

# Collective cyclotron motion of the relativistic plasma in graphene

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We present a theory of the finite temperature thermoelectric response functions of graphene in the hydrodynamic regime where electron-electron collisions dominate the scattering. In moderate magnetic fields, the Dirac particles undergo a collective cyclotron motion with a temperature-dependent relativistic cyclotron frequency proportional to the net charge density of the Dirac plasma. In contrast to the undamped cyclotron pole in Galilean-invariant systems (Kohn's theorem), here there is a finite damping induced by collisions between the counter-propagating particles and holes. This cyclotron motion shows up as a damped pole in the frequency-dependent conductivities and should be readily detectable in microwave measurements at room temperature. We also compute the large Nernst signal in the hydrodynamic regime, which is significantly bigger than that in ordinary metals.

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## I. INTRODUCTION

In the absence of gate voltage or external impurities, graphene is a quantum-critical system<sup>1-5</sup> whose quasiparticles are governed by a relativistic massless Dirac equation in two spatial dimensions.<sup>6-8</sup> [By “quantum critical” we mean here a system in which the inelastic scattering rate from electron-electron interactions is of order of  $k_B T/\hbar$  (up to logarithms). We are not referring to models of localization transitions of free electrons.<sup>9</sup>] Experimental realizations of relativistically invariant systems are rare, and so it is of interest to study physical phenomena, which rely on the Dirac nature of the electrons. In this paper we focus our attention on the possibility of observing a relativistic cyclotron resonance in the (collective) electronic motion in graphene. This resonance shows up as a bump in the frequency dependence of all thermoelectric response functions. As we will discuss in detail, the cyclotron resonance frequency has clear signatures in its dependence upon field ( $B$ ), temperature ( $T$ ), and electron density ( $\rho$ ), which distinguish it from the cyclotron resonance of nonrelativistic electrons. Moreover, the relativistic cyclotron resonance is intrinsically damped by electron-electron interactions. This damping arises from collisions between electrons and holes that execute cyclotron orbits in opposite directions. In contrast, the damping of the cyclotron resonance in metals arises primarily from impurity scattering: Kohn's theorem<sup>10</sup> implies that electron-electron interactions do not broaden the cyclotron resonance in Galilean-invariant systems. This theorem applies to metals with a single parabolic (nonrelativistic) band, which is a reasonable approximate description for many simple Fermi surfaces. However, it fails for semimetals such as graphene both due to the linear relativistic dispersion and the presence of two bands.

We shall be interested here in the cyclotron resonance in a hydrodynamic collision-dominated regime where disorder plays only a minor role. This regime exists at high temperature and is defined by the requirement that the vast majority of collisions arise from electron-electron interactions. This assures local equilibration before scattering from impurities occurs. The second requirement for hydrodynamics to apply

is the following: The rate of deflection of linearly propagating thermal quasiparticles due to the magnetic field must be small compared to the inelastic scattering rate,  $\tau_{\text{inel}}^{-1}$ , arising from interactions. The same restriction must hold for the frequency of the external driving fields,  $\omega \ll \tau_{\text{inel}}^{-1}$ . The quantum-critical nature of graphene<sup>1</sup> implies that the mean time between collisions is of order of  $\hbar/k_B T$ . Since we require that this inelastic collision time be shorter than the elastic scattering time from impurities, we cannot allow  $T$  to become too small, and we will find that room temperature is suitable for observing the physics we are interested in. Further conditions controlling the range of parameters are discussed below in Sec. III.

The cyclotron resonance is formally defined as the pole of the thermoelectric response functions closest to the real axis in the complex frequency plane. The expressions for the cyclotron frequency (its real part),  $\omega_c$ , and the damping frequency (its imaginary part),  $\gamma$ , are the same as those in Refs. 11 and 12,

$$\omega_c \equiv \frac{eB\rho v_F^2}{c(\varepsilon + P)}, \quad \gamma \equiv \frac{\sigma_Q B^2 v_F^2}{c^2(\varepsilon + P)}. \quad (1.1)$$

Here  $v_F$  is the “velocity of light” for the Dirac fermions, which was experimentally measured to be<sup>13-15</sup>  $v_F \approx 1.1 \times 10^8$  cm/s  $\approx c/300$ . The density of electrons  $\rho$  is defined so that  $\rho=0$  for undoped graphene.  $\varepsilon$  and  $P$  are the thermodynamic energy density and pressure of the Dirac plasma, respectively, which are also measured with respect to undoped graphene. In the relativistic regime where the temperature exceeds the chemical potential  $|\mu|$ , the energy density, and the pressure grow with the third power of temperature and assume typical values of  $\varepsilon, P \sim (k_B T)^3 / (v_F \hbar)^2 = 3.28 \times 10^{12} \times T_{300}^3$  meV/cm<sup>2</sup>, where  $T_{300} = T/300$  K. The coefficient  $\sigma_Q$  will be seen to arise as a transport parameter in the hydrodynamic description of a relativistic fluid, which cannot be fully determined by thermodynamics and hydrodynamics alone. It can, however, be computed in a microscopic approach, as recently carried out in Ref. 16. The parameter  $\sigma_Q$  has the units of an electrical conductivity and, as will become clear from later formulas, it describes the part of the dc

conductivity, which is independent of impurities, deriving solely from interactions, cf. Eq. (5.1). At particle-hole symmetry,  $\sigma_Q$  coincides with the finite conductivity of a clean system in zero field,  $B=0$ . In the relativistic collision-dominated regime, which is of foremost interest here, one has  $\sigma_Q \sim \frac{1}{\alpha^2} \frac{e^2}{h}$ ,<sup>17</sup> where  $\alpha = \frac{e^2}{\epsilon_r \hbar v_F} \approx 2/\epsilon_r$  is the fine-structure constant of graphene characterizing the strength of Coulomb interactions, with  $\epsilon_r$  as the dielectric constant due to the adjacent media. In general,  $\sigma_Q$  is a scaling function of  $\mu/T$ . The dependence of the thermodynamic variables and the transport coefficient  $\sigma_Q$  on temperature and chemical potential are further discussed in Sec. III A.

Physically, the cyclotron resonance is due to the tendency of fluid elements of the electron-hole plasma to undergo a circular motion at frequency  $\omega_c$ . This frequency results as an average over the left-circulating and right-circulating orbital motions of thermally excited electrons and holes, which collide with each other many times before they would be able to complete a (noninteracting) cyclotron orbit. For this reason,  $\omega_c$  is proportional to the excess charge density  $\rho$  and vanishes at the particle-hole symmetric Dirac point where  $\rho=0$ . Nevertheless, a bump in the frequency-dependent response around  $\omega=0$  survives also in this case. Note that it broadens rather rapidly with decreasing  $T$  as  $\gamma \sim T^{-3}$ .

