

Flux tubes and the type-I/type-II transition in a superconductor coupled to a superfluid

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We analyze magnetic-flux tubes at zero temperature in a superconductor that is coupled to a superfluid via both density and gradient (“entrainment”) interactions. The example we have in mind is high-density nuclear matter, which is a proton superconductor and a neutron superfluid, but our treatment is general and simple, modeling the interactions as a Ginzburg-Landau effective theory with four-fermion couplings, including only s -wave pairing. We numerically solve the field equations for flux tubes with an arbitrary number of flux quanta and compare their energies. This allows us to map the type-I/type-II transition in the superconductor, which occurs at the conventional $\kappa \equiv \lambda/\xi = 1/\sqrt{2}$ if the condensates are uncoupled. We find that a density coupling between the condensates raises the critical κ and, for a sufficiently high neutron density, resolves the type-I/type-II transition line into an infinite number of bands corresponding to “type-II(n)” phases, in which n , the number of quanta in the favored flux tube, steps from 1 to infinity. For lower neutron density, the coupling creates spinodal regions around the type-I/type-II boundary, in which metastable flux configurations are possible. We find that a gradient coupling between the condensates lowers the critical κ and creates spinodal regions. These exotic phenomena may not occur in nuclear matter, which is thought to be deep in the type-II region but might be observed in condensed-matter systems.

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I. INTRODUCTION

Superconductivity and superfluidity are well-studied phenomena, known to occur in many physical systems, from cold metals and cold atomic gases to nuclear matter and quark matter. In this paper, we investigate a system that has both a charged condensate, leading to superconductivity, and a neutral condensate, leading to superfluidity. We focus on the magnetic-flux tubes that are associated with the superconducting condensate and study how they are modified by the presence of the superfluid, assuming that the two condensates can interact with each other via density and gradient (“entrainment”) interactions.

An example of this type of system is nuclear matter, which at sufficiently high density undergoes Cooper pairing of both neutrons and protons. We will present our calculations in this context, referring to the charged condensate as the “proton condensate” and the neutral one as the “neutron condensate” and choosing values appropriate to nuclear matter for our parameters when presenting numerical results. In fact, the questions that we study in this paper were originally raised in investigations of the nature of the proton superconductivity in the nuclear matter in a neutron star. Although it is generally believed that the protons form a type-II superconductor,¹ there is evidence from long neutron star precession periods that seems to favor type-I superconductivity² (for contrary views see^{3,4}). This led Buckley *et al.*⁵ to suggest that if the density interaction between the magnitudes of the neutron and proton Cooper pair condensates is extremely strong, nuclear matter would be a type-I superconductor even if its penetration depth λ and coherence length ξ obey the conventional condition $\lambda/\xi > 1/\sqrt{2}$ for type-II superconductivity. We have argued elsewhere that the assumption of a strong coupling between the proton and neutron condensates is wrong for neutron star matter.⁶ However, Buckley *et al.*⁵ were correct in making the

point that a superconductor will be affected by interactions with a coexisting superfluid.

In this paper, we study the type-I versus type-II nature of a (proton) superconductor coupled to a (neutron) superfluid using an effective theory for the protons and neutrons that contains four-fermion interaction terms, which lead to s -wave pairing. We do not include higher-angular-momentum pairing, although that would be needed for a more realistic analysis of high-density nuclear matter. Our analysis extends that of Ref. 5 in the following ways: (a) Our model, like that of Ref. 5, contains a coupling a_{np} between the magnitudes of the neutron and proton condensates, and self-couplings a_{nn} and a_{pp} , but we survey the whole range of values of a_{np} from zero to of order a_{pp} ; (b) we also include “entrainment” interactions between the gradients of the proton and neutron condensates; (c) we use a simpler and more direct method to study the type-I/type-II phase boundary using the energetics of flux tube coalescence and/or fission: we calculate the energy of flux tubes with a wide range of magnetic fluxes—from one quantum to several hundred quanta—and find which one has the lowest energy per unit flux. As we will see, this has the additional benefit of allowing us to find exotic stable multiquantum flux tubes, such as have been found in systems of two coupled superconductors.⁷ However, as we discuss below, our analysis is not sensitive to minima in the interaction energy at finite separation between flux tubes.

Our analysis is entirely at zero temperature. This is a good approximation for neutron star matter near nuclear saturation density, where the critical temperatures for the superfluid and superconductor are of the order of MeV.^{8–10} The temperature of a compact star drops below this value within minutes of its formation in a supernova and is at or below the keV range after the first 1000 years.¹¹ When we discuss type-I versus type-II behavior, we are referring to the response of the system to a magnetic field at the lower critical value at $T=0$.

As far as we know, there has been no previous work on how a flux tube in a superconductor is affected by a gradient coupling to a coexisting superfluid. However, there has been work on possible knot solitons,¹² vortices in the $SO(5)$ model of high-temperature superconductivity,¹³ and on the complementary situation, a superfluid vortex with gradient coupling to a coexisting superconductor. There the coupling leads to the entrainment or Andreev-Bashkin effect,¹⁴ whereby the proton condensate is dragged along with the neutron condensate, producing a nonzero proton current around the vortex, dressing it with some magnetic flux.¹⁵ It is interesting to note that this flux is not a multiple of the flux quantum for proton flux tubes. This is possible because of the difference between the energetics of a neutron vortex and a proton flux tube. The flux tube has energy density localized to the vicinity of its core. Far from the core the energy density must vanish, which means the proton field must change in phase by a multiple of 2π , and the vector potential must cancel the resultant gradient, leading to a quantized magnetic flux. A neutron vortex, by contrast, has gradient energy that is not localized to the vicinity of the vortex, and the total energy per unit length diverges in the infinite volume limit. The vector potential is therefore not constrained to cancel any gradient in the proton field and takes on a value that minimizes the overall energy, with no quantization condition on the resulting magnetic flux.

Returning to the situation that we study, a proton flux tube in a neutron superfluid background, we do not expect a similar behavior. This is because the proton flux tube's energy density is localized around its core, giving it (unlike the neutron vortex) a finite energy per unit length. If the neutron condensate were entrained and developed nonzero circulation around the flux tube, it would acquire a nonlocalized energy density, leading to an infinite energy per unit length for the flux tube, which is clearly energetically disfavored. We will see below that the effect of gradient couplings on the proton superconductor is more subtle: it leads to metastable regions near the type-I/type-II boundary.

II. STABILITY OF FLUX TUBES

Our aim is to explore the response of the proton superconductor to an applied critical magnetic field at zero temperature. We will therefore construct a phase diagram in the space of the coupling constants of the Ginzburg-Landau (GL) effective theory. We would like to be able to specify when it is of type II (at the lower critical magnetic field, flux tubes appear and remain separated, i.e., they repel) and when it is of type I (at the critical magnetic field, macroscopic normal regions appear, i.e., the flux tubes attract and coalesce). The simplest way to do this is to calculate the energy per unit length E_n of a flux tube containing n flux quanta. The same approach has been used for vortices in the $SO(5)$ model.¹⁶ It is convenient to work in terms of the energy per flux quantum,

$$B_n = \frac{E_n}{n} - E_1. \quad (1)$$

When B_n is negative the n -quantum flux tube is stable against fission into many single-quantum flux tubes, and it is

energetically favorable for n single-quantum flux tubes to coalesce into one n -quantum flux tube. When B_n is positive the n -quantum flux tube is unstable against fission, and coalescence is energetically disfavored. If one calculates B_n for all n , then the energetically favored value of n is the one that minimizes B_n .

In a traditional type-I superconductor, small flux tubes attract each other and amalgamate into large ones and ultimately into macroscopic normal regions, so we would expect to find $B_n < 0$ with its value dropping monotonically as n rises. In a type-II superconductor, we would expect $B_n > 0$, with its value rising monotonically with n . Our calculations confirm these results for a single superconductor, but we will see that B_n shows more complicated behavior when the superconductor feels interaction with a coexisting superfluid.

