Classification of symmetric polynomials of infinite variables: Construction of Abelian and non-Abelian quantum Hall states

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The classification of complex wave functions of infinite variables is an important problem since it is related to the classification of possible quantum states of matter. In this paper, we propose a way to classify symmetric polynomials of infinite variables using the pattern of zeros of the polynomials. Such a classification leads to a construction of a class of simple non-Abelian quantum Hall states which are closely related to parafermion conformal field theories.

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I. INTRODUCTION

A. Functions of infinite variables

One of the most important problems in condensed matter physics is to understand how particles are organized in the ground state. Almost all the low energy properties of a system are determined by such an organization. Mathematically, the ground state of N particles is described by a wave function—a complex function of N variables $\Phi(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N)$, where $\boldsymbol{r}_i$ is the coordinate of the ith particle. Thus, the problem of understanding the patterns of the many-particle organizations (or in physical terms, of understanding the quantum phases of many-particle systems) is to classify the complex wave functions $\Phi(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N)$ in the $N\to\infty$ limit.

Such a classification problem is one of the most fundamental problems in physics since it determines the possible quantum phases of many-particle systems. Due to the success of the Landau symmetry breaking theory in describing phases and phase transitions,[1] for a long time physicists believe that the phases of matter are classified by their symmetry properties. Mathematically, this is equivalent to believing that the wave functions are classified by their symmetry properties, such as, for example, whether the wave function is invariant under translation $\Phi(\boldsymbol{r}_i)\to\Phi(\boldsymbol{r}_i+\boldsymbol{a})$ or not. Under such a belief, the wave functions with the same symmetries are grouped into one class and such a class represents a single phase of matter. This is why group theory becomes an important mathematical foundation in physics.

However, after the discovery of fractional quantum Hall (FQH) states,[2,3] it was realized that symmetry is not enough to classify all the possible organizations encoded in the wave functions $\Phi(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N)$. This is because the wave functions that describe different FQH states have exactly the same symmetry. Thus, the wave functions of FQH states contain new kinds of organizations of particles that has nothing to do with symmetry.[4,5] The new organizations of the particles are called topological orders.

Intuitively, what is new in the FQH wave functions is that the wave functions contain a long-range quantum entanglement.[6,7] This is why the FQH wave functions describe new states of matter that cannot be described by symmetries. The wave functions with long-range entanglements and the corresponding topological orders not only appear in FQH systems, but they also appear in various quantum spin systems. Understanding this new class of wave functions and the resulting new states of matter is currently a very active research direction in condensed matter physics.[8–28]

To gain a deeper and more precise understanding of topological orders and the associated long-range entanglements, we need to solve the related mathematical problem of classifying $\Phi(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N)$ in the $N\to\infty$ limit. This is a difficult problem which is not well studied in mathematics. The problem is not even well defined. However, this does not mean that the problem is not important. It is common not to have a well defined problem when we wander into an unknown territory. The first task of knowing the unknown is usually to come up with a proper definition of the problem.

In this paper, we will not attempt to classify generic complex wave functions $\Phi(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N)$. We limit ourselves to a simpler problem of trying to classify FQH states and their topological orders. (For a review on topological order in FQH states, see Refs. 29 and 30.) The corresponding mathematical problem is to classify symmetric and antisymmetry polynomials of N variables $\Phi(z_i,\ldots,z_N)$ in the $N\to\infty$ limit. We will first try to come up with a physically meaningful and mathematically rigorous definition of the problem. Then, we will solve the problem in some simple cases. This leads to a class of "simple" (anti)symmetric polynomials which corresponds to a class of simple FQH states. The constructed FQH states include both Abelian and non-Abelian FQH states.[14,15]

II. FRACTIONAL QUANTUM HALL STATES AND POLYNOMIALS

A. Fractional quantum Hall wave functions

First, we would like to give a brief review on FQH theory. A FQH state is a quantum ground state of two-dimensional electrons in a magnetic field. Such a quantum state is described by a complex wave function,

$$\Psi(x_1,y_1,x_2,y_2,\ldots,x_N,y_N),$$

where (x_i,y_i) are the coordinates of the ith electron and N is the total number of electrons. In the strong magnetic field limit, if the filling fraction ν of a FQH state is less than 1, then all the electrons are in the lowest Landau level. In this case, the ground state wave function has the following form:

$$\Psi = \Phi(z_1, \cdots, z_N)\exp\left(-\frac{1}{4}\sum_{i=1}^{N} |z_i|^2\right),$$

where $z_i = x_i + iy_i$ and $\Phi(z_1, \ldots, z_N)$ is a holomorphic function of z_i [i.e., $\Phi(z_1, \ldots, z_N)$ does not depend on z_i^*]. Since $\Phi(z_1, \ldots, z_N)$ has no poles, $\Phi(z_1, \ldots, z_N)$ is a polynomial of z_i's.

Due to the Fermi statistics of the electrons, $\Phi(z_1, \ldots, z_N)$ must be an antisymmetric polynomial. If we assume the electrons to have Bose statistics, then $\Phi(z_1, \ldots, z_N)$ must be a symmetric polynomial. Thus, to understand the phases of FQH systems is to classify antisymmetric or symmetric polynomials.

It turns out that for every antisymmetric polynomial $\Phi_{\text{anti-sys}}(z_1, \ldots, z_N)$, one can uniquely construct a symmetric polynomial $\Phi_{\text{sys}}(z_1, \ldots, z_N) = \frac{\Phi_{\text{anti-sys}}(z_1, \ldots, z_N)}{\prod_{i<j}(z_i - z_j)}$. Thus, classifying antisymmetric polynomials and classifying symmetric polynomials are almost identical problems. Therefore, in this paper, we will assume electrons to have Bose statistics and concentrate on classifying symmetric polynomials.

For a system of N bosonic electrons, which symmetric polynomial will represent the ground state of the system? It will depend on the interaction between the electrons. If the interaction potential between two electrons has a δ-function form

$$V_1(z_1, z_2) = \delta(z_1 - z_2), \tag{1}$$

then the ground state is described by the symmetric polynomial

$$\Phi_{1/2} = \prod_{i<j} (z_i - z_j)^2.$$

Such a state has a vanishing total potential energy $V_{\text{tot}} = 0$, where

$$V_{\text{tot}} \equiv \int \prod_i d^2 z_i \Psi^*(\{z_i\}) \sum_{i<j} V(z_i, z_j) \Psi(\{z_i\}).$$

The vanishing of the total potential energy V_{tot} requires that the wave function $\Psi(\{z_i\})$ to be zero as $z_1 \to z_2$. Since the average energy $V_{\text{tot}} \geq 0$ for any wave functions, the vanishing V_{tot} for $\Phi_{1/2}$ indicates that $\Phi_{1/2}$ is the ground state.

If the interaction potential between two electrons is given by[31]

$$V_2(z_1, z_2) = v_0 \delta(z_1 - z_2) + v_2 \partial_{z_1^*}^2 \delta(z_1 - z_2) \partial_{z_1}^2, \tag{2}$$

with $v_0 > 0$ and $v_2 > 0$, then the ground state will be

$$\Phi_{1/4} = \prod_{i<j} (z_i - z_j)^4.$$

For interaction (2), the vanishing of the total potential energy V_{tot} not only requires that the wave function $\Psi(\{z_i\})$ to be zero as $z_1 \to z_2$, but it also requires $\Psi(\{z_i\})$ to vanish faster than $(z_1 - z_2)^2$ as $z_1 \to z_2$. This means that the symmetric polynomial must have a fourth order zero as $z_i \to z_j$. One such polynomial is given by $\Phi_{1/4} = \prod_{i<j}(z_i - z_j)^4$, which has the lowest total power of z_i's.