In the nonrelativistic regime  $|\mu| \gg T$ , where graphene turns into an ordinary Fermi liquid, the cyclotron resonance tends to the semiclassical value  $\omega_c = ev_F^2 B / (c\mu)$  corresponding to the cyclotron frequency of noninteracting Dirac fermions at the chemical potential  $\mu$ . This should indeed be expected since all thermally excited quasiparticles share essentially the same noninteracting cyclotron frequency, which is not altered by the interactions.

From microscopic transport theory one finds that  $\sigma_Q$  decreases as  $(T/\mu)^2$  in the nonrelativistic Fermi-liquid regime.<sup>16</sup> Accordingly, the intrinsic damping  $\gamma$  decreases and the cyclotron resonance becomes increasingly sharp for small  $T/\mu$ . We note that in addition to the intrinsic damping represented by  $\gamma$ , there will also be extrinsic damping from impurity scattering, as is discussed in Sec. III. The complete expression for the frequency dependence of the conductivity across the cyclotron resonance is given in Eq. (5.6) and is illustrated in Fig. 2.

Recent experiments<sup>13,18</sup> have observed a “nonhydrodynamic” cyclotron resonance in a regime of strong magnetic fields in which the Landau levels can be resolved. As we will see below, we are discussing here the different regime of weak fields and high temperatures to which we hope the experiments will be extended.

## II. MODEL OF GRAPHENE

We consider a single sheet of graphene described by the Hamiltonian,

$$H = H_0 + H_1 + H_{\text{dis}}, \quad (2.1)$$

where the low-energy tight-binding part is

$$H_0 = - \sum_s \sum_{V=K,K'} \int d\mathbf{x} [\Psi_{s,V}^\dagger (iv_F \vec{\sigma} \cdot \vec{\nabla} + \mu) \Psi_{s,V}], \quad (2.2)$$

with the Fermi velocity  $v_F$  and the two component wave function  $\Psi_{s,V}$  describing the amplitude on the two sublattices for electrons with spin  $s$  and momenta close to one of the two Fermi points  $V=K, K'$ . A magnetic field is introduced as usual via minimal coupling.

The  $1/r$  Coulomb interactions take the form

$$H_1 = \frac{1}{2} \sum_{s,V} \sum_{s',V'} \int d\mathbf{x} d\mathbf{x}' \Psi_{s,V}^\dagger(\mathbf{x}) \Psi_{s',V'}(\mathbf{x}) \frac{e^2}{\epsilon_r |\mathbf{x} - \mathbf{x}'|} \times \Psi_{s',V'}^\dagger(\mathbf{x}') \Psi_{s',V'}(\mathbf{x}').$$

The term  $H_{\text{dis}}$  describes the presence of weak disorder, which induces elastic scattering and thus weak momentum relaxation at a rate  $\tau$  proportional to the impurity concentration. In the high-temperature regime, which we are focusing on, the elastic scattering rate is smaller than the inelastic scattering rate and can be taken into account as a perturbation.

## III. OBSERVABILITY OF THE COLLECTIVE CYCLOTRON RESONANCE IN GRAPHENE

As discussed in Ref. 11, the applicability of hydrodynamics to system (2.1) requires that the magnetic field  $B$  be weak so that the Landau quantization of thermal excitations is not discernible, i.e.,

$$E_1 \equiv \hbar v_F \sqrt{\frac{2eB}{\hbar c}} \ll k_B T, \quad (3.1)$$

where  $E_1$  is the first Landau level for graphene at the Dirac point.<sup>7,15,19,20</sup> This amounts to requiring that

$$B \ll B^*(T) \equiv \frac{\hbar c (k_B T)^2}{2e (\hbar v_F)^2} = (0.42 T_{300}^2) \text{ T}. \quad (3.2)$$

For the hydrodynamic analysis to hold, one needs a slightly more stringent condition,

$$E_1 \ll \hbar \tau_{\text{inel}}^{-1} \sim \alpha^2 k_B T, \quad (3.3)$$

where the last estimate applies to the relativistic regime  $T < |\mu|$ . This condition expresses the requirement that the inelastic scattering rate, and thus the equilibration rate should dominate the rate by which the magnetic field deflects electrons from their linear motion. In the following we will assume  $\alpha$  to be of order unity, but we nevertheless will retain  $\alpha$  in most scaling estimates.

It will also be useful to express our electron densities in terms of the characteristic density of thermal excitations,

$$\rho_{\text{th}} \equiv \left( \frac{k_B T}{\hbar v_F} \right)^2 = (1.27 \times 10^{11} \times T_{300}^2) \text{ cm}^{-2}. \quad (3.4)$$

This density should be compared to disorder-induced density variations, which are of order  $\delta\rho_{\text{dis}} \sim 10^{11} \text{ cm}^{-2}$  (Ref. 14)

varying on typical length scales  $\sim 30$  nm. To remain close to quantum criticality, ensuring a universal conductivity due to thermal pair creation and annihilation processes, the regime  $\delta\rho_{\text{dis}} < \rho_{\text{th}}$  is preferred. This suggests measurements at room temperature or above.

As mentioned above, the cyclotron resonance occurs at

$$\begin{aligned}\omega_c &= \frac{eB\rho v_F^2}{c(\varepsilon + P)} = \frac{1}{2\Phi_{\varepsilon+P}} \frac{k_B T}{\hbar} \frac{B}{B^*(T)} \frac{\rho}{\rho_{\text{th}}} \\ &= \frac{T_{300}}{\Phi_{\varepsilon+P}} \frac{B}{B^*(T)} \frac{\rho}{\rho_{\text{th}}} 1.96 \times 10^{13} \text{ s}^{-1},\end{aligned}\quad (3.5)$$

where we have used a free-electron equation of state to determine the value of  $\varepsilon + P$ , with  $\Phi_{\varepsilon+P} = (\varepsilon + P)(\hbar v)^2 T^{-3}$  being a dimensionless number  $\mathcal{O}(1)$ , as given in Eq. (3.13) below. We will also comment in Sec. III A on the effect of interactions on the equation of state.

The collective cyclotron frequency lies within the hydrodynamic frequency regime if  $\hbar\omega_c \ll \hbar\tau_{\text{inel}}^{-1} \sim \alpha^2 k_B T$ , where the latter estimate applies to the relativistic quantum-critical regime. This requires

$$\frac{\hbar\omega_c}{\alpha^2 k_B T} = \frac{1}{2\alpha^2 \Phi_{\varepsilon+P}} \frac{B}{B^*(T)} \frac{\rho}{\rho_{\text{th}}} \ll 1. \quad (3.6)$$

For room temperature and values of  $B$  and  $\rho$  as suggested below, this falls into the range of microwave frequencies. Thus the resonance should be readily detectable by measuring the real and imaginary part of  $\sigma_{xx}(\omega)$  of a graphene sample in a cavity.