Calculations of B_n are straightforward because they always occur in a cylindrically symmetric geometry, so the problem is one dimensional. For a more detailed understanding of flux tube interactions, one would have to consider two single-quantum flux tubes a distance d apart. Their total energy is $U(d)$, where $U(0) = E_2$ and $U(\infty) = 2E_1$, so $B_2 = \frac{1}{2}[U(0) - U(\infty)]$. As expected, $B_2 < 0$ means that the flux tubes have lower energy when they amalgamate, and $B_2 > 0$ means that the flux tubes have lower energy when they separate. If $U(d)$ is monotonic, we can conclude that flux tubes either coalesce ($B_2 < 0$) or repel to infinite separation ($B_2 > 0$), corresponding to type-I or type-II behavior, respectively. However, if there is a minimum in $U(d)$ at some favored intermediate separation $d = d^*$, then irrespective of the sign of B_n one has a new variety of type-II superconductor with some favored Abrikosov lattice spacing d^* . Such behavior has been found to arise from a ϕ^6 term¹⁷ and in the case of two charged condensates.⁷ Calculating $U(d)$ in the current context is an interesting but demanding problem, which we leave for future work. In this paper, we assume that $U(d)$ is monotonic, so to analyze the attractiveness and/or repulsiveness of the flux tube interactions it is sufficient to calculate B_n , or equivalently, E_n/n .

III. FLUX TUBES IN THE GINZBURG-LANDAU MODEL

A. Ginzburg-Landau model

We start by writing down the zero-temperature Ginzburg-Landau effective theory of proton and neutron condensates in the presence of a magnetic field.^{5,18} We denote the proton condensate field by ϕ_p , the neutron condensate field by ϕ_n , and the magnetic vector potential by \mathbf{A} . The free-energy density is

$$\mathcal{F} = \frac{\hbar^2}{2m_c} \left(\left| \left(\nabla - \frac{iq}{\hbar c} \mathbf{A} \right) \phi_p \right|^2 + |\nabla \phi_n|^2 \right) + \frac{|\nabla \times \mathbf{A}|^2}{8\pi} + U_{\text{ent}}(\phi_p, \phi_n) + V(|\phi_p|^2, |\phi_n|^2), \quad (2)$$

where m_c is twice the nucleon mass, q is twice the proton charge, U_{ent} is the entrainment free-energy density (see Ref. 18),

$$\begin{aligned}
 U_{\text{ent}} = & -\frac{\hbar^2}{2m_c} \frac{\sigma}{2\langle\phi_p\rangle\langle\phi_n\rangle} \left\{ \phi_p^* \phi_n^* \left[\left(\nabla - \frac{iq}{\hbar c} \mathbf{A} \right) \phi_p \cdot \nabla \phi_n \right] \right. \\
 & + \phi_p^* \phi_n \left[\left(\nabla - \frac{iq}{\hbar c} \mathbf{A} \right) \phi_p \cdot \nabla \phi_n^* \right] \\
 & + \phi_p \phi_n \left[\left(\nabla + \frac{iq}{\hbar c} \mathbf{A} \right) \phi_p^* \cdot \nabla \phi_n^* \right] \\
 & \left. + \phi_p \phi_n^* \left[\left(\nabla + \frac{iq}{\hbar c} \mathbf{A} \right) \phi_p^* \cdot \nabla \phi_n \right] \right\}, \quad (3)
 \end{aligned}$$

and

$$\begin{aligned}
 V(|\phi_p|^2, |\phi_n|^2) = & -\mu_p |\phi_p|^2 - \mu_n |\phi_n|^2 + \frac{a_{\text{pp}}}{2} |\phi_p|^4 + \frac{a_{\text{nn}}}{2} |\phi_n|^4 \\
 & + a_{\text{pn}} |\phi_p|^2 |\phi_n|^2, \quad (4)
 \end{aligned}$$

σ is a parameter characterizing the strength of the gradient coupling, μ_p and μ_n are the chemical potentials of the proton and neutron condensate excitations, and a_{pp} , a_{nn} , and a_{pn} are the GL quartic couplings.

In zero magnetic field, the condensates would have position-independent bulk densities $\langle\phi_p\rangle^2$ and $\langle\phi_n\rangle^2$ obtained by minimizing the free energy. This allows us to eliminate the chemical potentials μ_p, μ_n by writing

$$\begin{aligned}
 \mu_p & = a_{\text{pp}} \langle\phi_p\rangle^2 + a_{\text{pn}} \langle\phi_n\rangle^2, \\
 \mu_n & = a_{\text{nn}} \langle\phi_n\rangle^2 + a_{\text{pn}} \langle\phi_p\rangle^2, \quad (5)
 \end{aligned}$$

so up to constants involving $\langle\phi_p\rangle$ and $\langle\phi_n\rangle$, the potential V can be expressed in terms of the deviations of the condensate fields from their bulk values,

$$\begin{aligned}
 V(|\phi_p|^2, |\phi_n|^2) = & \frac{a_{\text{pp}}}{2} (|\phi_p|^2 - \langle\phi_p\rangle^2)^2 + \frac{a_{\text{nn}}}{2} (|\phi_n|^2 - \langle\phi_n\rangle^2)^2 \\
 & + a_{\text{pn}} (|\phi_p|^2 - \langle\phi_p\rangle^2) (|\phi_n|^2 - \langle\phi_n\rangle^2). \quad (6)
 \end{aligned}$$

In a neutron star, electrical neutrality keeps the proton fraction small in the 5%–10% range;^{19,20} we will take $\langle\phi_p\rangle^2/\langle\phi_n\rangle^2 \approx 0.05$. As we now argue, a typical value for the entrainment coupling is $\sigma \sim 10^{-1}$. We first relate our formalism to the hydrodynamic limit of the free energy, following Ref. 18. We focus on the phases of the fields, $\phi_p = \langle\phi_p\rangle \exp(i\chi_p)$ and $\phi_n = \langle\phi_n\rangle \exp(i\chi_n)$ and assume the fields have constant magnitude, and their phases have gradients

$$\mathbf{v}_p = \frac{\hbar}{2m_p} \nabla \chi_p - \frac{2e}{m_p c} \mathbf{A}, \quad \mathbf{v}_n = \frac{\hbar}{2m_n} \nabla \chi_n. \quad (7)$$

The free-energy density (2) then reduces to the hydrodynamic form

$$F = \frac{1}{2} \rho^{\text{pp}} \mathbf{v}_p^2 + \frac{1}{2} \rho^{\text{nn}} \mathbf{v}_n^2 + \rho^{\text{pn}} \mathbf{v}_p \cdot \mathbf{v}_n + V + \frac{\mathbf{B}^2}{8\pi}, \quad (8)$$

where the symmetric matrix ρ of superfluid densities has elements

$$\rho^{\text{pp}} = 2m_p \langle\phi_p\rangle^2 \approx m_c \langle\phi_p\rangle^2, \quad (9)$$

$$\rho^{\text{nn}} = 2m_n \langle\phi_n\rangle^2 \approx m_c \langle\phi_n\rangle^2, \quad \rho^{\text{pn}} = -2m_n \sigma \langle\phi_p\rangle \langle\phi_n\rangle.$$

Our entrainment parameter σ is therefore related to the parameter ϵ of Refs. 21–23 by $\sigma = \epsilon \langle\phi_n\rangle / \langle\phi_p\rangle$. Since ϵ is of the order of 0.03 and $\langle\phi_n\rangle^2 / \langle\phi_p\rangle^2 \sim 20$, we expect $\sigma \sim 10^{-1}$. This is consistent with the estimate $\rho^{\text{pn}} \approx -\frac{1}{2} \rho^{\text{pp}}$ used by Ref. 18. In terms of the Andreev-Bashkin parametrization,¹⁴ $\rho_{12} = -\rho^{\text{pn}}$, $\rho_1 = \rho^{\text{pp}} + \rho^{\text{pn}}$, and $\rho_2 = \rho^{\text{nn}} + \rho^{\text{pn}}$, so $\rho_1 / \rho_{12} \sim 1$ and $\rho_2 / \rho_{12} \sim 40$. All the interactions in Eq. (8), including the entrainment, have their ultimate origin in the strong interaction between the nucleons, which is isospin symmetric, and hence does not distinguish protons from neutrons.

