Extended scaling for the high-dimension and square-lattice Ising ferromagnets

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(Received 26 February 2008; revised manuscript received 2 June 2008; published 28 July 2008)

If scaling variables and scaling expressions are chosen judiciously, "critical" scaling analyses do not need to be restricted to a narrow range of temperatures near the critical point. The standard scaling variable $(T-T_c)/T_c$ is inadapted to wide ranges of temperature because it diverges at high temperatures. With the variable $\tau=(T-T_c)/T$, in the high-dimension (mean-field) limit, the reduced susceptibility and the second-moment correlation length of the Ising ferromagnet depend on temperature as $\chi(T)=\tau^{-1}$ and $\xi(T)=T^{-1/2}\tau^{-1/2}$ exactly over the entire temperature range above the critical temperature T_c. For the canonical two-dimensional square lattice near-neighbor Ising ferromagnet, it is shown that compact "extended scaling" expressions analogous to the high-dimensional limit form but with the appropriate exponents give accurate approximations to the true temperature dependencies, again over the entire temperature range from T_c to infinity. Within this approach, for near-neighbor interaction systems, there is no crossover temperature above which mean-field-like behavior sets in.

DOI: 10.1103/PhysRevB.78.024435 PACS number(s): 75.50.Lk, 05.50.+q, 75.40.Mg

I. INTRODUCTION

The remarkable critical behavior at second-order phase transitions has been intensively studied for many years. In the limit where the temperature T tends to the critical temperature T_c, observables $Q(T)$ diverge as

$$Q(T) \sim |T-T_c|^{-q}, \quad (1)$$

where q is the critical exponent. The common approach[1] is to use $t=(T-T_c)/T_c$ as the scaling variable. There are both analytic and nonanalytic corrections to the strict critical limiting form as soon as T is not infinitesimally close to T_c, leading to the Wegner[2] expansion, which can be written with t as the scaling variable as

$$Q(t) = C_q t^{-q}[1 + a_1 t^{\theta} + \ldots + b_1 t + b_2 t^2 + \ldots], \quad (2)$$

where C_q is the critical amplitude, θ is a universal nonanalytic correction exponent, a_n are the nonuniversal correction amplitudes, and b_n are the nonuniversal analytic correction amplitudes.

It is widely considered that there is only a narrow temperature range, the critical region, in the immediate vicinity of T_c, where these equations are valid and that there is a crossover toward mean-field-like behavior outside this region. Thus the standard protocol for estimating critical exponents from experimental or numerical data is to carry out analyses using Eq. (1) with the scaling variable t, together with finite size scaling (FSS) rules derived using these equations, over as narrow a range of temperature as possible around the critical point, introducing phenomenological nonanalytic corrections to scaling if the data require them. In fact it is legitimate *a priori* to choose any temperature-dependent normalization for the scaling variable for the analysis close to T_c as long as the normalization factor does not have critical behavior; the choice of scaling variable is however crucial if data are to be analyzed over a wide temperature range. Our aim is to demonstrate that if scaling variables and scaling expressions are chosen judiciously, an approximate but good quality "critical" analysis can be extended to the entire range of temperature above the critical point.

It is important to be explicit as to the definitions of the observables; to be specific we will use throughout a terminology corresponding to near-neighbor interaction spin-1/2 Ising ferromagnets with finite-ordering temperatures on hypercubic lattices. The arguments can be generalized to other systems, *mutatis mutandis*. With spins $S(x)$ at sites x and $\beta=1/T$, we will follow the standard convention[3–5] and discuss the reduced susceptibility

$$\chi(\beta) = \chi_{\text{th}}(\beta)/\beta = \sum_x \langle S(0)S(x)\rangle, \quad (3)$$

where $\chi_{\text{th}}(\beta)$ is the thermodynamic susceptibility $[\partial(M(\beta))/\partial(H)]$, the direct output of an experimental measurement of the magnetization $M(\beta)$ in a limitingly small magnetic field H. [Confusingly $\chi(\beta)$ is often referred to as "the susceptibility" in numerical studies].

The other observable that we will discuss is the second-moment correlation length $\xi_{sm}(\beta)$, which is not identical to the "true" correlation length[3] $\xi_{\text{true}}(\beta)$. The second moment of the correlation function is defined as

$$\mu_2(\beta) = \sum_x x^2\langle S(0)S(x)\rangle \quad (4)$$

and the second-moment correlation length is $\xi_{sm}(\beta)=[\mu_2(\beta)/z\chi(\beta)]^{1/2}$, where z is the number of neighbors. We will refer to $\xi_{sm}(\beta)$ as $\xi(\beta)$. This correlation length is the infinite size limit of the size-dependent correlation length calculated in numerical simulations following Ref. 6.

