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Nonperturbative functional renormalization group for random field models and related disordered systems. I. Effective average action formalism

Gilles Tarjus* and Matthieu Tissier†

LPTMC, CNRS-UMR 7600, Université Pierre et Marie Curie, Boîte Postale 121, 4 Place Jussieu, 75252 Paris Cedex 05, France

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We have developed a nonperturbative functional renormalization group approach for random field models and related disordered systems for which, due to the existence of many metastable states, conventional perturbation theory often fails. The approach combines an exact renormalization group equation for the effective average action with a nonperturbative approximation scheme based on a description of the probability distribution of the renormalized disorder through its cumulants. For the random field $O(N)$ model, the minimal truncation within this scheme is shown to reproduce the known perturbative results in the appropriate limits, near the upper and lower critical dimensions and at a large number N of components, while providing a unified nonperturbative description of the full (N,d) plane, where d is the spatial dimension.

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I. INTRODUCTION

The effect of quenched disorder on the long-distance physics of many-body systems largely remains an unsettled question, despite decades of intensive research. Ongoing controversies persist, for instance, on the equilibrium and out-of-equilibrium behavior of spin glasses and systems coupled to a random field.[1,2] Even though progress has been made, it has so far proven difficult to construct a proper renormalization group (RG) approach that provides a description of ordering transitions and criticality in these systems. A technical reason for this unsatisfactory situation is that quenched disorder makes the system intrinsically inhomogeneous and that one should, in principle, follow the renormalization of the whole probability distribution of the disorder. A physical reason is that the presence of disorder and of the resulting spatial inhomogeneity lead, for at least some range of the control parameters, to multiple "metastable states." (At this point, we use the term metastable state in a loose acceptance to describe configurations that minimize some energy or free energy, action or effective action in field-theoretical terminology, but differ from the true ground state.) How such metastable states evolve upon coarse graining under RG then represents the central issue: at large length scales, their influence could vanish, leaving only benign signatures in the thermodynamics, or else it could modify the critical behavior of the system, the nature of its phases, and, often in an even more spectacular way, the relaxation and out-of-equilibrium dynamical properties.

A well known example of the kind of puzzles associated with quenched disorder and metastable states is the failure of the so-called "dimensional reduction" property in the random field Ising model (RFIM).[3–6] Standard perturbation theory predicts to all orders that the critical behavior of the RFIM in dimension d is the same as that of the pure Ising model, i.e., in the absence of random field, in two dimensions less, $d-2$. The property has been shown in a compact and elegant manner by Parisi and Sourlas[7] by means of a supersymmetric formalism. However, dimensional reduction predicts a lower critical dimension for ferromagnetism in the RFIM of $d_{lc}=3$, which is in contradiction to rigorous results.[8,9] The dimensional reduction property must therefore break down in low enough dimension. The supersymmetric approach gives a hint at the origin of the breakdown, which appears to be related, yet in a somewhat obscure way, to the presence of multiple metastable states[10] (in this case, local minima of the Hamiltonian).

Over the years and on top of numerous computer simulations and scarce exact analytical results, theoretical approaches have been devised to cope with disordered systems characterized by multiple metastable states, such as spin glasses and random field models.[1] To list the main ones, we mention the following:

(i) phenomenological approaches such as the heuristic domain-wall arguments[11,12] and the "droplet" description,[13–15] in which one directly focuses on rare excitations and the associated low-energy metastable states;

(ii) mean-field theories combined with the replica formalism in order to handle the average over disorder; for models with spin-glass ordering, the potentially dramatic effect of the metastable states is captured through a spontaneous breaking of the replica symmetry;[1,2,16,17]

(iii) specific RG techniques for low-dimensional ($d=1,2$) systems, for instance, the Coulomb gas RG approach for two-dimensional disordered XY models[18,19] or the real space RG for strongly disordered one-dimensional systems;[20–22]

(iv) the perturbative functional RG for energy-dominated disordered models considered in the vicinity of a critical dimension at which the fundamental fields are dimensionless;[23–28] one must then follow the flow of a whole function, an appropriate renormalized cumulant of the disorder. As first shown by Fisher[28] for an elastic manifold pinned by a random potential, the long-distance physics is controlled by a zero-temperature fixed point at which the renormalized cumulant is a nonanalytic function of the fields, with the nonanalyticity encoding the effect of the many metastable states at zero temperature.

All of these approaches, however, are either questionable or not easily generalizable: on one hand, the phenomenological approaches lack rigorous foundations and the relevance of mean-field descriptions to finite-dimensional systems is, to say the least, far from guaranteed; on the other hand, the perturbative functional RG becomes extremely complex and soon untractable in practice for random field systems when going beyond one-loop calculations;[29–31] moreover, it does not allow one to study the RFIM. (As for specific RG techniques, they are not extendable by construction.)

The purpose of the present work, which is described here and in a companion paper,[32] is therefore to propose a general theoretical framework that leads to a consistent description of the equilibrium behavior of the random field models and related disordered systems. To achieve this, we rely on a version of Wilson's continuous RG via momentum shell integration.[33] Under various terminologies, "exact RG," "functional RG," and "nonperturbative RG," it has been developed in the past 15 years to become a powerful method for investigating both universal and nonuniversal properties in statistical physics and quantum field theory.[34–38] The approach is "exact" in the sense that the RG flow associated with the progressive account of the field fluctuations over larger and larger length scales is described through an exact functional differential equation. It is "functional" because through the exact equation, one follows the flow of an infinite hierarchy of functions of the fields in place of simply coupling constants. It is "nonperturbative" (beyond the mere tautology that an exact description automatically includes all of the perturbative as well as the nonperturbative effects) because it lends itself to efficient approximation schemes that are able to capture genuine nonperturbative phenomena:[36] to name a few, in the case of the standard $O(N)$ scalar model, (numerically) tractable approximations describe the Kosterlitz–Thouless transition of the XY model in $d=2$, which is known to be associated with the binding /unbinding of topological defects (vortices), as well as the convexity property of the thermodynamic potential in the case of spontaneous symmetry breaking, which is recovered in other treatments through nonperturbative configurations such as instantons.

To study the problem at hand, we combine the ideas of the perturbative functional RG for disordered systems with the general formalism of the exact/functional/nonperturbative RG. In the following, we shall denote our approach *nonperturbative functional RG* (NP-FRG). It provides a framework in which to study both the perturbative and the nonperturbative effects in any spatial dimension d and for any number of components of the fundamental fields N. From the scope of the present series of papers, we exclude the relaxation and out-of-equilibrium dynamic phenomena, as well as spin-glass ordering. We also postpone to a forthcoming publication the development of the NP-FRG in a superfield formalism able to directly address the failure of supersymmetry in connection with that of dimensional reduction. Short versions of the present work appeared in Refs. 30 and 39.

The present paper is organized as follows: In Sec. II, we present the models and the formalism. We first introduce the models and discuss their physical relevance and the main open questions. From the corresponding replica field theories, we then derive the exact RG equation for the effective average action, which is the generating functional of the one-particle irreducible correlation functions at the running scale. We next relate the replica formalism, in which the replica symmetry is explicitly broken through the application of sources, to the cumulants of the renormalized disorder. We close the section by writing down the exact RG flow equations for these cumulants.

In Sec. III, we introduce a systematic nonperturbative approximation scheme. After first discussing the symmetries of the problem and the way to implement them in the effective average action formalism, we introduce the nonperturbative truncation scheme of the exact RG equation: it relies on (i) an expansion in cumulants of the disorder and (ii) a well tested approximation of the nonperturbative RG, the "derivative expansion," which uses the fact that the relevant physics is dominated by long wavelength modes to perform an expansion in the number of spatial derivatives of the fundamental fields. Finally, we detail the minimal truncation that we use in our numerical investigation of the random field $O(N)$ model [RF$O(N)$M].

In Sec. IV, we specialize the formalism to the study of the RF$O(N)$M. We introduce the scaling dimensions suitable to a search for the putative zero-temperature fixed point controlling the ordering transition. We first consider the case of the RFIM and then extend our description to the RF$O(N)$M. With the help of these dimensions, the RG flow equations are then cast in a scaled form. We also briefly comment on the possible application to other disordered systems.

In Sec. V, we next discuss an important property of the truncations previously described: because of the one-loop structure of the exact flow equations and of the appropriate choice of the approximations, one recovers the perturbative results both near the upper critical dimension, $d_{uc}=6$, and in the $N\to\infty$ limit of the RF$O(N)$M. Even more interestingly, we also show that our minimal truncation near the lower critical dimension for ferromagnetism of the RF$O(N>1)$M, $d_{lc}=4$, reduces to the perturbative functional RG result (at one loop) obtained from the nonlinear sigma version of the model.[23] To the least, the truncated NP-FRG thus provides a nonperturbative interpolation in the whole (N,d) plane of the known perturbative results near $d=4$, $d=6$, as well as $N\to\infty$. Finally, the presentation and discussion of the results obtained for the RF$O(N)$M within the present NP-FRG approach will be described in the companion paper.[32]

II. MODELS AND FORMALISM

A. Models

We focus on the equilibrium long-distance behavior of a class of disordered models, in which N-component classical variables with $O(N)$ symmetric interactions are coupled to a random field. Depending on whether the coupling is linear or bilinear, the models belong to the "random field" (RF) or the "random anisotropy" (RA) subclasses. Such models with $N=1$, 2, or 3 are relevant to describe a variety of systems encountered in condensed matter physics or physical chemistry. To name a few, one can mention dilute antiferromagnets in a uniform magnetic field,[40] critical fluids and binary mixtures in aerogels (both systems being modeled by the $N=1$ RF Ising model),[41–43] vortex phases in disordered type-II superconductors (described in terms of an elastic glass model, whose simplest version is the $N=2$ RF XY model),[44–46] amorphous magnets, such as alloys of rare-earth compounds,[47,48] and nematic liquid crystals in disordered porous media (described by $N=2$ or $N=3$ RA models).[49]

Other related models can be described as well within the same formalism, but will only be alluded to: the "random elastic" model describing an elastic system, such as an interface or a vortex lattice, which is pinned by the presence of impurities; the "random temperature" model associated with impurity-generated bond or site dilution in a ferromagnetic Ising model. For reasons that will become clear further down in this section, from the present study we exclude spin-glass ordering and we rather concentrate on ferromagnetic ordering (in which the $O(N)$ symmetry is spontaneously broken) or "quasiordering" (phases with quasi-long-range order).

Our starting point is the field theoretical (coarse-grained) description of the systems in terms of an N-component scalar field $\boldsymbol{\chi}(\boldsymbol{x})$ in a d-dimensional space and an effective Hamiltonian or bare action,

$$S[\boldsymbol{\chi};\boldsymbol{h},\boldsymbol{\tau}] = \int_{\boldsymbol{x}} \Bigg\{ \frac{1}{2}\sum_{\mu=1}^{N} (|\partial\chi^{\mu}(\boldsymbol{x})|^2 + \tau\chi^{\mu}(\boldsymbol{x})^2) + \frac{u}{4!}\left(\sum_{\mu=1}^{N}\chi^{\mu}(\boldsymbol{x})^2\right)^2 - \sum_{\mu=1}^{N} h^{\mu}(\boldsymbol{x})\chi^{\mu}(\boldsymbol{x}) - \sum_{\mu,\nu=1}^{N}\tau^{\mu\nu}(\boldsymbol{x})\chi^{\mu}(\boldsymbol{x})\chi^{\nu}(\boldsymbol{x})\Bigg\}, \quad (1)$$

where $\int_{\boldsymbol{x}} \equiv \int d^d x$ and the superscript μ spans the N components of the field; $\boldsymbol{h}(\boldsymbol{x})$ is a random magnetic field and $\boldsymbol{\tau}(\boldsymbol{x})$ is a second-rank random anisotropy tensor, which, for simplicity, are both taken (see also the discussion below) with the Gaussian distributions characterized by zero means and the variances given by

$$\overline{h^{\mu}(\boldsymbol{x})h^{\nu}(\boldsymbol{y})} = \Delta\,\delta_{\mu\nu}\delta(\boldsymbol{x}-\boldsymbol{y}), \quad (2)$$

$$\overline{\tau^{\mu\nu}(\boldsymbol{x})\tau^{\rho\sigma}(\boldsymbol{y})} = \frac{\Delta_2}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho})\delta(\boldsymbol{x}-\boldsymbol{y}), \quad (3)$$

where the overbar generically denotes the average over quenched disorder. Higher-order random anisotropies could be included as well. They will indeed be generated along the RG flow. However, for symmetry reasons, when starting with only a second-rank, or more generally an even-rank, random anisotropy, only even-rank anisotropies are generated: this corresponds to what is called the RA model. The model with a nonzero Δ, for which anisotropies of both odd and even ranks are generated under RG flow, is the RF model.

The equilibrium properties of the model are obtained from the average over disorder of the logarithm of the partition function,

$$\mathcal{Z}[\boldsymbol{J}] = \int \mathcal{D}\boldsymbol{\chi}\,\exp\left[-S[\boldsymbol{\chi};\boldsymbol{h},\boldsymbol{\tau}] + \int_{\boldsymbol{x}} \boldsymbol{J}(\boldsymbol{x})\cdot\boldsymbol{\chi}(\boldsymbol{x})\right], \quad (4)$$

where $\boldsymbol{J}(\boldsymbol{x})$ is a source linearly coupled to the fundamental field and a (ultraviolet) momentum cutoff Λ, which is associated with an inverse microscopic length scale such as a lattice spacing, is implicitly considered in the functional integration over the field. With this definition, however, the partition function and the corresponding thermodynamic potential $W[\boldsymbol{J}]=\ln\mathcal{Z}[\boldsymbol{J}]$ are still functionals of the random fields: $W[\boldsymbol{J}]\equiv W[\boldsymbol{J};\boldsymbol{h},\boldsymbol{\tau}]$. As is well known from the theory of systems with quenched disorder, the thermodynamics is given by the average over disorder of the "free energy," i.e.,

$$\overline{W[\boldsymbol{J}]} = \overline{\ln\,\mathcal{Z}[\boldsymbol{J}]}. \quad (5)$$

Full information on the system, in particular, an access to the correlation (Green's) functions of the field, requires knowledge of the higher moments of $W[\boldsymbol{J}]$, which is viewed as a random functional.[50] As will be more thoroughly discussed further below, such information can be conveniently extracted by using the replica formalism, whose starting point is the replacement of $\ln\mathcal{Z}$ by the limit of $(\mathcal{Z}^n-1)/n$ when n, i.e., the number of replicas of the original system, goes to zero. Quite differently from the standard but controversial use of this replica trick, in which the analytic continuation for $n<1$ opens the possibility of a *spontaneous* breaking of the replica symmetry,[16] we will consider an *a priori* more benign procedure, in which the symmetry between replicas is *explicitly* broken by the introduction of external sources acting on each replica independently. This procedure will allow us to generate the cumulant expansion of the disorder-dependent functional $W[\boldsymbol{J}]$.

