16})$$

Thus, for each ρ , ribbon operators linearly generate an algebra closed under Hermitian conjugation. If ρ is dual and ρ' is direct then

$$F_\rho^{h,g} = \delta_{g,1} L_\rho^h, \quad F_{\rho'}^{h,g} = T_{\rho'}^g. \quad (\text{B17})$$

If $(\rho_1, \rho_2) <$ then

$$F_{\rho_1}^{h,g} F_{\rho_2}^{k,l} = F_{\rho_2}^{\bar{k}h, \bar{k}l} F_{\rho_1}^{h,g}. \quad (\text{B18})$$

If $(\rho_1, \rho_2) >$ then

$$F_{\rho_1}^{h,g} F_{\rho_2}^{k,l} = F_{\rho_2}^{k,l} F_{\rho_1}^{\bar{k}h, \bar{k}g}. \quad (\text{B19})$$

If $(\rho_1, \rho_2) <>$ then

$$F_{\rho_1}^{h,g} F_{\rho_2}^{k,l} = F_{\rho_2}^{\bar{k}h, \bar{k}l} F_{\rho_1}^{h,g}. \quad (\text{B20})$$

If $(\rho_1, \rho_2) +$ then

$$F_{\rho_1}^{h,g} F_{\rho_2}^{k,l} = F_{\rho_2}^{\bar{k}h, \bar{k}l} F_{\rho_1}^{h,g}. \quad (\text{B21})$$

If $(\sigma_1, \sigma_2) \circ$ then

$$F_{\sigma_1 \circ \sigma_2}^{k,l} F_{\sigma_1}^{h,g} = F_{\sigma_1 \circ \sigma_2}^{\bar{k}h, \bar{k}g} F_{\sigma_2}^{\bar{k}h, \bar{k}l}. \quad (\text{B22})$$

If $(\rho_1, \rho_2) \circ$ then

$$[F_{\rho_1}^{h,g}, F_{\rho_2}^{k,l}] = 0. \quad (\text{B23})$$

Proposition 2. Let ρ be a ribbon, $h, g \in G$.

(i) If ρ is dual

$$\text{Tr}(L_\rho^h) = \delta_{h,1} \text{Tr}(1). \quad (\text{B24})$$

(ii) If ρ is direct

$$|G| \text{Tr}(T_\rho^g) = \text{Tr}(1). \quad (\text{B25})$$

(iii) If ρ is proper

$$|G| \text{Tr}(F_\rho^{h,g}) = \delta_{h,1} \text{Tr}(1). \quad (\text{B26})$$

Proof. If ρ is not dual, choose any direct triangle τ in ρ , so that $\rho = \rho_1 \tau \rho_2$. Let $\rho' = \rho_1 \tau'$ with τ' dual. Then $(\rho', \rho)_<$ and thus using Eq. (B18)

$$\text{Tr}(F_\rho^{h,g}) = \text{Tr}(L_{\rho'}^k F_{\rho'}^{h,g} L_{\rho'}^{\bar{k}}) = \text{Tr}(F_{\rho'}^{kh\bar{k},kg} L_{\rho'}^k L_{\rho'}^{\bar{k}}) = \text{Tr}(F_{\rho'}^{kh\bar{k},kg}). \quad (\text{B27})$$

If ρ is not direct, choose any dual triangle τ in ρ , so that $\rho = \rho_1 \tau \rho_2$. Let $\rho' = \rho_1 \tau'$ with τ' as direct. Then $(\rho, \rho')_<$ and thus using Eq. (B18)

$$\begin{aligned} \text{Tr}(F_\rho^{h,g}) &= \sum_{k \in G} \text{Tr}(T_{\rho'}^k F_{\rho'}^{h,g} T_{\rho'}^k) \\ &= \sum_{k \in G} \text{Tr}(F_{\rho'}^{h,g} T_{\rho'}^{hk} T_{\rho'}^k) = \delta_{h,1} \text{Tr}(F_\rho^{h,g}). \end{aligned} \quad (\text{B28})$$

Equations (B27) and (B28) together with Eqs. (B16) and (B17) give the desired results.

As a consequence of the previous proposition and Eq. (B15) we have the following orthogonality results.

Corollary 3. Let ρ be a ribbon, $h, g \in G$.

(i) If ρ is dual

$$\text{Tr}(L_\rho^{h^\dagger} L_\rho^{h'}) = \delta_{h,h'} \text{Tr}(1). \quad (\text{B29})$$

(ii) If ρ is direct

$$|G| \text{Tr}(T_\rho^{g^\dagger} T_\rho^{g'}) = \delta_{g,g'} \text{Tr}(1). \quad (\text{B30})$$

(iii) If ρ is proper

$$|G| \text{Tr}(F_\rho^{h,g^\dagger} F_\rho^{h',g'}) = \delta_{h,h'} \delta_{g,g'} \text{Tr}(1). \quad (\text{B31})$$

Definition 4: Rotationally invariant ribbon operators.

Given a closed ribbon σ , we say that an operator $F = \sum_{h,g \in G} c_{h,g} F_\sigma^{h,g}$, $c_{h,g} \in \mathbf{C}$, is rotationally invariant if for any σ' with $(\sigma, \sigma')_\circ$ we have $F = \sum_{h,g \in G} c_{h,g} F_{\sigma'}^{h,g}$. In this regard, results (B15), (B16), and (B22) imply that if $(\sigma_1, \sigma_2)_\circ$ then

$$F_{\sigma_1}^{h,g} = \sum_{l \in G} F_{\sigma_1^{>\sigma_2}}^{h\bar{g}hl} F_{\sigma_2}^{\bar{h}l, \bar{g}l}. \quad (\text{B32})$$

5. Vertex and face operators

We now define vertex and face operators in terms of ribbon operators. Let α_s (β_s) be the unique dual (direct) closed ribbon with $\partial\alpha_s = s$ ($\partial\beta_s = s$). For any site $s = (v, f)$ let

$$A_s^h := F_{\alpha_s}^{h,1}, \quad B_s^g := F_{\beta_s}^{1,\bar{g}}. \quad (\text{B33})$$

Let s', s'' be sites with $(\alpha_s, \alpha_{s'})_\circ$, $(\beta_s, \beta_{s'})_\circ$. From Eqs. (B17) and (B32) we get

$$A_s^h = A_{s'}^h, \quad B_s^k = T_{\beta_{s,s'}}^g B_{s''}^{\bar{k}g}. \quad (\text{B34})$$

Thus, vertex operators are rotationally invariant and we can write $A_v^h := A_s^h$ for $v = v_s$. These definitions of A_v^h and B_s^g agree with those given in Eqs. (2) and (4).

Let us list several useful properties. If $s \neq s'$ then

$$A_s^h A_{s'}^{h'} = A_s^{hh'}, \quad A_s^1 = 1, \quad A_s^{h^\dagger} = \bar{A}_s^h, \quad (\text{B35})$$

$$B_s^g B_{s'}^{g'} = \delta_{g,g'} B_s^g, \quad \sum_{g \in G} B_s^g = 1, \quad B_s^{g^\dagger} = B_s^g. \quad (\text{B36})$$

$$A_s^h B_s^g = B_s^{hg\bar{h}} A_s^h, \quad (\text{B37})$$

$$[A_s^h, B_{s'}^g] = [A_s^g, A_{s'}^{g'}] = [B_s^h, B_{s'}^{h'}] = 0. \quad (\text{B38})$$

All these properties follow from the properties of ribbon operators. Note in particular that the well-known²² flux metamorphosis (B37) is a consequence of $(\alpha_s, \beta_s)_{<>}$.

For subgroups $H, H' \subset G$, H' normal, we define the operators

$$A_v^H := A_s^H := L_{\alpha_s}^H, \quad B_f^{H'} := B_s^{H'} := T_{\beta_s}^{H'}, \quad (\text{B39})$$

where $s = (v, f)$ is a site. We set $A_v := A_v^G$ and $B_f := B_f^1$. From Eqs. (B37) and (B38) we have for arbitrary vertices v, v' and faces f, f' ,

$$[A_v^H, A_{v'}^H] = [A_v^H, B_{f'}^{H'}] = [B_f^{H'}, B_{f'}^{H'}] = 0. \quad (\text{B40})$$

Of particular interest are the commutation rules between ribbon operators and vertex and face operators at their ends. We first consider nonclosed ribbons. Let $s_i = \partial_i \rho$. If $v_0 \neq v_1$ then $(\alpha_{s_0}, \rho)_<$, $(\alpha_{s_1}, \rho)_>$ and from Eqs. (B18) and (B19) we get

$$\begin{aligned} A_{s_0}^k F_\rho^{h,g} &= F_\rho^{kh\bar{k},kg} A_{s_0}^k, \\ A_{s_1}^k F_\rho^{h,g} &= F_\rho^{h,g\bar{k}} A_{s_1}^k, \end{aligned} \quad (\text{B41})$$

If $f_0 \neq f_1$ then $(\rho, \beta_{s_0})_<$, $(\rho, \beta_{s_1})_>$ and from Eqs. (B18) and (B19) we get

$$\begin{aligned} B_{s_0}^k F_\rho^{h,g} &= F_\rho^{h,g} B_{s_0}^{kh}, \\ B_{s_1}^k F_\rho^{h,g} &= F_\rho^{h,g} B_{s_1}^{\bar{g}hk}. \end{aligned} \quad (\text{B42})$$

If ρ is dual but not closed then $\alpha_{s_0} = \alpha_{s_0, s_1} \alpha_{s_1, s_0}$ and from Eqs. (B9), (B15), and (B23) we get

$$A_{s_i}^k F_\rho^{h,1} = F_\rho^{kh\bar{k},1} A_{s_i}^k, \quad (\text{B43})$$

and if ρ is direct but not closed then $\beta_{s_0} = \beta_{s_0, s_1} \beta_{s_1, s_0}$ and from Eqs. (B9), (B15), and (B23) we get

$$[B_{s_i}^k, F_\rho^{1,g}] = 0. \quad (\text{B44})$$

Now we consider closed ribbons. So let σ be a closed ribbon with $s = \partial\sigma$. If σ is a proper closed ribbon then

$(\alpha_s, \sigma)_{<>}$ and $(\sigma, \beta_s)_{<>}$ so that from Eq. (B20) we get

$$\begin{aligned} A_s^k F_\sigma^{h,g} &= F_\sigma^{k h k, k g k} A_s^k, \\ B_s^k F_\sigma^{h,g} &= F_\sigma^{h,g} B_s^{g h g k h}. \end{aligned} \quad (\text{B45})$$

If σ is closed but not proper, then either $\sigma = \alpha_s$ or $\sigma = \beta_s$, but for that case we already have Eq. (B37).