Long high-temperature series expansions (HTSE) have been calculated for both $\chi(\beta)$ and $\mu_2(\beta)$.[5] These are Taylor expansions in β where each individual term is strictly exact. Whatever the dimension the series begin by $\chi(\beta)=1+\ldots$ and $\mu_2(\beta)=z\beta+\ldots$, giving the strict high-temperature limits $\chi(\beta)\to 1$ and $\xi(\beta)\to\beta^{1/2}$. Based on this simple observation an "extended scaling" formulation was introduced,[7,8] in which the scaling variable is $\tau=(1-\beta/\beta_c)=(T-T_c)/T$;[9] to leading order the reduced susceptibility is written $\chi(\beta) \sim \tau^{-\gamma}$ and the second-moment correlation length is written as $\xi(\beta)/\beta^{1/2} \sim \tau^{-\nu}$. t is not a good variable in this context simply because it diverges in the high-temperature limit. The Wegner expansion for the reduced susceptibility was originally written with τ as the scaling variable,[2,5] so

$$\chi(\tau) = C_\chi \tau^{-\gamma}[1 + a_{\chi,1}\tau^{\theta} + \ldots + b_{\chi,1}\tau + b_{\chi,2}\tau^2 + \ldots] \quad (5)$$

with the same exponents as in Eq. (2) but different amplitudes a_n, b_n. For the correlation length we can write

$$\xi(\tau) = C_\xi(\beta/\beta_c)^{1/2}\tau^{-\nu}[1 + a_{\xi,1}\tau^{\theta} + \ldots + b_{\xi,1}\tau + b_{\xi,2}\tau^2 + \ldots]. \quad (6)$$

With this approach the leading critical term remains a good approximation over a wide temperature range with analytic correction terms, and nonanalytic terms when the factors a are nonzero; the analytic correction terms are however considerably weaker than when t is chosen as the scaling variable.

We will discuss two extreme canonical systems: the high-dimensional near-neighbor Ising ferromagnet on a hypercubic lattice and the two-dimensional (2D) Ising ferromagnet on a square lattice. For these two systems there are essentially no nonanalytic correction terms. In the high-dimensional (mean-field) case the critical behavior parameterized using the extended scaling protocol is strictly exact, with temperature-independent effective critical exponents over the entire temperature range from T_c to infinity. In the square lattice the protocol leads to compact high-quality approximations over the whole range, which can be made very accurate with minimal correction terms. In the temperature range close to T_c the expression for $\xi(T)$ can be linked to renormalization group theory (RGT) analytic corrections.

In the general case, when the extended scaling protocol is used, experimental or simulation data can be usefully analyzed over a wide temperature range in terms of the critical behavior rather than only in the very close neighborhood of T_c. It should be underlined that outside the strict critical- and high-temperature limits, the protocol is approximate; in particular nonanalytic terms are not included explicitly. The influence of these terms on the extended scaling analysis has been studied in the canonical three-dimensional (3D) systems.[8]

II. HIGH-DIMENSION ISING MODEL

It is instructive to first consider the high-dimension (or mean-field) limit of the HTSE for the spin-1/2 Ising near-neighbor ferromagnet on [hyper]cubic lattices. Let us denote by z the number of neighbors ($z=2d$ for hypercubes in dimension d). In order to discuss this limit it is convenient to scale the interaction strength by the factor z^{-1}. Then the HTSE for the reduced susceptibility[10] $\chi(\beta)$ and for the second moment of the correlation function $\mu_2(\beta)$ are given exactly by infinite series of terms in powers of $\tanh(\beta/z)$ with $\beta=1/T$:

$$\chi(\beta) = 1 + (z)\tanh(\beta/z) + (z^2 - z)\tanh(\beta/z)^2 + (z^3 - 2z^2 + z)\tanh(\beta/z)^3 + \ldots \quad (7)$$

and

$$\mu_2(\beta) = (z)\tanh(\beta/z) + (2z^2)\tanh(\beta/z)^2 + (3z^3 - 2z^2 + z)\tanh(\beta/z)^3 + \ldots . \quad (8)$$

The second-moment correlation length is defined by

$$\xi(\beta) = [\mu_2(\beta)/z\chi(\beta)]^{1/2}. \quad (9)$$

For high dimensions $z\to\infty$ the $[z \tanh(\beta/z)]^n$ contribution will dominate at each n. Hence

$$\chi(\beta) = 1 + z\tanh(\beta/z) + [z\tanh(\beta/z)]^2 + [z\tanh(\beta/z)]^3 + \cdots = [1 - z\tanh(\beta/z)]^{-1} = (1-\beta)^{-1} \quad (10)$$

so $\beta_c=1$.

Similarly in the high-dimension limit

$$\mu_2(\beta) = z\chi(\beta)\xi(\beta)^2 = z\tanh(\beta/z)[1 + 2z\tanh(\beta/z) + 3(z\tanh(\beta/z))^2 + 4(z\tanh(\beta/z))^3 + \ldots] = z\tanh(\beta/z)(1 - z\tanh(\beta/z))^{-2} = \beta(1-\beta)^{-2}, \quad (11)$$

i.e.,

$$\chi(\beta) = (1 - \beta/\beta_c)^{-1} = [(T - T_c)/T]^{-1} \quad (12)$$

and

$$\xi(\beta)/\beta^{1/2} = (1 - \beta/\beta_c)^{-1/2} = [(T - T_c)/T]^{-1/2} \quad (13)$$

exactly for all β less than $\beta_c=1$, i.e., all T greater than $T_c=1$.