Within the replica formalism, the original problem is replaced by one with n replica fields $\{\boldsymbol{\chi}_a(\boldsymbol{x})\}$, where $a=1,2,\ldots,n$, and the "replicated action," which is obtained after explicitly performing the average over the disorder in the partition function, reads

$$S_n[\{\boldsymbol{\chi}_a\}] = \int_{\boldsymbol{x}}\Bigg\{\frac{1}{2}\sum_{a=1}^{n}\left[|\partial\boldsymbol{\chi}_a(\boldsymbol{x})|^2 + \tau|\boldsymbol{\chi}_a(\boldsymbol{x})|^2 + \frac{u}{12}(|\boldsymbol{\chi}_a(\boldsymbol{x})|^2)^2\right] - \frac{1}{2}\sum_{a,b=1}^{n}[\Delta\,\boldsymbol{\chi}_a(\boldsymbol{x})\cdot\boldsymbol{\chi}_b(\boldsymbol{x}) + \Delta_2\,(\boldsymbol{\chi}_a(\boldsymbol{x})\cdot\boldsymbol{\chi}_b(\boldsymbol{x}))^2]\Bigg\}, \quad (6)$$

with the corresponding partition function,

$$\mathcal{Z}_n[\{\boldsymbol{J}_a\}] = \int\prod_{a=1}^{n}\mathcal{D}\boldsymbol{\chi}_a\,\exp\left(-S_n[\{\boldsymbol{\chi}_a\}] + \sum_{a=1}^{n}\int_{\boldsymbol{x}}\boldsymbol{J}_a(\boldsymbol{x})\cdot\boldsymbol{\chi}_a(\boldsymbol{x})\right), \quad (7)$$

where the linear sources $\boldsymbol{J}_a(\boldsymbol{x})$, $a=1,2,\ldots,n$, separately act on each replica. Associated with this partition function is the generating functional of the connected Green's functions, $W_n[\{\boldsymbol{J}_a\}]=\ln\mathcal{Z}_n[\{\boldsymbol{J}_a\}]$, and the effective action, $\Gamma_n[\{\phi_a\}]$, which is defined through a Legendre transform,

$$\Gamma_n[\{\phi_a\}] = -W_n[\{\boldsymbol{J}_a\}] + \sum_{a=1}^{n}\int_{\boldsymbol{x}}\boldsymbol{J}_a(\boldsymbol{x})\cdot\phi_a(\boldsymbol{x}). \quad (8)$$

The fields $\{\phi_a\}$ and the sources $\{\boldsymbol{J}_a\}$ are related by

$$\phi_a^{\mu}(\boldsymbol{x}) = \langle \chi_a^{\mu}(\boldsymbol{x}) \rangle = \frac{\delta W_n[\{\boldsymbol{J}_a\}]}{\delta J_a^{\mu}(\boldsymbol{x})}, \tag{9a}$$

where $\langle X \rangle$ represents the average of X with the weight given in Eq. (7), and

$$J_a^{\mu}(\boldsymbol{x}) = \frac{\delta \Gamma_n[\{\phi_a\}]}{\delta \phi_a^{\mu}(\boldsymbol{x})}. \tag{9b}$$

The effective action is the generating functional of the one-particle irreducible $(1-PI)$ correlation functions or proper vertices.

The formalism we are about to describe also applies to extensions of the replicated action of Eq. (6) that can be cast in the form

$$S_n[\{\boldsymbol{\chi}_a\}] = \int_{\boldsymbol{x}} \Bigg\{ \sum_{a=1}^{n} \left[\frac{1}{2}|\partial \boldsymbol{\chi}_a(\boldsymbol{x})|^2 + U_\Lambda(\boldsymbol{\chi}_a(\boldsymbol{x})) \right] - \frac{1}{2}\sum_{a,b=1}^{n} V_\Lambda(\boldsymbol{\chi}_a(\boldsymbol{x}), \boldsymbol{\chi}_b(\boldsymbol{x})) + \cdots \Bigg\}, \tag{10}$$

where the subscript Λ recalls that the various terms are at their bare value, which is defined at the microscopic scale Λ, and the dots indicate possible functions involving higher numbers of replicas. The functions $U_\Lambda, V_\Lambda, \ldots$ satisfy the $O(N)$ symmetry as well as the S_n permutational symmetry between replicas. Equation (1) is obviously a special case of the above expression, and higher-order anisotropies are included in a two-replica term, which is only function of $\boldsymbol{\chi}_a(\boldsymbol{x}) \cdot \boldsymbol{\chi}_b(\boldsymbol{x})$. RF and RA $O(N)$ models with non-Gaussian distributions of the random fields and anisotropies are described by terms involving higher number of replicas. [Note that the RA $O(N)$ model is defined as such for $N>1$; the Ising case, $N=1$, corresponds to another model, the random temperature model introduced hereafter.]

Other disordered systems are also described by the form of the replicated action in Eq. (10). For instance, the random temperature model corresponds to Eq. (10) with U_Λ and V_Λ functions of the fields only through the $O(N)$ invariants $\rho_a = \frac{1}{2}|\boldsymbol{\chi}_a|^2$, $\rho_b = \frac{1}{2}|\boldsymbol{\chi}_b|^2$. In the RF, RA, and random temperature models, the one-replica part of the bare action simply describes n copies of the standard ferromagnetic $O(N)$ model without disorder.

The random elastic model is also a special case of Eq. (10). However, contrary to the models just discussed, the one-replica potential U_Λ is absent (or reduced to a purely quadratic term) so that there is no mechanism triggering a paramagnetic-ferromagnetic phase transition. The two-replica potential V_Λ, which is the second cumulant of a random pinning potential, is now function of only the difference between the two replica fields, $\boldsymbol{\chi}_a(\boldsymbol{x}) - \boldsymbol{\chi}_b(\boldsymbol{x})$. As a result, the model has an additional symmetry, i.e., the statistical tilt symmetry,[51] which guarantees that the one-replica part of the action, including the kinetic term, is not renormalized: the effective action has thus the same one-replica part as the bare one. [Note that as shown in Ref. 30 and in the companion paper,[32] the random elastic model, although with an underlying periodicity, also emerges as a low-disorder approximation of the RF and RA XY $(N=2)$ models.]

B. Exact renormalization group equation for the effective average action

The exact RG in the effective average action formalism[34,36,52] relates the bare action [here Eq. (10)] to the full effective action [Eq. (8)] through a progressive inclusion of fluctuations of longer and longer wavelength. To do so, one introduces an infrared regulator, which is characterized by a scale k, which, in the functional integration leading to the partition function, suppresses the contribution of the low-energy modes with momentum $|\boldsymbol{q}| \lesssim k$, while including the high-energy modes with $|\boldsymbol{q}| \gtrsim k$. After Legendre transformation, this defines an "effective average action" at the running scale k, Γ_k, which continuously interpolates between the microscopic scale $k=\Lambda$, at which $\Gamma_{k=\Lambda}$ reduces to the bare action, and the macroscopic one, $k=0$, at which $\Gamma_{k=0}$ equals the full effective action.

More precisely, in the present context, a "masslike" quadratic term is added to the bare action [Eq. (10)],

$$\Delta \mathcal{S}_k[\{\boldsymbol{\chi}_a\}] = \frac{1}{2}\sum_{a,b=1}^{n}\sum_{\mu,\nu=1}^{N}\int_{\boldsymbol{q}} R_{k,ab}^{\mu\nu}(q^2)\chi_a^{\mu}(-\boldsymbol{q})\chi_b^{\nu}(\boldsymbol{q}), \tag{11}$$

where $\int_{\boldsymbol{q}} \equiv \int d^d q/(2\pi)^d$; $R_{k,ab}^{\mu\nu}(q^2)$ denotes infrared cutoff functions which, in order to enforce that the additional term satisfies the same $O(N)$ and S_n symmetries as the bare action (see above), must take the following form:

$$R_{k,ab}^{\mu\nu}(q^2) = [\hat{R}_k(q^2)\delta_{ab} + \tilde{R}_k(q^2)]\delta_{\mu\nu}. \tag{12}$$

The cutoff functions $\hat{R}_k(q^2)$ and $\tilde{R}_k(q^2)$ are chosen such as to realize the decoupling of the low- and high-momentum modes at the scale k: for this, they must decrease sufficiently fast for large momentum $|\boldsymbol{q}| \gg k$ and go to a constant value (a "mass") for small momentum $|\boldsymbol{q}| \ll k$. The presence of an off-diagonal component $\tilde{R}_k(q^2)$ is somewhat unusual and will be discussed later on. The cutoff functions must also satisfy the two constraints that (i) they go to zero when $k \to 0$ so that one indeed recovers the full effective action with all modes accounted for and (ii) $\hat{R}_k(q^2)$ diverges while $\tilde{R}_k(q^2)$ stays finite when $k \to \Lambda$ so that the effective average action does reduce to the bare action. (In what follows, we are only concerned with the long-distance behavior of the models and do not pay attention to the microscopic details; we thus let Λ go to ∞ in the cutoff functions.) Different choices have been proposed and tested in recent literature. Standard choices for $\hat{R}_k(q^2)$ are of the form

$$\hat{R}_k(q^2) = Z_k q^2 r(q^2/k^2), \tag{13}$$

where Z_k is a field renormalization constant yet to be specified and $r(y) = y^{-1}(1-y)\Theta(1-y)$,[53] where Θ is the Heaviside function and $r(y) = (e^y - 1)^{-1}$.[52]

From the partition function $\mathcal{Z}_k[\{\boldsymbol{J}_a\}]$ obtained from the bare action supplemented with the k-dependent regulator [Eq. (11)], one defines the generating functional of the

Green's functions $W_k[\{\boldsymbol{J}_a\}]=\ln \mathcal{Z}_k[\{\boldsymbol{J}_a\}]$ and through a Legendre transform, one has access to the effective average action at the running scale k, Γ_k,

$$\Gamma_k[\{\phi_a\}] + W_k[\{\boldsymbol{J}_a\}] = \sum_{a=1}^{n} \int_{\boldsymbol{x}} \boldsymbol{J}_a(\boldsymbol{x}) \cdot \phi_a(\boldsymbol{x}) - \Delta\mathcal{S}_k[\{\phi_a\}], \tag{14}$$

where the fields $\{\phi_a\}$ and the sources $\{\boldsymbol{J}_a\}$ are related by the (k dependent) expression

$$\phi_a^{\mu}(\boldsymbol{x}) = \langle \chi_a^{\mu}(\boldsymbol{x})\rangle = \frac{\delta W_k[\{\boldsymbol{J}_a\}]}{\delta J_a^{\mu}(\boldsymbol{x})}. \tag{15}$$

The Legendre transform is slightly modified by the addition of the last term in Eq. (14), which ensures that the effective average action Γ_k does reduce to the bare action at the microscopic scale, with no contribution from the infrared regulator. This addition does not change the behavior in the $k \to 0$ limit since the regulator identically goes to zero. Physically and to use the language of magnetic systems, the effective average action is a coarse-grained Gibbs free energy. It is the generating functional of the $1-PI$ correlation functions, from which one can derive all of the Green's functions of the modified system at the scale k. Note, that here and in the following, we omit the subscript n associated with the number of replicas in order to simplify the notations.

The evolution of the effective average action with the infrared cutoff k is governed by an exact flow equation,

$$\partial_k \Gamma_k[\{\phi_a\}] = \frac{1}{2}\int_{\boldsymbol{q}} \mathrm{Tr}\{\partial_k \boldsymbol{R}_k(q^2)[\boldsymbol{\Gamma}_k^{(2)} + \boldsymbol{R}_k]^{-1}_{\boldsymbol{q},-\boldsymbol{q}}\}, \tag{16}$$

where the trace involves a sum over both replica indices and N-vector components; $\boldsymbol{R}_k(q^2)$ is defined in Eq. (12) and $\boldsymbol{\Gamma}_k^{(2)}$ is the tensor formed by the second functional derivatives of Γ_k with respect to the fields $\phi_a^{\mu}(\boldsymbol{q})$,

$$(\Gamma_k^{(2)})_{ab}^{\mu\nu}(\boldsymbol{q},\boldsymbol{q}') = \frac{\delta^2 \Gamma_k}{\delta\phi_a^{\mu}(\boldsymbol{q})\,\delta\phi_b^{\nu}(\boldsymbol{q}')}. \tag{17}$$

The above RG flow equation is a complicated functional integrodifferential equation that cannot be exactly solved in general; however, due to its one-loop structure and its reasonably transparent physical content, it provides a convenient starting point for nonperturbative approximation schemes.

At this point, it is quite clear to see why we have excluded spin-glass ordering from our considerations. The quadratic form of the infrared regulator in Eq. (11) suppresses the fluctuations of the low-momentum modes of the fundamental fields $\boldsymbol{\chi}_a$. On the other hand, spin-glass ordering involves fluctuations of composite fields, which are associated, e.g., with the "overlap" between different replicas.[16] Proper RG treatment of such fluctuations implies the introduction of a masslike regulator for composite fields, i.e., in the simplest case, a functional that is quartic in the fundamental fields instead of the quadratic term used here. We do not consider this case in the present work.

C. Explicit replica symmetry breaking and cumulants of the renormalized disorder

Among the technical difficulties encountered when making use of the exact RG equation [Eq. (16)], there is one which is specific to disordered systems and to the present replica formalism: one must invert the matrix $\boldsymbol{\Gamma}_{k,ab}^{(2)}+\boldsymbol{R}_{k,ab}$ for arbitrary replica fields (since all replicas are different due to the independently applied sources). Before delving into this problem, it is worth giving some physical insight into the meaning of the explicit replica symmetry breaking used here.

As discussed in Sec. II A, after a full account of the fluctuations, the bare disorder is renormalized to a full random ("free energy") functional $W[\boldsymbol{J}]$, which, to make its dependence on the bare quenched disorder explicit, we now denote $W[\boldsymbol{J};\boldsymbol{h}]$. This random object can be characterized by the infinite set of its cumulants, $W_1[\boldsymbol{J}_1], W_2[\boldsymbol{J}_1,\boldsymbol{J}_2], W_3[\boldsymbol{J}_1,\boldsymbol{J}_2,\boldsymbol{J}_3],\ldots$, where

$$W_1[\boldsymbol{J}_1] = \overline{W[\boldsymbol{J}_1;\boldsymbol{h}]}, \tag{18}$$

$$W_2[\boldsymbol{J}_1,\boldsymbol{J}_2] = \overline{W[\boldsymbol{J}_1;\boldsymbol{h}]W[\boldsymbol{J}_2;\boldsymbol{h}]} - \overline{W[\boldsymbol{J}_1;\boldsymbol{h}]}\;\overline{W[\boldsymbol{J}_2;\boldsymbol{h}]}, \tag{19}$$

etc. The first cumulant W_1 gives access to the thermodynamics of the system and the higher-order cumulants describe the distribution of the renormalized disorder (we define, as in the bare action, a disorder with zero mean). Note that by construction, the cumulants are invariant under permutations of their arguments.