For inelastic scattering to be the dominant relaxation process, i.e., one certainly needs the impurity scattering time to satisfy  $\tau \gg \tau_{\text{inel}}$ . However, to observe the relativistic cyclotron resonance, one has to require the more stringent condition,

$$\tau \gg \frac{1}{\omega_c} = \left( \frac{k_B T}{\hbar\omega_c} \right) \frac{2.5 \times 10^{-14} \text{ s}}{T_{300}} = \frac{\Phi_{\varepsilon+P} B^*(T) \rho_{\text{th}}}{T_{300} B \rho} 5 \times 10^{-14} \text{ s}, \quad (3.7)$$

to ensure that the disorder-induced broadening does not wash out the resonance. Such long scattering times can indeed be achieved in high mobility graphene where  $\tau \sim 10^{-13}$  s is a typical value.<sup>14,21</sup>

Apart from a small impurity scattering one also needs the intrinsic broadening of the cyclotron resonance due to electron-electron interactions to be smaller than  $\omega_c$ ,

$$\frac{\gamma}{\omega_c} = \frac{\Phi_\sigma}{4\pi} \frac{B}{B^*(T)} \frac{\rho_{\text{th}}}{\rho} < 1, \quad (3.8)$$

where  $\Phi_\sigma = \sigma_Q / (e^2/h) = \mathcal{O}(1)$ .

Note that Eqs. (3.6) and (3.8) can be satisfied simultaneously, e.g., with a choice of magnetic field  $B/B^*(T) \sim \mathcal{O}(0.1)$  and charge density  $\rho/\rho_{\text{th}} \sim \mathcal{O}(1)$ .

### A. Scaling functions for thermodynamic variables

For the evaluation of thermodynamic state variables magnetic-field effects can be safely neglected since both

Zeeman energy and Landau-level splitting are significantly smaller than  $T$ . Further we will also neglect interactions and use the free theory to evaluate thermodynamic quantities. This can be justified by the fact that the Coulomb interactions are marginally irrelevant.<sup>2,3</sup> However, as emphasized by Son,<sup>1</sup> the bare values of the interactions are not small [ $\alpha = \mathcal{O}(1)$ ] and there may well be a significant regime of intermediate scales where the interactions remain significant, and the theory is characterized by a dynamic critical exponent  $z \neq 1$ . Over this intermediate regime, we have  $\varepsilon + P \sim T^{2+z}$ . Below, we neglect such effects, and we simply use the free theory to obtain a numerical estimate of parameters.

One obtains straightforwardly the scaling functions,

$$\varepsilon + P = \frac{T^3}{(\hbar v)^2} \Phi_{\varepsilon+P} \left( \tilde{\mu} = \frac{\mu}{T} \right) = T \rho_{\text{th}} \Phi_{\varepsilon+P} \left( \tilde{\mu} = \frac{\mu}{T} \right), \quad (3.9)$$

with

$$\begin{aligned}\Phi_{\varepsilon+P}(\tilde{\mu}) &= 4 \frac{3}{2} \int_0^\infty \frac{\tilde{k} d\tilde{k}}{2\pi} \left[ \frac{\tilde{k}}{e^{\tilde{k}-\tilde{\mu}} + 1} + \frac{\tilde{k}}{e^{\tilde{k}+\tilde{\mu}} + 1} \right] \\ &= \frac{9\zeta(3)}{\pi} + \frac{6 \log(2) \tilde{\mu}^2}{\pi} + \frac{\tilde{\mu}^4}{8\pi} + \mathcal{O}(\tilde{\mu}^6).\end{aligned}\quad (3.10)$$

The prefactor 3/2 derives from the relation  $P = \varepsilon/2$ , and the factor 4 accounts for spin and valley degeneracy. The electron density is

$$\rho = \frac{T^2}{(\hbar v)^2} \Phi_\rho \left( \tilde{\mu} = \frac{\mu}{T} \right) = \rho_{\text{th}} \Phi_\rho \left( \tilde{\mu} = \frac{\mu}{T} \right), \quad (3.11)$$

with

$$\begin{aligned}\Phi_\rho(\tilde{\mu}) &= 4 \int_0^\infty \frac{\tilde{k} d\tilde{k}}{2\pi} \left[ \frac{1}{e^{\tilde{k}-\tilde{\mu}} + 1} - \frac{1}{e^{\tilde{k}+\tilde{\mu}} + 1} \right] \\ &= \frac{4 \log(2) \tilde{\mu}}{\pi} + \frac{\tilde{\mu}^3}{6\pi} - \frac{\tilde{\mu}^5}{240\pi} + \mathcal{O}(\tilde{\mu}^7),\end{aligned}\quad (3.12)$$

and thus

$$\begin{aligned}\Phi_{\varepsilon+P} \left( \tilde{\rho} = \frac{\rho}{\rho_{\text{th}}} \right) &= \frac{9\zeta(3)}{\pi} + \frac{3\pi \tilde{\rho}^2}{8 \log(2)} - \frac{3\pi^3 \tilde{\rho}^4}{2048 \log^4(2)} + \mathcal{O}(\tilde{\rho}^6) \\ &= 3.444 + 1.700 \tilde{\rho}^2 - 0.1968 \tilde{\rho}^4 + \mathcal{O}(\tilde{\rho}^6).\end{aligned}\quad (3.13)$$

A plot of this function is shown in Fig. 1. We note that it has been computed in the weak-coupling limit; if  $\alpha = \mathcal{O}(1)$  there will be order unity corrections from the Coulomb interactions, which can be determined from the theory in Ref. 1.

For the results given in Sec. IV the scaling function for the entropy density  $s = (\varepsilon + P - \mu\rho)/T$  will also be useful;

$$\frac{s}{\rho_{\text{th}}} \equiv \Phi_s(\tilde{\mu}) = \Phi_{\varepsilon+P}(\tilde{\mu}) - \tilde{\mu} \Phi_\rho(\tilde{\mu}). \quad (3.14)$$

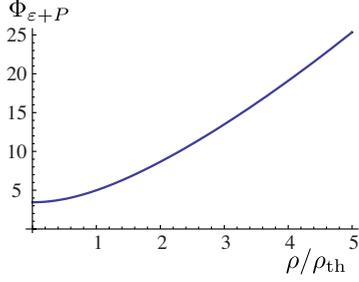


FIG. 1. (Color online) The dimensionless function  $\Phi_{\varepsilon+P}$  of the density  $\rho/\rho_{\text{th}}$  for noninteracting Dirac fermions.

Finally we mention that the transport coefficient  $\sigma_Q \equiv \frac{e^2}{h} \Phi_{\sigma}(\mu/T)$  is itself a scaling function of the ratio between chemical potential and temperature. In the relativistic regime,  $T > \mu$ , it can be roughly approximated by a constant, whereas in the nonrelativistic, Fermi-liquid regime,  $T < \mu$ , a microscopic study shows that  $\sigma_Q$  decreases as  $(T/\mu)^2$ . The precise scaling function has been obtained in Ref. 16.