B. Flux tube solutions

To study a flux tube containing n flux quanta, we assume a cylindrically symmetric field configuration in which the proton condensate field winds (in a covariantly constant way) around the z axis with a net phase $2\pi n$,

$$\phi_p = \langle\phi_p\rangle f(r) e^{in\theta}, \quad (10)$$

$$\phi_n = \langle\phi_n\rangle g(r), \quad (11)$$

$$\mathbf{A} = \frac{n\hbar c}{q} \frac{a(r)}{r} \hat{\theta}. \quad (12)$$

We have defined ϕ_n as a real field because, as noted above, any net phase change in the neutron condensate when it circles the flux tube would cost an infinite energy per unit length. Inserting the ansatz in Eq. (2), we obtain

$$\begin{aligned}
 \mathcal{F} = & \frac{\hbar^2}{2m_c} \left[\langle\phi_p\rangle^2 \left((f')^2 + \frac{n^2 f^2 (1-a)^2}{r^2} \right) \right. \\
 & \left. + \langle\phi_n\rangle^2 (g')^2 - 2\sigma \langle\phi_p\rangle \langle\phi_n\rangle f \cdot g \cdot f' \cdot g' \right] \\
 & + \frac{n^2 \hbar^2 c^2}{8\pi q^2} \frac{(a')^2}{r^2} + \frac{a_{\text{pp}} \langle\phi_p\rangle^4}{2} (f^2 - 1)^2 + \frac{a_{\text{nn}} \langle\phi_n\rangle^4}{2} (g^2 - 1)^2 \\
 & + a_{\text{pn}} \langle\phi_p\rangle^2 \langle\phi_n\rangle^2 (f^2 - 1)(g^2 - 1). \quad (13)
 \end{aligned}$$

Generating the Euler-Lagrange equations using the standard procedure, we obtain a set of coupled differential equations for f , g , and a ,

$$\begin{aligned}
 & \frac{\hbar^2}{2m_c a_{\text{pp}} \langle\phi_p\rangle^2} \left\{ f'' + \frac{f'}{r} - \frac{n^2 (1-a)^2 f}{r^2} \right. \\
 & \left. - \sigma \frac{\langle\phi_n\rangle}{\langle\phi_p\rangle} \left[f \cdot g \left(g'' + \frac{g'}{r} \right) + f (g')^2 \right] \right\} \\
 & = f (f^2 - 1) + \frac{a_{\text{pn}} \langle\phi_n\rangle^2}{a_{\text{pp}} \langle\phi_p\rangle^2} f (g^2 - 1),
 \end{aligned}$$

$$\begin{aligned}
 & \frac{\hbar^2}{2m_c a_{\text{pp}} \langle\phi_p\rangle^2} \left\{ g'' + \frac{g'}{r} - \sigma \frac{\langle\phi_p\rangle}{\langle\phi_n\rangle} \left[f \cdot g \left(f'' + \frac{f'}{r} \right) + g (f')^2 \right] \right\} \\
 & = \frac{a_{\text{nn}} \langle\phi_n\rangle^2}{a_{\text{pp}} \langle\phi_p\rangle^2} g (g^2 - 1) + \frac{a_{\text{pn}}}{a_{\text{pp}}} g (f^2 - 1)
 \end{aligned}$$

$$\frac{m_c c^2}{4\pi q^2 \langle \phi_p \rangle^2} \left(a'' - \frac{a'}{r} \right) = -(1-a)f^2. \quad (14)$$

At this point we recall the definition of the Ginzburg-Landau parameter $\kappa = \lambda / \xi$, where the London penetration depth λ and superconducting coherence length ξ are (see Ref. 24)

$$\lambda \equiv \sqrt{\frac{m_c c^2}{4\pi q^2 \langle \phi_p \rangle^2}} = \sqrt{\frac{m_c c^2}{16\pi \hbar c \alpha_{EM} \langle \phi_p \rangle^2}} \quad (15)$$

$$\xi \equiv \sqrt{\frac{\hbar^2}{2m_c a_{pp} \langle \phi_p \rangle^2}}.$$

To further simplify the equations, we then change variables to a dimensionless radial coordinate $\tilde{r} = r / \xi$, obtaining

$$\begin{aligned} f'' + \frac{f'}{\tilde{r}} - \frac{n^2(1-a)^2 f}{\tilde{r}^2} - \sigma \frac{\langle \phi_n \rangle}{\langle \phi_p \rangle} \left[f \cdot g \left(g'' + \frac{g'}{\tilde{r}} \right) + f(g')^2 \right] \\ = f(f^2 - 1) + \frac{a_{pn} \langle \phi_n \rangle^2}{a_{pp} \langle \phi_p \rangle^2} f(g^2 - 1) \quad (16) \\ g'' + \frac{g'}{\tilde{r}} - \sigma \frac{\langle \phi_p \rangle}{\langle \phi_n \rangle} \left[f \cdot g \left(f'' + \frac{f'}{\tilde{r}} \right) + g(f')^2 \right] \\ = \frac{a_{nn} \langle \phi_n \rangle^2}{a_{pp} \langle \phi_p \rangle^2} g(g^2 - 1) + \frac{a_{pn}}{a_{pp}} g(f^2 - 1) \\ a'' - \frac{a'}{\tilde{r}} = -\frac{1}{\kappa^2} (1-a)f^2. \end{aligned}$$

The free energy per unit length of the flux tube, in terms of the variable \tilde{r} , is

$$\begin{aligned} E_n = 2\pi a_{pp} \langle \phi_p \rangle^4 \xi^2 \int_0^\infty (\tilde{r} d\tilde{r}) \left\{ (f')^2 + \frac{n^2 f^2 (1-a)^2}{\tilde{r}} \right. \\ + \frac{\langle \phi_n \rangle^2}{\langle \phi_p \rangle^2} (g')^2 - 2\sigma \frac{\langle \phi_n \rangle}{\langle \phi_p \rangle} f \cdot g \cdot f' \cdot g' + n^2 \kappa^2 \frac{(a')^2}{\tilde{r}^2} \\ + \frac{1}{2} (f^2 - 1)^2 + \frac{1}{2} \frac{a_{nn} \langle \phi_n \rangle^4}{a_{pp} \langle \phi_p \rangle^4} (g^2 - 1)^2 \\ \left. + \frac{a_{pn} \langle \phi_n \rangle^2}{a_{pp} \langle \phi_p \rangle^2} (f^2 - 1)(g^2 - 1) \right\}. \quad (17) \end{aligned}$$

In addition to the system of equations, we require boundary conditions on the fields at the origin and at ∞ . Far from the flux tube core, the fields will go to their uniform condensate value, so $f(\infty) = g(\infty) = a(\infty) = 1$. Near the origin, $f(r) \propto r^n$, $a(r) \propto r^2$, and $g(r)$ are constant. Therefore we have the conditions $f(0) = 0$, $a(0) = 0$, and $g'(0) = 0$. To obtain the energy of a flux tube, we numerically solve the ODE system for the neutron and proton condensate and magnetic potential profile functions then calculate the free energy of the system by inserting the results into Eq. (17) and integrating.

The system has five independent parameters: a_{pp} , a_{nn}/a_{pp} , a_{pn}/a_{pp} , σ , and $\langle \phi_n \rangle / \langle \phi_p \rangle$. In neutral nuclear matter, the density of protons (neutrons) is proportional to $\langle \phi_p \rangle^2 (\langle \phi_n \rangle^2)$, and the proton density is approximately 5% of the total baryon

number density,¹⁸ so we set $\langle \phi_p \rangle^2 / \langle \phi_n \rangle^2 = .05$ in most of our analysis. Following Refs. 5 and 6, we set $a_{nn} = a_{pp}$, and use Eq. (15) to exchange the parameter a_{pp} for κ , which is the conventional parameter used in condensed-matter studies of superconductivity. Our reduced set of parameters is therefore κ , the proton-neutron gradient coupling σ , and the proton-neutron density coupling $\beta \equiv a_{pn}/a_{pp}$. We also study some effects of varying $\langle \phi_p \rangle^2 / \langle \phi_n \rangle^2$.