Thus using as the critical variable $\tau=(1-\beta/\beta_c)$ instead of t and introducing the prefactor $(\beta/z)^{-1/2}$ in the expression for $\xi(\beta)$, the critical regime as defined by the scaling expressions

$$\chi(\beta) = \tau^{-\gamma} \quad (14)$$

and

$$\xi(\beta)/\beta^{1/2} = \tau^{-\nu} \quad (15)$$

extends rigorously from T_c to infinite T, with temperature-independent mean-field exponents $\gamma=1$ and $\nu=1/2$.

This statement can be reformulated: in the "ideal" high-dimension (mean-field) limit, one can expect correction terms to be inexistent if the critical variable and observables are chosen correctly. The two observables which we have discussed, the reduced susceptibility $\chi(\beta)$ [i.e., $\chi_{\mathrm{th}}(\beta)/\beta$] and the "reduced" second-moment correlation length $\xi(\beta)/(\beta/z)^{1/2}$, show exact critical power-law behaviors (as $\tau^{-\gamma}$ and as $\tau^{-\nu}$, respectively) for all $T>T_c$. We can surmise that in finite dimensions, where the critical exponents are different, there will be correction terms but that the same critical variable and normalized observables will remain the most appropriate for expressing the critical behavior over a wide temperature range.

III. SQUARE LATTICE ISING MODEL

In finite dimensions the extreme simplicity of the mean-field case will be lost, but because of the generic structure of the HTSE shown above, the general form of Eqs. (3) and (15) [including the noncritical normalization $\beta^{1/2}$ in $\xi(\beta)/\beta^{1/2}$] can be expected to be rather robust.[7] The exact finite dimension leading critical behaviors for [hyper]cubic ferromagnets when $\beta \to \beta_c$ can be written as

$$\chi(\beta) \to C_\chi \tau^{-\gamma} \tag{16}$$

and

$$\xi(\beta) \to [C_\xi/\beta_c^{1/2}]\beta^{1/2}\tau^{-\nu} \tag{17}$$

with critical amplitudes C_χ, $C_\xi/\beta_c^{1/2}$. Extended scaling expressions can be written analogous to the infinite dimension form, linking the critical limit with the trivial high-temperature fixed point limit.[7,8] As a first step a minimal modification must be made in order to allow for the fact that in finite dimensions, C_χ and $C_\xi/\beta_c^{1/2}$ are not exactly equal to one. We write the extended scaling expressions $\chi^*(\beta)$ and $\xi^*(\beta)$ as

$$\chi^*(\beta) = C_\chi \tau^{-\gamma}[1 + \tau(1 - C_\chi)/C_\chi] \tag{18}$$

and

$$\xi^*(\beta) = \beta^{1/2}[(C_\xi/\beta_c^{1/2})\tau^{-\nu}][1 + \tau(\beta_c^{1/2} - C_\xi)/C_\xi], \tag{19}$$

These expressions, which depend only on the critical parameters β_c, γ, ν, C_χ, and C_ξ, are exact by construction in both the critical- and the high-temperature limits, and they provide compact approximate expressions for the behavior over the whole range in between. It has been demonstrated that for the standard three dimensional Ising, XY, and Heisenberg ferromagnets,[7,8] extended scaling expressions (defined entirely through the critical parameters appropriate for each particular case) agree with the true $\chi(T)$ and $\xi(T)$ to within better than about 2% over the entire range of temperature from T_c to infinity. Weak nonanalytic correction terms can be clearly seen.[8]

Here we will consider in more detail the particular case of the canonical two-dimension square lattice Ising ferromagnet, for which the critical temperature, the critical exponents, the temperature dependence of the "true" correlation length, and a number of other properties are known exactly from the original work by Onsager and others[1] and from more recent conformal field theory.[11] In the 2D Ising lattice there are no nonanalytic terms[4,12] (except for weak high-order terms due to the lattice breaking of rotational symmetry,[13] which can be ignored for present purposes). There are no known exact analytic expressions for the susceptibility or the second-moment correlation length, but many terms of the high-temperature series expansions have been calculated,[4,14–17] so $\chi(\beta)$ and $\xi(\beta)$ can be calculated to high precision over the entire range of β, from β_c to zero (i.e., from T_c to infinity). We have taken advantage of the exact knowledge of the critical temperature $\tanh(\beta_c)=\sqrt{2}-1$ (i.e., $\beta_c=0.440686794\cdots$) and of the critical exponents $\gamma=7/4$ and $\nu=1$ to form biased Padé approximants χ_p and ξ_p^2 of χ and ξ^2, using up to 48 HTSE coefficients. Using first-order inhomogeneous differential approximants, we would obtain an equivalent accuracy. Thus $C_\chi=0.962581\ldots$, and $C_\xi/\beta_c^{1/2}=0.854221175$. (The ratio of the exact "true" correlation length critical amplitude to the second-moment correlation length critical amplitude is 1.000 402 [18]). The extended scaling estimates $\chi^*(\beta)$ and $\xi^*(\beta)$ [Eqs. (18) and (19)] can be written down directly using these values.