#### IV. DERIVATION: HYDRODYNAMICS OF A RELATIVISTIC FLUID WITH COULOMB INTERACTIONS

Here we discuss the magnetohydrodynamics of a relativistic fluid<sup>11</sup> in the presence of Coulomb interactions. Our general results for response functions in the hydrodynamic regime are then readily applied to graphene close to its Dirac point. The characteristic velocity  $v_F \neq c$  determines the relativistic dispersion, and the charge of (antiparticles) particles in the fluid is  $\pm e$ . In the following we employ units with  $v_F = e = \hbar = 1$ .

Because the Coulomb interactions spread with the speed of light  $c \gg v_F$ , they can be considered instantaneous. They obviously break the relativistic invariance of the fluid by singling out the laboratory frame as a particular reference frame, which we will eventually work below.

The stress-energy tensor,  $T^{\mu\nu}$ , and the current,  $J^\mu$ , of a relativistic fluid are expressed in terms of the three velocity of the fluid element,  $u^\mu \equiv (1, \vec{v})/\sqrt{1-v^2}$  [ $u^\mu u_\mu = -1$ , with  $u^\mu = (1, 0, 0)$  in the local rest frame for which the energy flow vanishes];<sup>11,22</sup>

$$T^{\mu\nu} = (\varepsilon + P)u^\mu u^\nu + P g^{\mu\nu} + \tau^{\mu\nu}, \quad (4.1)$$

$$J^\mu = \rho u^\mu + v^\mu. \quad (4.2)$$

These expressions include dissipative terms  $v^\mu, \tau^{\mu\nu}$ , which account for heat currents and viscous forces, respectively, and will be derived below. Without the viscous terms, the stress-energy tensor is a diagonal matrix in the rest frame of the fluid element with the pressure in the spacelike entries and an energy density in the timelike entry. In the laboratory frame (to lowest order in  $v^i$ ) the components read

$$T^{00} = \varepsilon, \quad (4.3)$$

$$T^{0i} = T^{i0} = (\varepsilon + P)v^i, \quad (4.4)$$

$$T^{ij} = P\delta^{ij} + \tau^{ij}, \quad (4.5)$$

$$J^0 = \rho, \quad (4.6)$$

$$J^i = \rho v^i + v^i. \quad (4.7)$$

Here  $\varepsilon, P$ , and  $\rho$  are functions of the local chemical potential,  $\mu(r)$ , the local temperature  $T(r)$ , and the magnetic field  $B$ , as given in the previous section for graphene. Thereby the static Coulomb potential created by an inhomogeneous charge distribution is incorporated in  $\mu(r)$ .

The conservation laws for charge, energy, and momentum read

$$\partial_\beta J^\beta = 0, \quad (4.8)$$

$$\partial_\beta T^{\beta\alpha} = F^{\alpha\gamma} J_\gamma, \quad (4.9)$$

where the electromagnetic field tensor,

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & B \\ -E_y & -B & 0 \end{pmatrix}, \quad (4.10)$$

contains a self-generated spatially varying electric field due to the inhomogeneous charge density of the system itself;

$$\vec{E} = -\vec{\nabla} \phi,$$

$$\phi(x) = \int d^2y U(x-y)[\rho(y) - \rho], \quad (4.11)$$

with

$$U(r) = \frac{e^2}{r} = \int \frac{d^2k}{(2\pi)^2} U_k \exp[i\vec{k} \cdot \vec{r}], \quad U_k = \frac{2\pi\alpha}{|k|},$$

$$\vec{E}(\vec{k}) = -i\vec{k} U_k \rho(\vec{k}), \quad (4.12)$$

where the uniform background charge density  $\rho$  has been subtracted in Eq. (4.11).

In the laboratory frame the linearized conservation laws read more explicitly

$$\partial_t \rho + \vec{\nabla} \cdot \vec{J} = 0, \quad (4.13)$$

$$\partial_t \varepsilon + \vec{\nabla} \cdot [(\varepsilon + P)\vec{v}] = \rho \vec{v} \cdot \vec{E}, \quad (4.14)$$

$$(\partial_t + \tau^{-1})(\varepsilon + P)\vec{v} + \vec{\nabla} P = B \hat{e}_z \vec{J} + \rho \vec{E}, \quad (4.15)$$

where in addition we have included a momentum relaxation time  $\tau$  due to weak impurity scattering. Note that the latter also breaks the relativistic invariance, and accordingly the relativistic hydrodynamics should only be expected to hold as long as  $\tau^{-1} \ll \tau_{\text{inel}}^{-1}$ . The above set of equations is closed by the constitutive equation for the current,

$$\vec{J} = \rho \vec{v} + \vec{v}. \quad (4.16)$$

### A. Heat current and viscous terms in a magnetic field

With the help of the thermodynamic relations,

$$P + \varepsilon = sT + \mu\rho, \quad d\varepsilon = Tds + \mu d\rho, \quad (4.17)$$

and employing the conditions<sup>22</sup>  $u_\mu v^\mu = u_\mu \tau^{\mu\nu} = 0$ , the energy conservation law ( $u_\mu \partial_\nu T^{\mu\nu} = u_\mu F^{\mu\nu} J_\nu$ ) can be rewritten in the form of a law of entropy production,

$$\partial_\mu \left( su^\mu - \frac{\mu}{T} v^\mu \right) = -v^\mu \partial_\mu \left( \frac{\mu}{T} \right) + \frac{1}{T} F_{\mu\nu} u^\nu v^\mu - \frac{\tau^{\mu\nu}}{T} \partial_\mu u_\nu. \quad (4.18)$$

It is natural to interpret the left-hand side as the divergence of an entropy current,

$$S^\mu = su^\mu - \frac{\mu}{T} v^\mu, \quad (4.19)$$

which, by the second law of thermodynamics, has to be positive. For small velocity derivatives  $\partial_\mu u^\nu$  and electromagnetic fields  $B$  and  $E$ , the dissipative terms should be linear in these perturbations, and hence one deduces the form

$$v^\alpha = -\sigma_Q \left[ T(g^{\alpha\lambda} + u^\alpha u^\lambda) \partial^\alpha \left( \frac{\mu}{T} \right) - F^{\alpha\lambda} u_\lambda \right], \quad (4.20)$$

$$\tau^{\mu\nu} = -(g^{\mu\lambda} + u^\mu u^\lambda) [\eta (\partial_\lambda u^\nu + \partial^\nu u_{\lambda}) + (\zeta - \eta) \delta_\lambda^\nu \partial_a u^\alpha], \quad (4.21)$$

where  $\sigma_Q$  is a conductivity (of order of  $e^2/h$ ), and  $\eta$  and  $\zeta$  are the shear and bulk viscosities, respectively, as will be clear from expression (4.23) given below. For graphene at the Dirac point,  $\eta$ ,  $\zeta \sim T^2/\alpha^2$ . These can be computed by solving a linearized kinetic Boltzmann equation.<sup>17,23</sup>