Equations (12) and (14) can be generalized. Let ρ be a ribbon with two ends $i=0,1$ and set $s_i = \partial_i \rho$, $v_i = v_{s_i}$, and $f_i = f_{s_i}$. If $v_0 \neq v_1$ from Eq. (B41) we get

$$|G| \sum_{g \in G} T_\rho^g A_{v_i} T_\rho^g = 1, \quad (\text{B46})$$

and if $f_0 \neq f_1$ from Eq. (B42) we get

$$\sum_{h \in G} L_\rho^h B_{f_i} L_\rho^h = 1. \quad (\text{B47})$$

As explained in the main text (Sec. II B), these identities show how ribbon operators can be used to obtain arbitrary states with a number of excited spots from states with one excited spot less.

6. Edge operators

For subgroups $H \subset H' \subset G$, H normal in G , we define the operators

$$L_e^H := L_\tau^H, \quad T_{e'}^{H'} := T_{\tau'}^{H'}, \quad (\text{B48})$$

where $e = e_\tau$, $e' = e_{\tau'}$ with τ as a dual triangle and τ' as a direct one. Then from Eqs. (B5) and (B6) we have for arbitrary edges e, e' ,

$$[L_e^H, L_{e'}^{H'}] = [L_e^H, T_{e'}^{H'}] = [T_{e'}^{H'}, T_{e'}^{H'}] = 0. \quad (\text{B49})$$

In some particular cases, triangle operators and ribbons have nice commuting properties, but this is not always the case. If H, H', τ, τ' are as above, with τ and τ' either in ρ or with no overlap with it then from Eqs. (B9), (B15), and (B23) we have

$$[L_\tau^H, F_\rho^{h,g}] = [T_{\tau'}^{H'}, F_\rho^{h,g}] = 0. \quad (\text{B50})$$

Other triangles are more complicated. Let $\rho = \rho_1 \rho_2$, $(\tau_1, \rho_1)_{>}$, $(\tau_2, \rho_2)_{<}$, $(\rho_1, \tau_3)_{>}$, $(\rho_2, \tau_4)_{<}$, $h \in H$, and $k, l \in G$ with $ks\bar{k} = s$ for any $s \in H$. Then from Eqs. (B9), (B18), and (B19) we have

$$L_{\tau_1}^h F_\rho^{k,l} = \sum_{g \in G} T_{\rho_1}^g F_\rho^{k, g h \bar{g} l} L_{\tau_1}^h,$$

$$L_{\tau_2}^h F_\rho^{k,l} = \sum_{g \in G} T_{\rho_1}^g F_\rho^{k, g h \bar{g} l} L_{\tau_2}^h,$$

$$T_{\tau_3}^h F_\rho^{k,l} = \sum_{g \in G} T_{\rho_1}^g F_\rho^{k,l} T_{\tau_3}^{h \bar{g} k g},$$

$$T_{\tau_4}^h F_\rho^{k,l} = \sum_{g \in G} T_{\rho_1}^g F_\rho^{k,l} T_{\tau_4}^{g \bar{k} g h}. \quad (\text{B51})$$

As a result, if $N \subset M \subset G$ with N, M normal and N central in M we have for any $h, g \in G$ $\tau = \tau_1$ (or τ_2) and $\tau' = \tau_3$ (or τ_4) with τ_i as above,

$$L_\tau^N T_{\tau'}^M F_\rho^{h,g} L_\tau^N T_{\tau'}^M = \delta_{hM, M} \frac{1}{|N|} F_\rho^{h, Ng} L_\tau^N T_{\tau'}^M, \quad (\text{B52})$$

which then gives Eq. (58).

7. Algebra \mathcal{A}_ρ

We want to define the ribbon operator algebra \mathcal{F}_ρ , but as an intermediate step we introduce the algebra \mathcal{A}_ρ . If ε is a trivial ribbon then $\mathcal{A}_\varepsilon := \mathbf{C}$. If τ is a direct (dual) triangle then $\mathcal{A}_\tau := \text{Lin}\{T_\tau^g | g \in G\}$ ($\mathcal{A}_\tau := \text{Lin}\{L_\tau^h | h \in G\}$). Finally, if $\rho = (\tau_i)$ is an arbitrary ribbon $\mathcal{A}_\rho := \otimes_i \mathcal{A}_{\tau_i}$. That \mathcal{A}_τ is really an algebra follows from Eq. (B4).

We now proceed to show several results which are essential in order to characterize ribbon operator algebras in Appendixes B 8–B 11.

Lemma 5. Let ρ be a ribbon, $O \in \mathcal{A}_\rho$ an operator, H a subgroup of G , and s a site.

(i) If ρ is not a rotation or a complement of α_s then

$$O A_s^H = 0 \Rightarrow O = 0. \quad (\text{B53})$$

(ii) If ρ is not a rotation or a complement of β_s then

$$O B_s^H = 0 \Rightarrow O = 0. \quad (\text{B54})$$

(iii) If τ is a dual triangle such that neither it nor its complement belong to ρ ,

$$O L_\tau^H = 0 \Rightarrow O = 0. \quad (\text{B55})$$

(iv) If τ is a direct triangle such that neither it nor its complement belong to ρ ,

$$O T_\tau^H = 0 \Rightarrow O = 0. \quad (\text{B56})$$

Proof. First, note that $A_s^H A_s^G = A_s^G$ implies $O A_s^H = 0 \Rightarrow O A_s^G = 0$, and similarly for L_τ^H , so that it suffices to consider $H = G$ in these cases. Also, $B_s^H B_s^1 = B_s^1$ and similarly for T_τ^H , so that it suffices to consider $H = 1$ for them.

(i) There exists a direct triangle τ such that it overlaps with α_s but not with ρ , so that $[T_\tau^h, O] = 0$, and a site s' such that $(\alpha_{s'}, \alpha_s)_\circ$ and $(\tau, \alpha_{s'})_{<}$ or $(\tau, \alpha_{s'})_{>}$. In the first case $A_{s'}^G = A_s^G$, and from Eq. (B41) we have $0 = \sum_g T_\tau^g O A_{s'}^G T_\tau^g = |G|^{-1} \sum_{g,h} O A_{s'}^h T_\tau^{hg} T_\tau^g = O A_{s'}^1 T_\tau^G = O$ and the other case is similar.

(ii) There exists a dual triangle τ such that it overlaps with β_s but not with ρ , so that $[L_\tau^h, O] = 0$, and a site s' such that $(\beta_{s'}, \beta_s)_\circ$ and $(\beta_{s'}, \tau)_{<}$ or $(\beta_{s'}, \tau)_{>}$. In the first case $B_{s'}^1 = B_s^1$, from Eq. (B42) we have $0 = \sum_h L_\tau^h O B_{s'}^h L_\tau^h = \sum_h O B_{s'}^h L_\tau^h L_\tau^h = O B_{s'}^G L_\tau^1 = O$ and the other case is similar.

[(iii) and (iv)] The proofs are analogous to (i) and (ii). \square

Given a vertex v and a dual triangle τ we set for any $O \in \mathcal{A}_\rho$,

$$O_k := A_v^k O A_v^{\bar{k}}, \quad O_k := L_\tau^k O L_\tau^{\bar{k}}. \quad (\text{B57})$$

One can check that $O_k, O_k \in \mathcal{A}_\rho$. Then if ρ satisfies the conditions of lemma 5,

$$[O, A_v^H] = 0 \Leftrightarrow |H|O = \sum_{k \in H} O_k, \quad (\text{B58})$$

$$[O, L_\tau^H] = 0 \Leftrightarrow |H|O = \sum_{k \in H} O'_k \quad (\text{B59})$$

because for $k \in H$ we have $O_k A_v^H = O_k A^k A_v^H = A_v^k O A_v^H = A^k A_v^H O = A_v^H O = O A_v$ giving $O_k = O$ and similarly for L_τ^H . As a consequence, we also get under the same conditions and $k \in H$

$$[O, A_v^H] = 0 \Rightarrow [O, A_v^k] = 0. \quad (\text{B60})$$

Given a site s in ρ and a direct triangle τ , one can check that unless ρ is both a nonclosed and nondirect ribbon with $f_{\partial_i \rho} = f_s$, for any $O \in \mathcal{A}_\rho$ there exist $O_{k,k'}, O'_{k,k'} \in \mathcal{A}_\rho$ such that $O = \sum_{k,k' \in G} O_{k,k'} = \sum_{k,k' \in G} O'_{k,k'}$ with

$$B_s^g O_{k,k'} = O_{k,k'} B_s^{\bar{g}gk'}, \quad T_\tau^g O_{k,k'} = O_{k,k'} T_\tau^{\bar{g}gk'}. \quad (\text{B61})$$

Then if ρ satisfies the conditions of lemma 5 and H is normal in G

$$[O, B_f^H] = 0 \Leftrightarrow O = \sum_{k,k' \in G | \bar{k}k' \in H} O_{k,k'}, \quad (\text{B62})$$

$$[O, T_\tau^H] = 0 \Leftrightarrow O = \sum_{k,k' \in G | \bar{k}k' \in H} O'_{k,k'} \quad (\text{B63})$$

because $OB_f^H = OB_f^H B_f^H = B_f^H OB_f^H = \sum_{k,k'} B_s^H O_{k,k'} B_s^H = \sum_{k,k'} O_{k,k'} B_s^{\bar{H}k'} B_s^H = \sum_{\bar{k}k' \in H} O_{k,k'} B_f^H$ and similarly for T_τ^H . As a consequence, we also get under the same conditions and $C \in (G/H)_{\text{cj}}$,

$$[O, B_f^H] = 0 \Rightarrow [O, B_f^C] = 0, \quad (\text{B64})$$

where $B_f^C = \sum_{c \in C} \sum_{h \in H} B_f^{ch}$.

8. Algebra \mathcal{F}_ρ

In this section we characterize the ribbon operator algebra that has been introduced so far.

Definition 6. Let ρ be a ribbon. The ribbon operator algebra $\mathcal{F}_\rho \subset \mathcal{A}_\rho$ consists of those operators $F \in \mathcal{A}_\rho$ such that $[F, A_v] = [F, B_f] = 0$ for any vertex $v \neq v_{\partial_i \rho}$ and any face $f \neq f_{\partial_i \rho}$, $i=0,1$.