IV. NUMERICAL RESULTS

A. Flux tube solutions

For given values of $\langle \phi_p \rangle^2 / \langle \phi_n \rangle^2$, κ , the proton-neutron gradient coupling σ , and the proton-neutron amplitude coupling $\beta \equiv a_{pn}/a_{pp}$, we numerically solved the equations of motion (16) giving the field profiles for flux tubes with various numbers n of flux quanta. We obtained the solutions using a finite-element relaxation method, which is much less sensitive to initial conditions than the traditional ‘‘shooting’’ method, and better suited to repeatedly solving the equations for different sets of parameters. Next, we insert the solution for each profile into our expression for the free energy (17) and numerically integrate it to obtain a value for E_n .

To estimate the numerical errors in our results, we varied the convergence criterion in the finite-element relaxation calculation, the spacing of the radial grid of points, and the radius out to which the grid extended. We found that the resultant variation in E_n/n was of order 10^{-6} , so numerical errors are invisible on the scale of the plots shown in Fig. 3.

Having obtained E_n we then plot the series B_n to determine whether the system is type I or type II for the chosen point in parameter space. In this way we find the points in parameter space where the system changes from a type-I state to a type-II state. Taking various slices through the parameter space, we can generate phase diagrams that show the boundary curves between the various phases.

Figures 1 and 2 each show a profile for a flux tube with a single flux quantum $n=1$ on the top, and a profile for a flux tube with 100 flux quanta on the bottom. Figure 1 shows the effect of nonzero density coupling β and Fig. 2 shows the effect of nonzero gradient coupling σ . We have plotted the normalized difference in density of the pair fields from their condensate values,

$$\begin{aligned} \delta\rho_p(\tilde{r}) &\equiv \frac{\phi_p^2(\tilde{r}) - \langle \phi_p \rangle^2}{\langle \phi_p \rangle^2} = f^2(\tilde{r}) - 1, \\ \delta\rho_n(\tilde{r}) &\equiv \frac{\phi_n^2(\tilde{r}) - \langle \phi_n \rangle^2}{\langle \phi_n \rangle^2} = g^2(\tilde{r}) - 1. \quad (18) \end{aligned}$$

1. No coupling to neutrons

We do not show a plot of the flux tube profile for a simple superconductor since this is well known: in a core region whose area rises as the number of flux quanta n , the proton condensate is suppressed; in a wall region the condensate returns to its vacuum value. At the Bogomolnyi point,²⁵ $\kappa = 1/\sqrt{2}$, the energy per flux quantum is independent of n ,²⁶

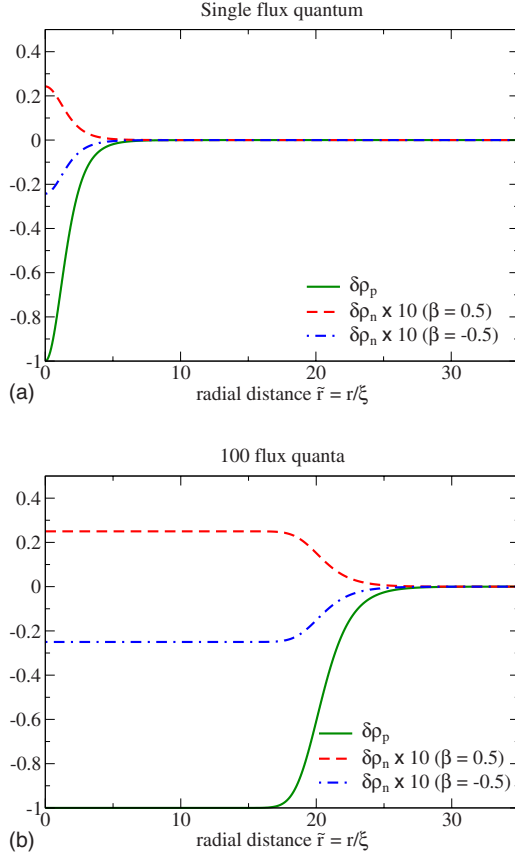


FIG. 1. (Color online) Profile of flux tube with $n=1$ units of flux (top) and $n=100$ units of flux (bottom) showing the effect of density coupling β between neutron and proton condensates. The plot shows the deviation $\delta\rho$ of the condensates from their vacuum values (18). With no coupling between the condensates ($\beta=\sigma=0$), the neutrons are undisturbed ($\delta\rho_n=0$). With a nonzero density coupling β , the neutron condensate (broken lines) is significantly perturbed by the flux tube. Note that the neutron $\delta\rho_n$'s are multiplied by 10 (not by 100 as in Fig. 2) to make them visible. The other parameters are $\kappa=3.0$, $\sigma=0.0$, and $\langle\phi_p\rangle^2/\langle\phi_n\rangle^2=0.05$.

but on either side of this value there are area and perimeter contributions to the energy,²⁷ so for κ close to $1/\sqrt{2}$ we expect the energy of a flux tube in a simple superconductor to have the following dependence on n :

$$E_n^{(\text{sc})}(\kappa) = nE_{\text{Bog}} + \delta\kappa M(n - c_{1/2}\sqrt{n} + c_1 + \dots). \quad (19)$$

This is an expansion around $n=\infty$, but our numerical results will show that it works down to $n=1$. We define $\delta\kappa \equiv \kappa - 1/\sqrt{2}$. E_{Bog} is the energy per unit flux at $\delta\kappa=0$. By convention we take the parameter M , which has dimensions of energy, to be positive. The value of $c_{1/2}$ is then positive, ensuring that for $\delta\kappa>0$, $n=\infty$ is disfavored (type II), and for $\delta\kappa<0$, $n=\infty$ is favored (type I). We will see this behavior in our numerical results (Sec. I and upper left plot of Fig. 3).

2. Density coupling to neutrons

For positive β , which corresponds to positive a_{np} , Eqs. (2) and (4) indicate that there is a repulsion between the neutron and proton condensates, so in the center of the flux tube,

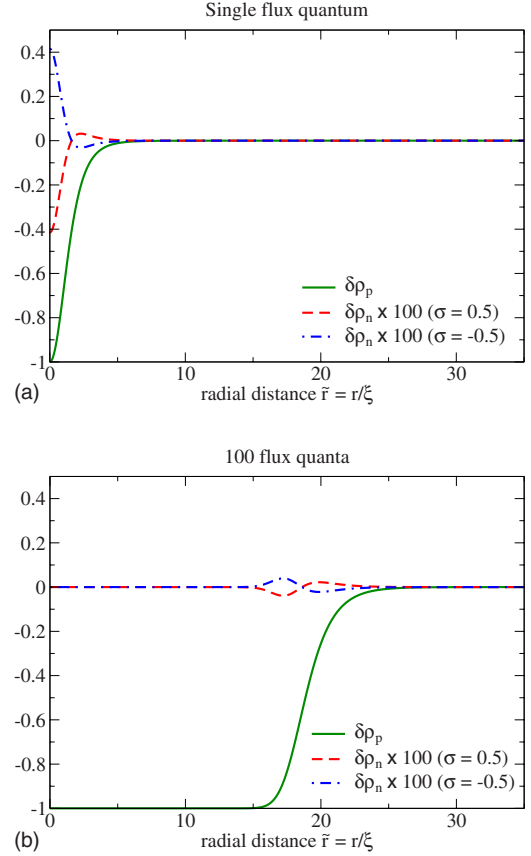


FIG. 2. (Color online) Profile of flux tube with $n=1$ units of flux (top) and $n=100$ units of flux (bottom) showing the effect of gradient coupling σ between neutrons and protons. The plot shows the deviation $\delta\rho$ of the condensates from their vacuum values (18). With no coupling between the condensates ($\beta=\sigma=0$), the neutrons are undisturbed ($\delta\rho_n=0$). With a nonzero gradient coupling σ , the neutron condensate (broken lines) is slightly perturbed by the flux tube. Note that the neutron $\delta\rho_n$'s are multiplied by 100 to make them visible. The other parameters are $\kappa=3.0$, $\beta=0.0$, and $\langle\phi_p\rangle^2/\langle\phi_n\rangle^2=0.05$.