More complicated ground states can be obtained through more complicated interactions. For example, consider the following three-body interaction between electrons:[32,33]

$$V_{\text{Pf}}(z_1, z_2, z_3) = \mathcal{S}[v_0 \delta(z_1 - z_2)\delta(z_2 - z_3) - v_1 \delta(z_1 - z_2)\partial_{z_3^*}\delta(z_2 - z_3)\partial_{z_3}], \tag{3}$$

where $\mathcal{S}$ is the total symmetrization operator between z_1, z_2, and z_3. Such an interaction selects the symmetric polynomial[14]

$$\Phi_{\text{Pf}} = \mathcal{A}\left(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots\frac{1}{z_{N-1} - z_N}\right)\prod_{i<j}(z_i - z_j)$$

to describe the ground state (which has a vanishing total potential energy V_{tot}). Here, $\mathcal{A}$ is the total antisymmetrization operator between $z_1, \ldots, z_N$.

Three symmetric polynomials $\Phi_{1/2}$, $\Phi_{1/4}$, and Φ_{Pf} contain different topological orders and correspond to three different phases of an N-electron system in the $N \to \infty$ limit. They are the filling fraction $\nu = 1/2$ Laughlin state,[3] the filling fraction $\nu = 1/4$ Laughlin state, and the filling fraction $\nu = 1$ Pfaffian state.[14] We would like to find a classification of symmetric polynomials such that the above three symmetric polynomials belong to three different classes.

B. Ideal Hamiltonian and zero-energy state

The above three examples share some common properties. The Hamiltonians described by the interaction potentials V_1, V_2, and V_{Pf} are all positive definite and contain zero-energy eigenstates. The zero-energy eigenstates of V_1, V_2, and V_{Pf} are known and are given by $\Phi_{1/2}$, $\Phi_{1/4}$, and Φ_{Pf}. Thus, $\Phi_{1/2}$, $\Phi_{1/4}$, and Φ_{Pf} are exact ground states of the corresponding Hamiltonians. Since the interaction potentials are constructed from δ functions and their derivatives, the exact ground states (the zero-energy states) for such type of potentials are characterized by the pattern of zeros, i.e., the orders of zeros of the ground state wave function, as we bring two or more electrons together.

In this paper, we will concentrate on such ideal Hamiltonians and their exact zero-energy ground states. From this point of view, classifying FQH states corresponds to classifying patterns of zeros in symmetric polynomials. In other words, for each pattern of zeros, we can define an ideal Hamiltonian such that the symmetric polynomials with the given pattern of zeros will be the zero-energy ground states of the Hamiltonian. Such symmetric polynomials will describe a phase of a FQH system *provided that the Hamiltonian has a finite energy gap.*

Clearly, for the ideal Hamiltonians, apart from the zero-energy ground states, other eigenstates of the Hamiltonian all have nonzero and positive energies. However, this does not imply the Hamiltonian to have a finite energy gap. Only when the minimal energy of the excitations has a finite nonzero limit as electron number approaches to infinity, does the Hamiltonian have a finite energy gap. Thus, to classify the

FQH states, we not only need to classify the patterns of zeros and the associated ideal Hamiltonians, we also need to judge if the constructed ideal Hamiltonian has a finite energy gap or not. At the moment, we do not have a good way to make such a judgment. Therefore, here, we will concentrate on classifying the patterns of zeros and the associated zero-energy states.

III. PATTERN OF ZEROS

A. Derived polynomials and their D_{ab} characterization

In order to classify translation invariant symmetric polynomials of N variables $\Phi(z_1,\ldots,z_N)$ in the $N\to\infty$ limit, we need to define the polynomials for any N. The key in our definition is to introduce "local conditions." These local conditions apply to polynomials of any numbers of variables.

From the discussion in Sec. II A, we see that one way to implement the local condition is to let one variable approach another and to specify the power of the zero as follows:

$$\Phi\,(z_1,\ldots,z_N)|_{z_1\to z_2}\sim(z_1-z_2)^{D_{11}}.$$

We would like to stress that the above local condition is consistent with the translation invariance of the polynomial.

However, D_{11} does not contain all the information that is needed to specify various interesting polynomials. To implement more general local conditions, we need to bring three or more variables together and specify the patterns of zeros.[19,32]

To describe the patterns of zeros in a systematic way, we obtain from a polynomial $\Phi(z_1,\ldots,z_N)$ another polynomial P' by letting $z_1\to z_2$ as follows:

$$\Phi\,(z_1,\ldots,z_N)|_{z_1\to z_2\equiv z^{(2)}}\sim(z_1-z_2)^{D_{11}}P'(z^{(2)},z_3,\ldots,z_N)$$

Here, $\sim$ means equal up to a nonzero complex constant. The value of D_{11} encodes a part of the local conditions. Then, we let $z_3\to z^{(2)}$ in $P'(z^{(2)},z_3,\ldots,z_N)$ as follows:

$$P'(z^{(2)},z_3,\ldots,z_N)\sim(z_3-z^{(2)})^{D_{12}}P''(z^{(3)},z_4,\ldots,z_N),$$

where $z^{(3)}=z^{(2)}$. In this way, we obtain a new polynomial $P''(z^{(3)},z_4,\cdots,z_N)$. In general, we obtain $P(\{z_i^{(a)}\})$, where $z^{(a)}$ is a type-a variable obtained by fusing a z_i variables together. Note that $z_i^{(1)}=z_i$ is the original variable. If we view $z_i=z_i^{(1)}$ as coordinates of electrons, then $z_i^{(a)}$ are coordinates of bound states of a electrons. We will call such a bound state a type-a particle.

$P(\{z_i^{(a)}\})$ is a symmetric polynomial that is symmetric between variables of the same type. It satisfies certain local conditions and forms a Hilbert space. The polynomial $P(\{z_i^{(a)}\})$ is also called a derived polynomial since it is obtained from $\Phi(\{z_i\})$ by fusing variables together.

The general local conditions on $\Phi(\{z_i\})$ are specified by pattern of zeros in its derived polynomial $P(\{z_i^{(a)}\})$,

$$P\,(z_1^{(a)},z_1^{(b)},\ldots)|_{z_1^{(a)}\to z_1^{(b)}\equiv z^{(a+b)}}\sim(z_1^{(a)}-z_1^{(b)})^{D_{ab}}\widetilde{P}(z^{(a+b)},\ldots)+o((z_1^{(a)}-z_1^{(b)})^{D_{ab}}),\qquad(4)$$

where D_{ab} satisfy

$$D_{ab}=D_{ba}\in Z,\quad D_{aa}=\text{even},\quad D_{ab}\geq 0.\qquad(5)$$

$\{D_{ab}\}$ is the set of data that specifies the local condition that $\Phi(\{z_i\})$'s must satisfy. Such a set of data is called a pattern of zeros.

We note that there are many different ways to fuse a z_i variables into a $z^{(a)}$ variable. The different ways of fusion may lead to different derived polynomials which are linearly independent. Here, we will impose a unique-fusion condition on the symmetric polynomial $\Phi(\{z_i\})$: *The derived polynomials obtained from different ways of fusion are always linearly dependent*; i.e., the derived polynomials form a one-dimensional linear space. In this paper, we will study symmetric polynomials $\Phi(\{z_i\})$ that satisfy this unique-fusion condition and are characterized by the data D_{ab}.