Proposition 7. Let ρ be a ribbon. The $|G|^2$ ribbon operators $F_\rho^{h,g}$, $h, g \in G$, linearly generate \mathcal{F}_ρ . Moreover, ρ is proper if and only if they form a basis of \mathcal{F}_ρ .

Proof. For ribbons ρ of length zero or one, $\mathcal{F}_\rho = \mathcal{A}_\rho$ because of Eq. (B23) and the first part of the statement follows since ribbon operators generate \mathcal{A}_ρ . For ribbons of length $l > 1$, we proceed inductively on l . So let ρ be such a ribbon and set $\rho =: \rho' \tau$, with τ as a triangle. Observe that e_τ is not part of ρ' and that ρ' and τ share vertices or faces only at

their ends, so that $\mathcal{F}_\rho \subset \mathcal{F}_{\rho'} := \mathcal{F}_{\rho'} \otimes \mathcal{F}_\tau = (F_{\rho'}^{h_1, g_1} F_\tau^{h_2, g_2} | h_i, g_i \in G)$, where (\cdot) is the subspace linearly generated by the set (\cdot) . In view of Eq. (B9), what we want to show is that

$$\mathcal{F}_\rho' = \left(\sum_{k \in G} F_{\rho'}^{h,k} F_\tau^{\bar{k}h, \bar{k}g} | h, g \in G \right) \quad (\text{B65})$$

is indeed equal to \mathcal{F}_ρ . We set $s = \partial_0 \tau$, $v = v_s$, $f = f_s$, and distinguish two cases.

(a) τ is direct. In this case, $\mathcal{F}_\rho \subset \mathcal{F}_{\rho'}$ is the subalgebra of operators commuting with A_v . Then from Eqs. (B41) and (B58) we get $\mathcal{F}_\rho = (\sum_{k \in G} F_{\rho'}^{h,k} F_\tau^{\bar{k}h, \bar{k}g} | h, h', g \in G)$. Applying $F_\tau^{h,g} = F_\tau^{h',g}$ here and in Eq. (B65) gives $\mathcal{F}_\rho = \mathcal{F}_{\rho'}$.

(b) τ is dual. In this case, $\mathcal{F}_\rho \subset \mathcal{F}_{\rho'}$ with \mathcal{F}_ρ the subalgebra of operators commuting with B_f . Then from Eqs. (B42) and (B62) we get $\mathcal{F}_\rho = (F_{\rho'}^{h,g} F_\tau^{\bar{g}h, g'} | h, g, g' \in G)$. Applying $F_\tau^{h,g} = F_\tau^{h,1} \delta_{g,1}$ here and in Eq. (B65) gives $\mathcal{F}_\rho = \mathcal{F}_{\rho'}$.

This completes the inductive step. The second part of the statement follows from Eq. (B17) and corollary 3. \square

We now construct an alternative basis for \mathcal{F}_ρ . For each conjugacy class $C \in (G)_{\text{cj}}$ we choose an element r_C and denote by $\mathbf{N}_C \subset G$ the subgroup of elements commuting with r_C and by \mathcal{Q}_C a set of representatives of G/\mathbf{N}_C . Then for each $C \in (G)_{\text{cj}}$ we set $C = \{c_i\}_{i=1}^{|C|}$ and $\mathcal{Q}_C = \{q_i\}_{i=1}^{|C|}$ so that $c_i = q_i r_C \bar{q}_i$. Any $g \in G$ can be written in a unique way as $g = q_i n$, with $n \in \mathbf{N}_C$. We introduce index functions as follows: $i(g) := i$ and $n(g) := n$. For each irreducible representation $R \in (\mathbf{N}_C)_{\text{ir}}$, we choose a particular basis and denote by $\Gamma_R(k)$, $k \in G$, the corresponding unitary matrices of the representation. The desired new basis is the following:

$$F_\rho^{RC; \mathbf{u}\mathbf{v}} := \frac{n_R}{|\mathbf{N}_C|} \sum_{n \in \mathbf{N}_C} \bar{\Gamma}_R^{j'j'}(n) F_\rho^{\bar{c}_i, q_i n \bar{q}_i'}, \quad (\text{B66})$$

where $\mathbf{u} = (i, j)$, $\mathbf{v} = (i', j')$ with $i, i' = 1, \dots, |C|$ and $j, j' = 1, \dots, n_R$. The inverse change is

$$F_\rho^{h,g} = \sum_{R \in (\mathbf{N}_C)_{\text{ir}}} \sum_{j, j'=1}^{n_R} \Gamma_R^{jj'}(n_{h,g}) F_\rho^{RC; \mathbf{u}\mathbf{v}}, \quad (\text{B67})$$

where $\bar{h} \in C \in (G)_{\text{cj}}$, $n_{h,g} = \bar{q}_i(\bar{h}) g q_i(\bar{g}\bar{h})$, $\mathbf{u} = [i(\bar{h}), j]$, and $\mathbf{v} = [i(\bar{g}\bar{h}), j']$ using the index functions for C . That Eq. (B66) is really a basis follows from

$$\text{Tr}(F_\rho^{RC; \mathbf{u}\mathbf{v}} F_\rho^{R'C': \mathbf{u}'\mathbf{v}'}) = \frac{|n_R|}{|\mathbf{N}_C| |G|} \delta_{R,R'} \delta_{C,C'} \delta_{\mathbf{u}, \mathbf{u}'} \delta_{\mathbf{v}, \mathbf{v}'} \text{Tr}(1). \quad (\text{B68})$$

Instead of Eqs. (B41) and (B42) we can now write for $D_s^{h,g} := A_s^h B_s^g$, $s_i = \partial_i \rho$,

$$D_{s_0}^{h,g} F^{RC; \mathbf{u}\mathbf{v}} = \sum_{s=1}^{n_R} \Gamma_R^{sj} [n(hq_i)] F^{RC; \mathbf{u}(s)\mathbf{v}} D_{s_0}^{h,g \bar{c}_i},$$

$$D_{s_1}^{h,g} F^{RC;uv} = \sum_{s=1}^{n_R} \bar{\Gamma}_R^{sj'} [n(hq_{i'})] F^{RC;uv(s)} D_{s_1}^{h,c_i'g}, \quad (\text{B69})$$

where $\mathbf{u}=(i,j)$, $\mathbf{v}=(i',j')$, $\mathbf{u}(s)=[i(hq_i),s]$, and $\mathbf{v}(s)=[i(hq_{i'}),s]$.

9. Algebra \mathcal{K}_σ

Here we discuss the algebra of operators that gives the projectors onto states of different topological charge in systems with Hamiltonian H_G (1).

Definition 8. Let σ be a closed ribbon. The closed ribbon operator algebra $\mathcal{K}_\sigma \subset \mathcal{A}_\sigma$ consists of those operators $K \in \mathcal{A}_\sigma$ such that $[K, A_v] = [K, B_f] = 0$ for every vertex v and face f .

Note that $\mathcal{K}_\sigma \subset \mathcal{F}_\sigma$. It is not difficult to check that \mathcal{K}_{α_s} is linearly generated by the operators A_s^h , $h \in G$, and \mathcal{K}_{β_s} is linearly generated by the operators B_s^C , with $C \in (G)_{\text{cj}}$ and $B_s^C = \sum_{g \in C} B_s^g$. Note that these are exactly the rotationally invariant subalgebras of \mathcal{F}_{α_s} and \mathcal{F}_{β_s} .

For any closed ribbon σ we define the operators

$$K_\sigma^{DC} := \sum_{q \in Q_C} \sum_{d \in D} F_\sigma^{qd\bar{q},qrC\bar{q}}, \quad (\text{B70})$$

where $C \in (G)_{\text{cj}}$ and $D \in (\mathbf{N}_C)_{\text{cj}}$. The point of these operators is that they are rotationally invariant,

$$(\sigma, \sigma')_e \Rightarrow K_{\sigma'}^{DC} = K_\sigma^{DC}, \quad (\text{B71})$$

as can be checked applying Eq. (B32). In fact, it can be shown that if σ is proper they form a basis of the subalgebra of rotationally invariant ribbon operators of \mathcal{F}_σ . From Eqs. (B15) and (B16) we get

$$\begin{aligned} K_\sigma^{DC} K_\sigma^{D'C'} &= \delta_{C,C'} \sum_{D''} N_{DD'}^{D''} K_\sigma^{D''C}, \\ K_\sigma^{DC\dagger} &= K_\sigma^{\bar{D}C}, \\ \sum_{C \in (G)_{\text{cj}}} K_\sigma^{1C} &= 1, \end{aligned} \quad (\text{B72})$$

where the sum runs over $D'' \in (\mathbf{N}_C)_{\text{cj}}$, $DD' = \sum_{D''} N_{D,D'}^{D''} D''$, and \bar{D} denotes the inverse class of D . Result (B26) implies for any proper σ ,

$$|G| \text{Tr}(K_\sigma^{DC}) = |C| \delta_{D,1} \text{Tr}(1), \quad (\text{B73})$$

which together with Eq. (B72) and $N_{DD'}^1 = \delta_{\bar{D},D'} |D|$ gives

$$\text{Tr}(K_\sigma^{DC\dagger} K_\sigma^{D'C'}) = \frac{|D||C|}{|G|} \delta_{D,D'} \delta_{C,C'} \text{Tr}(1). \quad (\text{B74})$$

Proposition 9. Let σ be a proper closed ribbon. The operators K_σ^{DC} , $C \in (G)_{\text{cj}}$, and $D \in (\mathbf{N}_C)_{\text{cj}}$, form a basis of \mathcal{K}_σ .

Proof. This is just a particular case of proposition 14.