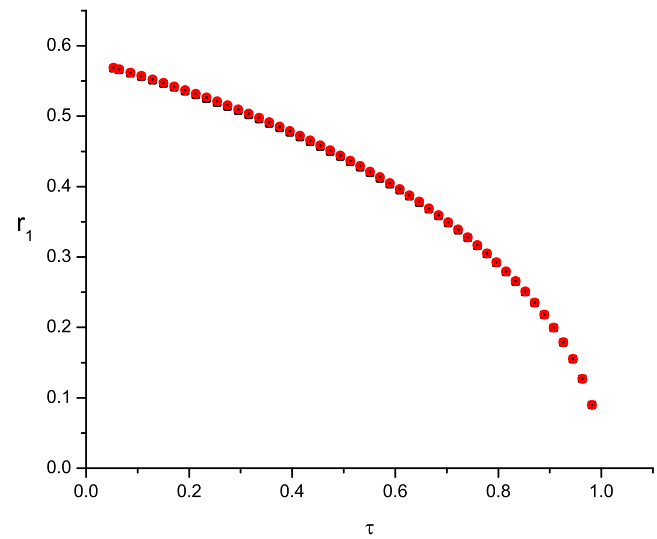

FIG. 1. (Color online) The Fisher-Burford "effective range of direct interaction" parameter $r_1(T)$ calculated from the high-precision square lattice Padé approximants values (squares, black online) and from the extended scaling expressions (circles, red online), as functions of $\tau=1-\beta/\beta_c$.

Fisher and Burford[3] some 40 years ago introduced a noncritical prefactor in the expression for $\xi(\beta)$ in Ising ferromagnets; they already noted that in the mean-field limit, the prefactor would be equal to $\beta^{1/2}$ as confirmed by the high-dimension limit discussion above. For Ising ferromagnets in 2D and 3D, they introduced an "effective range of direct interaction" parameter $r_1(T)$, which they defined through

$$[r_1(T)/\xi(T)]^{2-\eta} = 1/\chi(T). \tag{20}$$

The temperature variation of $r_1(T)$ is mainly due to the noncritical prefactor in the expression for the second-moment correlation length $\xi(T)$. Fisher and Burford did not give an explicit expression for $r_1(T)$ in finite dimensions, but they calculated it numerically over a wide range of temperature from the HTSE terms known at the time for five different Ising ferromagnets in 2D and 3D (their Fig. 6). We can recalculate the Fisher-Burford square lattice $r_1(T)$ as defined above from the high-precision square lattice Padé approximants values and also from the extended scaling expressions. In Fig. 1 we compare over a very wide temperature range

$$r_1(\tau) = \xi_p(\tau)/[\chi_p(\tau)]^{1/(2-\eta)} \tag{21}$$

obtained directly from the high precision values $\xi_p(\tau),\chi_p(\tau)$ with $r_1^*(\tau)$ estimated from the $\xi^*(\tau),\chi^*(\tau)$ of the extended scaling expressions. On the scale of this plot the two sets of points are almost indistinguishable, validating the assumption that the extended scaling expressions are accurate approximations to the exact behavior of the observables.

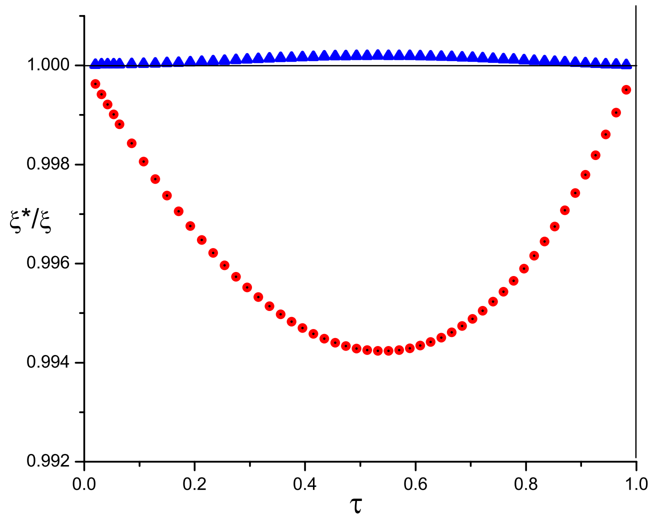

FIG. 2. (Color online) The ratio $\xi^*(\tau)/\xi_p(\tau)$ where the $\xi^*(\tau)$ are the extended scaling estimates of the correlation length (circles, red online: one correction term; triangles, blue online: two correction terms) and $\xi_p(\tau)$ the high precision data, as a function of $\tau=1-\beta/\beta_c$.