The relativistic expression [Eq. (4.20)] reduces to a spatial vector in the laboratory frame,

$$\vec{v} = -\sigma_Q \left( \vec{\nabla} \mu - \frac{\nabla T}{T} \mu - B \hat{\epsilon} \vec{v} - \vec{E} \right), \quad (4.22)$$

while the viscous contribution to the stress-energy tensor takes the familiar form

$$\tau^{ij} = [\eta (\partial_i u^j + \partial_j u^i) + (\zeta - \eta) \delta^{ij} \vec{\nabla} \cdot \vec{v}]. \quad (4.23)$$

### B. Linear thermoelectric response

The thermoelectric transport coefficients describing the current ( $\vec{J}$ ) and heat current ( $\vec{Q}$ ) response to electric fields and temperature gradients are defined by the relation,

$$\begin{pmatrix} \vec{J} \\ \vec{Q} \end{pmatrix} = \begin{pmatrix} \hat{\sigma} & \hat{\alpha} \\ T \hat{\alpha} & \hat{\kappa} \end{pmatrix} \begin{pmatrix} \vec{E} \\ -\vec{\nabla} T \end{pmatrix}, \quad (4.24)$$

where  $\hat{\sigma}$ ,  $\hat{\alpha}$ ,  $\hat{\alpha}$ , and  $\hat{\kappa}$  are  $2 \times 2$  matrices acting on the spatial indices  $x$  and  $y$ . Rotational invariance in the plane imposes the form

$$\hat{\sigma} = \sigma_{xx} \hat{1} + \sigma_{xy} \hat{\epsilon}, \quad (4.25)$$

where  $\hat{1}$  is the identity and  $\hat{\epsilon}$  is the antisymmetric tensor  $\hat{\epsilon}_{xy} = -\hat{\epsilon}_{yx} = 1$ .  $\sigma_{xx}$  and  $\sigma_{xy}$  describe the longitudinal and Hall conductivity, respectively. An analogous form holds for the thermoelectric conductivities  $\hat{\alpha}$  and  $\hat{\alpha}$ , which determine the Peltier, Seebeck, and Nernst effects, as well as for the matrix  $\hat{\kappa}$ , which governs thermal transport in the absence of electric fields. The latter applies to samples connected to conducting leads, allowing for a stationary current flow. In contrast, the thermal conductivity,  $\hat{\kappa}$ , is defined as the heat current response to  $-\vec{\nabla} T$  in the absence of an electric current (electrically isolated boundaries) and is given by

$$\hat{\kappa} = \hat{\kappa} - T \hat{\alpha} \hat{\sigma}^{-1} \hat{\alpha}. \quad (4.26)$$

### C. Response functions from hydrodynamics

We will now use the conservation laws [Eqs. (4.13)–(4.15)] to solve for the slow relaxation dynamics toward equilibrium, starting from a small initial long-wavelength perturbation. From the full solution of the relaxation dynamics in linear-response approximation one can then determine the thermoelectric response functions in the hydrodynamic regime.

As dynamic variables we choose  $T$ ,  $\mu$  (which includes the static Coulomb potential), and  $v^x$  and  $v^y$ , and write

$$\mu(r, t) = \mu + \delta\mu(r, t),$$

$$T(r, t) = T + \delta T(r, t). \quad (4.27)$$

The other variables,  $\varepsilon$ ,  $P$ , and  $\rho$  are constrained by local thermodynamic equilibrium to have the form

$$\rho(r, t) = \rho + \delta\rho \equiv \rho + \left. \frac{\partial\rho}{\partial\mu} \right|_T \delta\mu + \left. \frac{\partial\rho}{\partial T} \right|_\mu \delta T,$$

$$\varepsilon(r, t) = \varepsilon + \delta\varepsilon \equiv \varepsilon + \left. \frac{\partial\varepsilon}{\partial\mu} \right|_T \delta\mu + \left. \frac{\partial\varepsilon}{\partial T} \right|_\mu \delta T,$$

$$P(r, t) = P + \delta P \equiv P + \rho \delta\mu + s \delta T, \quad (4.28)$$

where  $\delta\mu$  is the deviation of the total electrochemical potential from the equilibrium value  $\mu$  and  $\partial\rho/\partial\mu|_T \equiv \chi$  is the susceptibility. Note, that even though for graphene at the Dirac point ( $\mu=0$ )  $\chi(k) \sim k \rightarrow 0$  at  $T=0$ , the susceptibility is always finite,  $\chi \sim T$ , for the hydrodynamic regime considered here ( $T > 0$ ).

Following the technique of Kadanoff-Martin<sup>24</sup> to derive hydrodynamic response functions, we prepare the system in a state of local equilibrium as characterized by slowly varying initial conditions  $T(\vec{r})$  and  $\mu(\vec{r})$ , as created by an external electric field. Local equilibrium implies that the screening of an inhomogeneous charge density must be self-consistently built into the initial conditions. The relation between the actual initial variation in the chemical potential,  $\delta\mu^0$ , and both an applied external field  $\vec{E}^{\text{ext}} \equiv -\vec{\nabla}(\delta\mu^{\text{ext}})$  and temperature deviations  $\delta T^0$  is therefore not entirely trivial. The total chemical potential  $\mu(r)$  is the sum of the externally applied

potential  $\delta\mu^{\text{ext}}$  and the Coulomb potential created by the total induced charge density. In  $k$  space, the Thomas-Fermi [random-phase approximation (RPA)] screened Coulomb potential induced by the two perturbations leads to the chemical potential,

$$\delta\mu^0(k) = \delta\mu^{\text{ext}}(k) - \frac{U_k}{1 + U_k \partial\rho/\partial\mu|_T} \times \left( \frac{\partial\rho}{\partial\mu} \Big|_T \delta\mu^{\text{ext}}(k) + \frac{\partial\rho}{\partial T} \Big|_\mu \delta T^0(k) \right). \quad (4.29)$$