For any proper closed ribbons σ , consider the subalgebra $\mathcal{K}_\sigma^C \subset \mathcal{K}_\sigma$ with basis $\{K_\sigma^{DC} | D \in (\mathbf{N}_C)_{\text{cj}}\}$. The point is that in view of Eqs. (A21) and (B72) we have $\mathcal{K}_\sigma^C \simeq \mathcal{Z}_C$, where \mathcal{Z}_C is

the center of the group algebra $\mathbf{C}[\mathbf{N}_C]$. In particular the isomorphism identifies K_σ^{DC} with $e_D := \sum_{h \in D} h$. Note that the isomorphism preserves adjoints as defined in Eq. (A18). This suggests the introduction of a different basis for \mathcal{K}_σ . We define in analogy with Eq. (A24),

$$K_\sigma^{RC} := \frac{n_R}{|\mathbf{N}_C|} \sum_{D \in (\mathbf{N}_C)_{\text{cj}}} \bar{\chi}_R(D) K_\sigma^{DC}, \quad (\text{B75})$$

where $R \in (\mathbf{N}_C)_{\text{ir}}$. Due to Eq. (A25), the reverse change of basis is

$$K_\sigma^{DC} = \sum_{R \in (\mathbf{N}_C)_{\text{ir}}} \frac{|D|}{n_R} \chi_R(D) K_\sigma^{RC}. \quad (\text{B76})$$

Due to Eq. (A23), the elements of the new basis are orthogonal projectors summing up to the identity

$$K_\sigma^{RC\dagger} = K_\sigma^{RC},$$

$$K_\sigma^{RC} K_\sigma^{R'C'} = \delta_{R,R'} \delta_{C,C'} K_\sigma^{RC},$$

$$\sum_{R,C} K_\sigma^{RC} = 1. \quad (\text{B77})$$

10. Algebra \mathcal{J}_ρ

We discuss now the algebra of operators that gives the projectors onto states with different domain-wall types in systems with Hamiltonian H_G^{NM} (35), with $N \subset M \subset G$ subgroups, N normal in G .

Definition 10. Let ρ be an open ribbon. The ribbon operator algebra $\mathcal{J}_\rho \subset \mathcal{F}_\rho$ consists of those operators $J \in \mathcal{F}_\rho$ such that $[J, A_v^M] = [J, B_f^N] = 0$ for every vertex v and face f .

We denote by r_T an arbitrarily chosen representative of a class T of the double coset $M \backslash G / M$, by $\mathbf{N}_T \subset M$ the subgroup of elements m such that $mr_T M = r_T M$, and by Q_T a set of representatives of M / \mathbf{N}_T . For any open ribbon ρ we define the operators

$$J_\rho^{CT} := \sum_{q \in Q_T} \sum_{c \in C} F_\rho^{qc\bar{q},qrT^M}, \quad (\text{B78})$$

where $C \in (N, \mathbf{N}_T)_{\text{cj}}$ and $T \in M \backslash G / M$. From Eq. (B15) we get

$$J_\rho^{CT} J_\rho^{C'T'} = \delta_{T,T'} \sum_{C''} N_{CC'}^{C''} J_\rho^{C''T},$$

$$J_\rho^{CT\dagger} = J_\rho^{\bar{C}T},$$

$$\sum_{T \in M \backslash G / M} J_\rho^{1T} = 1, \quad (\text{B79})$$

where the sum runs over $C'' \in (N, M)_{\text{cj}}$, $CC' = \sum_{C''} N_{C,C'}^{C''} C''$ and \bar{C} denotes the inverse class of C . From Eq. (B26) we get

$$|G| \text{Tr}(J_\rho^{CT}) = |T| \delta_{C,1} \text{Tr}(1), \quad (\text{B80})$$

which together with Eq. (B79) and $N_{CC'}^1 = \delta_{\bar{C},C'} |C|$ gives

$$|G|\text{Tr}(J_\rho^{CT\dagger} J_\rho^{C'T'}) = |C||T|\delta_{C,C'}\delta_{T,T'}\text{Tr}(1). \quad (\text{B81})$$

Proposition 11. Let ρ be an open ribbon. The operators J_ρ^{CT} , $C \in (N, \mathbf{N}_T)_{\text{cj}}$, $T \in M \setminus G/M$, form a basis of \mathcal{J}_ρ .

Proof. Set $(v_i, f_i) = \partial_i \rho$ and let $\mathcal{F}'_\rho \subset \mathcal{F}_\rho$ be the subalgebra of operators commuting with $A_{v_i}^M$. From Eqs. (B41) and (B58) we get $\mathcal{F}'_\rho = (\sum_{m \in M} F_\rho^{m\bar{h}\bar{m}, mgM} |h, g \in G)$. $\mathcal{J}'_\rho \subset \mathcal{F}'_\rho$ is the subalgebra of operators commuting with $B_{f_i}^N$. From Eqs. (B42) and (B62) we get $\mathcal{J}'_\rho = (\sum_{m \in M} F_\rho^{m\bar{h}\bar{m}, mgM} |h \in N, g \in G)$. Finally, if $h \in C \in (N, \mathbf{N}_T)_{\text{cj}}$ and $g \in T \in M \setminus G/M$ we have $\sum_{m \in M} F_\rho^{m\bar{h}\bar{m}, mgM} = \frac{|\mathbf{N}_T|}{|C|} J_\rho^{CT}$. The result follows in view of Eq. (B81). From the previous proposition and Eqs. (B9), (B51), and (B52) we get the following result, which is no longer true if the condition of N being Abelian is removed.

Corollary 12. Let ρ be an open ribbon and e an edge. If N is Abelian $[J, L_e^N] = [J, T_e^M] = 0$ for any $J \in \mathcal{J}_\rho$.

For any open ribbon ρ and $T \in M \setminus G/M$, consider the subalgebra $\mathcal{J}'_\rho \subset \mathcal{J}_\rho$ with basis $\{J_\rho^{CT} | C \in (N, \mathbf{N}_T)_{\text{cj}}\}$. The point is that in view of Eqs. (A21) and (B79) we have $\mathcal{J}'_\rho \cong \mathcal{Z}_{N, \mathbf{N}_T}$. In particular the isomorphism identifies J^{CT} with e_C^M . Note that the isomorphism preserves adjoints as defined in Eq. (A18). This suggests the introduction of a different basis for \mathcal{J}_ρ . We define in analogy with Eq. (A30),

$$J_\rho^{RT} := \frac{n_R |\tilde{R}| |M|}{|\mathbf{N}_T|^2} \sum_{C \in (N, \mathbf{N}_T)_{\text{cj}}} \bar{\chi}_R(C) J_\rho^{CT}, \quad (\text{B82})$$

where $R \in (N, \mathbf{N}_T)_{\text{ir}}$. Due to Eq. (A31), the reverse change of basis is

$$J_\rho^{CT} = \sum_{R \in (N, \mathbf{N}_T)_{\text{ir}}} \frac{|C|}{n_R} \chi_R(C) J_\rho^{RT}. \quad (\text{B83})$$

Due to Eq. (A29), the elements of the new basis are orthogonal projectors summing up to the identity

$$\begin{aligned} J_\rho^{RT\dagger} &= J_\rho^{RT}, \\ J_\rho^{RT} J_\rho^{R'T'} &= \delta_{R,R'} \delta_{T,T'} J_\rho^{RT}, \\ \sum_{R,T} J_\rho^{RT} &= 1. \end{aligned} \quad (\text{B84})$$

Two comments should be made here. First, in the particular case of M normal in G , $M \setminus G/M = G/M$ and for $T \in G/M$ we have $\mathbf{N}_T = M$, so that the two labels for the basis of \mathcal{J}_ρ are not related anymore. Second, although definition 10 only applies to open ribbons, the algebra \mathcal{J}_ρ can be extended to any ρ taking proposition 11 as a definition. As long as ρ is proper, properties (B79)–(B84) will still hold.

The special case of M normal and N central in M deserves special attention. Instead of Eqs. (B78) and (B82) we can write

$$J_\rho^{n,t} := F^{n,tM}, \quad J_\rho^{\chi,t} := \frac{1}{|N|} \sum_{n \in N} \bar{\chi}(n) J_\rho^{n,t}, \quad (\text{B85})$$

where $n \in N$, $\tilde{t} \in G/M$, and $\chi \in (N)_{\text{ch}}$, with $(N)_{\text{ch}}$ as the character group of N . Then, if $\rho = \rho_1 \rho_2$ is an open ribbon from Eq. (B9) we get Eq. (64).

11. Algebra \mathcal{K}'_σ

Here we discuss the algebra of operators that gives the projectors onto states of different charge, confined and topological, in systems with Hamiltonian H_G^{NM} (35), where $N \subset M \subset G$ are normal subgroups in G with N central in M .

Definition 13. Let σ be a closed ribbon. The closed ribbon operator algebra $\mathcal{K}'_\sigma \subset \mathcal{F}_\sigma$ consists of those operators $K \in \mathcal{F}_\sigma$ such that $[K, A_v^M] = [K, B_f^N] = [K, T_e^M] = [K, L_e^N] = 0$ for any vertex v , face f , and edge e .

Note that if $N=1$ and $M=G$, then $\mathcal{K}'_\sigma = \mathcal{K}_\sigma$. We extend our previous notation and set $(A, B)_{\text{cj}} \subset \{\{bab | b \in B\} | a \in A\}$ for two subgroups A, B of some other group. For each class $C \in (G/N, M/N)_{\text{cj}}$, we choose a representative $r_C \in G$. Let $\mathbf{N}'_C := \{m \in M | mr_C \bar{m} r_C \in N\}$ and choose a set $Q_C \subset M$ of representatives of M/\mathbf{N}'_C . For any closed ribbon σ we define the operators

$$K_\sigma^{DC} := \sum_{q \in Q_C} \sum_{d \in D} \sum_{n \in N} F_\sigma^{q\bar{d}\bar{q}, qr_C \bar{q}n}, \quad (\text{B86})$$

where $C \in (G/N, M/N)_{\text{cj}}$ and $D \in (\mathbf{N}'_C)_{\text{cj}}$. With this notation, results (B72) remain true, and Eqs. (B73) and (B88) only need a slight modification,

$$|G|\text{Tr}(K_\sigma^{DC}) = |C||M|\delta_{D,1}\text{Tr}(1), \quad (\text{B87})$$

$$\text{Tr}(K_\sigma^{DC\dagger} K_\sigma^{D'C'}) = \frac{|D||C||M|}{|G|} \delta_{D,D'} \delta_{C,C'} \text{Tr}(1). \quad (\text{B88})$$

Proposition 14. Let σ be a proper closed ribbon. The operators K_σ^{DC} , $C \in (G/N, M/N)_{\text{cj}}$, $D \in (\mathbf{N}'_C)_{\text{cj}}$, form a basis of \mathcal{K}'_σ .