The cumulants can be generated from an average involving copies, or "replicas," of the original disordered system as follows:

$$\begin{aligned}\overline{\exp\left(\sum_{a=1}^{n} W[\boldsymbol{J}_a;\boldsymbol{h}]\right)} &= \exp(W[\{\boldsymbol{J}_a\}]) \\ &= \exp\left(\sum_{a=1}^{n} W_1[\boldsymbol{J}_a] + \frac{1}{2}\sum_{a,b=1}^{n} W_2[\boldsymbol{J}_a,\boldsymbol{J}_b] \right. \\ &\quad \left. + \frac{1}{3!}\sum_{a,b,c=1}^{n} W_3[\boldsymbol{J}_a,\boldsymbol{J}_b,\boldsymbol{J}_c] + \cdots\right),\end{aligned} \tag{20}$$

where the n copies have *the same* bare disorder but are coupled to different external sources. To fully characterize the random functional $W[\boldsymbol{J};\boldsymbol{h}]$, it is indeed important to describe its cumulants for generic arguments, i.e., for different sources. (Be aware that the subscripts $1,2,\ldots$ used to denote the cumulants of W should not be confused with the subscript n that denotes the number of replicas in Sec. II A and is omitted since; here, for instance, W_1 denotes the one-replica component, which corresponds to the first cumulant, whereas in the previous notation, $W_{n=1}$ is given by the sum of all cumulants with all their arguments equal.)

A convenient trick to extract the cumulants with their full functional dependence is to let the number of replicas be arbitrary and to view the expansion on the right-hand side of Eq. (20) as an expansion of the functional $W[\{\boldsymbol{J}_a\}]$ defined below Eq. (7) in increasing number of "free," or unconstrained, sums over replicas. The term of order p in the expansion is a sum over p replica indices of a functional depending on exactly p replica sources; this functional is precisely equal to the pth cumulant of $W[\boldsymbol{J};\boldsymbol{h}]$. This procedure, which rests on an explicit breaking of the replica symmetry and an analytic continuation to arbitrary numbers of replicas (including the limit $n\to 0$ previously introduced), is *a priori* different from the standard use of replicas, in which all the sources are equal; it avoids the delicate handling of a spontaneous replica symmetry breaking.[1,2,16,17] It was used in a similar context by Le Doussal and Wiese.[54,55] The practical implementation of the expansion in free replica sums will be detailed in Sec. II D.

In our present NP-FRG approach, however, the central object is the effective action Γ, not W. The expansion of $\Gamma[\{\phi_a\}]$ in increasing number of free replica sums reads

$$\Gamma[\{\phi_a\}]=\sum_{a=1}^{n}\Gamma_1[\phi_a]-\frac{1}{2}\sum_{a,b=1}^{n}\Gamma_2[\phi_a,\phi_b]+\frac{1}{3!}\sum_{a,b,c=1}^{n}\Gamma_3[\phi_a,\phi_b,\phi_c]+\cdots, \tag{21}$$

where for later convenience, we have introduced a minus sign for all the even terms of the expansion. $\Gamma[\{\phi_a\}]$ and $W[\{\boldsymbol{J}_a\}]$ are related to a Legendre transform, so if one also expands the sources $\boldsymbol{J}_a[\{\phi_f\}]$ (where $\{\phi_f\}$ denotes the n replica fields to avoid confusion in the indices) in increasing number of free replica sums, one can relate the terms of the expansion of the effective action to the cumulants of the random functional $W[\boldsymbol{J};\boldsymbol{h}]$. The relation is straightforward for the first terms but gets more involved as the order increases.

More precisely, $\Gamma_1[\phi]$ is the Legendre transform of $W_1[\boldsymbol{J}]$, namely,

$$\Gamma_1[\phi]=-W_1[\boldsymbol{J}]+\int_{\boldsymbol{x}}\boldsymbol{J}(\boldsymbol{x})\cdot\phi(\boldsymbol{x}), \tag{22}$$

with

$$\phi^{\mu}(\boldsymbol{x})=\frac{\delta W_1[\boldsymbol{J}]}{\delta J^{\mu}(\boldsymbol{x})}, \tag{23}$$

and the second-order term is given by

$$\Gamma_2[\phi_1,\phi_2]=W_2[\boldsymbol{J}[\phi_1],\boldsymbol{J}[\phi_2]], \tag{24}$$

where $\boldsymbol{J}[\phi]$ is the *nonrandom* source defined via the inverse of the Legendre transform relation in Eq. (22), i.e., $J^{\mu}[\phi](\boldsymbol{x})=\delta\Gamma_1[\phi]/\delta\phi^{\mu}(\boldsymbol{x})$. [Note that the $\boldsymbol{J}(x)$ that is introduced here differs from the source $\boldsymbol{J}_a(x)$ introduced in Eq. (9b): through the Legendre relations, the latter depends on all the fields $\{\phi_a\}$, while the former depends on only one replica field.] The above expression motivates our choice of signs for the terms of the expansion in free replica sums of $\Gamma[\{\phi_a\}]$, Eq. (20): $\Gamma_2[\phi_1,\phi_2]$ is directly the second cumulant of $W[\boldsymbol{J};\boldsymbol{h}]$ (with the proper choice of $\boldsymbol{J}[\phi]$).

For the higher-order terms, one finds

$$\Gamma_3[\phi_1,\phi_2,\phi_3]=-W_3[\boldsymbol{J}[\phi_1],\boldsymbol{J}[\phi_2],\boldsymbol{J}[\phi_3]]+\int_{\boldsymbol{xy}}\{W_{2,\boldsymbol{x}}^{(10)}[\boldsymbol{J}[\phi_1],\boldsymbol{J}[\phi_2]](W_1^{(2)}[\boldsymbol{J}[\phi_1]])^{-1}_{\boldsymbol{xy}}\times W_{2,\boldsymbol{y}}^{(10)}[\boldsymbol{J}[\phi_1],\boldsymbol{J}[\phi_3]]+\text{perm}(123)\}, \tag{25}$$

etc., where perm(123) denotes the two additional terms obtained by cyclic permutations of the fields ϕ_1,ϕ_2,ϕ_3 and where we have used the following short-hand notation,

$$W_{1,\boldsymbol{x}_1\cdots\boldsymbol{x}_p}^{(p)}[\boldsymbol{J}_1]=\frac{\delta^{p}W_1[\boldsymbol{J}_1]}{\delta J_1(\boldsymbol{x}_1)\cdots\delta J_1(\boldsymbol{x}_p)}, \tag{26}$$

$$W_{2,\boldsymbol{x}_1\cdots\boldsymbol{x}_p\boldsymbol{y}_1\cdots\boldsymbol{y}_q}^{(pq)}[\boldsymbol{J}_1,\boldsymbol{J}_2]=\frac{\delta^{p+q}W_2[\boldsymbol{J}_1,\boldsymbol{J}_2]}{\delta J_1(\boldsymbol{x}_1)\cdots\delta J_1(\boldsymbol{x}_p)\delta J_2(\boldsymbol{y}_1)\cdots\delta J_2(\boldsymbol{y}_q)}, \tag{27}$$

etc. Note that for clarity, the $O(N)$ indices have been omitted in the above expressions.

We point out that $\Gamma_p[\phi_1,\ldots,\phi_p]$ for $p\geq 3$ cannot be directly taken as the pth cumulant of a physically accessible random functional, in particular, not of the disorder-dependent Legendre transform of $W[\boldsymbol{J};\boldsymbol{h}]$ (although it can certainly be expressed in terms of such cumulants of order equal to or lower than p). In the following and by an abuse of language, we will nonetheless generically call the Γ_p's "cumulants of the renormalized disorder" (which is true for $p=2$).

In complement to the above picture and more specifically for random field systems, it is also interesting to introduce a renormalized random field (or random force) $\breve{\boldsymbol{h}}[\phi](\boldsymbol{x})$, which is defined as the derivative of a random free-energy functional,

$$\breve{h}[\phi]^{\mu}(\boldsymbol{x})=-\frac{\delta}{\delta\phi^{\mu}(\boldsymbol{x})}(W[\boldsymbol{J}[\phi];\boldsymbol{h}]-\overline{W[\boldsymbol{J}[\phi];\boldsymbol{h}]}), \tag{28}$$

and whose first moment is equal to zero by construction. It is easy to derive that its pth cumulant ($p\geq 2$) is given by the derivative with respect to $\phi_1,\ldots,\phi_p$ of $W_p[\boldsymbol{J}[\phi_1],\ldots,\boldsymbol{J}[\phi_p]]$, which can then be related to derivatives of $\Gamma_2,\Gamma_3,\ldots$; for instance,

$$\overline{\breve{h}[\phi_1](\boldsymbol{x})\breve{h}[\phi_2](\boldsymbol{y})}=\Gamma_{2,\boldsymbol{xy}}^{(11)}[\phi_1,\phi_2], \tag{29}$$

where we have used a short-hand notation similar to that of Eqs. (26) and (27) and omitted the N-vector indices for simplicity. Terms of order 3 and higher are again given by more complicated expressions.

We close this discussion by noticing that in the simpler case of the random manifold model, where Γ_1 and W_1 are trivial and unrenormalized due to the statistical tilt symmetry (see above), $\boldsymbol{J}[\phi]$ has a simple explicit expression. For instance, if the bare action has a quadratic one-replica term, $\Gamma_1[\phi]$ is equal to this quadratic functional and $\boldsymbol{J}[\phi]$ is a known linear functional of ϕ, which further simplifies when considering uniform fields. This allows one to devise ways to directly measure the second cumulant of the renormalized disorder.[56,57] Nothing similar occurs in random field and random anisotropy models: the thermodynamics of such systems being highly nontrivial (with a phase transition and a critical point), the expression of $\boldsymbol{J}[\phi]$ is involved and *a priori* unknown.

D. Exact renormalization group equations for the renormalized disorder cumulants

The reasoning developed in Sec. II C can be applied to the effective average action Γ_k and its expansion in free replica sums. As a result, Eqs. (18)–(29) can be extended to any running scale k. Yet, to make the expansion in free replica sums an operational procedure, one needs to be able to perform systematic algebraic manipulations, as for instance the inversion of the matrix appearing on the right-hand side of the exact RG equation [Eq. (16)]. Here, we detail the method for matrices depending on two replica indices but functionals of the n replica fields. Extension to higher-order tensors is presented in Ref. 55.

A generic matrix $A_{ab}[\{\phi_f\}]$, where again $\{\phi_f\}$ denotes the n replica fields to avoid confusion in the indices, can be decomposed as

$$A_{ab}[\{\phi_f\}] = \hat{A}_a[\{\phi_f\}]\delta_{ab} + \widetilde{A}_{ab}[\{\phi_f\}]. \tag{30}$$

In the above expression, it is understood that the second term $\widetilde{A}_{ab}$ no longer contains an explicit Kronecker delta. Each component can now be expanded in increasing number of free replica sums,

$$\hat{A}_a[\{\phi_f\}] = \hat{A}^{[0]}[\phi_a] + \sum_{c=1}^{n} \hat{A}^{[1]}[\phi_a|\phi_c] + \cdots, \tag{31}$$

$$\widetilde{A}_{ab}[\{\phi_f\}] = \widetilde{A}^{[0]}[\phi_a,\phi_b] + \sum_{c=1}^{n} \widetilde{A}^{[1]}[\phi_a,\phi_b|\phi_c] + \cdots, \tag{32}$$

where the superscripts in square brackets denote the order in the expansion (and should not be confused with superscripts in parentheses indicating partial derivatives).

As an illustration, the expansion of the matrix $\boldsymbol{\Gamma}_k^{(2)}$ defined in Eq. (17) reads, in terms of the expansion of the effective average action itself,

$$\hat{\boldsymbol{\Gamma}}_k^{(2)}[\{\phi_f\}]_a = \boldsymbol{\Gamma}_{k,1}^{(2)}[\phi_a] - \sum_{c=1}^{n} \boldsymbol{\Gamma}_{k,2}^{(20)}[\phi_a,\phi_c] + \cdots, \tag{33}$$

$$\widetilde{\boldsymbol{\Gamma}}_k^{(2)}[\{\phi_f\}]_{ab} = -\boldsymbol{\Gamma}_{k,2}^{(11)}[\phi_a,\phi_b] + \sum_{c=1}^{n} \boldsymbol{\Gamma}_{k,3}^{(110)}[\phi_a,\phi_b,\phi_c] + \cdots, \tag{34}$$

where the permutational symmetry of the arguments of the $\Gamma_{k,p}$'s has been used.

Algebraic manipulations on such matrices can be performed by term-by-term identification of the orders of the expansions. For instance, the inverse $\boldsymbol{B}=\boldsymbol{A}^{-1}$ of the matrix $\boldsymbol{A}$ can also be put in the form of Eq. (30) and its components, $\hat{B}_a$ and $\widetilde{B}_{ab}$, can be expanded in a number of free replica sums. The term-by-term identification of the condition $\boldsymbol{A}\cdot\boldsymbol{B}=\boldsymbol{1}$ leads to a unique expression of the various orders, $\hat{B}^{[p]}$ and $\widetilde{B}^{[p]}$, of the expansion of $\boldsymbol{B}$ in terms of the $\hat{A}^{[q]}$'s and $\widetilde{A}^{[q]}$'s with $q\leq p$. The algebra becomes rapidly tedious, but the first few terms are easily derived,

$$\hat{B}^{[0]}[\phi_1] = \hat{A}^{[0]}[\phi_1]^{-1}, \tag{35}$$

$$\widetilde{B}^{[0]}[\phi_1,\phi_2] = -\hat{B}^{[0]}[\phi_1]\widetilde{A}^{[0]}[\phi_1,\phi_2]\hat{B}^{[0]}[\phi_2], \tag{36}$$

$$\hat{B}^{[1]}[\phi_1|\phi_2] = -\hat{B}^{[0]}[\phi_1]\hat{A}^{[1]}[\phi_1|\phi_2]\hat{B}^{[0]}[\phi_1], \tag{37}$$

$$\begin{aligned}\widetilde{B}^{[1]}[\phi_1,\ \phi_2|\phi_3] = -\hat{B}^{[0]}[\phi_1]\{&\widetilde{A}^{[1]}[\phi_1,\ \phi_2|\phi_3]\\ &-\widetilde{A}^{[0]}[\phi_1,\phi_3]\hat{B}^{[0]}[\phi_3]\widetilde{A}^{[0]}[\phi_3,\phi_2]\\ &-\hat{A}^{[1]}[\ \phi_1|\phi_3]\hat{B}^{[0]}[\phi_1]\widetilde{A}^{[0]}[\phi_1,\phi_2]\\ &-\widetilde{A}^{[0]}[\phi_1,\phi_2]\hat{B}^{[0]}[\phi_2]\hat{A}^{[1]}[\ \phi_2|\phi_3]\}\hat{B}^{[0]}[\phi_2],\end{aligned} \tag{38}$$

etc.