where the proton condensate is suppressed, the neutron condensate will be enhanced. That is exactly what we see in Fig. 1, where the dashed curve, showing the perturbation to the neutron density ρ_n , rises inside the flux tube. For negative β there is attraction between the two condensates, and the neutron condensate is suppressed inside the flux tube (dash-dotted line). We therefore expect that the leading correction due to the interaction will be proportional to the core area, i.e., proportional to n . The energy of an n -quantum flux tube is then

$$E_n(\kappa, \beta) \approx E_n^{(\text{sc})}(\kappa) + M_\beta(-n + b_{1/2}\sqrt{n} + b_1 + \dots), \quad (20)$$

where $E_n^{(\text{sc})}(\kappa)$ is the energy for an n -quantum flux tube in a pure superconductor, with no coupling to a superfluid (19). The leading correction is $-M_\beta n$, which should be negative and quadratic in β for small β (see Sec. IV), so the interaction energy parameter M_β is positive and proportional to β^2 . The subleading term proportional to \sqrt{n} arises from the energy cost of the gradient in ρ_n at the edge of the flux tube,

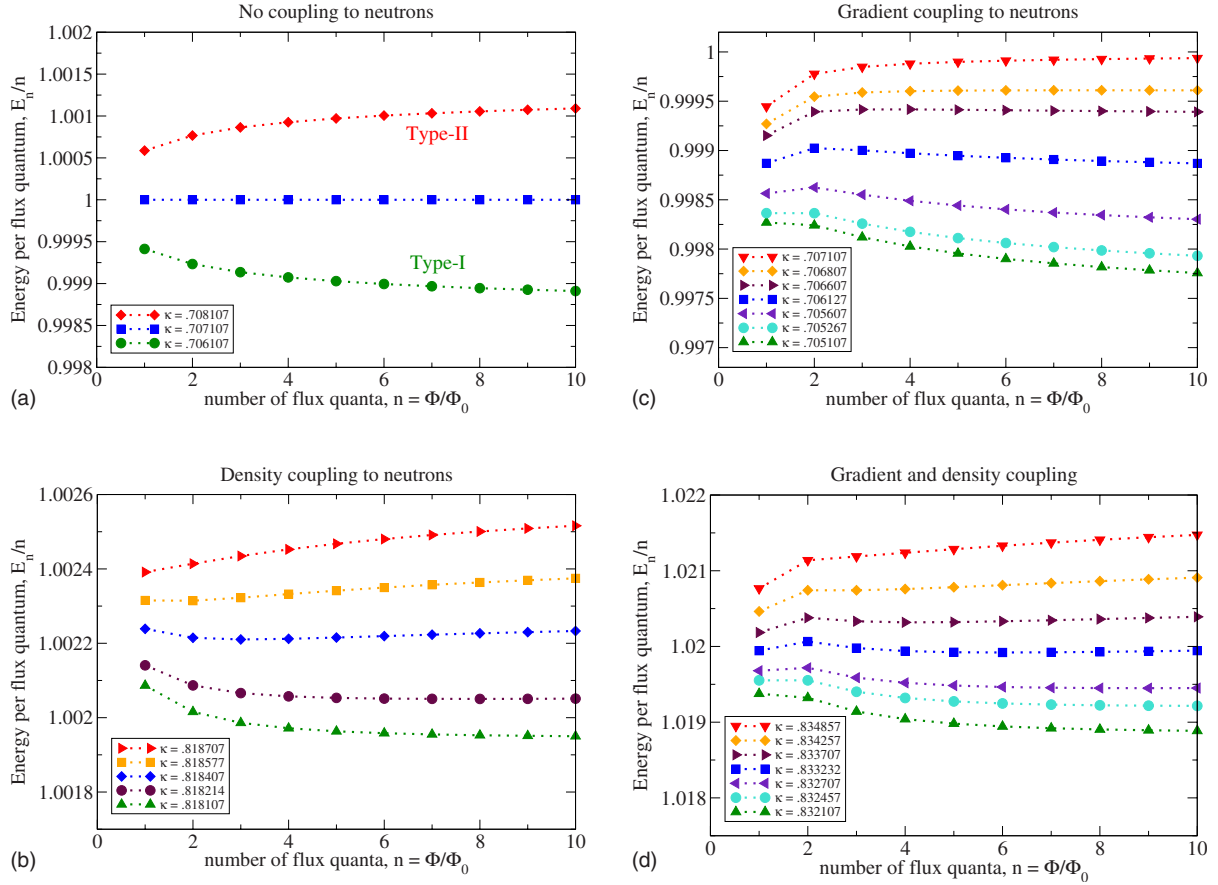


FIG. 3. (Color online) The energy per flux quantum E_n/n , in units of E_{Bog} [see Eq. (19)], as a function of the number n of units of flux in the flux tube. Top left, simple proton superconductor with neutrons completely decoupled ($\beta = \sigma = 0$); bottom left, density coupling between condensates ($\beta = 0.5, \sigma = 0$); top right, gradient coupling between condensates ($\beta = 0, \sigma = 0.5$); and bottom right, both couplings ($\beta = \sigma = 0.5$).

where it must return to its vacuum value, so we expect this term to be positive: $b_{1/2} > 0$. We do not have an *a priori* expectation for the sign of the subsubleading term b_1 .

3. Gradient coupling to neutrons

For positive σ , we expect from Eqs. (2) and (3) that the positive gradient in ρ_p at the wall of the flux tube will induce a positive gradient in ρ_n in the same range of radii, which lowers the energy of the system. This is exactly what we see in Fig. 2, where the dashed curve showing the perturbation to ρ_n has a positive slope in the range of radii where the solid curve (ρ_p) has the largest positive slope. On either side of that region, it has a negative slope, as it returns to its unperturbed value. For negative σ the effect is reversed: the dash-dotted curve shows ρ_n having a negative slope where ρ_p has the largest positive slope.

We therefore expect that in the presence of a gradient coupling, the correction to the energy of a flux tube has a dominant core-perimeter term proportional to \sqrt{n} ,

$$E_n(\kappa, \sigma) \approx E_n^{(\text{sc})}(\kappa) + M_\sigma (-s_{1/2} \sqrt{n} + s_1 + \dots). \quad (21)$$

The energy correction is negative and quadratic in σ for small σ (see Sec. IV), so the interaction energy parameter M_σ is proportional to σ^2 ; choosing it to be positive by con-

vention requires $s_{1/2}$ to be positive. We do not have an *a priori* expectation for the sign of s_1 .

4. Symmetry under change of sign of couplings

It is clear from Figs. 1 and 2 that for couplings β and σ of the order of 0.5, the modification of the field configuration due to the interaction between the condensates is extremely small, so it is reasonable to treat its effects perturbatively. (At the end of Sec. IV C will discuss the limit of small neutron condensate, where the perturbative approach becomes questionable.) When we evaluate the perturbative correction to the energy of the flux tube, there is no linear term in β and σ . Such a term would arise from evaluating the β and σ terms from the Hamiltonian in the unperturbed field configuration. However, in that configuration the neutron condensate sits at its vacuum value, so both terms evaluate to zero [$g = 1$ and $g' = 0$ in Eq. (17)].

We therefore expect the change in the energy of the flux tube to be quadratic in the couplings β and σ . First, this correction must be negative. This is a well-known result from perturbation theory: the second-order correction arises from the change in the configuration in response to the perturbation, which only occurs because it is driven by a resultant lowering of the energy. Second, the change in the energy

will in general contain β^2 , σ^2 , and $\beta\sigma$ terms. This means it will be even in β when $\sigma=0$ and even in σ when $\beta=0$, so we expect $M_\beta \propto \beta^2$ and $M_\sigma \propto \sigma^2$ in Eqs. (20) and (21).

However, if both β and σ are nonzero, then the $\beta\sigma$ terms spoil the symmetry of the energy under negation of the couplings. This is clear from Figs. 1 and 2. For example, suppose that as well as nonzero β we have a very small nonzero σ . Now consider sending $\beta \rightarrow -\beta$. From Fig. 1 we see that this changes the sign of the slope of ρ_n in the wall region where ρ_p has positive slope. If σ is nonzero, then these two configurations will have different energies since the gradient of ρ_n is then coupled to the gradient of ρ_p .