Not all possible choices of $\{D_{ab}\}$ are consistent. Only certain choices of $\{D_{ab}\}$ correspond to symmetric polynomials $\Phi(\{z_i\})$. The key is to find those $\{D_{ab}\}$'s that can be realized by some polynomials $\Phi(\{z_i\})$.

To get a feeling what a consistent set of $\{D_{ab}\}$ may look like, let us consider the following symmetric polynomials (the Laughlin state[3]):

$$\Phi_{1/q}(\{z_i\})=\prod_{i<j}(z_i-z_j)^q,\qquad(6)$$

where q is an even integer. Such a symmetric polynomial leads to the following derived polynomial:

$$P_{1/q}(\{z_i^{(a)}\})=\left\{\prod_{a<b}\left[\prod_{i,j}(z_i^{(a)}-z_j^{(b)})^{qab}\right]\right\}\times\left\{\prod_{a}\left[\prod_{i<j}(z_i^{(a)}-z_j^{(a)})^{qa^2}\right]\right\}.\qquad(7)$$

Thus, the symmetric polynomial $\Phi_{1/q}$ is specified by the pattern of zeros,

$$D_{ab}=qab,\quad a,b\in\{1,2,3,\ldots\},$$

where q is a positive even integer.

B. S_a characterization of polynomials

There is another way to implement local conditions on a translation invariant symmetric polynomial $\Phi(\{z_i\})$. We introduce a sequence of integers S_a, where $a=0,1,2,\ldots$, and require that the minimal total powers of $z_1,\ldots,z_a$ in $\Phi(\{z_i\})$ is given by S_a.[19] (Here, S_0 is defined as 0.) Thus, in addition to $\{D_{ab}\}$, we can also use $\{S_a\}$ to characterize a symmetric polynomial. For a translation invariant symmetric polynomial, $\Phi(0,z_2,\ldots,z_N)\neq 0$. Thus, $S_1=0$.

The two characterizations, $\{D_{ab}\}$ and $\{S_a\}$, are closely related. One way to see the relation is to put the symmetric polynomial $\Phi(z_1,\ldots,z_N)$ on a sphere as discussed in Appendix A. Let N_ϕ be the maximum power of z_1 in $\Phi(z_1,\ldots,z_N)$. Then, $\Phi(z_1,\ldots,z_N)$ can be put on a sphere with N_ϕ flux quanta and each variable z_i carries an angular momentum $J=N_\phi/2$.

From the discussion near the end of Appendix A, we find that each type-a particle described by $z_i^{(a)}$ in $P(\{z_i^{(a)}\})$ carries a definite angular momentum, which is denoted as J_a. Since the lowest total power of $z_1,\ldots,z_a$ is S_a, the minimal total L^z

quantum number for those variables is $-aJ+S_a$. Therefore, the angular momentum of the $z_i^{(a)}$ variable is

$$J_a = aJ - S_a. \tag{8}$$

Since $z_i^{(1)}=z_i$, we find that $J_1=J$.

Again, according to the discussion near the end of Appendix A, if we fuse two variables $z^{(a)}$ and $z^{(b)}$ into $z^{(a+b)}$, the type-$(a+b)$ particle described by $z^{(a+b)}$ will carry an angular momentum

$$J_{a+b} = J_a + J_b - D_{ab}. \tag{9}$$

We see that D_{ab} can be expressed in terms of S_a as follows:

$$D_{ab} = S_{a+b} - S_a - S_b. \tag{10}$$

The conditions on D_{ab} [Eq. (5)] can be translated into the conditions on S_a,

$$S_{2a} = \text{even}, \quad S_{a+b} \geq S_a + S_b. \tag{11}$$

From the recursive relation $J_{a+1}=J_a+J_1-D_{a,1}$, we find $S_{a+1}=S_a+D_{a,1}$. Using $S_1=0$, we see that S_a can also be calculated from D_{ab},

$$S_a = \sum_{b=1}^{a-1} D_{b,1}. \tag{12}$$

Due to the one-to-one correspondence between $\{D_{ab}\}$ and $\{S_a\}$, we will also call the sequence $\{S_a\}$ a pattern of zeros.

C. Boson occupation characterization

The symmetric polynomial $\Phi(z_1,\ldots,z_N)$ can be written as a sum of polynomials described by boson occupations

$$\Phi(\{z_i\}) = \sum_{\{\tilde{n}_l\}} C_{\{\tilde{n}_l\}} \Phi_{\{\tilde{n}_l\}}(\{z_i\}),$$

where $\Phi_{\{\tilde{n}_l\}}$ is a boson occupation state with $\tilde{n}_l$ bosons occupying the z^l orbital. Mathematically, $\Phi_{\{n_l\}}(\{z_i\})$ is given by

$$\Phi_{\{n_l\}}(z_1, \ldots, z_N) = \sum_P \prod_{i=1}^{N} z_{P(i)}^{l_i}, \tag{13}$$

where P is a one-to-one mapping from $\{1,\ldots,N\} \to \{1,\ldots,N\}$; Σ_P is the sum over all those one-to-one mapping; and l_i, where $i=1,2,\ldots$, is a sequence of ordered integers such that the number of l valued l_i's is n_l.

What kinds of boson occupations $\{\tilde{n}_l\}$ appear in the above sum? Let us set $z_1=0$ in $\Phi(\{z_i\})$. Since $\Phi(0,z_2,\ldots,z_N)\neq 0$ due to the translation invariance, there must be a boson occupation $\{\tilde{n}_l\}$ in the above sum that contains one boson occupying the $z^{l=0}$ orbital. Now let us assume that a boson occupies $z^{l=0}$ and bring the second particle z_2 to 0; the minimal power of z_2 in $\Phi(0,z_2,\ldots,z_N)$ is D_{11}:

$$\Phi(0,z_2,\ldots,z_N) \sim z_2^{D_{11}} P_2(z_3,z_4,\ldots) + o(z_2^{D_{11}}).$$

Thus, among those $\{\tilde{n}_l\}$ which have one boson occupying the $z^{l=0}$ orbital, there must an $\{\tilde{n}_l\}$ that contains a second boson occupying the z^{l_2} orbital where $l_2=D_{11}=S_2-S_1$. Next, let us assume that two bosons occupy the $z^{l=0}$ and z^{l_2} orbitals and we bring the third particle z_3 to 0; the minimal power of z_3 is D_{21}:

$$P_2(z_3,z_4,\ldots) \sim z_3^{D_{21}} P_3(z_4,z_5,\ldots) + o(z_3^{D_{21}}).$$

Thus, among those $\{\tilde{n}_l\}$ which have two bosons occupying the $z^{l=0}$ and z^{l_2} orbitals, there must be an $\{\tilde{n}_l\}$ that contains a third boson occupying the z^{l_3} orbital where $l_3=D_{21}=S_3-S_2$. This way we can show that there must an $\{\tilde{n}_l\}$ such that the ath boson occupies the orbital z^{l_a} with $l_a=S_a-S_{a-1}$. Here, $a=1,2,\ldots$ and $l_1=0$. Let n_l be the numbers of $l_a=S_a-S_{a-1}$ that satisfy $l_a=l$. We see that the boson occupation state $\Phi_{\{n_l\}}(\{z_i\})$ happens to be the state with its ath boson occupying the orbital z^{l_a}. This allows us to show that $\Phi(z_1,\ldots,z_N)$ has the form

$$\Phi(\{z_i\}) = \Phi_{\{n_l\}}(\{z_i\}) + \sum_{\{\tilde{n}_l\}} C_{\{\tilde{n}_l\}} \Phi_{\{\tilde{n}_l\}}(\{z_i\}), \tag{14}$$

or in other words,

$$\langle \Phi_{\{n_l\}} | \Phi \rangle \neq 0. \tag{15}$$

The two sequences, $\{S_a\}$ and $\{n_l\}$, have a one-to-one correspondence. We will call $\{n_l\}$ the boson occupation description of the pattern of zeros $\{S_a\}$.