After a Fourier transform in space and a Laplace transform in time, Eqs. (4.13)–(4.15) together with Eq. (4.12) take the form

$$\begin{aligned} & \omega \left( \frac{\partial\varepsilon}{\partial\mu} \Big|_T \delta\mu + \frac{\partial\varepsilon}{\partial T} \Big|_\mu \delta T \right) - k(\varepsilon + P)v_{\parallel} \\ &= i \left[ \frac{\partial\varepsilon}{\partial\mu} \Big|_T \delta\mu^0 + \frac{\partial\varepsilon}{\partial T} \Big|_\mu \delta T^0 \right], \\ & \omega \left( \frac{\partial\rho}{\partial\mu} \Big|_T \delta\mu + \frac{\partial\rho}{\partial T} \Big|_\mu \delta T \right) - k(\rho v_{\parallel} + \sigma_Q[Bv_{\perp} + E_k]) \\ &+ i\sigma_Q k^2 \left( \delta\mu - \frac{\mu}{T} \delta T \right) \\ &= i \left[ \frac{\partial\rho}{\partial\mu} \Big|_T \delta\mu^0 + \frac{\partial\rho}{\partial T} \Big|_\mu \delta T^0 \right], \\ & \left( \omega + \frac{i}{\tau} \right) (\varepsilon + P)v_{\parallel} - k(\rho\delta\mu + s\delta T) - i\rho Bv_{\perp} + i\sigma_Q B^2 v_{\parallel} \\ &- i\rho E_k + ik^2(\eta + \zeta)v_{\parallel} = i(\varepsilon + P)v_{\parallel}^0, \\ & \left( \omega + \frac{i}{\tau} \right) (\varepsilon + P)v_{\perp} + k\sigma_Q B \left( \delta\mu - \frac{\mu}{T} \delta T \right) + i\rho Bv_{\parallel} \\ &+ i\sigma_Q B[Bv_{\perp} + E_k] + ik^2\eta v_{\perp} = i(\varepsilon + P)v_{\perp}^0, \quad (4.30) \end{aligned}$$

where

$$E_k = -ikU_k \left( \frac{\partial\rho}{\partial\mu} \Big|_T \delta\mu + \frac{\partial\rho}{\partial T} \Big|_\mu \delta T \right). \quad (4.31)$$

In these expressions,  $\delta\mu^0$ ,  $\delta T^0$ ,  $\delta v_{\parallel}^0$ , and  $\delta v_{\perp}^0$  are the initial values (depending on the wave vector  $k$ ), while  $\delta\mu$ ,  $\delta T$ ,  $\delta v_{\parallel}$ , and  $\delta v_{\perp}$  are functions of  $k$  and  $\omega$ . The projections of  $\vec{v}$  parallel and orthogonal to  $\vec{k}$  are  $v_{\parallel} = \vec{v} \cdot \vec{k}/k$  and  $v_{\perp} = \vec{k}/k \cdot \hat{e}\vec{v}$ .

As shown in detail in Refs. 11 and 24, the linear-response functions can be obtained in full generality and in closed form from the solution of the above equations, e.g.,  $\sigma_{xx}(\omega, k) = \omega J_{\parallel}(\omega, k) / \delta\mu^0$ , where  $J_{\parallel} = \vec{J} \cdot \vec{k}/k$ .

We have restricted ourselves to the lowest nontrivial order in an expansion in  $k/\omega$ , where Coulomb effects become visible. This will be sufficient to describe the response to external perturbations of electromagnetic origin (e.g., microwaves) for which one always has  $ck/\omega < \sqrt{\varepsilon_r}$ . When applying the relativistic hydrodynamics to graphene with a characteristic velocity  $v/c \approx 1/300$ , this implies the relation  $vk/\omega$

$< (v/c)\sqrt{\varepsilon_r} \ll 1$ , which justifies the expansion to lowest order in  $k/\omega$ . The full  $k, \omega$  dependence can be obtained in closed form. However, it is very involved and does not contain much more physical information, so we do not report it explicitly here.

## V. RESULTS FOR THE THERMOELECTRIC RESPONSE

### A. Limit of vanishing magnetic field

In the limit of vanishing magnetic field  $B \rightarrow 0$ , the transverse response vanishes for symmetry reasons. The longitudinal transport coefficients take the relatively simple forms,

$$\begin{aligned} \sigma_{xx}(\omega, k; B=0) &= \left( \sigma_Q + \frac{\rho^2}{P + \varepsilon} \frac{\tau}{1 - i\omega\tau} \right) \left[ 1 - 2\pi\alpha \frac{ik}{\omega} \right. \\ &\quad \left. \times \left( \sigma_Q + \frac{\tau}{1 - i\omega\tau} \frac{\rho^2}{P + \varepsilon} \right) \right] + \mathcal{O}(k^2), \quad (5.1) \end{aligned}$$

$$\begin{aligned} \alpha_{xx}(\omega, k; B=0) &= \tilde{\alpha}_{xx}(\omega, k; B=0) + \mathcal{O}(k^2) \\ &= \left( -\sigma_Q \frac{\mu}{T} + \frac{s\rho}{P + \varepsilon} \frac{\tau}{1 - i\omega\tau} \right) \\ &\quad \times \left[ 1 - 2\pi\alpha \frac{ik}{\omega} \left( \sigma_Q + \frac{\tau}{1 - i\omega\tau} \frac{\rho^2}{P + \varepsilon} \right) \right] \\ &\quad + \mathcal{O}(k^2), \quad (5.2) \end{aligned}$$

$$\begin{aligned} \bar{\kappa}_{xx}(\omega, k; B=0) &= \left( \sigma_Q \frac{\mu^2}{T} + \frac{s^2 T}{P + \varepsilon} \frac{\tau}{1 - i\omega\tau} \right) \\ &\quad \times \left[ 1 - 2\pi\alpha \frac{ik}{\omega} \left( \sigma_Q + \frac{\tau}{1 - i\omega\tau} \frac{\rho^2}{P + \varepsilon} \right) \right] \\ &\quad + 2\pi\alpha i \frac{k}{\omega} \sigma_Q \frac{P + \varepsilon}{T} \frac{\tau}{1 - i\omega\tau} + \mathcal{O}(k^2), \quad (5.3) \end{aligned}$$

$$\begin{aligned} \kappa_{xx}(\omega, k; B=0) &= \sigma_Q \frac{(P + \varepsilon)^2}{T} \frac{1}{\rho^2 + (\sigma_Q/\tau)(P + \varepsilon)(1 - i\omega\tau)} \\ &\quad + \mathcal{O}(k^2). \quad (5.4) \end{aligned}$$

Note that all response functions contain a piece proportional to  $\sigma_Q$ , which is independent of the impurity scattering time  $\tau$  and thus is solely governed by the universal Coulomb interactions. A second term proportional to  $\tau/(1 - i\omega\tau)$  has the form of a classical Drude-type term, which is due to the slow relaxation of the ‘‘momentum mode.’’<sup>17</sup> This is an excitation of the electron-hole liquid that cannot relax due to Coulomb interactions because of their translational invariance. As one may expect for weak impurity concentration, the term contributed by the momentum mode grows linearly with the impurity scattering time  $\tau$ .