As a more stringent test of the extended scaling, in Figs. 2 and 3 we present the ratios $\xi^*(\tau)/\xi_p(\tau)$ and $\chi^*(\tau)/\chi_p(\tau)$, where again $\xi^*(\tau)$, $\chi^*(\tau)$ are the extended scaling estimates from Eqs. (19) and (18), and $\xi_p(\tau)$, $\chi_p(\tau)$ are the high precision calculated Padé values. It can be seen already that without further correction factors, the extended scaling values represent the high precision data for $\chi(\tau)$ and for $\xi(\tau)$ to better than 1% over the entire temperature range from $\tau=0$ to $\tau=1$. The figures also include $\xi^*(\tau)/\xi_p(\tau)$ and $\chi^*(\tau)/\chi_p(\tau)$ ratios where the expressions for $\xi^*(\tau)$ and $\chi^*(\tau)$ include second correction terms from Eqs. (28) and (29), as will be discussed later.

An alternative manner in which to present the data is to express temperature dependencies of $\chi(\tau)$ and $\xi(\tau)$ in terms

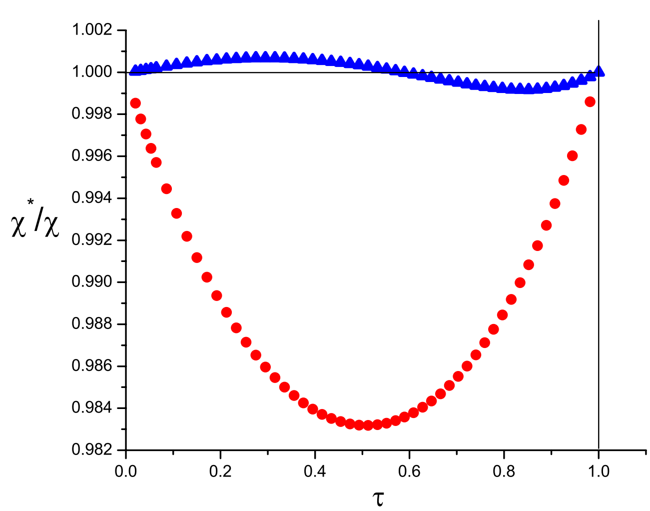

FIG. 3. (Color online) The ratio $\chi^*(\tau)/\chi_p(\tau)$ where the $\chi^*(\tau)$ are the extended scaling estimates of the reduced susceptibility (circles, red online: one correction term; triangles, blue online: two correction terms) and $\chi_p(\tau)$ the high precision data, as a function of $\tau=1-\beta/\beta_c$.

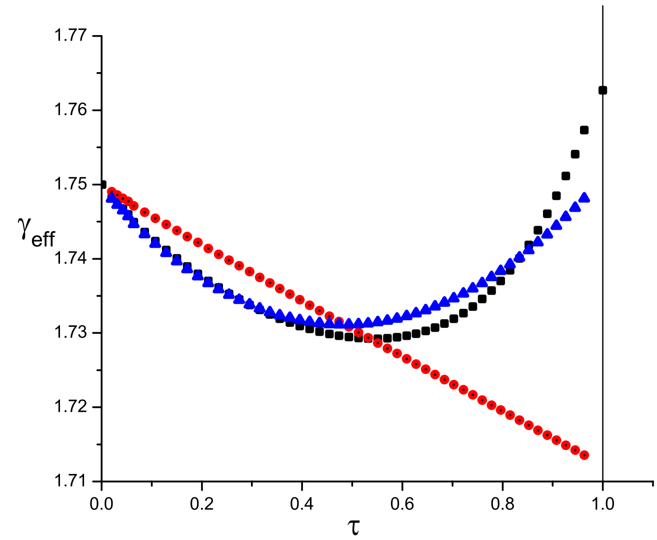

FIG. 4. (Color online) The effective exponent $\gamma_{\rm eff}$ calculated from the high precision $\chi_p(\tau)$ data (squares, black online) and from the extended scaling estimates (circles, red online: one correction term; triangles, blue online: two correction terms), as functions of $\tau=1-\beta/\beta_c$.

of temperature-dependent "effective" critical exponents.[5,19,20] In the spirit of the previous discussion, quite generally for any ferromagnet the temperature-dependent effective exponents can be rigorously defined as

$$\gamma_{\rm eff}(\tau) = -d\log[\chi(\tau)]/d\log(\tau), \tag{22}$$

$$\nu_{\rm eff}(\tau) = -d\log[\xi(\tau)/\beta^{1/2}]/d\log(\tau), \tag{23}$$

and

$$\eta_{\rm eff}(\tau) = 2 - d\log[\chi(\tau)]/d\log[\xi(\tau)/\beta^{1/2}]. \tag{24}$$

The definition of $\gamma_{\rm eff}(\tau)$ is the same as in references,[5,19] but because of the prefactor $\beta^{1/2}$ normalizing $\xi(\tau)$ the other two definitions are not standard. This prefactor is essential to ensure sensible high-temperature limits in Eqs. (23) and (24). We can note that for the first two parameters, an explicit choice must be made for β_c, while $\eta_{\rm eff}(\tau)$ (obviously linked to the two others) can be calculated from $\chi(\tau)$ and $\xi(\tau)$ data sets without any *a priori* knowledge of or estimate for β_c as Eq. (24) does not involve τ. The effective exponents tend to the critical exponent values at β_c. There are simple exact results for the high-temperature limits. Thus the leading terms in the HTSE for the reduced susceptibility are $\chi(\tau)=1+2d\beta+\cdots$, so in the high-temperature limit

$$\begin{aligned}\gamma_{\rm eff}(\tau\to 1) &= -d\log[\chi(\tau)]/d\log(\tau)\\ &= -d\log(1+2d\beta+\cdots)/d\log(1-\beta/\beta_c)\\ &= 2d\beta_c.\end{aligned} \tag{25}$$

Similarly $\nu_{\rm eff}(\tau\to 1)=d\beta_c$ and $\eta_{\rm eff}(\tau\to 1)=0$ in all dimensions for hypercubic lattices. In the square lattice case $\gamma_{\rm eff}(1)=1.7627\cdots$ and $\nu_{\rm eff}(1)=0.8814\cdots$.