The boson occupation distributions $\{\tilde{n}_l\}$ that appear in the sum in Eq. (14) satisfy certain conditions. First, the boson occupation $\{\tilde{n}_l\}$ can be described by a pattern of zeros $\{\tilde{S}_a\}$. Then, the conditions on $\{\tilde{n}_l\}$ can be stated as $\tilde{S}_a \geq S_a$. Thus, the minimal total power of $z_1,\ldots,z_a$ in $\Phi_{\{\tilde{n}_l\}}(\{z_i\})$ is $\tilde{S}_a$, which is equal or bigger than S_a.

Haldane[34] conjectured that $\tilde{n}_l$'s in expression (14) can be obtained from n_l by one or many squeezing operations. A squeezing operation is a two-particle operation that moves one particle from the orbital z^{l_1} to the orbital $z^{l_1'}$ and the other from z^{l_2} to $z^{l_2'}$, where $l_1 < l_1' \leq l_2' < l_2$, and $l_1+l_2=l_1'+l_2'$. We can show that if $\tilde{n}_l$ is obtained from n_l by squeezing operations, then the minimal total power of $z_1,\ldots,z_a$ in $\Phi_{\{\tilde{n}_l\}}(\{z_i\})$ is equal or bigger than S_a. This is consistent with the above discussion.

Let P_{a,J_a} be a projection operator acting on the state Φ on a sphere. P_{a,J_a} projects into the subspace where a particles in Φ have a total angular momentum equal to J_a or less. We see that for a symmetric polynomial $\Phi(z_1,\ldots,z_N)$ described by a pattern of zero S_a, it satisfies

$$P_{N,J_N} \cdots P_{3,J_3} P_{2,J_2} \Phi(z_1,\ldots,z_N) = \Phi(z_1,\ldots,z_N),$$

where $J_a=aJ-S_a$. This allows us to obtain

$$P_{N,J_N} \cdots P_{3,J_3} P_{2,J_2} \Phi_{\{n_l\}}(z_1,\ldots,z_N) \neq 0, \tag{16}$$

where n_l is the boson occupation description of S_a.

IV. CONSISTENT CONDITIONS ON THE PATTERN OF ZEROS

For a translation invariant symmetric polynomial $\Phi(\{z_i\})$, the corresponding pattern of zeros $\{D_{ab}\}$ and $\{S_a\}$ satisfies

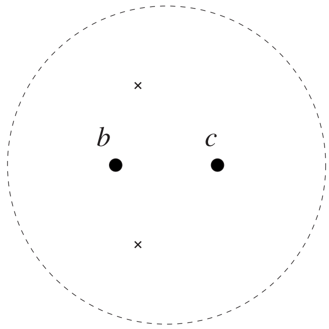

FIG. 1. $W_{a,bc}$ obtained by moving $z_1^{(a)}$ along a large loop around $z_1^{(b)}$ and $z_1^{(c)}$ counts the total numbers of zeros of $f(z_1^{(a)})$ in the loop. The crosses mark the zeros of $f(z_1^{(a)})$ not at $z_1^{(b)}$ and $z_1^{(c)}$.

some special properties. Here, we would like to find those properties as much as possible. Those properties will be called consistent conditions on the pattern of zeros. If we find all the consistent conditions, a set of integers $\{D_{ab}\}$ or $\{S_a\}$ that satisfies those consistent conditions will correspond to a translation invariant symmetric polynomial $\Phi(\{z_i\})$. We have already found some consistent conditions (11) and (16) on $\{S_a\}$. Here, we would like to find more conditions.

A. Concave condition

If we fix all variables $z_i^{(a)}$ except $z_1^{(a)}$, then the derived polynomial $P_\alpha(\{z_i^{(a)}\})$ gives us a complex function $f(z_1^{(a)})$. The complex function $f(z_1^{(a)})$ has isolated zeros at $z_i^{(b)}$'s and possibly also at some other points.

Let us move $z_1^{(a)}$ around two points $z_1^{(b)}$ and $z_1^{(c)}$. The phase of the complex function $f(z_1^{(a)})$ will change by $2\pi W_{a,bc}$, where $W_{a,bc}$ is an integer (see Fig. 1). Since $f(z_1^{(a)})$ has an order D_{ab} zero at $z_1^{(b)}$ and an order D_{ac} zero at $z_1^{(c)}$, the integer $W_{a,bc}$ satisfies

$$W_{a,bc} \geq D_{ab} + D_{ac}$$

because $f(z_1^{(a)})$ has no poles. Now, let $z_1^{(b)} \to z_1^{(c)}$ to fuse into $z^{(b+c)}$. In this limit, $W_{a,bc}$ becomes the order of zeros between $z_1^{(a)}$ and $z^{(b+c)}$: $W_{a,bc}=D_{a,b+c}$. Thus, we obtain the following conditions on D_{ab}:

$$D_{a,b+c} \geq D_{ab} + D_{ac}. \tag{17}$$

Concave condition (17) is equivalent to a condition on S_a,

$$S_{a+b+c} + S_a + S_b + S_c \geq S_{a+b} + S_{b+c} + S_{a+c}. \tag{18}$$

We note that the Laughlin state $\Phi_{1/q}(\{z_i\})=\Pi_{i<j}(z_i-z_j)^q$ saturates the above conditions: $D_{a,b+c}=D_{ab}+D_{ac}$ or $S_{a+b+c}+S_a+S_b+S_c=S_{a+b}+S_{b+c}+S_{a+c}$.

B. Symmetry condition

If we fix all variables $z_i^{(a)}$ except $z_1^{(a)}$, $z_2^{(b)}$, and $z_3^{(c)}$, then the derived polynomial $P_\alpha(\{z_i^{(a)}\})$ gives us a complex function $f(z_1^{(a)},z_2^{(b)},z_3^{(c)})$. Let us assume $z_1^{(a)}$, $z_2^{(b)}$, and $z_3^{(c)}$ are very close to each other and far away from all other variables. $f(z_1^{(a)},z_1^{(b)},z_1^{(c)})$ has a D_{ab}th order zero as $z_1^{(a)} \to z_2^{(b)}$ and a D_{ab}th order zero as $z_1^{(a)} \to z_3^{(c)}$. Thus, $D_{a,b+c}-D_{ab}-D_{ac}$ is the number of zeros of $f(z_1^{(a)})$ in the same neighborhood that are not at $z_2^{(b)}$ and $z_3^{(c)}$ (see Fig. 2).