Proof. Set $(v, f) = \partial \sigma$ and let $\mathcal{F}'_\sigma \subset \mathcal{F}_\sigma$ be the subalgebra of operators commuting with A_v^M and B_f^N . From Eqs. (B45), (B58), and (B62) we get $\mathcal{F}'_\sigma = (\sum_{m \in M} F_\sigma^{m\bar{h}\bar{m}, mgm} |h, g \in G, hg\bar{h}\bar{g} \in N)$. $\mathcal{K}'_\sigma \subset \mathcal{F}'_\sigma$ is the subalgebra of operators commuting with L_e^N and T_e^M for every edge e . From Eqs. (B50), (B51), (B59), and (B63) it follows that $\mathcal{K}_\sigma = (\sum_{m \in M} \sum_{n \in N} F_\sigma^{m\bar{h}\bar{m}, mg\bar{m}n} |h \in M, g \in G, hg\bar{h}\bar{g} \in N)$. But given such h and g there exists a class $C \in (G/N, M/N)_{\text{cj}}$ and $q \in Q_C$ with $\bar{q}gq\bar{r}_C \in N$ and a class $D \in (\mathbf{N}'_C)_{\text{cj}}$ with $\bar{q}h\bar{q} \in D$, so that $\sum_{m \in M} \sum_{n \in N} F_\sigma^{m\bar{h}\bar{m}, mg\bar{m}n} = \frac{|\mathbf{N}'_C|}{|D|} K_\sigma^{DC} = \frac{|M|}{|C||D|} K_\sigma^{DC}$. The result follows in view of Eq. (B88).

The change in basis (B75) that leads to relations (B77) is possible for \mathcal{K}'_σ just as it was for \mathcal{K}_σ , with the only difference that now the representations R belong to $(\mathbf{N}'_C)_{\text{ir}}$. For any proper closed ribbon we have $\mathcal{J}_\sigma \subset \mathcal{K}'_\sigma$. In fact

$$J_\sigma^{nt} = \sum_{C \subset tM} K_\sigma^{nC}, \quad (\text{B89})$$

where $n \in N$, $t \in G/M$, $C \in (G/N, M/N)_{\text{cj}}$, and $K_\sigma^{nC} = K_\sigma^{DC}$ with $D = \{n\} \in (N, \mathbf{N}'_C)_{\text{cj}} \subset (\mathbf{N}'_C)_{\text{cj}}$. From Eqs. (B76), (B82),

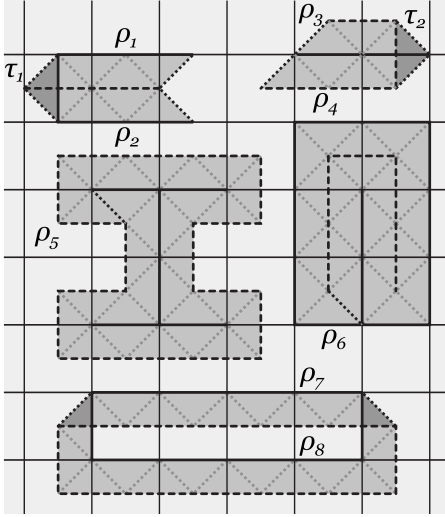


FIG. 14. Several constructions with nice strips and ribbons. All elements are displayed as in Fig. 13. The ρ_i are ribbons, except ρ_5 and ρ_6 which are just nice strips. We have $(\rho_1, \tau_1, \rho_2)_{\triangleleft}$ and $(\rho_3, \tau_2, \rho_4)_{\triangleright}$. ρ_5 is a dual block and ρ_6 is a direct block. ρ_7 and ρ_8 form a simple deformation, $(\rho_7, \rho_8)_{=}$.

and (B89) we get the following relation between the corresponding projector bases:

$$J_{\sigma}^{\chi} = \sum_{C \subset M} \sum_{R \in (N_C)_{\text{tr}}} \frac{(\chi_R, \chi)_N}{n_R} K_{\sigma}^{RC}, \quad (\text{B90})$$

where $\chi \in (N)_{\text{ch}}$, $t \in G/M$, and $C \in (G/N, M/N)_{\text{cj}}$.

APPENDIX C: RIBBON TRANSFORMATIONS

In this appendix we discuss several transformations that can be applied to ribbons. These transformations are interesting because they leave invariant the action of certain ribbon operator algebras on suitable subspaces of \mathcal{H}_G . In order to prove the desired properties, we need some preliminary results. We will find useful the notation \cdot_{ψ} for $|\psi\rangle = \cdot|\psi\rangle$. Also, for strips ρ, ρ' and a triangle τ , we write $(\rho, \tau, \rho')_{\triangleleft}$ if τ is direct, $(\rho, \tau)_{<}$, $(\rho', \tau)_{>}$, and we write $(\rho, \tau, \rho')_{\triangleright}$ if τ is dual, $(\tau, \rho)_{<}$, $(\tau, \rho')_{>}$ (see Fig. 14).

Lemma 15. Let ρ, ρ' be ribbons, $|\psi\rangle \in \mathcal{H}_G$ and $H, H' \subset G$ normal subgroups with $hh' = h'h$ for any $h \in H, h' \in H'$. (i) If $(\rho, \rho')_{\triangle}$, there exist a direct triangle τ such that $(\rho, \tau, \rho')_{\triangleleft}$ and for any $f \in F_{\rho} - \{f_{\partial_0 \rho}, f_{\partial_1 \rho}\}$ we have $B_f^H =_{\psi} 1$ then for $h' \in H'$

$$L_{\rho}^{h'} =_{\psi} \sum_{k, l \in G} F_{\rho'}^{\bar{k}h' \bar{k}l, l} T_{\tau}^k. \quad (\text{C1})$$

(ii) If $(\rho, \rho')_{\nabla}$ then for $g \in G$

$$T_{\rho}^g =_{\psi} T_{\rho'}^{\bar{g}}. \quad (\text{C2})$$

Proof. (i) Let $p_{\rho}^* = (f_0^*, \dots, e_r^*, f_r^*)$ and set $s_i = (\partial_0 e_{i+1}, f_i)$ for $i = 1, \dots, r-1$. Consider the states $|\psi^{g, h}\rangle := L_{e_1}^{H'} T_{\tau}^g L_{e_2}^{H'} B_{s_1}^{h_1} \dots L_{e_r}^{H'} B_{s_{r-1}}^{h_{r-1}} |\psi\rangle$, $g \in G$, $h \in H^{r-1}$. Then $L_{e_i}^{H'} =_{\psi^{g, h}} 1$ and thus $L_{\rho}^{h'} =_{\psi^{g, h}} L_{\rho'}^{H'} =_{\psi^{g, h}} 1$. But $|\psi\rangle$

$= \sum_{g \in G} T_{\tau}^g \sum_{h \in H^{r-1}} \prod_{i=1}^{r-1} B_{s_{i-1}}^{h_i} |\psi^{g, h}\rangle$ and the result follows using Eq. (B51) and the fact that $[L_{\rho}^k, B_{s_i}^h] = [L_{\rho'}^k, B_{s_i}^h] = 0$ for $k \in H'$, $h \in H$. (ii) The proof is dual to (i). Just note that one must use $\sum_{k \in G} L_{\tau}^k T_{e_1}^k L_{\tau}^k = 1$ instead of $\sum_{k \in G} T_{\tau}^k L_{e_1}^{H'} T_{\tau}^k = 1$ and $\sum_{k \in G} A_{v_{i-1}}^k T_{e_i}^k A_{v_{i-1}}^k = 1$ instead of $\sum_{k \in H} B_{f_{i-1}}^{H'} L_{e_i}^{H'} B_{s_{i-1}}^k = B_{f_{i-1}}^{H'}$. Alternatively, the result is trivial in terms of Wilson loops.

When working with ribbon deformations, it is useful to consider triangle strips that are more general than ribbons but still allow to introduce operators. We say that a strip ρ is nice if no two of its triangles overlap or, equivalently, if $\rho = \rho_1 \cdots \rho_n$ with ρ_i ribbons such that $(\rho_i, \rho_j)_{\circ}$ for $i \neq j$. Then ribbon operators can be generalized for nice strips using Eq. (B9). Although such nice strip operators still commute with all vertex operators A_v and face operators B_f except those in their ends, they can no longer be characterized by this property. A direct (dual) block ρ is a nice closed strip such that $(\rho, \rho)_{\triangle} [(\rho, \rho)_{\nabla}]$ (see Fig. 14).

Lemma 16. Let ρ be a nice closed strip, $|\psi\rangle \in \mathcal{H}_G$ and $H \subset G$ a normal subgroup. (i) If ρ is a dual block and for any $v \in V_{\rho}$ we have $A_v^H =_{\psi} 1$ then

$$L_{\rho}^H =_{\psi} 1. \quad (\text{C3})$$

(ii) If ρ is a direct block and for any $f \in F_{\rho}$ we have $B_f^H =_{\psi} 1$ then

$$T_{\rho}^H =_{\psi} 1. \quad (\text{C4})$$

Proof. (i) We proceed recursively on $|V_{\rho}|$. For $|V_{\rho}| = 0, 1$ the result is trivial. So let $|V_{\rho}| > 1$. Note that if $(\rho, \rho')_{\circ}$ then $L_{\rho}^H = L_{\rho'}^H$ due to Eq. (B32). Also, the path $p_{\rho} = (v_0, \dots, e_q, v_r)$ forms a tree. Thus, w.l.o.g. we can choose ρ such that $v_1 = v_{r-1}$ and there exists a dual triangle τ such that $\rho = \rho_1 \rho_2$ with $\alpha := \alpha_{\partial_0 \rho} = \rho_1 \tau$ and $(\rho_2, \tau, \rho_2)_{\triangleleft}$. Set $\rho_2 = \tau' \rho_3 \bar{\tau}'$, with τ' as a direct triangle. Then $L_{\rho}^{h'} = L_{\rho_1}^{h'} L_{\rho_2}^{h'} = L_{\rho_1}^{h'} \sum_{k \in G} T_{\tau'}^k L_{\rho_3}^{\bar{k}h' k} =_{\psi} L_{\rho_1}^{h'} \sum_{k \in G} T_{\tau'}^k L_{\tau}^{\bar{k}h' k} = L_{\rho_1}^{h'} L_{\tau}^{h'} = L_{\alpha}^{h'} =_{\psi} 1$, where we have used the fact that $\rho_3 \bar{\tau}'$ is a block and Eq. (C1) for $\tau, \bar{\tau}$. (ii) Again the proof is dual to (i) or, alternatively, trivial in terms of Wilson loops. \square

Corollary 17. Let $\rho = \rho_1 \rho_2$ be a nice strip. Under the same conditions of the previous lemma we have, respectively, (i) for $h \in H$

$$L_{\rho_1}^h =_{\psi} \sum_{g \in G} T_{\rho_1}^g L_{\rho_2}^{\bar{g}h}, \quad (\text{C5})$$

(ii) for $g \in G$

$$T_{\rho_1}^{gH} =_{\psi} T_{\rho_2}^{\bar{g}H}. \quad (\text{C6})$$

Proof. Apply Eqs. (B9) and (B15) to (i) Eq. (C3) and (ii) Eq. (C4). \square

For a region R we will understand a collection of faces f . We also consider dual regions R^* , collections of dual faces v^* .