Figures 4–6 show $\gamma_{\rm eff}(\tau)$, $\nu_{\rm eff}(\tau)$, and $\eta_{\rm eff}(\tau)$, comparing

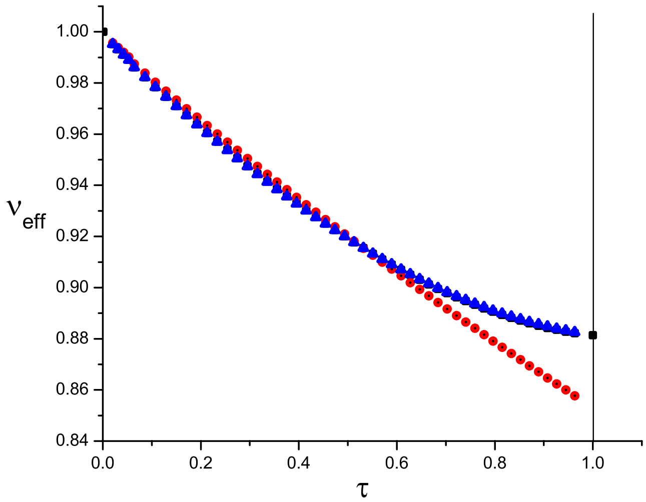

FIG. 5. (Color online) The effective exponent $\nu_{\rm eff}$ calculated from the high precision $\xi_p(\tau)$ data (squares, black online) and from the extended scaling estimates (circles, red online: one correction term; triangles, blue online: two correction terms), as functions of $\tau=1-\beta/\beta_c$.

the high precision values from HTSE with the extended scaling values, again with either one or two correction terms.

A number of remarks can be made. First, looking only at the high precision data it can be seen that the effective exponents defined through Eqs. (22)–(24) change smoothly and gradually with temperature over the whole range of temperature from the critical temperature to infinity. For the square lattice $\gamma_{\rm eff}(\tau)$ changes little with temperature while $\nu_{\rm eff}(\tau)$ varies rather more. In the 2D case $\eta_{\rm eff}(\tau)$ must necessarily change quite strongly, from the critical value 0.25 to the infinite temperature value of 0. In 3D the absolute value of the change is much weaker.

Second, presenting the data in this way is a very sensitive test of the extended scaling expressions; it can be seen that

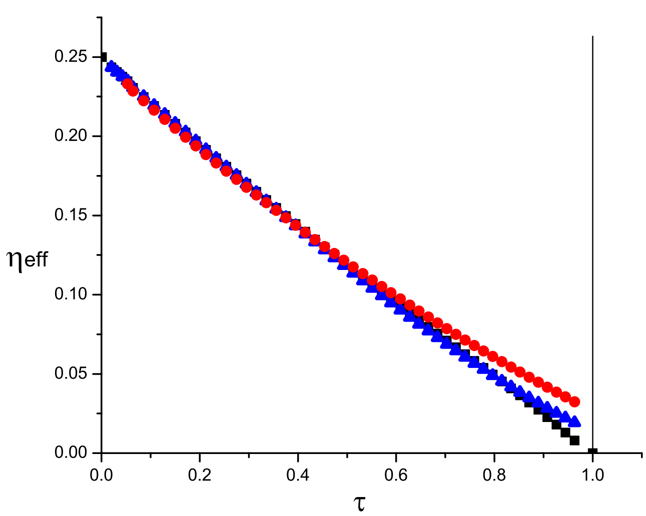

FIG. 6. (Color online) The effective exponent $\eta_{\rm eff}$ calculated from the high precision $\chi_p(\tau)$ and $\xi_p(\tau)$ data (squares, black online) and from the extended scaling estimates (circles, red online: one correction term; triangles, blue online: two correction terms), as functions of $\tau=1-\beta/\beta_c$.

with one correction term agreement with the high precision data is to within better than about 5% for each of the effective exponents over the entire temperature range. With two correction terms the agreement is considerably improved, particularly for $\nu_{\rm eff}$. From the general argument given above, in other ferromagnets one should expect qualitatively similar behavior, but with complications due to weak irrelevant operator terms in addition to the analytic correction terms.