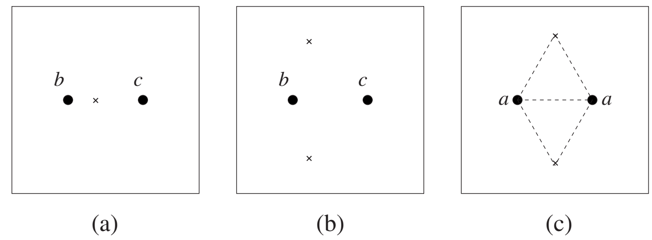

FIG. 2. Pattern of zeros of $f(z_1^{(a)},z_2^{(b)},z_3^{(c)})$ (when viewed as a function of $z_1^{(a)}$): (a) $D_{a,b+c}-D_b-D_c=1$, (b) $D_{a,b+c}-D_b-D_c=2$, and (c) $a=b=c$ and $D_{a,a+a}-D_a-D_a=2$. The dashed lines form two equilateral triangles. The zeros that are not located at any variable are marked by crosses.

Here, we would like to assume that $\Phi(\{z_i\})$ satisfies the unique-fusion condition. In this case, the type-a variables in the derived polynomials have "no shapes" and can be treated as points. Since other variables are far away, the zeros of $f(z_1^{(a)},z_2^{(b)},z_3^{(c)})$ must satisfy certain symmetry conditions (see Fig. 2).

We see that when $a=b=c$, $f(z_1^{(a)},z_2^{(a)},z_3^{(a)})$ can be zero only when $z_i^{(a)} \to z_j^{(a)}$ or when $z_1^{(a)}$, $z_2^{(a)}$, and $z_3^{(a)}$ form an equilateral triangle. Thus, the zeros of $f(z_1^{(a)},z_2^{(a)},z_3^{(a)})$ (when viewed as a function of $z_1^{(a)}$) that are marked by the crosses must appear in pairs. We find that $D_{a,a+a}-D_a-D_a$ must be even, or equivalently

$$S_{3a} - S_a = \text{even}. \tag{19}$$

C. *n*-cluster condition

The structure of symmetric polynomials with infinite variables is very complicated and hard to manage. Here, we would like to introduce an n-cluster condition that makes a polynomial with infinite variables behave more like a polynomial with a finite number of variables. A symmetric polynomial satisfies the n-cluster condition if after we fuse the variables of $\Phi(\{z_i\})$ into n-variable clusters, the derived polynomial

$$P_c(z_1^{(n)}, \ldots, z_{N_c}^{(n)}) \sim \prod_{i<j} (z_i^{(n)} - z_j^{(n)})^l \tag{20}$$

has a simple Laughlin form where l is a positive integer.

To see the structure of cluster form more clearly, let us assume that the polynomial $\Phi(z_1, \ldots, z_N)$ describes a FQH state with filling fraction ν. This means that as a homogenous polynomial, the total order of the z_i, S_N, in $\Phi(z_1, \ldots, z_N)$ satisfies

$$S_N = \frac{1}{2\nu}N^2 + O(N).$$

This motivates us to write $\Phi(z_1, \ldots, z_N)$ as

$$\Phi(\{z_i\}) = G(\{z_i\})\Phi_\nu(\{z_i\}),$$

$$\Phi_\nu(\{z_i\}) = \prod_{i<j} (z_i - z_j)^{\nu^{-1}}. \quad (21)$$

Here, $G(\{z_i\})$ satisfies

$$G(\lambda z_1, \ldots, \lambda z_N) = \lambda^{s_N} G(z_1, \ldots, z_N),\ s_N = O(N) \ldots .$$

Note that $G(\{z_i\})$ is, in general, not a single-valued function since $\Phi_\nu(\{z_i\})$ is, in general, not single valued. However, the product of $G(\{z_i\})$ and $\Phi_\nu(\{z_i\})$ is a single-valued symmetric polynomial.

We can fuse the variables in $G(\{z_i\})$ to obtain a derived function $G(\{z_i^{(a)}\})$ [just as how we obtain the derived polynomial $P(\{z_i^{(a)}\})$ from the original symmetric polynomial $\Phi(\{z_i\})$]. Similarly, we can also fuse the variables in $\Phi_\nu(\{z_i\})$ to obtain a derived function $\Phi_\nu(\{z_i^{(a)}\})$ as follows:

$$\Phi_\nu(\{z_i^{(a)}\}) = \prod_{i,j;a<b} (z_i^{(a)} - z_j^{(b)})^{ab/\nu} \prod_{i<j;a} (z_i^{(a)} - z_j^{(a)})^{a^2/\nu}. \quad (22)$$

Thus, the derived polynomial $P(\{z_i^{(a)}\})$ can be expressed as

$$P(\{z_i^{(a)}\}) = G(\{z_i^{(a)}\})\Phi_\nu(\{z_i^{(a)}\}). \quad (23)$$

Equation (23) can be viewed as a definition of $G(\{z_i^{(a)}\})$.

Assuming $\Phi(\{z_i\})$ has an n-cluster form, then if we fuse the variables of $G(\{z_i\})$ into n-variable clusters, the derived function

$$G(\{z_i^{(n)}\}) = 1.$$

Here, we will require $G(\{z_i^{(a)}\})$ to satisfy more strict conditions,

$$G(\{z_i^{(a)}\}) = G(\{z_i^{(a\%n)}\}),$$

$$G(\ldots; z_i^{(n)}; \ldots) = G(\ldots, \ldots). \quad (24)$$

The second condition states that $G(\ldots, z_i^{(a)}, \ldots)$ does not depend on $z_i^{(a)}$ if $a\%n=0$. If G satisfies Eq. (24), we will say the corresponding symmetric polynomial $\Phi(\{z_i\}) = G(\{z_i\})\Phi_\nu(\{z_i\})$ to have an n-cluster form.

For a symmetric polynomial $\Phi(\{z_i\})$ of an n-cluster form, its pattern of zeros D_{ab} can be written as

$$D_{ab} = \nu^{-1}ab + d_{ab}, \quad (25)$$

where d_{ab} satisfy

$$d_{ab} = d_{ba},$$

$$d_{ab} = 0 \quad \text{if } b\%n = 0,$$

$$d_{a,b+n} = d_{ab}. \quad (26)$$

The pattern of zeros D_{ab} that satisfies the above conditions is said to have an n-cluster form. Note that $\nu^{-1}ab$ in Eq. (25) describe the pattern of zeros in the derived function $\Phi_\nu(\{z_i^{(a)}\})$ [see Eq. (22)] and d_{ab} describes the pattern of zeros in the derived function $G(\{z_i^{(a)}\})$.

Setting $(a,b)=(n,n)$ and $(a,b)=(1,n)$ in Eq. (25), we find that

$$\nu^{-1}n^2 = \text{even}, \quad \nu^{-1}n = \text{integer}.$$

or

$$\nu^{-1} = \frac{m}{n}, \quad mn = \text{even}. \quad (27)$$

We also find that

$$D_{a,b+n} = D_{a,b} + am. \quad (28)$$

Let

$$s_a = S_a - \frac{1}{2\nu}a(a-1). \quad (29)$$

We find that [see Eq. (10)]

$$d_{ab} = s_{a+b} - s_a - s_b. \quad (30)$$

Cluster conditions (26) become

$$s_{a+n} - s_a - s_n = 0,$$

$$s_{a+n} - s_a = s_{b+n} - s_b. \quad (31)$$

Since $S_1=s_1=0$ [see Eq. (8)], we find that $s_{n+1}=s_n$ and $s_{a+n} - s_a = s_n$. Thus,

$$s_{a+kn} = ks_n + s_a, \quad \text{where } a = 1,2,\ldots,\infty. \quad (32)$$

This allows us to obtain s_a for any $a>0$ from $s_1, s_2, \ldots, s_n$. Similarly, all the S_a's can be determined from $S_1, S_2, \ldots, S_n$:

$$\begin{aligned} S_{a+kn} &= s_{a+kn} + \frac{m}{2n}(a+kn)(a+kn-1) \\ &= ks_n + s_a + \frac{m}{2n}(a+kn)(a+kn-1) \\ &= S_a - \frac{m}{2n}a(a-1) + k\left[S_n - \frac{m}{2}(n-1)\right] \\ &\quad + \frac{m}{2n}(a+kn)(a+kn-1) \\ &= S_a + kS_n + \frac{k(k-1)nm}{2} + kma. \end{aligned} \quad (33)$$

The above result is actually valid for any positive integer a.