B. Energetic stability of flux tubes

In Fig. 3, the energy per flux unit (E_n/n) is plotted against n for various values of the Ginzburg-Landau parameters, namely, κ , the density coupling β , and the gradient coupling σ . We fixed $\langle \phi_p \rangle^2 / \langle \phi_n \rangle^2 = 0.05$ (Sec. III A)

1. No coupling to neutrons

The upper left plot of Fig. 3 shows $E_n^{(sc)}(\kappa)/n$, the energy per flux quantum when there are no interactions between the neutron and proton pairs. We see that the only possible phases are the standard type I and type II, with a transition at the Bogomolnyi point, $\kappa=1/\sqrt{2}$, where the favored value of n jumps from 1 to infinity. The lower line (κ just below $1/\sqrt{2}$) corresponds to type I, where the lowest energy and/or flux is at $n=\infty$, so flux tubes attract. The upper line (κ just above $1/\sqrt{2}$) corresponds to type II, where the lowest energy and/or flux is at $n=1$, so flux tubes always repel each other. The middle line corresponds to the transition point ($\kappa=1/\sqrt{2}$), where there is no interaction between flux tubes.²⁵ Our numerical results are consistent with the expected form (19): when $\delta\kappa > 0$ the asymptotic value of E_n/n is increased, and E_n/n rises monotonically toward that asymptotic value, and conversely when $\delta\kappa < 0$ the asymptotic value of E_n/n is decreased, and E_n/n falls monotonically toward that asymptotic value. It is clear that $c_{1/2}$ in Eq. (19) must be positive to obtain this behavior at large n . From fits to our numerical calculations we find that c_1 is always positive, so it ‘‘fights against’’ the leading $c_{1/2}/\sqrt{n}$ term, but for all $n \geq 1$ it is overwhelmed. In fact, we find that Eq. (19) gives an excellent fit to our results down to $n=1$, without any higher order terms. In the remaining panels of Fig. 3, we explore the effect of density and gradient couplings between the proton superconductor and the neutron superfluid.

2. Density coupling to neutrons

The lower left panel of Fig. 3 shows the effect of a density coupling between the condensates. From Eqs. (19) and (20) we expect

$$E_n/n = E_{\text{Bog}} + (M\delta\kappa - M_\beta) + \frac{M_\beta b_{1/2} - \delta\kappa M c_{1/2}}{\sqrt{n}} + \frac{M_\beta b_1 + \delta\kappa M c_1}{n} + \dots \quad (22)$$

The first point to notice is that the density coupling shifts the

critical κ to a larger value. The transition between type I and type II occurs when the asymptotic behavior at large n changes from rising to falling, i.e., when the coefficient of the $1/\sqrt{n}$ term changes sign. This occurs for some positive value of $\delta\kappa$,

$$\delta\kappa_{\text{crit}}(\beta) = \frac{M_\beta b_{1/2}}{M c_{1/2}} \propto \beta^2, \quad (23)$$

which rises as β^2 because M , M_β , $b_{1/2}$, and $c_{1/2}$ are all positive, and $M_\beta \propto \beta^2$ when $\sigma=0$ (Sec. IV). Thus in the lower left panel of Fig. 3 we had to increase κ from around 0.707 to around 0.818 in order to find the transition.

The other important point is the presence of a minimum in E_n/n when κ is just above the new type-I/type-II boundary, indicating that the favored value of n may be neither 1 (standard type II) nor infinity (type I) but some intermediate value. This is consistent with Eq. (22), as long as we assume that the coefficient b_1 from Eq. (20) is either positive or negative and of sufficiently small magnitude so that the $1/n$ term in Eq. (22) has a positive coefficient (recall that M_β , M , and c_1 are all positive, and $\delta\kappa$ is also positive in this region). The minimum will then arise from competition between the positive $1/n$ term, which dominates at smaller n , giving a negative slope, and the $1/\sqrt{n}$ term which has a negative coefficient (because $\delta\kappa$ is just above the new critical value) and dominates at larger n giving a positive slope. However, as $\delta\kappa$ is reduced the negative coefficient of $1/\sqrt{n}$ becomes smaller and smaller, and the minimum moves out to arbitrarily large n , so the energetically favored value of n does not jump suddenly from 1 to ∞ as in the standard case but increases in steps from 1 to infinity as we lower κ through a range of values down to the new critical value. This creates an infinite number of ‘‘type-II(n)’’ phases, each with a different flux in the favored flux tube, and when that flux becomes infinite the superconductor becomes type I. This behavior is seen in our numerical results (Fig. 4).

3. Gradient coupling to neutrons

The upper right panel of Fig. 3 shows the effect of a gradient interaction with the superfluid. From Eqs. (19) and (20) we expect

$$E_n/n = E_{\text{Bog}} + M\delta\kappa + \frac{-M_\sigma s_{1/2} - \delta\kappa M c_{1/2}}{\sqrt{n}} + \frac{-M_\sigma s_1 + \delta\kappa M c_1}{n} + \dots \quad (24)$$

Here we see that the gradient coupling shifts the critical κ to a smaller value. The transition between type I and type II occurs when the coefficient of the $1/\sqrt{n}$ term changes sign, which in this case happens for small negative $\delta\kappa$,

$$\delta\kappa_{\text{crit}}(\sigma) = -\frac{M_\sigma s_{1/2}}{M c_{1/2}} \propto -\sigma^2, \quad (25)$$

which is proportional to $-\sigma^2$ because M , M_σ , $s_{1/2}$, and $c_{1/2}$ are all positive, and $M_\sigma \propto \sigma^2$ when $\beta=0$ (Sec. IV A 4).

The other important feature of this plot is the presence of a maximum in E_n/n when κ is close to the type-I/type-II

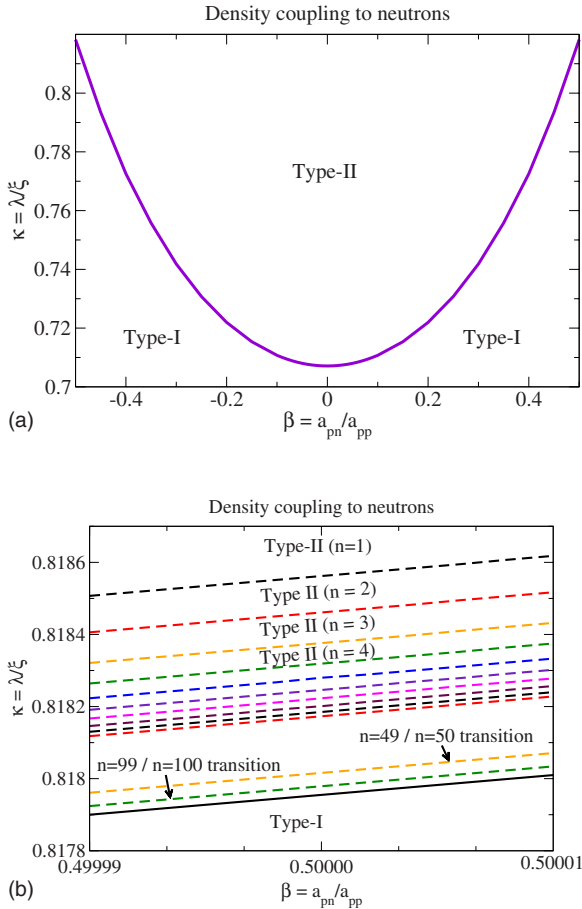


FIG. 4. (Color online) Effect on the superconductor of density coupling β to a superfluid, displayed as a phase diagram in the $\kappa - \beta$ plane, with no gradient coupling ($\sigma=0$) and $\langle \phi_p \rangle^2 / \langle \phi_n \rangle^2 = 0.05$. The top panel shows how nonzero β causes an increase in κ_{critical} . In the bottom panel we magnify the transition region near $\beta=0.5$, illustrating that on the type-II side there is a sequence of type-II(n) bands in which the number of flux quanta in the favored flux tube rises, reaching infinity when the superconductor becomes type I.

boundary. This is consistent with Eq. (24), as long as we assume that the coefficient s_1 from Eq. (21) is either positive or negative and of sufficiently small magnitude so that the $1/n$ term in Eq. (24) has a negative coefficient. The maximum will then arise from competition between the negative $1/n$ term, which dominates at smaller n , giving a positive slope, and the $1/\sqrt{n}$ term, which dominates at larger n giving a negative slope.