It is widely considered that there is a "crossover" to mean-field-like effective exponents ($\gamma=1, \nu=1/2$) outside a critical region in the neighborhood of β_c (see, e.g., Ref. 21). As we have noted above, the exact high-temperature limits are $\chi(\beta)=1+\ldots$ and $\mu_2(\beta)=z\beta+\ldots$ for all dimensions. Suppose t is chosen as scaling variable and the effective exponents defined through $\gamma_{\rm eff}(t)=-d\log[\chi(t)]/d\log(t)$ and $\nu_{\rm eff}(t)=-d\log[\xi(t)]/d\log(t)$. Then the high-temperature limits will always be $\gamma_{\rm eff}(t=\infty)=0$ and $\nu_{\rm eff}(t=\infty)=1/2$. This is simply an automatic consequence of the choice of t as scaling variable. [Only if $\gamma_{\rm eff}$ is defined using the thermodynamic susceptibility $\chi_{\rm th}(t)$, i.e., $\gamma_{\rm eff}(t)=-d\log[\chi_{\rm th}(t)]/d\log(t)$ will the high-temperature limit be $\gamma_{\rm eff}(t=\infty)=1$ as in Ref. 21]. With the present definitions of the temperature-dependent effective exponents in terms of τ [Eqs. (22) and (23)], there is clearly no such crossover in near-neighbor interaction ferromagnets; $\gamma_{\rm eff}$ and $\nu_{\rm eff}$ only vary weakly with temperature and tend to be system-dependent values which are not mean-field-like.

IV. ANALYTIC CORRECTIONS

The general formalism for corrections to scaling (both analytic and nonanalytic) within the renormalization group theory (RGT) is well established (see Refs. 13 and 22). In the square lattice Ising model for present purposes, it can be considered that there are no nonanalytic corrections.[12,13] This means that one can write for instance the ratio of the inverse of the exact "true" correlation length $1/\xi_{\rm true}(\tau)$ $=\{\ln[\coth(\tau)]-2\tau\}$ to the ideal pure critical power law value τ/C_ξ^t as an analytic correction factor $g(\tau)$, which is a smooth function having an asymptotic Taylor expansion form, i.e.,

$$C_\xi^t/\xi_{\rm true}(\tau) = \tau g(\tau) = \tau[1 + a_1\tau + a_2\tau^2 + a_3\tau^3 + \ldots]. \quad (26)$$

The angular dependence of the exact leading coefficients of this "large distance two-point correlation length" analytic correction series has been calculated.[13,22,23] If $g(\tau)$ is truncated at low order, it only gives a useful representation of $\xi_{\rm true}(\tau)$ very close to T_c.

For ξ (it is important to keep in mind that this second-moment correlation length is not the "true" correlation length), we choose to define a correction factor $g^*(\tau)$ in the extended scaling form:

$$C_\xi/\xi(\tau) = (1-\tau)^{-1/2}\tau^\nu g^*(\tau) \quad (27)$$

as $(\beta/\beta_c)^{1/2}=(1-\tau)^{1/2}$. Empirical fits can be made to the high-precision $\xi_p(\tau)$ square lattice data. We assume a Taylor series for $g^*(\tau)$, keeping only two terms (up to order τ^2) with the restriction that $g^*(1)=C_\xi$. We find that

$$g^*(\tau) = 1 - 0.19055\tau + 0.04476\tau^2 \qquad (28)$$

provides a fit accurate to within less than 0.02% over the entire temperature range from $\tau=0$ to 1, thus including both the "critical region" and the high-temperature region. If the standard $g(\tau)$ correction factor had been used, the coefficient for the term in τ would have become $0.5-0.19055 = 0.30945$. This value is very close to the value of the equivalent coefficient in the $g(\tau)$ series for the "true" correlation length, which is $0.3116\cdots$.[22]

The large coefficient of the τ term in $g^*(\tau)$ can be understood as correcting for the difference between $C_\xi/\beta_c^{1/2} = 0.854221$ and 1. The small second coefficient corresponds to a further adjustment, which leads to a very significant improvement in the fit. It is remarkable that excellent precision over the entire temperature range is obtained with only two correction terms, validating the use of the $g^*(\tau)$ parametrization rather than the $g(\tau)$ form for $\xi_{sm}(\tau)$. We can note that when $g^*(\tau)$ is used rather than $g(\tau)$, there are no correction terms to $\xi_{sm}(\tau)$ in the mean-field limit.

The square lattice correction terms for $\chi(\tau)$ have been carefully studied.[4,16,17] The coefficients for a large number of leading correction terms are known, of which two are exact, with additional terms over and above Taylor series terms. An empirical correction factor for $\chi(\tau)$ with two correction terms gives

$$\chi(\tau) = C_\chi \tau^{-7/4}(1 + 0.0779\tau - 0.03903\tau^2). \qquad (29)$$

The leading term in Eq. (29) has a coefficient essentially identical to the exact value [$0.0779032\cdots$ (Refs. 4, 16, and 17)], and the phenomenological inclusion of a single further term provides an overall fit up to infinite temperature which is accurate to better than 0.1%.