We can apply the above procedure to the exact RG equation for the effective average action. For convenience, we introduce the modified propagator at the scale k,

$$\boldsymbol{P}_k[\{\phi_f\}] = [\boldsymbol{\Gamma}_k^{(2)} + \boldsymbol{R}_k]^{-1}, \tag{39}$$

with

$$\boldsymbol{P}_{k,ab}[\{\phi_f\}] = \hat{\boldsymbol{P}}_{k,a}[\{\phi_f\}]\delta_{ab} + \widetilde{\boldsymbol{P}}_{k,ab}[\{\phi_f\}], \tag{40}$$

where $\hat{\boldsymbol{P}}_{k,a}$ and $\widetilde{\boldsymbol{P}}_{k,ab}$ are still tensors with respect to the momenta and vector component indices. Equation (16) then leads to an infinite hierarchy of flow equations for the cumulants of the renormalized disorder,

$$\begin{aligned}\partial_k\Gamma_{k,1}[\phi_1] = \frac{1}{2}\int_{\boldsymbol{q}} \{&\partial_k[\hat{R}_k(q^2) + \widetilde{R}_k(q^2)]\mathrm{tr}\ \hat{\boldsymbol{P}}_{k,\boldsymbol{q}-\boldsymbol{q}}^{[0]}[\phi_1]\\ &+\partial_k\hat{R}_k(q^2)\mathrm{tr}\ \widetilde{\boldsymbol{P}}_{k,\boldsymbol{q}-\boldsymbol{q}}^{[0]}[\phi_1,\phi_1]\},\end{aligned} \tag{41}$$

$$\begin{aligned}\partial_k\Gamma_{k,2}[\phi_1,\phi_2] = -\frac{1}{2}\int_{\boldsymbol{q}} \{&\partial_k[\hat{R}_k(q^2) + \widetilde{R}_k(q^2)]\mathrm{tr}\ \hat{\boldsymbol{P}}_{k,\boldsymbol{q}-\boldsymbol{q}}^{[1]}[\phi_1|\phi_2]\\ &+\partial_k\hat{R}_k(q^2)\mathrm{tr}\ \widetilde{\boldsymbol{P}}_{k,\boldsymbol{q}-\boldsymbol{q}}^{[1]}[\phi_1,\phi_1|\phi_2]\\ &+\partial_k\widetilde{R}_k(q^2)\mathrm{tr}\ \widetilde{\boldsymbol{P}}_{k,\boldsymbol{q}-\boldsymbol{q}}^{[0]}[\phi_1,\phi_2] + \mathrm{perm}(12)\},\end{aligned} \tag{42}$$

and so on, where tr indicates a trace over N-vector components and perm(12) denotes the expression obtained by permuting ϕ_1 and ϕ_2. (Some care is needed in the term by term identification in order to properly symmetrize the expressions and satisfy the permutational property of the various arguments of the cumulants.)

By expressing the higher-order terms $\hat{\boldsymbol{P}}_k^{[p]}$ and $\widetilde{\boldsymbol{P}}_k^{[p]}$ with $p\geq 1$ only by means of $\hat{\boldsymbol{P}}_k^{[0]}$ and the derivatives of the $\Gamma_{k,p}$'s and by introducing the short-hand notation $\widetilde{\partial}_k$ to indicate a derivative acting only on the cutoff functions, i.e., $\tilde{\partial}_k \equiv \partial_k \hat{R}_k\, \delta/\delta\hat{R}_k + \partial_k \tilde{R}_k\, \delta/\delta\tilde{R}_k$, Eq. (42) can be rewritten as

$$\partial_k \Gamma_{k,2}[\phi_1,\phi_2] = \frac{1}{2}\tilde{\partial}_k \,\mathrm{Tr}\Big\{ \hat{\boldsymbol{P}}_k^{[0]}[\phi_1](\boldsymbol{\Gamma}_{k,2}^{(20)}[\phi_1,\phi_2] - \boldsymbol{\Gamma}_{k,3}^{(110)}[\phi_1,\phi_1,\phi_2]) + \tilde{\boldsymbol{P}}_k^{[0]}[\phi_1,\phi_1]\boldsymbol{\Gamma}_{k,2}^{(20)}[\phi_1,\phi_2] + \frac{1}{2}\tilde{\boldsymbol{P}}_k^{[0]}[\phi_1,\phi_2] \times (\boldsymbol{\Gamma}_{k,2}^{(11)}[\phi_1,\phi_2] - \tilde{R}_k \mathbf{1}) + \mathrm{perm}(12)\Big\}, \tag{43}$$

and similarly for higher-order cumulants, where $1_{qq'}^{\mu\nu} = (2\pi)^d \delta(\boldsymbol{q}+\boldsymbol{q}')\delta_{\mu\nu}$ and the trace Tr is now over both momenta and N-vector components; the modified propagators $\hat{\boldsymbol{P}}_k^{[0]}$ and $\tilde{\boldsymbol{P}}_k^{[0]}$ are explicitly given by

$$\hat{\boldsymbol{P}}_k^{[0]}[\phi_1] = (\boldsymbol{\Gamma}_{k,1}^{(2)}[\phi_1] + \hat{R}_k \mathbf{1})^{-1}, \tag{44}$$

$$\tilde{\boldsymbol{P}}_k^{[0]}[\phi_1,\phi_2] = \hat{\boldsymbol{P}}_k^{[0]}[\phi_1](\boldsymbol{\Gamma}_{k,2}^{(11)}[\phi_1,\phi_2] - \tilde{R}_k \mathbf{1})\hat{\boldsymbol{P}}_k^{[0]}[\phi_2]. \tag{45}$$

This provides a hierarchy of exact RG equations for the cumulants of the renormalized disorder (including the first one that leads to a description of the thermodynamics). One should note that (i) the cumulants are functionals of the fields and contain full information on the complete set of $1-PI$ correlation functions and (ii) the flow equations are coupled, with the $(p+1)$th cumulant appearing on the right-hand side of the equation for the pth cumulant. As such, these RG equations remain untractable and their resolution requires approximations.

III. NONPERTURBATIVE APPROXIMATION SCHEME

A. Symmetries in the effective average action formalism

When writing the RG flow for the effective average action and when devising an approximation scheme to solve it, one should as far as possible make sure that the symmetries of the theory are not explicitly violated at any scale. Such a requirement is easily implemented as far as elementary symmetries, such as invariance by translation and rotation in Euclidean space, $O(N)$ symmetry, and S_n replica permutational symmetry, are concerned: the infrared regulator ΔS_k added to the bare action must be chosen such that it is invariant under the appropriate transformations, which is indeed guaranteed by the expressions in Eqs. (11) and (12). The exact effective average action at any scale k then also possesses the symmetries of the bare action, and one just has to be careful that the truncations do not explicitly break the symmetries, which is easily implemented.[36]

A similar treatment can be applied to most additional symmetries of the disordered systems under consideration. For instance, the "statistical tilt symmetry" of the random manifold model is easily extended to a k-dependent statistical tilt symmetry with any regulator of the form given in Eqs. (11) and (12), which implies that the one-replica part (first cumulant) of the effective average action is unrenormalized along the flow. Similarly, the additional inversion symmetries of the random anisotropy ($\boldsymbol{\chi}_a \cdot \boldsymbol{\chi}_b \to -\boldsymbol{\chi}_a \cdot \boldsymbol{\chi}_b$) and the random temperature ($\chi_a, \{\chi_b\}_{b\neq a} \to -\chi_a, \{\chi_b\}_{b\neq a}$) models are readily accounted for with the choice $\tilde{R}_k \equiv 0$. Truncation schemes naturally follow.

Taking into account the underlying supersymmetry that characterizes the random field model for a Gaussian distribution of the random field[7] is much more involved. First, because one knows that the supersymmetry, which goes with the dimensional reduction property, must be broken in low enough dimensions (at least, in $d=3$) so that, even if the RG flow is started with an initial condition obeying supersymmetry, a mechanism should be provided to describe a spontaneous breaking of the supersymmetry. Second, the supersymmetry shows up in a superfield formalism built with auxiliary fermionic and bosonic fields, but it is far from transparent in the present framework based on the fundamental fields. (This is already true at the level of the initial condition of the RG flow.) We shall therefore defer the proper resolution of this problem to a forthcoming publication.[58] Note that an underlying supersymmetry is also present in the random manifold model, where it also (incorrectly) leads to the $d \to d-2$ dimensional reduction. However, the pure model with no disorder is merely a free field theory, and this is easily accounted for.[59]

B. Truncation schemes

We have already stressed that solving the exact RG equation for the effective average action requires approximations. The general framework has proven quite versatile for devising efficient and numerically tractable approximations that are able to describe both universal and nonuniversal properties in any spatial dimension and to capture genuine nonperturbative phenomena (see Sec. I). Such approximations generally amount to truncating the functional form of the effective average action, which results in a self-consistent flow that preserves the fundamental structure of the theory (as the symmetries, see above).

If one is interested in the long-distance physics of a system and in observables at small momenta, a systematic truncation scheme is provided by the so-called *derivative expansion*.[34,36] It consists in expanding the effective average action in increasing number of derivatives of the field(s) and retaining only a limited number of terms. The lowest order is the "local potential approximation" (LPA),[60] in which one only considers the flow of the effective average potential, i.e., the effective average action for a uniform field configuration. The field is not renormalized and the associated anomalous dimension η is equal to zero. Field renormalization, which is important in the present problem wherein one expects the anomalous dimension to be quite sizeable in low dimensions (e.g., numerical estimates give $\eta \simeq 0.5$ for the RFIM in $d=3$), requires to go beyond the LPA and to consider the first order of the derivative expansion. Previous studies on a variety of systems, including the pure $O(N)$ model, have shown that the system's behavior is quantitatively very well described at this level of approximation.[36,37,61,62] Higher-order terms improve the accuracy,[63,64] but they rapidly become untractable except in simple models.

For the disordered systems considered here, one more step is needed. We have seen in Sec. II C that an *expansion in number of free replica sums* can be used to generate the cumulants of the renormalized disorder. Keeping only a limited number of terms in the expansion therefore leads to a systematic truncation scheme. To describe both the thermodynamics and the renormalized probability distribution of the disorder, one must consider at least the first two cumulants, or equivalently, the second order in the expansion in free replica sums.

Finally, on top of the two previous approximations, it may be useful, and numerically more tractable, to expand the functions appearing in the truncated effective average action in powers of the field considered around a given (uniform) configuration. This configuration can be taken either as zero everywhere or as a nontrivial configuration that minimizes the effective average potential (here, more precisely, its one-replica component that gives access to the thermodynamics). Again, the accuracy and convergence properties of such field expansions have been widely tested for many different models. In the present case, wherein nonanalyticities in the field dependences will be encountered, field expansions should be used with great caution.

C. Minimal truncation

Given the general scheme presented above, the choice of a minimal nonperturbative truncation is guided by a combination of factors: experience gained from studies on other models, constraints associated with the symmetries of the full theory, intuition or previous knowledge concerning the physics of the problem at hand, requirement of being able to recover as much as possible exact and perturbative results in the appropriate limits, and of course a practical limitation coming with the numerical capability to actually solve the set of RG flow equations.

As we have already alluded to, a description of the long-distance physics of random field models and related disordered systems at least requires to keep the first two cumulants of the disorder, i.e., the first two terms $\Gamma_{k,1}$ and $\Gamma_{k,2}$ of the expansion of the effective average action in free replica sums. Because of the anticipated non-negligible value of the anomalous dimension of the field η, in the description one must also include at least the first order of the derivative expansion of the first cumulant $\Gamma_{k,1}$. The resulting truncated functional form of the effective average action then reads

$$\Gamma_k[\{\phi_a\}] = \int_{\boldsymbol{x}} \Bigg\{ \sum_{a=1}^{n} \Bigg[U_k(\rho_a(\boldsymbol{x})) + \frac{1}{2} Z_k(\rho_a(\boldsymbol{x})) |\partial \phi_a(\boldsymbol{x})|^2 + \frac{1}{4} Y_k(\rho_a(\boldsymbol{x}))(\partial \rho_a(\boldsymbol{x}))^2 \Bigg] - \frac{1}{2} \sum_{a,b=1}^{n} V_k(\phi_a(\boldsymbol{x}), \phi_b(\boldsymbol{x})) \Bigg\}, \quad (46)$$

where, as before, $\rho_a(\boldsymbol{x}) = \frac{1}{2}|\phi_a(\boldsymbol{x})|^2$. In the above expressions, $U_k(\phi_1) \equiv U_k(\rho_1)$ is the effective average potential, which is equal to the one-replica component $\Gamma_{k,1}$ evaluated for a uniform field and will hereafter be simply denoted as the one-replica potential; $V_k(\phi_1, \phi_2) \equiv V_k(\rho_1, \rho_2, \phi_1 \cdot \phi_2)$ is the two-replica potential and is equal to the two-replica component $\Gamma_{k,2}$ evaluated for a uniform field configuration. Physically, $U_k(\phi_1)$ is a coarse-grained Gibbs free energy and $V_k(\phi_1, \phi_2)$ is the second cumulant of the renormalized disorder evaluated for uniform fields [see Eqs. (22) and (24)]. The two terms $Z_k(\rho_1)$ and $Y_k(\rho_1)$ correspond to field renormalization functions for the Goldstone and massive modes, respectively.

We note in passing that the fact that only the first two cumulants of the disorder have been kept in the truncation does not imply that the probability distribution of the renormalized disorder is actually taken as Gaussian. Indeed, as will be discussed in the companion paper,[32] the probability is not Gaussian in general. The truncation means that we have neglected the contribution coming from the third cumulant in the RG flow of the second cumulant and have therefore decoupled the hierarchy of flow equations for the cumulants.

Being interested in the description of the models in the full (N,d) diagram, we will have recourse to further approximations that make the numerical resolution of the flow equations easier. More specifically, we consider the lowest-order term of the field expansion of the field renormalization functions around a nontrivial configuration, $\rho_{m,k} = \frac{1}{2}|\phi_{m,k}|^2$, which minimizes the one-replica potential $U_k(\rho)$: $Y_k \equiv 0$ and $Z_k(\rho) \equiv Z_{m,k}$, with $Z_{m,k} = Z_k(\rho_{m,k})$ and $U'_k(\rho_{m,k}) = 0$. Physically, $\phi_{m,k}$ is the magnetization (order parameter) at the scale k. [If $\phi_{m,k\to 0} = 0$, the system is in an $O(N)$ symmetric phase, whereas if $\phi_{m,k\to 0} \neq 0$, the system is in the phase with broken symmetry.] $Z_{m,k}$ is chosen as the field renormalization in the cutoff function $\hat{R}_k(q^2)$ [see Eq. (13)]. Finally, we simplify the resulting RG flow equations by setting the off-diagonal cutoff function to zero, $\widetilde{R}_k \equiv 0$.