1. Transformation rules

a. Deformations

Before we define general ribbon deformations, such as the one in Fig. 4, we have to introduce certain simpler ones which are easier to manage in proofs, as the one depicted in Fig. 14. Then simple deformations can be combined together to give the general ones. We say that the ribbons ρ, ρ' form a simple deformation, denoted $(\rho, \rho')_{=}$, if (i) they are open, (ii) they share no triangles, (iii) $(\rho, \rho')_{<>}$, and (iv) for any $e \in E_{\rho}^{\Delta}$ we have $\partial_1 e \in V_{\rho'}$. The dual of (iv) is automatically true: for any $e \in E_{\rho'}^{\nabla}$, we have $f \in F_{\rho}$ for $f^* = \partial_1 e^*$. We will use the notation $F_{\rho, \rho'} = F_{\rho} - \{f_{s_0}, f_{s_1}\}$ and $V_{\rho, \rho'} = V_{\rho'} - \{v_{s_0}, v_{s_1}\}$, where $s_i = \partial_i \rho = \partial_i \rho'$.

Let $R = (R_1, R_2^*)$ with R_1 as a region and R_2^* as a region of the dual lattice. We introduce the relation between ribbons \simeq_R as the minimal equivalence relation such that $\rho'_1 \simeq_R \rho'_1$ if the following conditions are all true: $\rho_1 = \rho_2 \rho \rho_3$, $\rho'_1 = \rho_2 \rho' \rho_3$, $(\rho, \rho')_{=}$, $F_{\rho, \rho'} \subset R_1$, and $V_{\rho, \rho'}^* \subset R_2^*$. Thus, two ribbons are equivalent in the sense of \simeq_R if they can be transformed one into the other through simple deformations within R . Given a state $|\psi\rangle \in \mathcal{H}_G$ and subgroups $H, H' \subset G$, H normal, we define $R_{\psi}^{HH'} = (R_1, R_2^*)$ with R_1 as the region such that $f \in R_1$ if $B_f^H = \psi 1$ and R_2^* as the dual region such that $v^* \in R_2^*$ if $A_v^{H'} = \psi 1$. Then we write $\simeq_{\psi}^{HH'}$ for $\simeq_{R_{\psi}^{HH'}}$.

Proposition 18. Let $|\psi\rangle \in \mathcal{H}_G$ and $H, H' \subset G$ normal subgroups with $hh' = h'h$ for any $h \in H$, $h' \in H'$. If ρ, ρ' are ribbons with $\rho \simeq_{\psi}^{H, H'} \rho'$ then

$$F_{\rho}^{h', S} =_{\psi} F_{\rho'}^{h', S}, \quad (\text{C7})$$

where $h' \in H'$, $S \in G/H$.

Proof. Using Eq. (B9) for $\rho = \rho_1 \rho_2 \rho_3$ we can write

$$F_{\rho}^{h', gH} = \sum_{l, m \in G} F_{\rho_1}^{h', l} F_{\rho_2}^{\bar{h}l, l} F_{\rho_3}^{\bar{m}gh, gm, \bar{m}}, \quad (\text{C8})$$

and thus it is enough to consider simple deformations $(\rho, \rho')_{=}$. In that case, we can set $\rho = \tau_1 \rho_1 \tau_1'$ with τ_1, τ_1' dual triangles and there exists a ribbon ρ_2 such that $\rho_1 \rho_2$ is a block and the conditions of lemma 16 (ii) are satisfied, so that Eq. (C6) applies. But $(\rho_2, \rho')_{\nabla}$, so that using Eq. (C2) we get $T_{\rho}^S = T_{\rho_1}^S =_{\psi} T_{\rho_2}^S = T_{\rho'}^S$. We can write $\rho_2 = \bar{\tau}_2' \rho_2' \bar{\tau}_2$ and $\rho' = \tau_2 \rho_3 \tau_2'$ with τ_2, τ_2' direct triangles and set $\rho_2'' = \bar{\tau}_1 \rho_2' \bar{\tau}_1'$. Then $(\rho, \rho_2'')_{\Delta}$ and Eq. (C1) apply. Also, $\rho_2'' \rho_3$ is a block and the conditions of lemma 16 (i) are satisfied, so that Eq. (C5) applies (for H'). Putting everything together we get $L_{\rho}^{h'} =_{\psi} \sum_{k, l \in G} F_{\rho_2''}^{kh', kl} T_{\tau_2}^k =_{\psi} \sum_{k \in G} L_{\rho_2''}^{kh', k} T_{\tau_2}^k =_{\psi} L_{\rho'}^{h'}$, where we have used also Eqs. (B9) and (B16).

b. Extensions, contractions, and rotations

We also want to consider deformations in which the ends of ribbons are not fixed. Let $Q = (Q_1, Q_2^*)$ with $Q_1, Q_2 \subset E_{\text{ext}}$. We introduce the relation between ribbons \simeq_Q as the minimal equivalence relation such that $\rho \simeq_Q \rho'$ if $\rho = \rho_1 \rho' \rho_2$, $E_{\rho}^{\nabla} \subset Q_1$, and $E_{\rho'}^{\Delta} \subset Q_2$. Thus, two ribbons are equivalent in

the sense of \simeq_Q if they can be transformed from one into the other through extensions or contractions within Q . We also introduce an equivalence relation \simeq_Q for closed ribbons, the minimal such that $\sigma \simeq_Q \sigma'$ if $(\sigma, \sigma')_{\circ}$, $E_{\sigma \triangleright \sigma'}^{\nabla} \subset Q_1$, and $E_{\sigma \triangleright \sigma'}^{\Delta} \subset Q_2$. Thus, two closed ribbons are equivalent in the sense of \simeq_Q if they can be transformed one into the other through rotations within Q . Given a state $|\psi\rangle \in \mathcal{H}_G$ and subgroups $H, H' \subset G$, H' normal, we set $Q_{\psi}^{HH'} := (Q_1, Q_2^*)$ with Q_1 as the collection of edges e with $T_e^H |\psi\rangle = |\psi\rangle$ and Q_2 as the collection of edges e' with $L_{e'}^{H'} |\psi\rangle = |\psi\rangle$. Then we write $\simeq_{\psi}^{H, H'}$ for $\simeq_{Q_{\psi}^{HH'}}$ and similarly for \simeq .

Proposition 19. Let $|\psi\rangle \in \mathcal{H}_G$ and $H, H' \subset G$ subgroups with H' normal. (i) If ρ, ρ' are ribbons with $\rho \simeq_{\psi}^{H, H'} \rho'$ then

$$\sum_{k \in H} F_{\rho}^{kh', \bar{k}, kgH} =_{\psi} \sum_{k \in H} F_{\rho'}^{kh', \bar{k}, kgH}, \quad (\text{C9})$$

where $h' \in H'$, $g \in G$. (ii) If σ, σ' are closed ribbons with $\sigma \simeq_{\psi}^{H, H'} \sigma'$ then

$$\sum_{k \in H} F_{\sigma}^{khk, \bar{k}, g\bar{k}} =_{\psi} \sum_{k \in H} F_{\sigma'}^{khk, \bar{k}, g\bar{k}}. \quad (\text{C10})$$

where $h, g \in G$, $hg\bar{h}\bar{g} \in H'$.

Proof. (i) It is enough to consider $\rho = \rho' \tau$ or $\rho = \tau \rho'$ with τ as a triangle and then apply Eq. (B9). (ii) It is enough to consider that $\sigma \triangleright \sigma' = \tau$ and then apply Eq. (B32).

c. Inversions

We finally consider other kind of ribbon transformations in which basically ribbons are reversed. As in the other cases, we start introducing suitable relations. For open ribbons ρ, ρ' and $R = (R_1, R_2^*)$ as above, we write $\rho \doteq_R \rho'$ if $\partial_0 \rho = \partial_1 \rho'$, $\partial_1 \rho = \partial_0 \rho'$ and either (i.a) $(\rho, \rho')_{\nabla}$ and $V_{\rho, \rho'}^* \subset R_2^*$ or (i.b) $(\rho, \rho')_{\Delta}$ and $F_{\rho} \subset R_1$. For closed ribbons σ, σ' and a triangle τ , we write $\sigma \doteq_{R, \tau} \sigma'$ if either (ii.a) $(\sigma, \sigma')_{\nabla}$, τ is dual, $(\sigma, \tau, \sigma')_{\triangleright}$ and $V_{\rho}^* \subset R_2^*$ or (ii.b) $(\sigma, \sigma')_{\Delta}$, τ is direct, $(\sigma, \tau, \sigma')_{\triangleleft}$ and $F_{\rho} \subset R_1$. For $|\psi\rangle \in \mathcal{H}_G$, we write $\doteq_{\psi}^{HH'}$ for $\doteq_{R_{\psi}^{HH'}}$ and also $\doteq_{\psi}^{HH'}$ for $\doteq_{R_{\psi}^{HH'}, \tau}$ if either τ is dual and $L_{\tau}^H = \psi 1$ or τ is direct and $T_{\tau}^{H'} = \psi 1$.

Proposition 20. Let $|\psi\rangle \in \mathcal{H}_G$ and $H, H' \subset G$ normal subgroups with $hh' = h'h$ for any $h \in H$, $h' \in H'$. (i) If ρ, ρ' are open ribbons with $\rho \doteq_{\psi}^{H, H'} \rho'$ then

$$F_{\rho}^{h', S} =_{\psi} F_{\rho'}^{\bar{h}' S, \bar{S}}, \quad (\text{C11})$$

where $h' \in H'$, $s \in S \in G/H$. (ii) If σ, σ' are closed ribbons with $\sigma \doteq_{\psi}^{H, H'} \sigma'$ then

$$\sum_{k \in H'} F_{\rho}^{\bar{k}h'k, \bar{k}, kS} =_{\psi} \sum_{k \in H'} F_{\rho'}^{\bar{k}S\bar{h}'s, \bar{k}, kS}, \quad (\text{C12})$$

where $h' \in H'$ and $s \in S \in G/H$ with $sg\bar{s}\bar{g} \in H$.