The presence of this maximum allows for the possibility of metastable flux configurations. If we scan down in κ , we start in a type-II region where E_n/n has its minimum at $n=1$ and rises monotonically with n . However, at some point a metastable minimum at $n=\infty$ appears, which drops to become degenerate with the minimum at $n=1$. At this point there is a first-order transition: at the critical field, $n=1$ flux tubes would coexist with macroscopic normal regions (i.e., flux tubes with $n=\infty$) but not with flux tubes of intermediate size. Reducing κ further, the $n=1$ flux tube becomes energetically metastable and finally unstable.

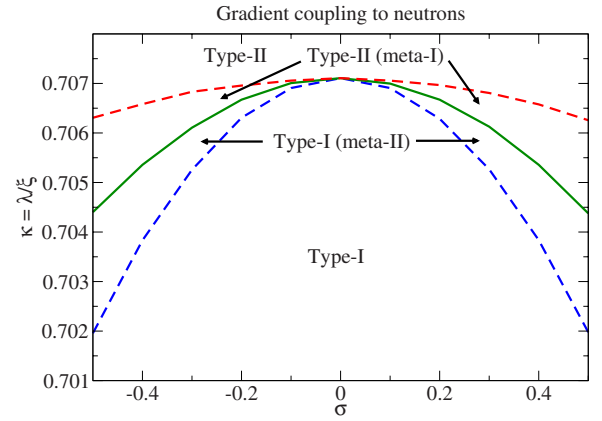


FIG. 5. (Color online) Effect on the superconductor of gradient coupling σ to a superfluid, displayed as a phase diagram in the $\kappa - \sigma$ plane, with no density coupling ($\beta=0$) and $\langle \phi_p \rangle^2 / \langle \phi_n \rangle^2 = 0.05$. The gradient coupling causes a decrease in κ_{critical} and creates metastable states on either side of the transition with spinodal lines as shown.

4. Density and gradient coupling to neutrons

The lower right panel of Fig. 3 shows the effect of a combination of gradient and density interactions. As κ is decreased, a metastable energy minimum emerges at finite n ; it drops and becomes a new global minimum at $n=n^*$, yielding a sharp transition from $n=1$ type II to $n=n^*$ type II. As κ is reduced further the favored number of flux quanta in a flux tube rises in integer steps from n^* to infinity, at which point the superconductor becomes type I.

C. Phase diagrams

Figures 4–7 illustrate the additional structure in the phase diagram of the superconductor induced by the couplings to a superfluid. Each diagram is a two-dimensional slice through the parameter space.

Figure 4 shows the consequences of a density coupling β between the superfluid and superconductor. We see that the density coupling, irrespective of its sign, favors type-I superconductivity, pushing the critical κ for the type-I/type-II transition up to higher values, forming a parabolic phase boundary in the $\beta - \kappa$ plane, as expected from Eq. (23). This can be thought of as arising from the fact that nonzero β lowers the energy per flux of the core of large flux tubes [see Eq. (22)], which favors type-I superconductivity.

In the bottom panel we zoom in on the transition line near $\beta=0.5$ to show the substructure in the phase-transition region that is invisibly small in the top panel. As one would expect from our discussion of Fig. 3 (lower left panel), on the type-II side of the transition there is a series of bands distinguished by the number of flux quanta n in the energetically favored flux tube. Type-II ($n=1$) is the standard type-II superconductor. With decreasing κ we find transitions to type-II ($n=2$), type-II ($n=3$), and on up to $n=\infty$, which is a type-I superconductor.

In Fig. 5 we show the consequences of a gradient coupling σ between the superfluid and superconductor. We see that the gradient coupling, irrespective of its sign, favors

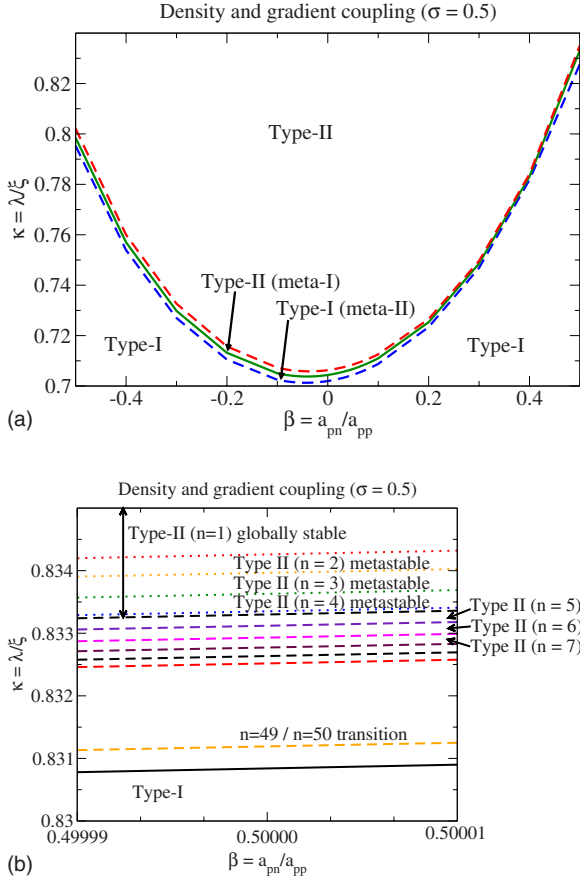


FIG. 6. (Color online) Phase diagram for combined density and gradient interactions: the κ - β plane for $\sigma=0.5$ and $\langle\phi_p\rangle^2/\langle\phi_n\rangle^2=0.05$. The type-I/type-II boundary is no longer symmetric under $\beta\rightarrow-\beta$. In the bottom panel we magnify the transition region near $\beta=0.5$, illustrating that on the type-II side as κ decreases the number of flux quanta in the favored flux tube jumps from 1 to a finite value (in this case $n=5$) and then there is a sequence of bands in which n rises to infinity, at which point the superconductor becomes type I.

type-II superconductivity, pushing the critical κ for the type-I/type-II transition down to lower values, forming an inverted parabolic phase boundary in the σ - κ plane, as expected from Eq. (25). It also makes the phase-transition first order, with spinodal lines where the unfavored phase becomes metastable. Both these effects arise from the lowering of the energy of the wall of the vortex, as explained in Sec. IV B 3.

In Fig. 6 we show phase diagrams for the combination of both density and gradient couplings, fixing $\sigma=0.5$ and varying β . As discussed in Sec. IV A 4, we expect that when $\sigma\neq 0$ the $\beta\rightarrow-\beta$ symmetry is now broken. In the bottom panel we magnify the transition region near $\beta=0.5$, illustrating that on the type-II side as κ decreases the number of flux quanta in the favored flux tube jumps from 1 to a finite value $n=5$, and then there is a sequence of bands in which n rises, reaching infinity when the superconductor becomes type I. This is the expected behavior, based on our discussion in Sec. IV B 4.