With the above approximations, which we shall refer to as the minimal truncation, the self-consistent NP-FRG equations can be derived from Eqs. (41)–(43). The flows of the one- and two-replica potentials read

$$\partial_k U_k(\rho_1) = \frac{1}{2} \int_{\boldsymbol{q}} \partial_k \hat{R}_k(q^2) \mathrm{tr}\{\hat{\boldsymbol{P}}_k^{[0]}(q^2;\rho_1) - \hat{\boldsymbol{P}}_k^{[0]}(q^2;\rho_1) V_k^{(11)}(\phi_1,\phi_1) \hat{\boldsymbol{P}}_k^{[0]}(q^2;\rho_1)\}, \quad (47)$$

$$\partial_k V_k(\phi_1,\phi_2) = -\frac{1}{2} \int_{\boldsymbol{q}} \partial_k \hat{R}_k(q^2) \mathrm{tr}\{\hat{\boldsymbol{P}}_k^{[0]}(q^2;\rho_1)[V_k^{(20)}(\phi_1,\phi_2) + \hat{\boldsymbol{P}}_k^{[0]}(q^2;\rho_2) V_k^{(11)}(\phi_1,\phi_2)^2 + V_k^{(20)}(\phi_1,\phi_2) \hat{\boldsymbol{P}}_k^{[0]}(q^2;\rho_1) V_k^{(11)}(\phi_1,\phi_1) + V_k^{(11)}(\phi_1,\phi_1) \hat{\boldsymbol{P}}_k^{[0]}(q^2;\rho_1) V_k^{(20)}(\phi_1,\phi_2)] \times \hat{\boldsymbol{P}}_k^{[0]}(q^2;\rho_2) + \mathrm{perm}(12)\}, \quad (48)$$

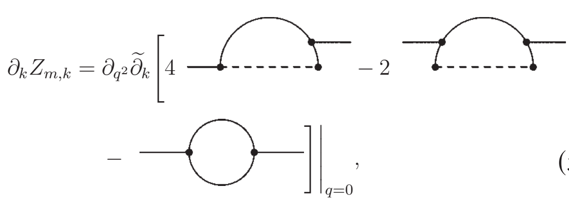
where the trace is over the N-vector components and, due to the $O(N)$ symmetry, $V_k(\phi_1,\phi_2) \equiv V_k(\rho_1,\rho_2,z)$ where z

$=\phi_1\cdot\phi_2/\sqrt{4\rho_1\rho_2}$; the (modified) propagator $\hat{\boldsymbol{P}}_k^{[0]}(q^2;\rho)$ is given by

$$\hat{\boldsymbol{P}}_k^{[0]}(q^2;\rho)^{\mu\nu}=\left[\frac{(1-\delta_{\mu 1})}{Z_{m,k}q^2+\hat{R}_k(q^2)+U_k'(\rho)}+\frac{\delta_{\mu 1}}{Z_{m,k}q^2+\hat{R}_k(q^2)+U_k'(\rho)+2\rho U_k''(\rho)}\right]\delta_{\mu\nu}, \tag{49}$$

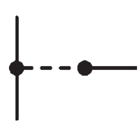
where $\mu=1$ is chosen to be the direction of the field ϕ and therefore corresponds to the massive mode while the $(N-1)$ remaining components represent the Goldstone modes.

The flow of the field renormalization constant $Z_{m,k}$ is obtained from the prescription $Z_k(\rho)=\partial_{q^2}\Gamma_{k,1}^{(2)}(q^2;\rho)^{\mu\mu}|_{q^2=0}$, where μ is chosen as a Goldstone mode $(\mu\neq 1)$[36] and from the condition $U_k'(\rho_{m,k})=0$. It can be explicitly written as



(50)

where a line denotes the Goldstone propagator and the dots represent vertices obtained from derivatives of either the one-replica potential (single dots) or the two-replica potential (dots linked by a dashed line); for instance,



represents the three-point vertex $\Gamma_k^{(21)}\equiv V_k^{(21)}$. We did not include the graphs containing four-point vertices because in the truncation considered here, they do not contribute to the flow of $Z_{m,k}$. From the above flow equation [Eq. (50)], one extracts a running anomalous exponent,

$$\eta_k=-k\partial_k Z_{m,k}. \tag{51}$$

The initial conditions for the RG flow equations are obtained from the bare action [Eq. (10)]. The RG flow equations form a closed set of coupled nonlinear integrodifferential equations for two functions, $U_k(\rho_1)$ and $V_k(\rho_1,\rho_2,z)$, and a constant, $Z_{m,k}$. The numerical task of solving these equations is still arduous and when needed for reducing the difficulty of the computations, we will also consider truncated expansions of the one- and two-replica potentials in some or all of their field arguments (see below).

The present approach represents a nonperturbative but of course approximate RG description. Already at the minimal truncation discussed above, one includes all operators previously suggested to be important for capturing the long-distance behavior of the present disordered models, namely operators involving 1- and 2-replica terms. As will be shown further below, it also reduces to the leading results of perturbative RG analyses near the upper critical dimension, d_{uc} $=6$, near the lower critical dimension for ferromagnetism when $N>1$, $d=4$, and when the number of components N becomes infinite. One of its main advantages is that it provides a unified framework to describe models in any spatial dimension d and for any number N of field components. As such, it guarantees a consistent interpolation of all known results in the whole (N,d) plane in addition to allowing the study of genuine nonperturbative phenomena. If a greater accuracy is needed, the truncation scheme proposed in Sec. III B gives a systematic means to refine the description by including, e.g., the third cumulant or a more detailed account of the momentum dependence of the $1-PI$ vertices. In the following, we more specifically focus on the random field $O(N)$ model.

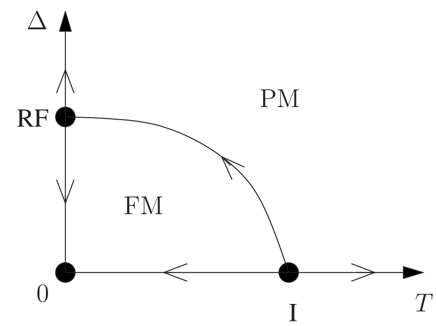


FIG. 1. Schematic phase diagram of the RFIM in the disorder strength Δ-temperature T plane above the lower critical dimension $d_{lc}=2$ (temperature can be introduced at the bare level through the Boltzmann weight). At low disorder and low temperature, the system is ferromagnetic, and it is paramagnetic otherwise. The arrows describe how the renormalized parameters evolve under the RG flow at long distance, and I and RF denote the critical fixed points of the pure and random field Ising models, respectively.

IV. RANDOM FIELD MODEL

A. Scaling dimensions near a zero-temperature fixed point

For the RFIM, it has been proposed,[65,66] and convincingly supported by numerical and experimental results,[6,40,67] that the fixed point controlling the critical behavior associated with the transition between a high-temperature—or a large-disorder strength—disordered (paramagnetic) phase and a low-temperature—or a small-disorder strength—ordered (ferromagnetic) phase is at zero temperature (see Fig. 1).

The existence of such a zero-temperature fixed point around which the temperature is dangerously irrelevant leads to a somewhat anomalous scaling at the critical point.[65,66] The two independent critical exponents characterizing the scaling behavior of the pure Ising model should *a priori* be supplemented by an additional exponent θ that describes the vanishing of the (renormalized) temperature as the fixed point is approached. This exponent θ leads to a modification of the so-called hyperscaling relation, which becomes $2-\alpha=(d-\theta)\nu$, where the critical exponents α and ν have their usual meaning, and to a new scaling of the correlation functions. In particular, the so-called "connected" and "disconnected" components of the pair correlation function (or two-point Green's function) behave at the critical point as

$$G_c(q) = \overline{\langle\chi(-\boldsymbol{q})\chi(\boldsymbol{q})\rangle} - \overline{\langle\chi(-\boldsymbol{q})\rangle\langle\chi(\boldsymbol{q})\rangle} \sim q^{-(2-\eta)}, \quad (52)$$

$$G_d(q) = \overline{\langle\chi(-\boldsymbol{q})\rangle\langle\chi(\boldsymbol{q})\rangle} \sim q^{-(4-\bar{\eta})}, \quad (53)$$

where η is the usual anomalous dimension of the field and $\bar{\eta}$ is related to the temperature exponent θ through

$$\bar{\eta} = 2 - \theta + \eta. \quad (54)$$

Above the upper critical dimension $d_{uc}=6$, the exponents take their classical (mean-field) values, i.e., $\eta=0$, $\alpha=0$, $\nu=1/2$, and $\theta=2$, leading to $\bar{\eta}=0$. The dimensional reduction property leads to a constant shift of dimension, $d \to d-2$, i.e., to $\theta=2$ and $\bar{\eta}=\eta$, and all the exponents are given by those of the pure model in dimension $d-2$. Whether the scaling behavior around the critical point is described by the three independent exponents, or only two, has been a long-time issue, with suggestions that an additional relation applies, $\theta=2-\eta$, or equivalently, $\bar{\eta}=2\eta$.[68] We shall address and answer this question in the following paper.[32]

To search for a zero-temperature fixed point, it is convenient to introduce a renormalized temperature. Actually, one could add an explicit temperature T in the Landau–Ginzburg–Wilson description of the model considered here: multiplying the argument of the exponential in the partition function [Eq. (4)] by a factor T^{-1} to make the correspondence with the Boltzmann factor of statistical physics leads to a bare replicated action in Eqs. (6) and (10), in which the one-replica part, which includes the kinetic term, is multiplied by a factor T^{-1}, the two-replica part by T^{-2}, etc. Generally speaking, one can use this temperature T as a bookkeeping device to sort the orders in the expansions in number of free replica sums. As a result, for instance, the modified propagator $\hat{P}_k^{[0]}[\phi_1]$ is proportional to T, whereas $\tilde{P}_k^{[0]}[\phi_1,\phi_2]$ is independent of T. One can use this bookkeeping trick to devise ways to define a renormalized temperature at running scale k, T_k, which reduces to the "bare" temperature T at the microscopic scale $k=\Lambda$. To this end, we first define the renormalized disorder strength at scale k, $\Delta_{m,k}$, as

$$\Delta_{m,k} = \Delta_k(\phi_1 = \phi_{m,k}, \phi_2 = \phi_{m,k}), \quad (55)$$

where, as before, $\phi_{m,k}$ is a field configuration that minimizes the (one-replica) potential $U_k(\phi)$ and $\Delta_k(\phi_1,\phi_2)$ is the second cumulant of the renormalized effective random field defined as in Eq. (29), namely,

$$\Delta_k(\phi_1,\phi_2) = V_k^{(11)}(\phi_1,\phi_2). \quad (56)$$

In the present truncation, the second cumulant is considered only for homogeneous field configurations and $\Gamma_{k,2}^{(11)}$ reduces to $V_k^{(11)}$ with the same notations for partial derivatives as in Eqs. (26) and (27) [e.g., $V_k^{(11)}(\phi_1,\phi_2) = \partial_{\phi_1}\partial_{\phi_2}V_k(\phi_1,\phi_2)$]. At the microscopic scale Λ, $\Delta_{m,k}$ reduces to Δ_Λ/T^2, where Δ_Λ is the bare variance of the random field and the factor T^{-2} comes for reasons just explained above.

A running temperature can now be defined by

$$T_k = \frac{Z_{m,k}(k/\Lambda)^2}{(\Delta_{m,k}/\Delta_\Lambda)}. \quad (57)$$

One checks that since $Z_{m,\Lambda}=T^{-1}$ [see Eq. (10) and the discussion above], T_k indeed reduces to T when $k=\Lambda$. An associated running exponent is obtained from

$$\theta_k = k\partial_k \ln T_k. \quad (58)$$

By using the definition of η_k, one may alternatively introduce a running exponent $\bar{\eta}_k = 2-\theta_k+\eta_k$, which converges to the critical exponent $\bar{\eta}$ defined in Eqs. (53) and (54) if the relevant fixed point is reached, and compute it from

$$\bar{\eta}_k - 2\eta_k = k\partial_k\Delta_{m,k}. \quad (59)$$

On top of the usual scaling dimensions, $U_k, V_k \sim k^d$ and $\phi \sim (Z_{m,k}^{-1}k^{d-2})^{1/2}$, one can use the running temperature to define dimensionless quantities (denoted by lowercase letters) suitable for looking for a zero-temperature fixed point,

$$\phi = \left(\frac{k^{d-2}}{Z_{m,k}T_k}\right)^{1/2}\varphi, \quad (60a)$$

$$U_k(\phi_1) = \frac{k^d}{T_k}u_k(\varphi_1), \quad (60b)$$

$$V_k(\phi_1,\phi_2) = \frac{k^d}{T_k^2}v_k(\varphi_1,\varphi_2), \quad (60c)$$

$$\Delta_k(\phi_1,\phi_2) = \frac{Z_{m,k}k^2}{T_k}\delta_k(\varphi_1,\varphi_2), \quad (60d)$$

with $\delta_k(\varphi_1,\varphi_2) = v_k^{(11)}(\varphi_1,\varphi_2)$. Note that with the definitions of $\Delta_{m,k}$ and T_k, $\delta_{m,k} \equiv \delta_k(\varphi_{m,k},\varphi_{m,k})$ is constant along the RG flow and equal to its initial value Δ_Λ/Λ^2 (in practice and since we are not interested here in making a precise connection to the microscopic scale, we will set $\delta_{m,k}=1$).