Note the simple structure of the leading  $k$  dependence of the response functions. It only depends on Coulomb interactions (via  $\alpha$ ), while the viscosities  $\eta$  and  $\zeta$  do not appear at this order. Including finite  $k$  introduces a simple factor,

$$1 - 2\pi\alpha \frac{ik}{\omega} \left( \sigma_Q + \frac{\tau}{1 - i\omega\tau P + \varepsilon} \frac{\rho^2}{\omega} \right) = 1 - 2\pi\alpha \frac{ik}{\omega} \sigma_{xx}(\omega, k=0), \quad (5.5)$$

which multiplies the  $k \rightarrow 0$  result. Interestingly, the thermal conductivity  $\kappa$  (in the absence of currents) is not affected by Coulomb interactions to lowest order.

### B. Magnetotransport

Most interesting for our purpose is the response in a weak magnetic field for which we obtain

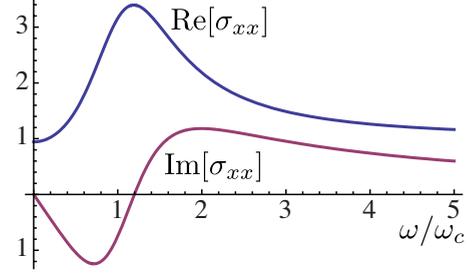


FIG. 2. (Color online) The real imaginary and imaginary parts of  $\sigma_{xx}$  in units of  $\sigma_Q$  for  $\gamma/\omega_c=0.3$  and  $\omega_c\tau=3$ .

$$\sigma_{xx}(\omega, k) = \sigma_Q \frac{\left( \omega + \frac{i}{\tau} \right) \left( \omega + \frac{i}{\tau} + i\gamma + \frac{i\omega_c^2}{\gamma} \right)}{\left( \omega + \frac{i}{\tau} + i\gamma \right)^2 - \omega_c^2} \left[ 1 - 2\pi\alpha \frac{ik}{\omega} \sigma_Q \frac{\left( \omega + \frac{i}{\tau} \right) \left( \omega + \frac{i}{\tau} + i\gamma + \frac{i\omega_c^2}{\gamma} \right)}{\left( \omega + \frac{i}{\tau} + i\gamma \right)^2 - \omega_c^2} \right] + \mathcal{O}(k^2), \quad (5.6)$$

$$\sigma_{xy}(\omega, k) = -\frac{\rho}{B} \frac{\omega_c^2 + \gamma^2 + 2\gamma \left( \frac{1}{\tau} - i\omega \right)}{\left( \omega + \frac{i}{\tau} + i\gamma \right)^2 - \omega_c^2} \left[ 1 - 2\pi\alpha \frac{ik}{\omega} \sigma_Q \frac{\left( \omega + \frac{i}{\tau} \right) \left( \omega + \frac{i}{\tau} + i\gamma + \frac{i\omega_c^2}{\gamma} \right)}{\left( \omega + \frac{i}{\tau} + i\gamma \right)^2 - \omega_c^2} \right] + \mathcal{O}(k^2). \quad (5.7)$$

We plot the frequency dependence of the longitudinal conductivity in Fig. 2. The collective cyclotron frequency  $\omega_c$  and the intrinsic interaction-induced damping frequency  $\gamma$  were given in Eq. (1.1).

For the thermoelectric response functions, we have

$$\begin{aligned} \tilde{\alpha}_{xx}(\omega, k) &= -\frac{\left( \omega + \frac{i}{\tau} \right) \left[ \sigma_Q \left( \frac{\mu}{T} \right) \left( \omega + \frac{i}{\tau} + i\gamma \right) - is \frac{\rho}{(\varepsilon + P)} \right]}{\left( \omega + \frac{i}{\tau} + i\gamma \right)^2 - \omega_c^2} \left[ 1 - 2\pi\alpha \frac{ik}{\omega} \sigma_Q \frac{\left( \omega + \frac{i}{\tau} \right) \left( \omega + \frac{i}{\tau} + i\gamma + \frac{i\omega_c^2}{\gamma} \right)}{\left( \omega + \frac{i}{\tau} + i\gamma \right)^2 - \omega_c^2} \right] + \mathcal{O}(k^2) \\ &= i \frac{\left( \omega + \frac{i}{\tau} \right) \left[ \frac{s\omega_c}{B} + i\sigma_Q \left( \frac{\mu}{T} \right) \left( \omega + \frac{i}{\tau} + i\gamma \right) \right]}{\left( \omega + \frac{i}{\tau} + i\gamma \right)^2 - \omega_c^2} \left[ 1 - 2\pi\alpha \frac{ik}{\omega} \sigma_Q \frac{\left( \omega + \frac{i}{\tau} \right) \left( \omega + \frac{i}{\tau} + i\gamma + \frac{i\omega_c^2}{\gamma} \right)}{\left( \omega + \frac{i}{\tau} + i\gamma \right)^2 - \omega_c^2} \right] + \mathcal{O}(k^2), \end{aligned} \quad (5.8)$$

$$\alpha_{xx}(\omega, k) = \tilde{\alpha}_{xx}(\omega, k) + \mathcal{O}(k^2), \quad (5.9)$$

$$\tilde{\alpha}_{xy}(\omega, k) = -\frac{s}{B} \frac{\omega_c^2 + \gamma^2 - i\gamma \left( \omega + \frac{i}{\tau} \right) \left[ 1 - \frac{\mu\rho}{(sT)} \right]}{\left( \omega + \frac{i}{\tau} + i\gamma \right)^2 - \omega_c^2} \left[ 1 - 2\pi\alpha \frac{ik}{\omega} \sigma_Q \frac{\left( \omega + \frac{i}{\tau} \right) \left( \omega + \frac{i}{\tau} + i\gamma + \frac{i\omega_c^2}{\gamma} \right)}{\left( \omega + \frac{i}{\tau} + i\gamma \right)^2 - \omega_c^2} \right] + \mathcal{O}(k^2), \quad (5.10)$$

$$\alpha_{xy}(\omega, k) = \tilde{\alpha}_{xy}(\omega, k) - 2\pi\alpha \frac{k}{\omega} \frac{B\sigma_Q^2}{T} \frac{\left( \omega + \frac{i}{\tau} \right)}{\left( \omega + \frac{i}{\tau} + i\gamma \right)^2 - \omega_c^2} + \mathcal{O}(k^2). \quad (5.11)$$

Notice that the Onsager reciprocity  $\alpha = \tilde{\alpha}$  only holds for the response to homogeneous perturbations (i.e.,  $k=0$ ), while there are deviations at finite  $k$ .