V. FINITE SIZE SCALING

An immediate practical consequence that follows from the discussion above concerns the extraction of critical parameters from numerical studies of systems other than these canonical ferromagnets. If numerical data have been obtained over a wide temperature range and not only in the region very close to T_c, direct plots of the effective exponents as defined through Eqs. (22)–(24) for the largest sample sizes available can be extrapolated to estimate critical exponents. The temperature dependence of the curves, including the temperature range well above T_c, can then give useful indications concerning critical exponents and corrections. Finite size scaling analyses should be made using appropriate expressions derived from the leading extended scaling form. The widely used finite size scaling relation

$$Q(L,T) \sim F[L^{1/\nu}(T-T_c)] \qquad (30)$$

is derived from the Fisher finite size scaling *ansatz* $Q(L,T) = F[L/\xi(T)]$ on the assumption that the correlation length behaves as $\xi(T) \sim t^{-\nu}$; as we have seen the latter is always a poor approximation for $\xi(T)$ except extremely close to T_c. Using the Fisher *ansatz*, together with the extended scaling rule, $\xi^*(\beta) \sim \beta^{1/2}\tau^{-\nu}$ leads to finite size scaling expressions,[7,8] which remain much better approximations over a considerably wider temperature range. If T_c is known to reasonable precision, such finite size scaling analyses from numerical data taken over a wide temperature range should give reliable and unbiased estimates for the critical exponents. Ideally an allowance for the correction factor $g^*(\tau)$ of the preceding section should also be included in the analysis, but this would require very high quality numerical data.

It is important that for each particular system the appropriate extended scaling form should be used. For instance, in spin glasses with symmetrical interaction distributions, the relevant scaling variable is[7,24] $\tau_{SG} = 1-(\beta/\beta_c)^2$.

VI. CONCLUSION

One should expect "ideal" critical behavior in the high-dimension (mean-field) limit ferromagnet, meaning that if the scaling variable and the normalizations of the observables are chosen appropriately, all observables should show pure critical power law behavior over the entire temperature range above T_c. In the high-dimension limit both the reduced susceptibility $\chi(\tau)$ and the "reduced" second-moment correlation length $\xi(\tau)/\beta^{1/2}$ indeed show pure critical power law behaviors (as $\tau^{-\gamma}$ and as $\tau^{-\nu}$, respectively, with temperature-independent mean-field exponents) for all $T > T_c$, validating the use of the scaling variable $\tau = (1-\beta/\beta_c)$ and the normalization for $\xi(\tau)$ through the $\beta^{1/2}$ noncritical prefactor. In other words, in terms of this scaling variable and these observables behavior is "always critical" for the mean-field system over the whole temperature range.

For systems in finite dimensions one can no longer expect ideal behavior, and there will always be deviations from the pure critical power laws as soon as $T-T_c$ is finite; these deviations can be expressed in terms of correction factors. The use of the standard scaling variable $t = (T-T_c)/T_c$ is poorly adapted to the analysis of data over a wide temperature range because this variable diverges at high temperatures. In order to reduce the importance of the necessary corrections (and to have sensible high temperature limits), it is judicious to base the choice of scaling variable and scaling expressions on those appropriate for the mean-field limit. In agreement with arguments from the general form of the HT series expansions, this leads to extended scaling expressions with a single adjustment term which are $\chi^*(\beta) = C_\chi[1-\beta/\beta_c]^{-\gamma}[1+\tau(1-C_\chi)/C_\chi]$ for the reduced susceptibility and $\xi^*(\beta) = \beta^{1/2}[C_\xi(1-\beta/\beta_c)^{-\nu}][1+\tau(1-C_\xi)/C_\xi]$ for the second-moment correlation length. The adjustment terms have been introduced in order that the expressions tend to their known exact high-temperature limits.

For the canonical square lattice Ising ferromagnet with its known critical temperature, exponents, and critical amplitudes C_χ and C_ξ, the expressions $\chi^*(\beta)$ and $\xi^*(\beta)$ are compact approximations to the exact behavior, which are accurate to within about 1% over the entire temperature range. Further correction terms can be included to represent analytic scaling corrections (there are essentially no nonanalytic corrections in this system). If the correction series is truncated after only one more term (so that it takes the form $[1+a_1\tau + a_2\tau^2]$, Eqs. (29) and (28)) the precision improves to better than 0.1% for $\chi(\beta)$ and better than 0.02% for $\xi(\beta)$ over the entire temperature range. The temperature dependence of the observables can be expressed in terms of strictly defined effective exponents, $\gamma_{\text{eff}}(\tau)$, $\nu_{\text{eff}}(\tau)$, and $\eta_{\text{eff}}(\tau)$, Eqs. (22)–(24), which vary smoothly and weakly with temperature and do not tend to mean-field values at high temperature. In the square lattice system the leading correction terms for $\xi^{*}(\beta)$ can be compared to the critical analytic correction terms for the "true" correlation length, for which the exact leading terms in the Taylor expansion have been discussed in detail in the RGT formalism.[13,22]

The present conclusions confirm those already given for general cases of ferromagnets and spin glasses.[7,8] For practical purposes the extended scaling protocol can be usefully applied to the analysis of experimental results or when extracting critical parameters from finite size scaling analyses on numerical simulation data. In the general case allowance should be made for nonanalytical correction terms.

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