B. Scaled form of the exact renormalization group equations for the random field Ising model

With the use of the above defined dimensionless renormalized quantities, the flow equations can be expressed in a scaled form. Specifically, one can recast Eqs. (47) and (48) for $N=1$ in the form

$$\partial_t u_k(\varphi) = -(d-2+\bar{\eta}_k-\eta_k)u_k(\varphi) + \frac{1}{2}(d-4+\bar{\eta}_k)\varphi u_k'(\varphi) + 2v_d\{l_1^{(d)}(u_k''(\varphi))\delta_k(\varphi,\varphi) + T_k l_0^{(d)}(u_k''(\varphi))\}, \quad (61)$$

$$\partial_t v_k(\varphi_1,\varphi_2) = -(d-4+2\bar{\eta}_k-2\eta_k)v_k(\varphi_1,\varphi_2) + \frac{1}{2}(d-4+\bar{\eta}_k)(\varphi_1\partial_{\varphi_1}+\varphi_2\partial_{\varphi_2})v_k(\varphi_1,\varphi_2)$$
$$- 2v_d\{l_{1,1}^{(d)}(u_k''(\varphi_1),u_k''(\varphi_2))\delta_k(\varphi_1,\varphi_2)^2 + l_2^{(d)}(u_k''(\varphi_1))\delta_k(\varphi_1,\varphi_1)v_k^{(20)}(\varphi_1,\varphi_2) + l_2^{(d)}(u_k''(\varphi_2))\delta_k(\varphi_2,\varphi_2)v_k^{(02)}(\varphi_1,\varphi_2)$$
$$+ T_k[l_1^{(d)}(u_k''(\varphi_1))v_k^{(20)}(\varphi_1,\varphi_2) + l_1^{(d)}(u_k''(\varphi_2))v_k^{(02)}(\varphi_1,\varphi_2)]\}, \tag{62}$$

where ∂_t is a derivative with respect to $t=\ln(k/\Lambda)$, a prime denotes a derivative with respect to the field (when only one argument is present), $v_d^{-1}=2^{d+1}\pi^{d/2}\Gamma(d/2)$, and we recall that $\delta_k(\varphi_1,\varphi_2)=v_k^{(11)}(\varphi_1,\varphi_2)$; $l_n^{(d)}(w)$ and $l_{n_1,n_2}^{(d)}(w_1,w_2)$ are the "dimensionless threshold functions" defined from the infrared cutoff function [Eq. (13)] as[36,52]

$$l_n^{(d)}(w) = -\frac{1}{2}(n+\delta_{n,0})\int_0^\infty dy y^{d/2}\frac{\eta_k r(y)+2yr'(y)}{[p(y)+w]^{n+1}}, \tag{63}$$

$$l_{n_1,n_2}^{(d)}(w_1,w_2) = -\frac{1}{2}\int_0^\infty dy y^{d/2}[\eta_k r(y) + 2yr'(y)]\frac{1}{[p(y)+w_1]^{n_1}[p(y)+w_2]^{n_2}} \times\left[\frac{n_1}{p(y)+w_1}+\frac{n_2}{p(y)+w_2}\right], \tag{64}$$

where $p(y)=y[1+r(y)]$ and $y=q^2/k^2$. The properties of these threshold functions, whose detailed behavior depends on the choice of the infrared cutoff function $r(y)$, have been extensively discussed.[36,52] They rapidly decay when $w\gg 1$, which, since $u_k''(\varphi)=U_k''(\phi)/(Z_k k^2)$ is the square of a renormalized mass, ensures that only modes with mass smaller than k contribute to the flow in Eqs. (61) and (62). As an illustration, the use of the so-called "optimized" cutoff function, $r(y)=y^{-1}(1-y)\Theta(1-y)$,[53] leads to explicit expressions, namely,

$$l_{n_1,n_2}^{(d)}(w_1,w_2) = \frac{2}{d}\left(1-\frac{\eta_k}{d+2}\right)\frac{1}{(1+w_1)^{n_1}(1+w_2)^{n_2}} \times\left(\frac{n_1}{1+w_1}+\frac{n_2}{1+w_2}\right). \tag{65}$$

The threshold functions essentially encode the nonperturbative effects beyond the standard one-loop approximation. Note that, although not shown in the notation, the threshold functions depend on the scale k via the running exponent η_k.

The above flow equations for $u_k(\varphi_1)$ and $v_k(\varphi_1,\varphi_2)$ are supplemented by equations for η_k and $\bar{\eta}_k$, i.e., for $Z_{m,k}$ and T_k or $\Delta_{m,k}$. (Note that the equation for $\bar{\eta}_k$ is actually redundant as it is a consequence of the other equations; it is nonetheless convenient to introduce and use it.) The flow equation for $Z_{m,k}$ follows from Eq. (50) and one finds

$$\eta_k = \frac{4v_d}{d}\{4m_{3,2}^{(d)}(u_{m,k}'',u_{m,k}'')u_{m,k}'''^2 - 2m_{2,2}^{(d)}(u_{m,k}'',u_{m,k}'')u_{m,k}'''\delta_{m,k}' + T_k m_{2,2}^{(d)}(u_{m,k}'',u_{m,k}'')u_{m,k}'''^2\}, \tag{66}$$

where we have used the short-hand notation $\delta_k'(\varphi)\equiv\partial_\varphi\delta_k(\varphi,\varphi)=\delta_k^{(10)}(\varphi,\varphi)+\delta_k^{(01)}(\varphi,\varphi)$ and the subscript m,k indicates that the functions are evaluated for fields equal to $\varphi_{m,k}$; we have also introduced the additional (dimensionless) threshold function,

$$m_{n_1,n_2}^{(d)}(w_1,w_2) = -\frac{1}{2}\int_0^\infty dy y^{d/2}(1+r(y)+yr'(y)) \times\frac{1}{(p(y)+w_1)^{n_1}(p(y)+w_2)^{n_2}} \times\Bigg\{(1+r(y)+yr'(y))(\eta_k r(y)+2yr'(y)) \times\left(\frac{n_1}{p(y)+w_1}+\frac{n_2}{p(y)+w_2}\right) - 2\eta_k(r(y)+yr'(y)) - 4y(2r'(y)+yr''(y))\Bigg\}, \tag{67}$$

whose properties are discussed in Refs. 36 and 52. For instance, with the "optimized" regulator introduced above,[53] one finds that

$$m_{n_1,n_2}^{(d)}(w_1,w_2) = \frac{1}{(1+w_1)^{n_1}(1+w_2)^{n_2}}. \tag{68}$$

Finally, the flow equation for $\Delta_{m,k}$ [or equivalently the flow of the constraint $\delta_{m,k}=1$ discussed below Eqs. (60a)–(60d)] leads to the following equation:

$$2\eta_k - \bar{\eta}_k = 2v_d\Bigg\{l_4^{(d)}(u_{m,k}'')u_{m,k}'''^2 - 4l_3^{(d)}(u_{m,k}'')u_{m,k}'''\delta_{m,k}' + l_2^{(d)}(u_{m,k}'') \times\left(\delta_{m,k}''+\frac{3}{2}\delta_{m,k}'^2-\frac{u_{m,k}'''}{u_{m,k}''}-\frac{1}{4}\Sigma_{m,k}\right) + l_1^{(d)}(u_{m,k}'')\frac{\delta_{m,k}'^2}{u_{m,k}''} - T_k\left[l_2^{(d)}(u_{m,k}'')u_{m,k}'''\delta_{m,k}' - l_1^{(d)}(u_{m,k}'') \times\left(\frac{1}{2}\delta_{m,k}''-\frac{u_{m,k}'''}{u_{m,k}''}\delta_{m,k}'+\frac{1}{2}\tilde{\Sigma}_{m,k}\right)\right]\Bigg\}, \tag{69}$$

where, as before, $\delta_k'(\varphi)\equiv\partial_\varphi\delta_k(\varphi,\varphi)$ and similarly for $\delta_k''(\varphi)$, and we have introduced

$$\Sigma_k(\varphi_1) = \lim_{\varphi_2\to\varphi_1} (\partial_{\varphi_1} - \partial_{\varphi_2})^2[\delta_k(\varphi_1,\varphi_2) - \delta_k(\varphi_1,\varphi_1)]^2, \tag{70}$$

and

$$\tilde{\Sigma}_k(\varphi_1) = \lim_{\varphi_2\to\varphi_1} (\partial_{\varphi_1} - \partial_{\varphi_2})^2\delta_k(\varphi_1,\varphi_2). \tag{71}$$

All other notations are as before.

Before extending the results to the RF$O(N)$M, we point out important features of the above equations. First, we have kept terms proportional to T_k but, provided one reaches a fixed point with an exponent $\theta=\theta_{k\to 0}>0$ where temperature is thus irrelevant, those terms are subdominant in the scaling region $k\to 0$. In particular, the fixed point is attained by following the flow with an initial temperature T equal to zero.

Second, "anomalous" terms, $\Sigma_{m,k}$ and $T_k\tilde{\Sigma}_{m,k}$, appear in the expression of $2\eta_k-\bar{\eta}_k$. As can be inferred from Eqs. (70) and (71), $\Sigma_{m,k}$ can only differ from zero, and $\tilde{\Sigma}_{m,k}$ become infinite, when a nonanalyticity (a "cusp") in $(\varphi_1-\varphi_2)$ appears in the (dimensionless) renormalized disorder function $\delta_k(\varphi_1,\varphi_2)$ when $\varphi_2\to\varphi_1$ (and both go to $\varphi_{m,k}$). If $\delta_k(\varphi_1,\varphi_2)$ is analytic, no signature of such an anomalous behavior is found. (We have implicitly assumed that no stronger nonanalyticity appears, which means that a fixed point can be reached and that the theory is renormalizable; this has to be checked in actual computations.) We shall come back in more detail to these two important aspects of the NP-FRG approach in the following paper.[32] Finally, one may notice that because of the $Z_2\equiv O(1)$ symmetry, the potential u_k is an even function of φ and because of the additional permutation symmetry, $v_k(\varphi_1,\varphi_2)=v_k(\varphi_2,\varphi_1)=v_k(-\varphi_1,-\varphi_2)=v_k(-\varphi_2,-\varphi_1)$.

C. Generalization to the RF$O(N)$M

The preceding treatment can be extended to the RF$O(N)$M. The variable $\rho=\frac{1}{2}|\phi|^2$ is written in terms of a dimensionless variable, $\rho=k^{d-2}T_k^{-1}Z_{m,k}^{-1}\tilde{\rho}$, where the tilde will be dropped in the following when no confusion is possible between dimensionless and dimensionfull quantities. The variable $z=\phi_1\cdot\phi_2/(2\sqrt{\rho_1\rho_2})$ is already dimensionless.

For the one-replica second-order tensors (in N-vector components) evaluated for a uniform field configuration, e.g., for $\hat{\boldsymbol{P}}_k^{[0]}(q^2;\phi_1)$ or for $\Delta_k(\phi_1,\phi_1)\equiv V_k^{(11)}(\phi_1,\phi_1)$, the $O(N)$ symmetry reduces the number of terms to a "longitudinal" component [corresponding to the massive mode; see Eq. (49)] and $N-1$ identical "transverse" components [corresponding to the Goldstone modes; see Eq. (49)]. We therefore introduce

$$\delta_k^{\mu\nu}(\rho,\rho,z=1) = \delta_{\mu\nu}[\delta_{\mu 1}\delta_{k,L}(\rho) + (1-\delta_{\mu 1})\delta_{k,T}(\rho)], \tag{72}$$

where

$$\delta_{k,L}(\rho) = 2\rho\partial_{\rho_1}\partial_{\rho_2}v(\rho_1,\rho_2,z=1)|_{\rho_1=\rho_2=\rho}, \tag{73}$$

$$\delta_{k,T}(\rho) = \frac{1}{2\rho}\,\partial_z v(\rho,\rho,z)|_{z=1}, \tag{74}$$

and we define the longitudinal, $w_{k,L}(\rho)$, and transverse, $w_{k,T}(\rho)$, masses as

$$w_{k,L}(\rho) = u_k'(\rho) + 2\rho u_k''(\rho), \tag{75}$$

$$w_{k,T}(\rho) = u_k'(\rho), \tag{76}$$

where the primes now denote derivatives with respect to ρ.

The renormalized disorder strength at the running scale k can be characterized, e.g., through the transverse component, $\Delta_{k,T}(\rho,\rho,z=1)$, evaluated for $\rho=\rho_{m,k}=\frac{1}{2}|\phi_{m,k}|^2$, and T_k is accordingly introduced. Expressing the $O(N)$ symmetry in the two-replica second-order tensors is a little more tedious but nonetheless straightforward.

The resulting flow equations in scaled form read [where for the ease of notation, we drop the subscript k on the right-hand sides, i.e., up to a sign, the beta functions, for all quantities except T_k, and also drop the argument of $v(\rho_1,\rho_2,z)$]

$$\begin{aligned}\partial_t u_k(\rho) = &-(d-2+\bar{\eta}-\eta)u(\rho) + (d-4+\bar{\eta})\rho u'(\rho)\\ &+ 2v_d\{(N-1)l_1^{(d)}(w_T(\rho))\delta_T(\rho) + l_1^{(d)}(w_L(\rho))\delta_L(\rho)\}\\ &+ 2T_k v_d\{(N-1)l_0^{(d)}(w_T(\rho)) + l_0^{(d)}(w_L(\rho))\},\end{aligned} \tag{77}$$

$$\begin{aligned}\partial_t v_k(\rho_1,\rho_2,z) = &-(d-4+2\bar{\eta}-2\eta)v + (d-4+\bar{\eta})(\rho_1\partial_{\rho_1}+\rho_2\partial_{\rho_2})v - \frac{v_d}{4\rho_1\rho_2}\{(N-1)[4\rho_2 l_2^d(w_T(\rho_1))\delta_T(\rho_1)(2\rho_1\partial_{\rho_1}v - z\partial_z v)\\ &+ l_{1,1}^{(d)}(w_T(\rho_1),w_T(\rho_2))(\partial_z v)^2] + (1-z^2)[4\rho_2 l_2^{(d)}(w_T(\rho_1))\delta_T(\rho_1)\partial_z^2 v + 8\rho_2^2 l_{1,1}^{(d)}(w_T(\rho_1),w_L(\rho_2))(\partial_{\rho_2}\partial_z v)^2\\ &- l_{1,1}^{(d)}(w_T(\rho_1),w_T(\rho_2))((\partial_z v)^2 + 2z\partial_z v\partial_z^2 v - (1-z^2)(\partial_z^2 v)^2)] + 8\rho_1\rho_2[l_2^{(d)}(w_L(\rho_1))\delta_L(\rho_1)(\partial_{\rho_1}v + 2\rho_1\partial_{\rho_1}^2 v)\\ &+ 2\rho_1\rho_2 l_{1,1}^{(d)}(w_L(\rho_1),w_L(\rho_2))(\partial_{\rho_1}\partial_{\rho_2}v)^2] + \text{perm}(12)\} - T_k\frac{v_d}{\rho_1\rho_2}\{(N-1)\rho_2 l_1^d(w_T(\rho_1))(2\rho_1\partial_{\rho_1}v - z\partial_z v)\\ &+ (1-z^2)\rho_2 l_1^d(w_T(\rho_1))\partial_z^2 v + 2\rho_1\rho_2 l_1^d(w_L(\rho_1))(\partial_{\rho_1}v + 2\rho_1\partial_{\rho_1}^2 v) + \text{perm}(12)\},\end{aligned} \tag{78}$$

$$\eta_k = \frac{v_d}{d}\Bigg\{ 8[m_{2,3}^{(d)}(w_L(\rho_m),0) + m_{3,2}^{(d)}(w_L(\rho_m),0)]\delta_T(\rho_m)\frac{w_L(\rho_m)^2}{\rho_m} + 8m_{3,1}^{(d)}(w_L(\rho_m),0)w_L(\rho_m)[\delta_T(\rho_m) - \delta_L(\rho_m)]\Bigg\}, \quad (79)$$

$$2\eta_k - \bar{\eta}_k = \frac{2v_d}{\rho_m u''(\rho_m)}\{(N-1)\rho_m l_1^d(0)\delta_T'(\rho_m)^2 + l_2^d(0)u''(\rho_m) + l_2^{(d)}(w_L(\rho_m))\delta_L(\rho_m)[(1+2\rho_m\delta_T'(\rho_m) + 2\rho_m^2\delta_T''(\rho_m))u''(\rho_m) - 2\rho_m^2 u'''(\rho_m)\delta_T'(\rho_m)] - 2l_{1,1}^{(d)}(0,w_L(\rho_m))u''(\rho_m)[1+\rho_m\delta_T'(\rho_m)]^2 + \rho_m l_1^{(d)}(w_L(\rho_m))\delta_T'(\rho_m)\delta_L'(\rho_m)\} + \cdots, \quad (80)$$

where all symbols have the same meaning as in the previous equations and, by construction, $w_L(\rho_m)=2\rho_m u''(\rho_m)$, $w_T(\rho_m)=0$, and $\delta_T(\rho_m)=1$. Note that in the last two equations, for simplicity we have omitted the (subdominant) terms involving T_k in the beta functions and that in Eq. (80), the dots denote "anomalous" terms, which generalize those found for the RFIM [see Eq. (69)] and vanish when the function $v_k(\rho_1,\rho_2,z)$ is analytic in all its arguments; their expression is lengthy and will be discussed in the companion paper.[32]

When $N=1$ and $z=\pm 1$, Eqs. (77) and (78) reduce to the previous equations for the RFIM [Eqs. (61) and (62)], when expressed with ρ as a variable instead of ϕ: $v_k(\rho_1,\rho_2,z=+1)$ is equal to $v_k(\varphi_1,\varphi_2)$ for $\varphi_1\varphi_2>0$ and $v_k(\rho_1,\rho_2,z=-1)$ is equal to $v_k(\varphi_1,\varphi_2)$ for $\varphi_1\varphi_2<0$; $\delta_{k,L}(\rho)\equiv\delta_k(\varphi)$ and $w_{k,L}(\rho)\equiv u''(\varphi)$.[69] Finally, the comments made about the important features of the flow equations for the RFIM carry over to the equations for the RF$O(N)$M.