It is convenient to introduce

$$h_a^{\text{sc}} \equiv s_a - \frac{a}{n}s_n = S_a - \frac{aS_n}{n} + \frac{am}{2} - \frac{a^2m}{2n}. \quad (34)$$

From Eq. (32), we can show that h_a^{sc} is periodic:

$$h_a^{\text{sc}} = h_{a+n}^{\text{sc}}.$$

Since $s_1=0$, we see that $h_1^{\text{sc}} = -s_n/n$ and $s_a = h_a^{\text{sc}} - ah_1^{\text{sc}}$. From Eq. (29), we see that S_a can be calculated from h_a^{sc}:

$$S_a = h_a^{\text{sc}} - ah_1^{\text{sc}} + \frac{a(a-1)m}{2n}. \quad (35)$$

Equations (34) and (35) imply that the two sequences of numbers, $\{S_a\}$ and $\{h_a^{\text{sc}}\}$, have a one-to-one correspondence and can faithfully represent each other. In this paper, we will use both sequences to characterize the symmetric polynomials. The $h_a^{\rm sc}$ characterization turns out to have a close relation to the conformal field theory (CFT) description of the FQH states (see Appendix B).[14,35–37]

If S_a has n-cluster form (33), then the corresponding boson occupation numbers n_l have some nice properties. From Eq. (33), we see that $S_{a+n}-S_{a-1+n}=S_a-S_{a-1}+m$. Thus, $l_i=S_i-S_{i-1}$ satisfy $l_{i+n}=l_i+m$. This means that the boson occupation in the orbitals z^l has a periodic structure: every time we skip n bosons, we skip m orbitals. Or, in other words, if we know the occupation distribution of the first n bosons, the occupation distribution of second n bosons can be obtained from that of first n bosons by shifting the orbital index l by m. Thus, the occupation numbers n_l satisfy $n_l=n_{l\%m}$ [see Eq. (52)]. Also, each m orbital contain n bosons. Due to the one-to-one correspondence between S_a and n_l, we can also use $n_0,\ldots,n_{m-1}$ to describe the pattern of zeros in $\Phi(\{z_i\})$.

D. Translation invariance

To study the translation invariance of the symmetric polynomial $\Phi(z_1,\ldots,z_N)$, let us put the polynomial on a sphere (see Appendix A) and study its rotation invariance. In fact, in this paper, when we mention translation invariance, we actually mean rotation invariance on a sphere.

Let N_ϕ be the number of flux quanta going through the sphere. Then, each variable z_i in $\Phi(z_1,\ldots,z_N)$ carries an angular momentum $J=N_\phi/2$. What is the total angular momentum of $\Phi(z_1,\ldots,z_N)$? In general, $\Phi(z_1,\ldots,z_N)$ does not carry a definite angular momentum. Therefore, here we will calculate the maximum angular momentum of $\Phi(z_1,\ldots,z_N)$ from the pattern of zeros D_{ab}.

The maximum angular momentum is nothing but the angular momentum of $z^{(N)}$—the particle obtained by fusing all the N electrons together. The angular momentum of $z^{(N)}$ is given by [see Eqs. (8) and (12)]

$$J_N=J_{\rm tot}=NJ-\sum_{a=1}^{N-1}D_{a,1}=NJ-S_N. \tag{36}$$

If $J_N=0$ for a symmetric polynomial $\Phi(\{z_i\})$, then $\Phi(\{z_i\})$ is invariant under the $O(3)$ rotation of the sphere. In other words, $\Phi(\{z_i\})$ is translation invariant.

However, for an arbitrary choice of N and J, J_N is not zero in general. J_N can be zero only for certain combinations of (N,J). For the filling fraction $\nu=1/q$ Laughlin state, $J_N=NJ-q\frac{N(N-1)}{2}$. We find $J_N=0$ if

$$2J=N_\phi=qN-q. \tag{37}$$

This is the relation between the number of magnetic flux quanta, N_ϕ, and the number of electrons, N, of the $\nu=1/q$ Laughlin state if the Laughlin state is to fill the sphere completely (which gives rise to a rotation invariant state).

Assume that the symmetric polynomial $\Phi(z_1,\ldots,z_N)$ has an n-cluster form described by the data $(m;\,S_a|_{a=1,\ldots,n})$. If we put the polynomial on a sphere, the maximum total angular momentum of $\Phi(z_1,\ldots,z_N)$ is given by Eq. (36). If $N=nN_c$, we find from Eq. (33) that

$$S_{nN_c}=S_{n+(N_c-1)n}=N_cS_n+\frac{mnN_c(N_c-1)}{2},$$

$$J_{\rm tot}=JnN_c-N_cS_n-\frac{mnN_c(N_c-1)}{2}. \tag{38}$$

When $N=nN_c$, $\Phi(z_1,\ldots,z_N)$ can give rise to the Laughlin wave function (20) after fusing z_i's into N_c $z_i^{(n)}$'s [see Eq. (20)]. Since it is always possible to fill the sphere with the Laughlin state, this implies that there exists an integer $2J$ to make $J_{\rm tot}=0$. Such an integer is given by

$$2J=N_\phi=\frac{2S_{nN_c}}{nN_c}=\frac{2S_n}{n}+m(N_c-1). \tag{39}$$

This requires that

$$2S_n=0 \bmod n. \tag{40}$$

To summarize, the n-cluster condition requires that if $N\%n=0$ and (N_ϕ,N) satisfies Eq. (39), then the symmetric polynomial $\Phi(z_1,\ldots,z_N)$ must represent a rotation invariant state on sphere. The existence of such rotation invariant state requires S_n to satisfy Eq. (40).

V. CONSTRUCTION OF IDEAL HAMILTONIANS

We have seen that the pattern of zeros in an electron wave function $\Phi(\{z_i\})$ can be described by a set of integers $S_2,S_3,\ldots$. In this section, we are going to construct an ideal Hamiltonian on sphere to realize such a kind of electron wave function as a ground state of the Hamiltonian.

On a sphere, the set of integers S_a also has a very physical meaning. For an electron system on a sphere with N_ϕ flux quanta, each electron carries an orbital angular momentum $J=N_\phi/2$ if the electrons are in the first Landau level.[31] For a cluster of a electrons, the maximum allowed angular momentum is aJ. However, for the wave function $\Phi(\{z_i\})$ described by S_a, the maximum allowed angular momentum is only $J_a=aJ-S_a$. The pattern of zeros forbids the appearance of angular momenta $aJ-S_a+1,aJ-S_a+2,\ldots,aJ$ for any a-electron clusters in $\Phi(\{z_i\})$.