Proof. (i.a) This case follows from Eqs. (C2) and (C5).

(i.b) There exists ribbons ρ_i and direct ribbons ρ'_i, ρ''_i , $i = 1, 2$, such that $\rho = \rho'_1 \rho_1 \rho''_1$ and $\rho = \rho'_2 \rho_2 \rho''_2$. Then there exists a direct triangle τ so that Eq. (C1) applies to ρ_1, ρ_2 . Moreover, for $s = \partial_0 \rho_1$ we have $\beta_s = \tau \rho'_2 \rho'_1$ and $B_s^H = \psi 1$. Then using also Eq. (B9) we have $L_{\rho'}^{h'} = \psi \sum_{m \in G} T_{\rho'_1}^m L_{\rho_1}^{\bar{m} h' m} = \psi \sum_{k, l \in G} F_{\rho_2}^{k h' k l} T_{\rho'_1}^k = \psi \sum_{l \in G} F_{\rho_2}^{l h' l, l} T_{\rho'_1}^H = \psi \sum_{l \in G} F_{\rho_2}^{l h' l, l} T_{\rho'_1}^H$. Together with Eq. (C5), this gives Eq. (C11). (ii.a) From Eq. (C2) we have $T_{\sigma}^g = \psi T_{\sigma'}^g$. We can set $\sigma = \tau' \rho$, $\sigma' = \rho' \tau'$ with τ' as a direct triangle. The strip $\sigma_0 = \bar{\tau} \rho \tau \rho'$ is a nice closed strip and indeed a block. Then Eqs. (B9) and (C5) give $L_{\sigma}^{h'} = \psi \sum_{k \in G} T_{\sigma}^k L_{\tau \rho'}^{\bar{k} h' k}$. But Eq. (C1) implies $L_{\tau}^g = \sum_{k \in G} T_{\tau}^k L_{\tau}^{\bar{k} g k}$ for any $g \in G$ and then $L_{\sigma}^{h'} = \psi \sum_{k \in G} T_{\sigma}^k L_{\sigma'}^{\bar{k} h' k}$. The result follows. (ii.b) From Eq. (C1) we have $L_{\rho'}^{h'} = \psi \sum_{k, l \in G} F_{\rho'}^{k h' k l} T_{\tau}^k$. We can set $\sigma = \tau' \rho$, $\sigma' = \rho' \tau'$ with τ' as a dual triangle. The strip $\sigma_0 = \bar{\tau} \rho \tau \rho'$ is a nice closed strip and indeed a block. Then Eqs. (B9) and (C6) give $T_{\sigma}^{gH} = \psi T_{\rho}^{gH}$. Using Eq. (B13) the result follows.

2. Deformations in $\mathcal{F}_{\rho}, \mathcal{K}_{\sigma}, \mathcal{J}_{\rho'}$ and \mathcal{K}'_{σ}

We are now in position to discuss the transformation properties of the ribbon operator algebras introduced in Appendix B. We distinguish three cases, which depend on the values of the subgroups N, M that label Hamiltonian (35).

a. Original Kitaev model: $N=1, M=G$

In this case we are interested in the algebras \mathcal{F}_{ρ} and \mathcal{K}_{σ} . As for the first, open ribbons can be deformed so that if $\rho \simeq_{\psi}^{1G} \rho'$ then $F_{\rho}^{h, g} = \psi F_{\rho'}^{h, g}$. That is, the action of \mathcal{F}_{ρ} is invariant as long as ribbons are deformed without crossing any excitation. They can also be reversed: if $\rho \simeq_{\psi}^{1G} \rho'$ then $F_{\rho}^{h, g} = \psi F_{\rho'}^{\bar{g} h, \bar{g}}$. Regarding closed ribbons, the action of \mathcal{K}_{σ} is invariant under deformations (\simeq_{ψ}^{1G}) or rotations (\simeq_{ψ}^{G1}). Closed ribbon inversions give charge inversion: if $\sigma \simeq_{\psi}^{1G} \sigma'$ then $K_{\sigma}^{RC} = \psi K_{\sigma'}^{R^C \bar{C}}$, where $R^C := R^g$ (as defined in Appendix A 2) with $r_{\bar{C}} = \bar{g} r_C g$ for some $g \in G$.

b. String tension: N and M normal, N central in M

In this case we are interested in the algebras \mathcal{J}_{ρ} , which gives the domain-wall fluxes, and \mathcal{K}'_{σ} , which gives the charges. The action of \mathcal{J}_{ρ} is invariant under deformations which do not cross confined excitations (\simeq_{ψ}^{MN}), even if ends change as long as they do not cross a domain wall (\simeq_{ψ}^{MN}). Inversions (\simeq_{ψ}^{MN}) give domain flux inversion: (χ, t) goes to $(\bar{\chi}', \bar{t})$. The action of \mathcal{K}'_{σ} is invariant under deformations (\simeq_{ψ}^{NM}) or rotations in which the end of σ does not cross domain walls (\simeq_{ψ}^{MN}). Charge inversion (\simeq_{ψ}^{NM}) is as follows: (R, C) goes to (\bar{R}^C, \bar{C}) where $R^C := R^m$ with $r_{\bar{C}} = \bar{m} r_C m$ for some $m \in M$.

c. Domain walls: N normal and Abelian

In this case we are only interested in domain-wall fluxes, that is, in \mathcal{J}_{ρ} . Its action is invariant under deformations (those allowed by \simeq_{ψ}^{NN}), even if ends change as long as they do not cross a domain wall (\simeq_{ψ}^{MN}). Domain-wall flux inversion (\simeq_{ψ}^{NN}) is as follows: (R, T) goes to (\bar{R}^T, \bar{T}) , where $R^T := R^{r^T m}$ with $r_{\bar{T}} M = \bar{m} r_T M$ for some $m \in M$, so that $\mathbf{N}_{\bar{T}} = \bar{m} r_T \mathbf{N}_{r_T m}$.

3. Charge types

The previous results about closed ribbon transformations must be complemented with the following one, which relates proper closed ribbon operators with local vertex and face operators. Let $N, M \subset G$ be normal subgroups in G with N central in M and define for $R \in (\mathbf{N}'_C)_{\text{ir}}$ and $C \in (G/N)_{\text{cj}}$,

$$D_s^{RC} := \frac{n_R}{|\mathbf{N}'_C|} \sum_D \sum_{q \in Q_C} \sum_{d \in D} \bar{\chi}_R(d) A_s^{qd\bar{q}} B_s^{qrC\bar{q}}, \quad (\text{C13})$$

where D runs over $(\mathbf{N}'_C)_{\text{cj}}$.

Proposition 21. Let s be a site, σ a closed ribbon, and τ a dual triangle with $\beta_s \simeq_{\psi} v_{\sigma} \sigma$. If $|\psi\rangle \in \mathcal{H}_G$ is such that $A_v^M = \psi 1$ for any vertex $v \neq v_s$ in f_s and $L_{\tau}^N = \psi 1$ then

$$K_{\sigma}^{RC} = D_s^{RC}, \quad (\text{C14})$$

where $R \in (\mathbf{N}'_C)_{\text{ir}}$, $C \in (G/N)_{\text{cj}}$.

Proof. Let s' be the second site of σ , so that $v_{s'} = v_s$ and e_{τ} does not belong to $f_{s'}$. The states $|\psi^g\rangle := A_v^M B_{s'}^g$, $g \in G$, are such that $\beta_s \simeq_{\psi}^{NM} \sigma$. Then with the notation of Eq. (C12) we have $\sum_{k \in H'} F_{\sigma}^{\bar{k} h' k, k S k} B_{s'}^g = \sum_{k \in H'} B_{s'}^{\bar{k} h' k g k h' k} F_{\sigma}^{\bar{k} h' k, k S k} = \sum_{k \in H'} B_{s'}^{\bar{k} h' k g k h' k} B_{s'}^{k S k} = \sum_{k \in H'} A_s^{k h' k} B_{s'}^{k S k} B_{s'}^g$. Since $|\psi\rangle = \sum_g B_{s'}^g |\psi^g\rangle$, the result follows. \square

APPENDIX D: LOCAL DEGREES OF FREEDOM

In this appendix we give the details of the results indicated in Sec. II D. Choose any $C \in (G)_{\text{cj}}$ and two indices i, i' and define

$$|n\rangle := |n; i, i'\rangle := F_{\rho}^{\bar{c}; \bar{q}; n q i'} |\psi_G\rangle, \quad (\text{D1})$$

where $|\psi_G\rangle$ is a ground state of H_G (1). Let V be the space with basis $|n\rangle$, $n \in \mathbf{N}_C$. Then there exists an evident isomorphism $p: \mathbf{C}[\mathbf{N}_C] \rightarrow V$. For $n, n' \in \mathbf{N}_C$ and $s = \partial_0 \rho$, $s' = \partial_1 \rho$, consider the operators

$$a_{n, n'} := A_s^{\bar{q}; n q i} A_{s'}^{\bar{q}; n' q i'}. \quad (\text{D2})$$

They give a representation $a, \mathbf{N}_C \times \mathbf{N}_C \rightarrow \mathbf{GL}(V)$ because

$$a_{n_1, n_2} |n\rangle = |n_1 n \bar{n}_2\rangle, \quad (\text{D3})$$

so that if $\mathcal{R}: \mathbf{N}_C \times \mathbf{N}_C \rightarrow \mathbf{GL}(\mathbf{C}[\mathbf{N}_C])$, is the representation of Appendix A, we have $a_{n_1, n_2} p = p \mathcal{R}_{n_1, n_2}$. This has several consequences. First, if we define in accordance with Eq. (A15) a basis for V with elements,

$$|R; jj'\rangle := \sum_{n \in \mathbf{N}_C} \bar{\Gamma}_R^{jj'}(n) |n\rangle, \quad (\text{D4})$$

then in the new basis

$$a_{n,n'} |R; jj'\rangle = \sum_{k,k'=1}^{n_R} \Gamma_R^{kj}(n) \bar{\Gamma}_R^{k'j'}(n') |R; kk'\rangle. \quad (\text{D5})$$

In $\mathbf{C}[\mathbf{N}_C]$ from Eq. (A19) we get

$$e_R^{uv} e_R^{jj'} e_R^{v'u'} = \delta_{R,R'} \delta_{v,j} \delta_{v',j'} e_R^{uu'}, \quad (\text{D6})$$

$$e_R e_R^{jj'} = e_R^{jj'} e_R = \delta_{R,R'} e_R^{jj'}, \quad (\text{D7})$$

which through the isomorphism p give

$$a_R^{uv} a_R^{u'v'} |R'; jj'\rangle = \delta_{R,R'} \delta_{v,j} \delta_{v',j'} |R; uu'\rangle, \quad (\text{D8})$$

$$a_R |R'; jj'\rangle = a_R' |R'; jj'\rangle = \delta_{R,R'} |R; jj'\rangle, \quad (\text{D9})$$

where

$$a_R^{uv} := \frac{n_R}{|\mathbf{N}_C|} \sum_{n \in \mathbf{N}_C} \bar{\Gamma}_R^{uv}(n) a_{n,1}, \quad (\text{D10})$$

$$a_R^{u'v'} := \frac{n_R}{|\mathbf{N}_C|} \sum_{n \in \mathbf{N}_C} \bar{\Gamma}_R^{u'v'}(n) a_{1,n}, \quad (\text{D11})$$

$$a_R = \sum_{u=1}^{n_R} a_R^{uu}, \quad a_R' = \sum_{u=1}^{n_R} a_R'^{uu}. \quad (\text{D12})$$

Note that $a_R^{uv}, a_R \in \mathcal{D}_s, a_R'^{uv}, a_R' \in \mathcal{D}_{s'}$.