D. Application to related disordered models

Even though we have chosen to more specifically focus on the random field model, it is worth sketching at this point the relevance of the NP-FRG equations derived in this section to other disordered systems. (As already stressed several times, we exclude spin-glass ordering from our considerations.)

The flow equations obtained for the RF$O(N)$M [Eqs. (77)–(79)] directly apply to the RA$O(N)$M for describing the long-distance physics associated with ferromagnetic ordering. The putative fixed points are also expected to be at zero temperature so that similar scaling dimensions need be introduced. The specificity of the random anisotropy model comes in the initial conditions (see Sec. II A) and in the additional symmetry of the two-replica potential, namely, $v_k(\rho_1,\rho_2,z)=v_k(\rho_1,\rho_2,-z)$.

Similarly, the flow equations for the RFIM [Eqs. (61), (62), and (66)] can be applied to the random elastic model. In this case, one can check that, owing to the statistical tilt symmetry, $u_k'(\varphi)\equiv 0$ and $\eta_k\equiv 0$, while $v_k(\varphi_1,\varphi_2)\equiv v_k(\varphi_1-\varphi_2)$. After introducing the variable $y=\varphi_1-\varphi_2$ and dropping the temperature, Eq. (62) can be rewritten as

$$-\partial_t v_k(y) = (d-4+2\bar{\eta}_k)v_k(y) - \frac{1}{2}(d-4+\bar{\eta}_k)yv_k'(y) + 2v_d l_2^{(d)}(0)[v_k''(y) - 2v_k''(0)]v_k''(y), \quad (81)$$

where a prime denotes a derivative with respect to y. The roughness exponent is defined through $\zeta=-(d-4+\bar{\eta})/2$, and one can then see that the above equation reduces to the one-loop FRG equation for a disordered elastic medium.[28,44] Going beyond this level of description requires the consideration of the next orders of the truncation scheme, in particular, the inclusion of the three-replica potential and application of the next order of the derivative expansion for the two-replica effective average action.

Finally, Eqs. (61), (62), and (66) can be used in the case of the random temperature model with an appropriate account of the symmetry: $u_k\equiv u_k(\rho)$ and $v_k\equiv v_k(\rho_1,\rho_2)$, with $\rho=\varphi^2/2$. However, the scaling dimensions introduced to search for a zero-temperature fixed point are not appropriate in the present case wherein one anticipates a fixed point at a nonzero temperature (for a preliminary nonperturbative treatment, see Ref. 70).

V. RECOVERING THE PERTURBATIVE RESULTS

A. Analysis of the nonperturbative functional renormalization group equations near $d=6$ and for $N\to\infty$

For ease of notation, we consider only the RFIM, but a similar analysis holds for the RF$O(N)$M. It is easy to check that the flow equations [Eqs. (61), (62), (66), and (69)] admit for fixed-point solution the Gaussian fixed point characterized by $\eta_*^{(G)}=\bar{\eta}_*^{(G)}=0$, $u_*^{(G)}(\varphi)=2v_d l_1^{(d)}(0)/(d-2)$, and $\delta_*^{(G)}(\varphi_1,\varphi_2)=1$. The Gaussian fixed point is once unstable for dimensions larger than 6, but the coupling constant associated with the φ^4 term in $u(\varphi)$ also becomes relevant for dimensions less than 6 so that the Gaussian fixed point becomes unstable for $d<6$, as already well known.

The first order in $\epsilon=6-d$ can be derived by a direct expansion of the fixed-point solution, with $u_*(\varphi)=u_*^{(G)}+\epsilon u_1(\varphi)$ and $\delta_*(\varphi_1,\varphi_2)=1+\epsilon\delta_1(\varphi_1,\varphi_2)$. One easily finds that at this order, one still has $\eta_*=\bar{\eta}_*=0$. After inserting these results in Eqs. (61) and (62), deriving the equation for v_k with respect to φ_1 and φ_2, and setting the left-hand sides to zero, one obtains the following equations for $u_1(\varphi)$ and $\delta_1(\varphi_1,\varphi_2)$:

$$0 = 4u_1(\varphi) - \varphi u_1'(\varphi) - \frac{v_6}{2}[l_1^{(6)}(0) - 4l_2^{(6)}(0)u_1''(\varphi) + 4l_1^{(6)}(0)\delta_1(\varphi,\varphi)], \quad (82)$$

$$0 = (\varphi_1\partial_{\varphi_1} + \varphi_2\partial_{\varphi_2})\delta_1(\varphi_1,\varphi_2) - 2v_6 l_2^{(6)}(0)(\partial_{\varphi_1}+\partial_{\varphi_2})^2\delta_1(\varphi_1,\varphi_2). \quad (83)$$

By introducing the variables $x=(\varphi_1+\varphi_2)/2$ and $y=(\varphi_1-\varphi_2)/2$, the latter equation can be rewritten as

$$(x\partial_x + y\partial_y)\delta_1(x,y) = 2v_6 l_2^{(6)}(0)\partial_x^2\delta_1(x,y). \quad (84)$$

The symmetry of $v(\varphi_1,\varphi_2)$ with respect to the exchange of φ_1 and φ_2 and to changes in the sign of φ_1 and φ_2 (see Sec.

IV B) translates into the fact that δ_1 is an even function of x and y. Provided one requires that $\delta_1(x,y\to 0)$ is finite (which is needed for a well defined renormalizable theory), the only acceptable solution satisfying this property is a constant; due to the constraint $\delta_{m,k}\equiv\delta_k(\varphi_{m,k})=1$, it is equal to zero, i.e., $\delta_1(x,y)=0$. In addition to the now unstable Gaussian fixed point, Eq. (82) has then for solution $u_1(\varphi)=(\lambda_{1*}/8)(\varphi^2-\varphi_{m*}^2)^2+\text{const}$, where $\varphi_{m*}^2=6v_6 l_2^{(6)}(0)$. One also finds $\lambda_{1*}=[36v_6 l_3^{(6)}(0)]^{-1}$ so that, up to irrelevant constant factors, the solution corresponds to the fixed point of the pure Ising model (no random field) at first order in $\epsilon=4-d$. The fixed point is found once unstable and the associated exponents, e.g., $\nu=1/2+\epsilon/12$, satisfy the $d\to d-2$ dimensional reduction.

Equivalently, one can make a more direct connection to standard perturbation analysis by reframing the above results in a double expansion in ϵ and in the φ^4 coupling constant defined through $\lambda_k=u_k''''(\varphi_{m,k})$. Introducing, as before, $\rho_{m,k}=(1/2)\varphi_{m,k}^2$, one obtains from Eqs. (77)–(80) that $\eta,\bar{\eta}=O(\lambda^2)$, $\delta=1+O(\lambda^2)$, and

$$\partial_t\lambda_k=-\epsilon\lambda_k+36v_d l_3^{(d)}(0)\lambda_k^2+O(\lambda_k^3,T_k\lambda_k^2), \tag{85}$$

$$\partial_t\rho_{m,k}=-(2-\epsilon)\rho_{m,k}+6v_d[l_2^{(d)}(0)-4l_3^{(d)}(0)\lambda_k\rho_{m,k}] + O(\lambda_k^3,T_k\lambda_k^2), \tag{86}$$

where we have used the Taylor expansion of the threshold functions for small arguments. [The fixed-point solution of Eqs. (85) and (86) is, of course, equal to that obtained above with $\lambda_*=\epsilon\lambda_{1*}$ and $\rho_{m*}=\varphi_{m*}^2/2$.] Again, up to irrelevant factors, this gives back the one-loop perturbative result for the pure Ising model obtained in a weak-coupling expansion in $d=4-\epsilon$.

The above result is derived through an expansion in a single coupling constant λ_k associated with the one-replica part of the effective action. It was argued by Brezin and De Dominicis[71,72] that one should consider an expansion involving all φ^4 coupling constants associated with multiple replicas instead. In the present formalism, we can perform a more careful analysis by using the φ^4 coupling constants associated with the two-replica part of the effective action, coupling constants that are considered as potentially relevant in Refs. 71 and 72. We find that this does not change the conclusion and, as previously obtained in Ref. 73, that the fixed point corresponding to dimensional reduction is still once unstable at first order in ϵ. This is discussed in more detail in the Appendix.

The above analysis is extended to the $O(N)$ version in a straightforward way. The property that the perturbative result at first order in $\epsilon=6-d$ is recovered within our nonperturbative approximation scheme is actually a consequence of the one-loop-like structure of the exact flow equation for the effective average action [Eq. (16)]. For the very same reason, the large N limit can also be easily recovered.

Rescaling the variables as $\rho\to N\rho$ and $z\to z$ and the potentials as $u\to Nu$ and $v\to Nv$ and retaining only the dominant terms when $N\to\infty$, one finds that $\eta=O(1/N)$ and $\bar{\eta}=O(1/N)$ and that the "longitudinal" contributions drop out from the RG flow equations. As a consequence, Eqs. (77) and (78) can be recast as

$$\partial_t u_k(\rho)=-(d-2)u_k(\rho)+(d-4)\rho u_k'(\rho) + 2v_d l_1^{(d)}(u_k'(\rho))\delta_{k,T}(\rho), \tag{87}$$

$$\begin{aligned}\partial_t\delta_{k,T}(\rho_1,\rho_2,z) &= (d-4)(\rho_1\partial_{\rho_1}+\rho_2\partial_{\rho_2})\delta_{k,T}(\rho_1,\rho_2,z)\\ &\quad -2v_d\Bigg\{\frac{l_{11}^{(d)}(u_k'(\rho_1),u_k'(\rho_2))}{\sqrt{\rho_1\rho_2}}\delta_{k,T}(\rho_1,\rho_2,z)\partial_z\delta_{k,T}(\rho_1,\rho_2,z)\\ &\quad +\frac{l_2^{(d)}(u_k'(\rho_1))}{2\rho_1}\delta_{k,T}(\rho_1)(2\rho_1\partial_{\rho_1}-z\partial_z)\delta_{k,T}(\rho_1,\rho_2,z)\\ &\quad +\frac{l_2^{(d)}(u_k'(\rho_2))}{2\rho_2}\delta_{k,T}(\rho_2)(2\rho_2\partial_{\rho_2}-z\partial_z)\delta_{k,T}(\rho_1,\rho_2,z)\Bigg\},\end{aligned} \tag{88}$$

where we have defined a generalized "transverse" disorder cumulant $\delta_{k,T}(\rho_1,\rho_2,z)$ via an extension of Eq. (74), namely,

$$\delta_{k,T}(\rho_1,\rho_2,z)=\frac{1}{2\sqrt{\rho_1\rho_2}}\partial_z v_k(\rho_1,\rho_2,z), \tag{89}$$

which reduces to $\delta_{k,T}(\rho)$ when $\rho_1=\rho_2=\rho$ and $z=1$. Equation (88) is obtained by deriving the flow equation for $v_k(\rho_1,\rho_2,z)$.

If one starts the flow equations with an initial condition $v_\Lambda(\rho_1,\rho_2,z)=2\sqrt{\rho_1\rho_2}z$ (corresponding to $\delta_{\Lambda,T}=1$), the beta function is identically zero and one therefore finds that the solution of Eq. (88) at all scales remains $\delta_{k,T}(\rho_1,\rho_2,z)=1$.[74] The resulting equation for the one-replica potential is then very similar to its counterpart for the pure $O(N)$ model with $N\to\infty$ in dimension $d-2$ (the flow equation is then simply given by the LPA[36]).

To more explicitly see the connection, one can follow the flow of the φ^4 coupling constant $\lambda_k=u_k''(\rho_{m,k})$ as well as that of $\rho_{m,k}$, which, we recall, satisfies $u_k'(\rho_{m,k})=0$ and is akin to a (dimensionless) order parameter at the running scale k. One finds

$$\partial_t\lambda_k=-(6-d)\lambda_k+4v_d l_3^{(d)}(0)\lambda_k^2 + [(d-4)\rho_{m,k}-2v_d l_2^{(d)}(0)]u_k'''(\rho_{m,k}), \tag{90}$$

$$\partial_t\rho_{m,k}=-(d-4)\rho_{m,k}+2v_d l_2^{(d)}(0), \tag{91}$$

which results in the nontrivial fixed point $\rho_{m*}=2v_d l_2^{(d)}(0)/(d-4)$, $\lambda_*=(6-d)/[4v_d l_3^{(d)}(0)]$. This fixed point is once unstable (and it remains so when considering the additional directions associated with the two-replica potential, see above) and is characterized by critical exponents that satisfy the dimension reduction property, e.g., $\nu=1/(d-4)$ to be compared to $\nu=1/(d-2)$ for the pure model. Note that the above perturbative expressions are recovered from the truncated NP-FRG equations even with an additional approximation using a field expansion around the minimum of the one-replica potential.