Finally, for the thermal conductivities, we have

$$\bar{\kappa}_{xx}(\omega, k) = \frac{-\gamma \frac{P+\varepsilon}{T} - i \frac{s^2 T}{P+\varepsilon} \left(\omega + \frac{i}{\tau}\right) + \sigma_Q \frac{\mu^2}{T} \left(\omega + \frac{i}{\tau}\right) \left(\omega + \frac{i}{\tau} + i\gamma\right)}{\left(\omega + \frac{i}{\tau} + i\gamma\right)^2 - \omega_c^2} \left[ 1 - 2\pi\alpha \frac{ik}{\omega} \sigma_Q \frac{\left(\omega + \frac{i}{\tau}\right) \left(\omega + \frac{i}{\tau} + i\gamma + \frac{i\omega_c^2}{\gamma}\right)}{\left(\omega + \frac{i}{\tau} + i\gamma\right)^2 - \omega_c^2} \right] - 2\pi\alpha \frac{k}{\omega} \sigma_Q \frac{P+\varepsilon}{T} \frac{\omega + \frac{i}{\tau}}{\left(\omega + \frac{i}{\tau} + i\gamma\right)^2 - \omega_c^2} + \mathcal{O}(k^2), \quad (5.12)$$

$$\kappa_{xx}(\omega, k) = i \frac{(\varepsilon + P)}{T} \frac{\left(\omega + \frac{i}{\tau} + \frac{i\omega_c^2}{\gamma}\right)}{\left(\omega + \frac{i}{\tau} + \frac{i\omega_c^2}{\gamma}\right)^2 - \omega_c^2} + \mathcal{O}(k^2), \quad (5.13)$$

$$\begin{aligned} \bar{\kappa}_{xy}(\omega, k) &= -\frac{\frac{s^2 T^2 \rho}{(P+\varepsilon)^2} - \mu \sigma_Q \left[ \gamma \frac{\mu \rho}{P+\varepsilon} - 2i \frac{sT}{P+\varepsilon} \left(\omega + \frac{i}{\tau} + i\gamma\right) \right]}{T \frac{\left(\omega + \frac{i}{\tau} + i\gamma\right)^2 - \omega_c^2}} \\ &\times \left[ 1 - 2\pi\alpha \frac{ik}{\omega} \sigma_Q \frac{\left(\omega + \frac{i}{\tau}\right) \left(\omega + \frac{i}{\tau} + i\gamma + \frac{i\omega_c^2}{\gamma}\right)}{\left(\omega + \frac{i}{\tau} + i\gamma\right)^2 - \omega_c^2} \right] \\ &+ 2\pi\alpha \frac{k}{\omega} \frac{B \mu \sigma_Q^2}{T} \frac{\left(\omega + \frac{i}{\tau}\right)}{\left(\omega + \frac{i}{\tau} + i\gamma\right)^2 - \omega_c^2} + \mathcal{O}(k^2), \quad (5.14) \end{aligned}$$

$$\kappa_{xy}(\omega, k) = \frac{(\varepsilon + P)}{T} \frac{\omega_c}{\left(\omega + \frac{i}{\tau} + \frac{i\omega_c^2}{\gamma}\right)^2 - \omega_c^2} + \mathcal{O}(k^2). \quad (5.15)$$

Note that the interaction-induced damping frequency for  $\kappa$  is  $\omega_c^2/\gamma$  and not  $\gamma$ . The former damping frequency also appears<sup>11</sup> in the response functions for  $\rho_{xx}$  and  $\rho_{xy}$ , as can be easily checked by inverting Eqs. (5.6) and (5.7).

As in Eq. (5.4), in the presence of a magnetic field the thermal conductivity in absence of currents is again independent of Coulomb interactions to lowest order in  $k/\omega$ .

### C. Nernst effect

An important thermoelectric response is the Nernst effect, which measures the transverse electric field  $E_y$  that is estab-

lished as a consequence of an applied longitudinal thermal gradient  $\nabla_x T$  in the absence of electrical currents. The ratio  $e_N = E_y / (-\nabla_x T)$  is called Nernst signal and is easily obtained from the coefficients determined above as  $e_N = (\sigma^{-1} \alpha)_{xy}$ . It vanishes in the absence of a magnetic field and grows linearly for small fields  $B$ . The Nernst effect has become a popular measurement to characterize nonstandard metals, such as in bismuth<sup>25</sup> (where it was originally discovered<sup>26</sup>) in other semimetals,<sup>27</sup> and in materials close to quantum-critical points,<sup>28</sup> as well as in superconductors.<sup>29</sup> All these systems share with undoped graphene the property of being far from a simple Fermi liquid. In the latter the so-called Sondheimer cancellation suppresses the Nernst signal while it becomes very large in the systems mentioned above. By far the strongest Nernst signals (on the order of 1 mV/K for fields of 1 T) have been observed in bismuth whose band structure exhibits close similarities with graphene with which it shares the presence of nearly massless Dirac fermions.

From the above formalism one easily obtains the full expression for the Nernst signal (at  $k=0$ ),

$$e_N = \frac{k_B \varepsilon + P}{e} \frac{\frac{\omega_c}{\tau}}{k_B T \rho \left(\frac{\omega_c^2}{\gamma} + \frac{1}{\tau}\right)^2 + \omega_c^2}, \quad (5.16)$$

where  $k_B/e = 86.17 \mu\text{V/K}$  is its natural quantum unit. For small doping  $\rho$ , such that  $\omega_c \tau \ll \min(\gamma/\omega_c, 1)$ , this result simplifies to

$$e_N(\rho \rightarrow 0) = \frac{k_B \varepsilon + P}{e} \frac{1}{k_B T \rho} \omega_c \tau = \frac{k_B \tau T}{e \hbar} \frac{B}{B^*}. \quad (5.17)$$

In relatively clean samples close to quantum criticality this may exceed the quantum unit without violating the conditions for the applicability of hydrodynamics.

In clean samples with  $\omega_c \tau \gg 1$  and in the limit of large fields  $\omega_c \ll \gamma$ , one obtains the result

$$e_N = \frac{k_B \varepsilon + P}{e} \frac{1}{k_B T \rho \omega_c \tau} = \frac{k_B}{e} \frac{\Phi_{\varepsilon+P}^2}{\left(\frac{\rho}{\rho_{\text{th}}}\right)^2 \left(\frac{\tau T}{\hbar}\right) \left(\frac{B}{B^*}\right)}, \quad (5.18)$$

which decays inversely proportional to the field strength.

## VI. CONCLUSIONS

Our main experimental predictions for the hydrodynamic cyclotron resonance in graphene are given by the frequency-dependent conductivities in Eqs. (5.6) and (5.7) with the frequencies  $\omega_c$  and  $\gamma$  specified in Eqs. (3.5) and (3.8) and the dimensionless function of the density  $\Phi_{\varepsilon+p}$  estimated in Fig. 1. This resonance occurs in a regime of weak magnetic fields where the Landau levels are not yet formed, and the dynamics is dominated by inelastic electron-electron collisions, which occur at a rate  $\sim h/k_B T$ . The electronic dynamics is quantum critical, and the observation of such a resonance will offer a valuable opportunity to explore quantum critical-

ity. As has been argued elsewhere,<sup>11</sup> similar physics applies to a variety of systems in the vicinity of a quantum phase transition, including the superconductor-insulator transition in the cuprate superconductors.

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