Such a condition can be easily enforced by a Hamiltonian. Let $P_S^{(a)}$ be a projection operator that acts on a-electron Hilbert space. $P_S^{(a)}$ projects onto the subspace of a electrons with total angular momenta $aJ-S+1,\ldots,aJ$. Now consider the Hamiltonian[19,38,39]

$$H_{\{S_a\}}=\sum_a\sum_{a\text{-electron clusters}}P_{S_a}^{(a)}, \tag{41}$$

where $\sum_{a\text{-electron clusters}}$ sum over all a-electron clusters. The wave function $\Phi(\{z_i\})$ with a pattern of zeros described by S_a, if it exists, will be the zero-energy ground state of the above Hamiltonian.

We note that the Hamiltonian $H_{\{S_a\}}$ is well defined for any choice of S_a. However, for a generic choice of S_a, the zero-energy ground state of $H_{\{S_a\}}$ may not be the one with a pattern of zeros described by S_a. This is because when we say that the wave function $\Phi(\{z_i\})$ has a pattern of zeros described by S_a, we mean two things:

(a) The angular momenta $aJ-S_a+1,\cdots aJ$ do not appear for any a-electron clusters in $\Phi(\{z_i\})$.

(b) The angular momenta $aJ-S_a$ does appear for a-electron clusters in $\Phi(\{z_i\})$.

The zero-energy ground state of $H_{\{S_a\}}$ satisfies condition (a). However, sometimes, we may find that condition (a) also implies that $aJ-S_a$ does not appear for a-electron clusters in $\Phi(\{z_i\})$ for certain values of a. This means that the zero-energy ground state of $H_{\{S_a\}}$ is actually described by a pattern of zeros $\tilde{S}_a$ which satisfy $\tilde{S}_a \geq S_a$. However, for a certain special set of $\{S_a\}$ that describe the pattern of zeros of an existing symmetric polynomial, we have $\tilde{S}_a = S_a$. For those S_a, the zero-energy ground state of $H_{\{S_a\}}$ is described by the pattern of zeros of $\{S_a\}$ itself.

We have seen that for a FQH state described by a pattern of zeros $\{S_a\}$, a state of a-electron clusters has a nonzero projection into the space $\mathcal{H}_{a,S_a}$, where $\mathcal{H}_{a,S_a}$ is a space with a total angular momentum $aJ-S_a$. However, different positions of other electrons may lead to different images in the space $\mathcal{H}_{a,S_a}$. Let $\mathcal{H}_a$ be the subspace of $\mathcal{H}_{a,S_a}$ that is spanned by those images. In general, $\mathcal{H}_{a,S_a} \neq \mathcal{H}_a$. So, in general, the zero-energy ground state of the ideal Hamiltonian $H_{\{S_a\}}$ may not be unique. In an attempt to construct an ideal Hamiltonian for which the FQH state Φ is the unique ground state, we can add additional projection operators and introduce a new ideal Hamiltonian

$$H_{\{S_a\}} = \sum_a \sum_{a\text{-electron clusters}} (P^{(a)}_{S_a} + P_{\bar{\mathcal{H}}_a}), \tag{42}$$

where $P_{\bar{\mathcal{H}}_a}$ is a projection operator into the space $\bar{\mathcal{H}}_a$ and $\bar{\mathcal{H}}_a$ is a subspace of $\mathcal{H}_{a,S_a}$ formed by vectors that is perpendicular to $\mathcal{H}_a$.

VI. SUMMARY OF GENERAL RESULTS

In Secs. III and V, we have considered a subclass of symmetric polynomials $\Phi(\{z_i\})$ (of infinity variables) that satisfy (a) a unique-fusion condition (see discussion in Sec. III A), (b) an n-cluster condition (see discussion in Sec. IV C), and (c) the translation invariance

$$\Phi(\{z_i\}) = \Phi(\{z_i - z\}).$$

The unique-fusion condition requires that when we fuse the variables z_i together to obtain new polynomials, we will always get the same polynomial no matter how we fuse the variables together. The n-cluster condition requires that if we fuse all the variables z_i into clusters of n variables each, the resulting polynomial of the clusters has the Jastrow form $\prod_{i<j}(z_i^{(n)}-z_j^{(n)})^q$.

We find that each translation invariant symmetric polynomial $\Phi(\{z_i\})$ of the n-cluster form and satisfying the unique-fusion condition is characterized by a set of non-negative integers $(m;S_2,\ldots,S_n)$. However, not all sets of non-negative integers $(m;S_2,\ldots,S_n)$ can be realized by such symmetric polynomials. The $(m;S_2,\ldots,S_n)$ that correspond to existing translation invariant symmetric polynomials (that satisfy the n-cluster and the unique-fusion conditions) must satisfy certain conditions.

First, m and S_n must satisfy [see Eqs. (27) and (40)]

$$m > 0, \quad mn = \text{even},$$

$$2S_n = 0 \bmod n. \tag{43}$$

From $m,S_2,\ldots,S_n$ and $S_1=0$, we can determine S_a for any $a>1$ [see Eq. (33)]:

$$S_{a+kn} = S_a + kS_n + \frac{k(k-1)nm}{2} + kma. \tag{44}$$

Those S_a must satisfy [see Eqs. (11) and (18)]

$$\Delta_2(a,a) = \text{even},$$

$$\Delta_2(a,b) \geq 0, \quad \Delta_3(a,b,c) \geq 0, \tag{45}$$

where

$$\Delta_2(a,b) \equiv S_{a+b} - S_a - S_b,$$

$$\Delta_3(a,b,c) \equiv S_{a+b+c} - S_{a+b} - S_{b+c} - S_{a+c} + S_a + S_b + S_c. \tag{46}$$

S_a's also satisfy another condition which is harder to describe. To describe the new condition, we first note that the sequence $\{S_a\}$ can be encoded by another sequence of non-negative integers n_l, where $l=0,1,\ldots$. To obtain n_l from S_a, we introduce $l_a \equiv S_a - S_{a-1}$ for $a=1,2,\ldots$. Then, n_l is the number of l_a's that satisfy $l_a=l$. The two sequences, $\{S_a\}$ and $\{n_l\}$, have a one-to-one correspondence and can faithfully represent each other. The number n_l can be regarded as the boson occupation number that was used to characterize FQH states in the thin cylinder limit.[40–42] n_l is also used to label Jack polynomials that describe FQH states.[43,44]

Now, let us introduce $2J+1$ orbitals $|m_z\rangle$, $m_z=-J,-J+1,\ldots,J-1,J$, which form a representation of SU(2) with an angular momentum J. (Here $2J$ is an integer.) We can create a many-boson state $|\{n_l\}\rangle$ by putting n_l bosons into the $m_z=l-J$ orbitals. Then, S_a must be such that [see Eq. (16)]

$$P_{N,NJ-S_N} \cdots P_{3,3J-S_3} P_{2,2J-S_2} |\{n_l\}\rangle \neq 0, \tag{47}$$

where P_{a,J_a} is a projection operator that projects into the subspace where any a particles have a total angular momentum equal to J_a or less and N is the number of particles in $|\{n_l\}\rangle$.

We will call $(m;S_2,\ldots,S_n)$ an $\boldsymbol{S}$ vector and denote it as

$$\boldsymbol{S} = (m;S_2,\ldots,S_n).$$

We find that translation invariant symmetric polynomials (that satisfy the n-cluster and the unique-fusion conditions) are labeled by the $\boldsymbol{S}$ vectors that satisfy Eqs. (43), (45), and (47).