Finally, from Eq. (B42) we have

$$B_s^{ck} \bar{B}_{s'}^{c'k'} |n; i, i'\rangle = \delta_{k,i} \delta_{k',i'} |n; i, i'\rangle \quad (\text{D13})$$

and from Eq. (B41)

$$A_s^{\bar{q}kq_i} \bar{A}_{s'}^{\bar{q}'k'q'_i} |n; i, i'\rangle = |n; k, k'\rangle. \quad (\text{D14})$$

Note that $|R; jj'\rangle$ is just a shorthand for Eq. (21). Finally, these results must be complemented with proposition 21.

APPENDIX E: SINGLE-QUASIPARTICLE STATES

Only in a surface of nontrivial topology can we find two closed ribbons σ, σ' such that $(\sigma, \sigma')_+$ (see Fig. 15). When such ribbons exist, we can construct for any $h, g \in G$ the state

$$|\psi_{hg}\rangle := F_\sigma^{hg} L_\sigma^{\bar{g}} \prod_v A_v |\mathbf{1}\rangle. \quad (\text{E1})$$

The state $|\psi\rangle$ is not zero because in Eqs. (B21) and (B45)

$$L_\sigma^g L_\sigma^{\bar{h}} |\psi_{hg}\rangle = \prod_v A_v |\mathbf{1}\rangle. \quad (\text{E2})$$

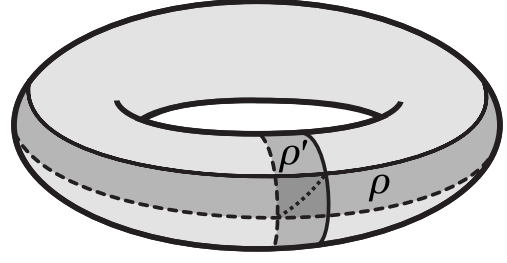


FIG. 15. In a torus we can find a pair of closed ribbons σ, σ' such that they form a crossed joint, $(\sigma, \sigma')_+$. This is not possible in a sphere.

At most, it can have an excitation at $(v, f) = \partial\sigma = \partial\sigma'$. In fact, in Eq. (B45)

$$B_f |\psi_{hg}\rangle = \delta_{g,hg} |\psi_{hg}\rangle, \quad (\text{E3})$$

showing that for non-Abelian groups single-quasiparticle excitations exist.

APPENDIX F: CONDENSATION

In this appendix we give the details of the calculations of certain expected values for ribbon operators $\langle F \rangle$ for a ground state of Hamiltonian (35) for $N \subset M \subset G$ subgroups of G , N normal. Such ground states are characterized by conditions (36). For $S \subset G, g \in G$ we introduce the notation

$$\delta_{g,S} := \delta_{g,S}. \quad (\text{F1})$$

Proposition 22. Let $h, g \in G, n \in N$ and $|\psi\rangle, |\psi'\rangle \in \mathcal{H}_G$ satisfy Eq. (36).

(i) For an arbitrary ribbon ρ

$$F_\rho^{h,g} |\psi\rangle = \delta_{g,M} F_\rho^{hg} |\psi\rangle, \quad (\text{F2})$$

$$\langle \psi' | F_\rho^{h,g} | \psi \rangle = \delta_{h,N} \langle \psi' | F_\rho^{hg} | \psi \rangle. \quad (\text{F3})$$

(ii) If ρ is an open ribbon

$$F_\rho^{NM} |\psi\rangle = |\psi\rangle, \quad (\text{F4})$$

$$\langle \psi' | F_\rho^{h,g} | \psi \rangle = \delta_{h,N} \delta_{g,M} \frac{1}{|M|} \langle \psi' | \psi \rangle. \quad (\text{F5})$$

(iii) If σ is a boundary ribbon

$$F_\sigma^{MN} |\psi\rangle = |\psi\rangle, \quad (\text{F6})$$

$$\langle \psi' | F_\sigma^{h,g} | \psi \rangle = \delta_{h,M} \delta_{g,N} \frac{1}{|N|} \langle \psi' | \psi \rangle. \quad (\text{F7})$$

Proof. (i) If ρ is a triangle this is a direct consequence of the identities $L_\tau^n L_\tau^N = L_\tau^N, T_\tau^g T_\tau^M = \delta_{g,M} T_\tau^g T_\tau^M, T_\tau^M L_\tau^h T_\tau^M = \delta_{h,M} T_\tau^M L_\tau^h T_\tau^M$, and $L_\tau^N T_\tau^g L_\tau^N = L_\tau^N T_\tau^{gn} L_\tau^N$. For general ribbons, just apply Eq. (B9). (ii) From (i) we get $F_\sigma^{NM} |\psi\rangle = F_\sigma^{1G} |\psi\rangle = |\psi\rangle$ using Eq. (B16). Let $s_i = \partial_i \rho$ and set $\langle \cdot \rangle := \langle \psi' | \cdot | \psi \rangle$. Then from Eqs. (36) and (B42) we have $\langle F_\rho^{h,g} \rangle = \langle B_{s_0}^N F_\rho^{h,g} \rangle = \langle F_\rho^{hg} B_{s_0}^N \rangle = \delta_{h,N} \langle F_\rho^{hg} \rangle$ and for $m \in M$ from Eqs. (36) and (B41) we have $\langle F_\rho^{h,g} \rangle = \langle F_\rho^{hg} A_{s_1}^m \rangle = \langle A_{s_1}^m F_\rho^{hg} \rangle = \langle F_\rho^{hg} \rangle$. Thus

$\langle F_{\sigma}^{h,g} \rangle = \delta_{g,M} \delta_{h,N} \langle F_{\sigma}^{1,1} \rangle$ and the result follows since $\langle F_{\rho}^{1,M} \rangle = \langle F_{\rho}^{1,G} \rangle = \langle 1 \rangle$. (iii) Using the notation of Appendix C, p_{σ} encloses a disk $R \subset R_{\psi}^N$ so that $F_{\sigma}^{MN}|\psi\rangle = |\psi\rangle$. Also, $L_{\sigma}^m|\psi\rangle = |\psi\rangle$ for any $m \in M$. To check this, suppose, for example, that the edges E_{ρ}^{Δ} lie outside R and choose for each vertex v in R a ribbon ρ_v with p_{ρ_v} a path inside R from $v_0 = v_{\partial_0 \rho}$ to v . If we set $A_{\rho}^m = A_{v_0}^m \prod_{v \neq v_0} \sum_k T_{\rho_v}^k A_{v_0}^{\bar{m}k}$, with the product running over all vertices in R , one can check that $|\psi\rangle = A^m|\psi\rangle = L_{\sigma}^m|\psi\rangle$. The other case is similar. Thus, for $m \in M$ we get $\langle F_{\sigma}^{h,g} \rangle = \delta_{g,N} \langle F_{\sigma}^{hm,g} \rangle$, so that $\langle F_{\sigma}^{h,g} \rangle = \delta_{g,N} \delta_{h,M} \langle F_{\sigma}^{1,1} \rangle$. The result follows since $\langle F_{\rho}^{1,N} \rangle = \langle F_{\rho}^{1,G} \rangle = \langle 1 \rangle$.

A state satisfying Eqs. (F4) and (F6) for all open ribbons ρ and boundary ribbons σ cannot contain vertex, face, or edge excitations. Therefore, these conditions characterize ground states. We proceed to check Eq. (49); the derivation of Eq. (56) is similar. From Eq. (B70) and (F7) we get

$$\langle K_{\sigma}^{DC} \rangle = \frac{|C \cap M|}{|M||G|} \sum_{g \in G} |D \cap \bar{g}Mg|, \quad (F8)$$

where $D \in (\mathbf{N}_C)_{\text{cj}}$, $C \in (G)_{\text{cj}}$. This together with Eq. (B75) give Eq. (49) because if $e_{M\uparrow}$ is the induced representation in G of the identity representation in M ,

$$\chi_{e_{M\uparrow}}(g) = \frac{1}{|M|} \sum_{k \in G} \delta_{g, \bar{k}Mk}. \quad (F9)$$

As for Eq. (55), from Eq. (F5) we have

$$|M| \left\langle \sum_{n \in \mathbf{N}_C} \Gamma_R^{jj'}(n) F_{\rho}^{\bar{c}_i, \bar{q}_i n q_i} \right\rangle = \delta_{c_i, N} \sum_{n \in M_c^{i,i'}} \Gamma_R^{jj'}(n), \quad (F10)$$

where $M_c^{i,i'} := \mathbf{N}_C \cap q_i M \bar{q}_i$. If $M_c^{i,i'}$ is empty, we are done. $M_c^{i,i'} = M_c^{i,i} s$ for some $s \in \mathbf{N}_C$, so that $\sum_{n \in M_c^{i,i'}} \Gamma_R(n) = \sum_{n \in M_c^{i,i}} \Gamma_R(n) \Gamma_R(s)$. But⁶³ $\sum_{n \in M_c^{i,i}} \Gamma_R(n) = 0$ if $(\chi_R, 1)_{M_c^{i,i}} = 0$.

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