B. Recovering the perturbative functional renormalization group near $D=4$

A strong property of the minimal nonperturbative truncation described above is that it also reduces, in the appropriate limit and for the RF$O(N>1)$M, to the perturbative FRG equations at first order in $\epsilon=d-4$, which were derived by Fisher.[23] The latter are obtained from a low-disorder loop expansion of the nonlinear sigma model associated with the RF$O(N)$M. It is therefore quite remarkable that our formalism, in which no hard constraint is enforced, leads to the proper result within the minimal approximation scheme.

For the RF$O(N)M$ with $N>1$, $d=4$ is the lower critical dimension for ferromagnetism. (Here, we mean long-range ferromagnetic order with a nonzero order parameter; the case of quasi-long range order will be discussed later on.) As a result, the critical point and the associated fixed point occur near $d=4$ for a value of ρ_m that diverges as $1/\epsilon$ with $\epsilon=d-4$. As in the case of the pure $O(N)$ model near $d=2$,[37] one can therefore organize a systematic expansion in powers of $1/\rho_m$.

At the minimum of the one-replica potential ($\rho=\rho_m$), the transverse mass, which is associated with the Goldstone modes, is zero, whereas the longitudinal mass is very large and scales as ρ_m [anticipating that $u''(\rho_m)$ does not vanish]. One can then use the asymptotic properties of the threshold functions for large arguments,

$$l_n^{(d)}(w \to \infty) \sim w^{-(n+1)}, \quad l_{n_1,n_2}^{(d)}(w \to \infty,0) \sim w^{-(n_1+1)}, \tag{92}$$

$$m_{n_1,n_2}^{(d)}(w \to \infty,0) \sim w^{-n_1}, \tag{93}$$

which encodes the decoupling of the massive mode.

In addition, we assume that as $\rho_m \to \infty$, $\delta_{L,T}(\rho_m)$ stay finite [recall that actually, $\delta_T(\rho_m)=1$] and that their derivatives, $\delta'_{L,T}(\rho_m)$, etc., go to zero at least as fast as $1/\rho_m$; on the other hand, ρ_m is a singular point for $u(\rho)$ (the location of its minimum) so that even when $\rho_m \to \infty$ we expect that $u''(\rho_m)$, $u'''(\rho_m)$, etc., stay of $O(1)$. The consistency of these assumptions is easily checked *a posteriori*. Inserting the above results and assumptions in Eqs. (79) gives

$$\eta_k \simeq \frac{8v_d}{d\rho_{m,k}}, \tag{94}$$

which shows that η is of order $1/\rho_m$.

Deriving once the flow equation for the one-replica potential $u_k(\rho)$ leads to

$$\partial_t u'_k(\rho) = -(2-\eta_k)u'_k(\rho) + (\epsilon + \bar{\eta}_k)\rho u''_k(\rho) - 2v_d(N-1)l_2^{(d)}(u'_k(\rho))\delta_{k,T}(\rho), \tag{95}$$

from which one obtains the flow equation for the running order parameter $\rho_{m,k}$,

$$\partial_t \rho_{m,k} = -(\epsilon + \bar{\eta}_k)\rho_{m,k} + 2v_d(N-1)l_2^{(d)}(0), \tag{96}$$

where $\epsilon=d-4$. (Note that we have again omitted the subscript k on the right-hand sides and dropped the subdominant terms involving the renormalized temperature T_k.) The last equation shows that the fixed point value of $\rho_{m,k}$ satisfies, as anticipated, $\rho_{m*}=O(1/\epsilon)$, which results in $\eta, \bar{\eta}=O(\epsilon)$.

One can now apply a similar treatment to the flow equation for the two-replica potential evaluated for $\rho_1=\rho_2=\rho_{m,k}$. For convenience, we introduce the function

$$R_k(z) = \frac{v_k(\rho_{m,k},\rho_{m,k},z)}{(2\rho_{m,k})^2}, \tag{97}$$

which, due to Eq. (74) and the constraint $\delta_{k,T}(\rho_{m,k})=1$, satisfies $R'_k(z=1)=1/(2\rho_{m,k})$.[75] The flow equation for $R_k(z)$ can be expressed as

$$\partial_t R_k(z) = \frac{1}{(2\rho_{m,k})^2}\partial_t v_k(\rho,\rho,z)\Big|_{\rho=\rho_{m,k}} + \partial_t\rho_{m,k}\,\partial_\rho\left[\frac{v_k(\rho,\rho,z)}{(2\rho)^2}\right]\Bigg|_{\rho=\rho_{m,k}}, \tag{98}$$

which with the help of Eq. (96) finally leads to

$$\partial_t R_k(z) \simeq (\epsilon + 2\eta_k)R(z) - 2v_d l_2^{(d)}(0)\{(N-1)[4R_k(z)R'_k(1) + R'_k(z)(R'_k(z) - 2zR'_k(1))] + (1-z^2)[-R'_k(z)^2 + 2[R'_k(1) - zR'_k(z)]R''_k(z) + (1-z^2)R''_k(z)^2]\}. \tag{99}$$

To dominant order in ϵ, one can set $d=4$ in v_d and $l_2^{(d)}(0)$ in all equations and in v_d/d in Eq. (94). By using the property of the threshold function $l_2^{(d=4)}(0)=1+O(\eta)$ and by discarding the subdominant terms, one finally arrives at

$$\eta_k = 4v_4R'_k(1), \bar{\eta}_k = -\epsilon + 4(N-1)v_4R'_k(1), \tag{100}$$

$$\partial_t R_k(z) = \epsilon R_k(z) - 2v_4\{4(N-2)R'_k(1)R_k(z) + (N-1) \times[R'_k(z) - 2zR'_k(1)]R'_k(z) + (1-z^2)[-R'_k(z)^2 + 2[R'_k(1) - zR'_k(z)]R''_k(z) + (1-z^2)R''_k(z)^2]\}, \tag{101}$$

where $v_4^{-1}=32\pi^2$ and $R_k(z)$ is of order ϵ near its fixed point. The above equations coincide with the one-loop perturbative FRG equations.[23] Note that this result is independent of the choice of the infrared cutoff function $\hat{R}_k(q^2)$: indeed, one easily checks that not only $l_2^{(4)}(0)=1+O(\eta)$ but also $\lim_{w\to\infty}[m_{2,3}^{(4)}(w,0)w^2]=1+O(\eta)$, irrespective of the regulator.

Finally, we note that setting $N=2$ and introducing the variable $\phi=\cos^{-1}(z)$ in Eq. (101) leads to

$$\partial_t R_k(\phi) = \epsilon R_k(\phi) - 2v_4[R''_k(\phi) - 2R''_k(0)]R''_k(\phi), \tag{102}$$

which, after the use of Eq. (100) for η_k and $\bar{\eta}_k$, coincides with the one-loop perturbative FRG equation for a disordered periodic elastic system with a one-component displacement field: compare, for instance, to Eq. (81), in which one should set $\zeta=0$ due to the periodicity.[76] (Be careful, however, that η_k and $\bar{\eta}_k$ denote different sets of exponents in the formalism leading to Eq. (81) and in the present one.[77])

VI. CONCLUDING REMARKS

In this work, which is described in the present paper and in the following one,[32] we have developed a theoretical approach that is able to describe the long-distance physics, criticality, phase ordering, or "quasi-ordering" of systems in the presence of quenched disorder, in particular, random field models for which standard perturbation theory is known to fail. The approach is based on an exact renormalization group equation for the effective average action (the generating functional of $1-PI$ vertices) and on a nonperturbative truncation scheme. This nonperturbative RG formalism has recently been applied with success to a variety of systems. The key point in the present problem is to provide a proper account of the renormalized distribution of the quenched disorder, and we have shown that this can be conveniently done through a cumulant expansion and the use of a replica method in which the permutational symmetry among replicas is explicitly broken.

We have stressed that any relevant treatment of random field models and related disordered systems must include the second cumulant of the renormalized disorder, i.e., at least a function of two (replica) field arguments. Accordingly, we have proposed a nonperturbative approximation scheme. Within this scheme, the minimal truncation for the RF$O(N)M$ already reproduces the leading results of perturbative RG analyses near the upper critical dimension, $d_{uc}=6$, and when the number of components N becomes infinite. More importantly, it gives back the perturbative FRG equations near the lower critical dimension for ferromagnetism when $N>1$, $d=4$.

One of the main advantages of the present approach, which will be illustrated in the following paper, is that it provides a unified framework in which to describe models in any spatial dimension d and for any number N of field components. As such, it guarantees a consistent interpolation of all known results in the whole (N,d) plane in addition to allowing the study of genuine nonperturbative phenomena.

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APPENDIX: EXPANSION IN SEVERAL COUPLING CONSTANTS NEAR $d=6$

Near the upper critical dimension $d_{uc}=6$, the flow equations for the RFIM derived within the minimal nonperturbative truncation [Eqs. (61), (62), (66), and (69)] can be expanded in several φ^4 coupling constants in order to make the connection with recent one-loop studies of the RFIM.[71–73] On top of the one-replica φ^4 coupling constant already used in Sec. V A, $\lambda_k=u_k''''(\varphi_{m,k})$, we introduce two additional coupling constants obtained from the two-replica potential,

$$u_{k,2}=-\frac{1}{2}\,(\partial^2_{\varphi_1}+\partial^2_{\varphi_2})\partial_{\varphi_1}\partial_{\varphi_2}v_k(\varphi_1,\varphi_2)|_{\varphi_1=\varphi_2=\varphi_{m,k}}, \tag{A1}$$

$$u_{k,3}=-\partial^2_{\varphi_1}\partial^2_{\varphi_2}v_k(\varphi_1,\varphi_2)|_{\varphi_1=\varphi_2=\varphi_{m,k}}, \tag{A2}$$

which amounts to consider a two-replica potential of the form

$$v_k(\varphi_1,\varphi_2)=\varphi_1\varphi_2\left[\delta_{m,k}-\frac{u_{k,2}}{6}(\varphi_1^2+\varphi_2^2-6\varphi_{m,k}^2)-\frac{u_{k,3}}{4}(\varphi_1\varphi_2-4\varphi_{m,k}^2)+\cdots\right], \tag{A3}$$

where the dots denote higher-order terms in the field expansion around the minimum of the one-replica potential and, as before, $\delta_{m,k}=\partial_{\varphi_1}\partial_{\varphi_2}v_k(\varphi_1,\varphi_2)|_{\varphi_{m,k}}\equiv 1$. The present description is thus very similar to that used in Refs. 71 and 72 except that we do not include three- and four-replica terms. However, the issues raised by Brezin and De Dominicis[71,72] can already be addressed by considering the two-replica term.

By expanding the flow equations for the one- and two-replica potentials in powers of the coupling constants, which we generically denote $u_{k,\alpha}$ with $u_{k,1}=\lambda_k$, one finds that $\eta=O(u^2)$ and that up to a $O(u^3)$,

$$\partial_t u_{k,1}=(d-6)u_{k,1}+2v_d\{6l_3^{(d)}(0)u_{k,1}^2+12l_2^{(d)}(0)u_{k,1}\times(u_{k,2}+u_{k,3})+3T_k l_2^{(d)}(0)u_{k,1}^2\}, \tag{A4}$$

$$\partial_t u_{k,2}=(d-4)u_{k,2}+2v_d\{6l_3^{(d)}(0)u_{k,1}(u_{k,2}+u_{k,3})+6l_2^{(d)}(0)u_{k,2}(u_{k,2}+u_{k,3})+3T_k l_2^{(d)}(0)u_{k,1}u_{k,2}\}, \tag{A5}$$

$$\partial_t u_{k,3}=(d-4)u_{k,3}+2v_d\{l_4^{(d)}(0)u_{k,1}^2+4l_3^{(d)}(0)u_{k,1}(u_{k,2}+u_{k,3})+2l_2^{(d)}(0)[(u_{k,2}+u_{k,3})^2+3u_{k,3}^2]+2T_k l_2^{(d)}(0)u_{k,1}u_{k,3}\}, \tag{A6}$$

$$\partial_t\left(\frac{\lambda_k\rho_{m,k}}{3}\right)=-2\left(\frac{\lambda_k\rho_{m,k}}{3}\right)+2v_d\left\{l_2^{(d)}(0)u_{k,1}+2l_1^{(d)}(0)(u_{k,2}+u_{k,3})+T_k l_1^{(d)}(0)u_{k,1}+\left(\frac{\lambda_k\rho_{m,k}}{3}\right)[2l_3^{(d)}(0)u_{k,1}+8l_2^{(d)}(0)(u_{k,2}+u_{k,3})+T_k l_2^{(d)}(0)u_{k,1}]\right\}, \tag{A7}$$

where, we recall, $\rho_{m,k}=\varphi_{m,k}^2/2$. In addition, by using $\partial_t T_k=(2+\eta_k-\bar{\eta}_k)T_k$ and the equation for $2\eta_k-\bar{\eta}_k$ [Eq. (69)] one obtains to a $O(u^2)$,

$$\partial_t T_k=2T_k+4v_d T_k\left\{2l_2^{(d)}(0)(u_{k,2}+u_{k,3})+6l_1^{(d)}(0)\frac{(u_{k,2}+u_{k,3})^2}{u_{k,1}}+T_k l_1^{(d)}(0)(2u_{k,2}+3u_{k,3})\right\}. \tag{A8}$$

It can now be checked that the above equations coincide with those derived by Mukaida and Sakamoto[73] with the introduction of new running coupling constants: $g_0=T_k\delta_k(0)^{-1}$, $g_1=u_{k,1}\delta_k(0)$, $g_2=u_{k,2}$, $g_3=u_{k,3}$, where $\delta_k(0)\equiv\delta_k(\varphi_1=0,\varphi_2=0)=1+2(u_{k,2}+u_{k,3})\rho_{m,k}$ (and, of course, the three- and four-replica contributions missing). As in Ref. 73, we therefore obtain that the dimensional reduction fixed point is once unstable at first order in $\epsilon=6-d$. On the other hand, the analysis performed by Brezin and De Dominicis[71,72] requires the introduction of different scaling dimensions, corresponding to new coupling constants, $\hat{g}_0=g_0$, $\hat{g}_1=g_1$, $\hat{g}_{2,3}=g_{2,3}g_0^{-1}$. The beta functions we now obtain for $\hat{g}_0$, $\hat{g}_1$, and $\hat{g}_2$ coincide with those of Refs. 71 and 72, but that for $\hat{g}_3$ is ill defined as it contains a term that blows up as $k\to 0$. The scaling dimensions suggested by Brezin and De Dominicis[71,72] are thus not compatible with our approach.

*tarjus@lptl.jussieu.fr
†Present address: Instituto de Física, Facultad de Ingenería, Universidad de la República, J. H. y Reissig 565, 11000 Montevideo, Uruguay. tissier@lptmc.jussieu.fr
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