Finally, in Fig. 7, we anticipate one direction in which this work could be extended by exploring the consequences of

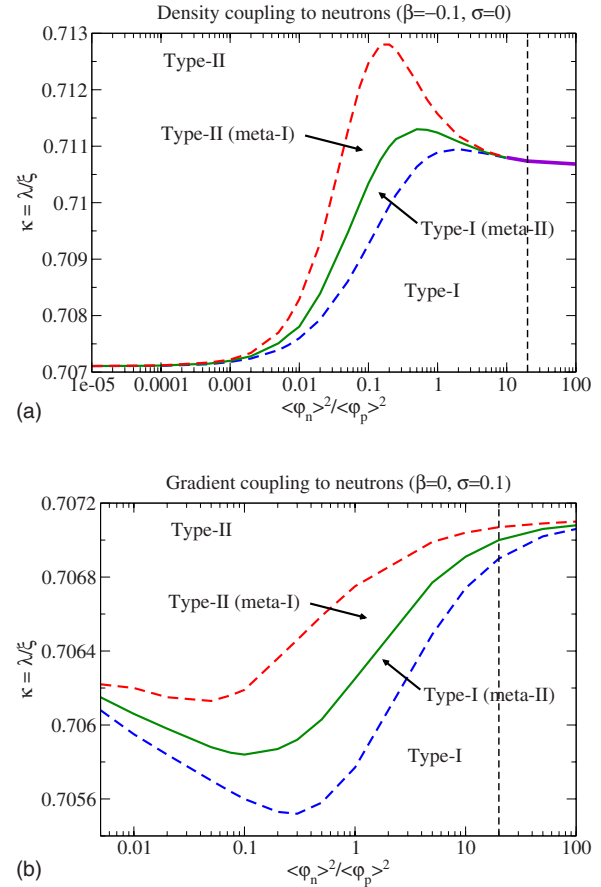


FIG. 7. (Color online) Phase diagrams in the κ vs $\langle\phi_n\rangle^2/\langle\phi_p\rangle^2$ plane. The vertical dashed lines show $\langle\phi_n\rangle^2/\langle\phi_p\rangle^2=20$, the value used for other figures in this paper. The top panel is for density coupling $\beta=-0.1$, but no gradient coupling ($\sigma=0$). The bottom panel is for gradient coupling $\sigma=0.1$ but no density coupling ($\beta=0$). In both cases, we see that the type-I/type-II transition converges to $\kappa=1/\sqrt{2}$ as the neutron condensate disappears. For the case of a density coupling, as the neutron condensate decreases, the type-I/type-II boundary changes at $\langle\phi_n\rangle^2/\langle\phi_p\rangle^2\sim 10$ from a narrow region of type-II(n) phase bands (thick line) to wider metastable regions.

varying the ratio of the superfluid density to the superconductor density, which up to now was fixed to $\langle\phi_n\rangle^2/\langle\phi_p\rangle^2=20$, an appropriate value for neutral beta-equilibrated nuclear matter, of the type we expect to find inside neutron stars. Figure 7 shows phase diagrams in the plane of κ and $\langle\phi_n\rangle^2/\langle\phi_p\rangle^2$ for a system with a density coupling (top panel) and with a gradient coupling (bottom panel).

For the case of a density coupling we use a negative value of the coupling, because this corresponds to an attractive interaction, which gives smooth behavior in the limit where the neutron condensate disappears, $\langle\phi_n\rangle^2/\langle\phi_p\rangle^2\rightarrow 0$. As is clear from the plot, the type-I/type-II transition then converges to the standard value for a single-component superconductor, $\kappa=1/\sqrt{2}$. For a repulsive interaction, the $\langle\phi_n\rangle^2/\langle\phi_p\rangle^2\rightarrow 0$ limit is singular: we discuss this in more detail below. It is interesting to note that the effects of the density coupling change dramatically with the relative densities of the neutrons and protons. At $\langle\phi_n\rangle^2/\langle\phi_p\rangle^2\geq 10$ the

density coupling produces a thin region of multiflux-quantum type-II(n) phases, as was illustrated in Fig. 4. However, for lower values, it has a similar effect to a gradient coupling, inducing metastable regions on either side of the type-I/type-II boundary. This should be understandable in terms of the dependence of the coefficients $b_{1/2}$ and b_1 [Eq. (20)] on $\langle\phi_n\rangle^2/\langle\phi_p\rangle^2$. In Sec. II we argued that if b_1 is large enough then the E_n/n curve has a minimum at finite n , yielding a type-II(n) phase. We conjecture that as $\langle\phi_n\rangle^2/\langle\phi_p\rangle^2$ gets smaller, b_1 becomes sufficiently negative that this is no longer the case, and instead there is a maximum, leading to metastability of the $n=0$ and $n=\infty$ states in spinodal regions around the type-I/type-II boundary. This is a topic for future investigation.

For the case of a gradient coupling (right panel of Fig. 7), the effects of varying $\langle\phi_n\rangle^2/\langle\phi_p\rangle^2$ are less dramatic. It is interesting that, as for a density coupling, the variation is nonmonotonic. Again, we conjecture that this could be understood in terms of variation of the coefficients $s_{1/2}$ and s_1 [Eq. (21)] with $\langle\phi_n\rangle^2/\langle\phi_p\rangle^2$. As the superfluid density drops to zero, its effects become negligible, and the critical value of κ converges toward $1/\sqrt{2}$ as one would expect.

Finally, we discuss the singularity of the $\langle\phi_n\rangle^2/\langle\phi_p\rangle^2 \rightarrow 0$ limit for a positive β , i.e., a repulsive density coupling between the neutron and proton condensates. From Eq. (6) we see that the expectation value of the neutron condensate is $\langle\phi_n\rangle + \frac{1}{2}\beta(\langle\phi_p\rangle - \phi_p)$, so far from the flux tube, where ϕ_p is $\langle\phi_p\rangle$, it is $\langle\phi_n\rangle$. However, in the core of the condensate it is larger (there is less proton condensate to repel it). In fact, even if the parameter $\langle\phi_n\rangle^2$ were zero or slightly negative, there would be a positive neutron condensate in the core of the flux tube. This shows that for positive β the neutrons do *not* decouple and become irrelevant in the limit $\langle\phi_n\rangle \rightarrow 0$. We note two consequences of this. First, for small $\langle\phi_n\rangle$ the $\beta \rightarrow -\beta$ symmetry discussed in Sec. IV is no longer present because the effect of the flux tube on the neutron condensate is no longer a small perturbation. Second, in a system where $\langle\phi_n\rangle^2$ is small and negative (i.e., the neutrons just barely fail to condense in the presence of the proton condensate) flux tubes could have superfluid cores, which is another topic that we leave for future investigation.

V. CONCLUSION

We conclude that coupling a superconductor to a coexisting superfluid causes significant modification of the energetics of the flux tubes. On the basis of calculations restricted to the cylindrical geometry of n -quantum flux tubes, we conclude that a coupling between the densities of the condensates shifts the type-I/type-II boundary to larger κ and, if the superfluid density is high enough, appears to create an infinite number of new type-II(n) phases whose most stable flux tubes contain multiples of the basic flux quantum. A gradient coupling between the condensates leads to metastable re-

gions surrounding the transition between type-I and type-II superconductivity.

As discussed in Sec II, our calculation corresponds to comparing the energy at zero and infinite separation of flux tubes with varying numbers of flux quanta. This leaves open the possibility that there might be additional minima at finite separation. It is therefore possible that in parts of the phase diagram there might be a different phase from the ones we identify, namely, an alternative type of type-II superconductor in which the spacing between flux tubes is fixed by the microscopic physics rather than by the strength of the applied field. To resolve this question will require calculation of the free energy of a pair of flux tubes at arbitrary separation. Such calculations have been performed for large separation²⁸⁻³⁰ and by perturbing about the Bogomolnyi point¹⁷ and by numerical computation.³¹ In particular, the numerical methods that have been used recently to follow the interaction and annihilation or vortex-antivortex pairs³² would be readily applicable to the simpler time-independent calculation of the interaction potential of flux tubes. Another natural generalization of our calculation would be to allow for non- s -wave pairing, such as the 3P_2 pairing that is believed to occur in the neutron superfluid in the core of a neutron star.

Our results add another example to the class of two-component Ginzburg-Landau models with nonstandard superconducting behavior. Previous work in this area includes the $SO(5)$ model of high-temperature superconductivity, which has flux tubes described by a two-component GL model, where each component carries a different $U(1)$ charge, and only one of them condenses in the vacuum.¹³ Another example is the case of a two-component GL model where both components have electric charge, very different mass, and nearly the same Fermi energy. This system was found to have nonmonotonic $E(n)/n$ and intermediate minima in the interaction potential.⁷

The exotic phenomena that we predict are localized to the region around the type-I/type-II transition, so they may not turn out to be relevant for the inner core of a neutron star, which is believed to be well inside the type-II regime.¹ However, given the extremely impressive recent progress in creating exotic systems such as multicomponent superfluids of trapped cold atoms, it seems quite conceivable that a material that is both a superconductor and a superfluid might be created in the laboratory and could be studied under controlled conditions. Our results would be directly relevant to such a material.

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