We would like to stress that Eqs. (43), (45), and (47) are only necessary conditions for $(m;S_2,\ldots,S_n)$ to describe a translation invariant symmetric polynomial of the n-cluster form and satisfying the unique-fusion condition. We do not know if those conditions are sufficient or not. Some $\boldsymbol{S}$ vectors that satisfy Eqs. (43), (45), and (47) may not correspond to an existing symmetric polynomial. Also, there may be more than one symmetric polynomial that are described by the same $\boldsymbol{S}$ vectors that satisfy Eqs. (43), (45), and (47). Each such symmetric polynomial corresponds to a FQH state. By solving Eqs. (43), (45), and (47), we can obtain $(m;S_2,\ldots,S_n)$'s that correspond to the Laughlin states, the Pfaffian state,[14] the parafermion states,[19] and many new non-Abelian states.

We also obtained some additional results. Our numerical studies of Eqs. (43) and (45) suggest that all solutions of the equations satisfy

$$h_a^{\mathrm{sc}} = h_{n-a}^{\mathrm{sc}}, \tag{48}$$

where

$$h_a^{\mathrm{sc}} = S_a - \frac{aS_n}{n} + \frac{am}{2} - \frac{a^2 m}{2n}, \tag{49}$$

although we cannot derive Eq. (48) analytically. Such a relation implies that

$$S_{n-a} = S_a + \frac{n-2a}{n} S_n.$$

h_a^{sc}'s also satisfy

$$h_a^{\mathrm{sc}} = h_{a\%n}^{\mathrm{sc}}, \quad h_a^{\mathrm{sc}} \geq 0,$$

where $a\%n \equiv a \bmod n$.

From Eq. (49), we find that $(h_1^{\mathrm{sc}},\ldots,h_n^{\mathrm{sc}})$ and $(S_2,\ldots,S_n)$ have a one-to-one correspondence. They can faithfully represent each other. Due to the one-to-one relation between S_a and h_a^{sc}, we can also use n, m, and $h_1^{\mathrm{sc}},\ldots,h_n^{\mathrm{sc}}$ to characterize the pattern of zero in the symmetric polynomial $\Phi(\{z_i\})$. We will package the data in the form

$$\boldsymbol{h} = \left(\frac{m}{n}; h_1^{\mathrm{sc}}, \ldots, h_n^{\mathrm{sc}}\right),$$

and call $\boldsymbol{h}$ an $\boldsymbol{h}$ vector. We see that patterns of zeros in a symmetric polynomial can also be described by the $\boldsymbol{h}$ vectors.

Each symmetric polynomial described by the pattern of zeros $\{S_a\}$ is related to a CFT generated by simple-current operators which have an Abelian fusion rule (see Appendix B). $(h_1^{\mathrm{sc}},\cdots,h_n^{\mathrm{sc}})$ turn out to be the scaling dimensions of those simple-current operators. Since $\Delta_3(a,b,c)$ only depends on h_a^{sc},

$$\Delta_3(a,b,c) = h_{a+b+c}^{\mathrm{sc}} - h_{a+b}^{\mathrm{sc}} - h_{b+c}^{\mathrm{sc}} - h_{a+c}^{\mathrm{sc}} + h_a^{\mathrm{sc}} + h_b^{\mathrm{sc}} + h_c^{\mathrm{sc}},$$

and $\Delta_3(a,b,c) \geq 0$ is a property of the simple-current CFT.

Condition (47) is hard to check. So let us consider Eqs. (43) and (45) only. One class of solutions of Eqs. (43) and (45) is given by

$$h_a^{\mathrm{sc}} = h_a^{Z_n} \equiv \frac{(a\%n)[n-(a\%n)]}{n}.$$

This class of solutions corresponds to the Z_n parafermion CFT which is generated by simple-current operators ψ_a that satisfy an Abelian fusion rule $\psi_a\psi_b = \psi_{a+b}$ and $\psi_0 = \psi_n = 1$. The scaling dimensions of ψ_a is given by the above $h_a^{Z_n}$. The parafermion states introduced in Ref. 19 are related to such a class of solutions.

A more general class of solutions of Eqs. (43) and (45) corresponds to generalized parafermion CFTs. A generalized parafermion CFT is generated simple-current operators that have the following dimensions:

$$h_a^{\mathrm{sc}} = h_a^{Z_n^k} \equiv \frac{(ka\%n)[n-(ka\%n)]}{n}.$$

Those solutions represent a new class of non-Abelian FQH states, which will be called generalized parafermion states.

It turns out that all the solutions of Eqs. (43) and (45) are closely related to parafermion CFTs; i.e., a solution h_a^{sc} satisfies

$$h_a^{\mathrm{sc}} = \sum_i \frac{k_i}{2} h_a^{\mathrm{PF}_i} \bmod 1, \tag{50}$$

where k_i's are positive or negative integers and $h_a^{\mathrm{PF}_i}$'s are the scaling dimensions of the parafermion operators in some parafermion CFTs labeled by i. They are given by

$$h_a^{\mathrm{PF}} = h_a^{Z_{n'}^k}$$

for certain integers k and n', where n' is a factor of n.

If $h_a^{\mathrm{sc}} = h_a^{\mathrm{PF}}$, then the solution corresponds to an existing symmetric polynomial generated by a (generalized) parafermion CFT. If $h_a^{\mathrm{sc}} = \frac{1}{2} h_a^{\mathrm{PF}}$, then the solution corresponds to the square root of a symmetric polynomial generated by a (generalized) parafermion CFT. Therefore, the later solution does not correspond to any existing symmetric polynomials. Numerical experiments suggest that the later cases always have $\Delta_3(a,b,c) = \text{odd}$ for some a, b, and c. This motivates us to introduce the new condition

$$\Delta_3(a,b,c) = \text{even} \tag{51}$$

to exclude those illegal cases. Conditions (43), (45), and (51) provide an easy way to obtain S_a's that may correspond to existing symmetric polynomials. The new condition (51) is a generalization of necessary conditions $\Delta_3(a,a,a) = \text{even}$ [see Eq. (19)], (B8), and $S_a \neq 1$.

Our numerical studies suggest that the solutions of Eqs. (43), (45), and (51) give rise to h_a^{sc} that satisfy

$$h_a^{\mathrm{sc}} = \sum_i k_i h_a^{\mathrm{PF}_i} \bmod 2.$$

Those solutions also have the properties that $m = \text{even}$ and $S_a = \text{even}$.

We also find that for S_a satisfying Eqs. (43) and (45), the corresponding n_l is a periodic function of l (for $l \geq 0$) with a period m:

$$n_l = n_{l\%m}. \tag{52}$$

The n_l's satisfy

$$\sum_{l=0}^{2J} n_l = nN_c, \quad \sum_{l=0}^{2J} (l-J)n_l = 0, \tag{53}$$

for any (J,N_c) satisfying

$$2J = \frac{2S_n}{n} + m(N_c - 1) \tag{54}$$

VII. GENERAL STRUCTURE OF THE SOLUTIONS

The $\boldsymbol{S}$ vectors that satisfy Eqs. (43) and (45) have some general properties. In this section, we will discuss those properties.

A. *n*-cluster polynomial as *κn*-cluster polynomial

Let $P(\{z_i^{(a)}\})$ be a derived symmetric polynomial of n-cluster form described by $(m;S_2,\ldots,S_n)$. From Eq. (26), we see that $P(\{z_i^{(a)}\})$ can also be viewed as a symmetric polynomial of κn-cluster form where κ is a positive integer. When viewed as a κn-cluster polynomial, $P(\{z_i^{(a)}\})$ is described by $(\kappa m,S_2,\ldots,S_{\kappa n})$, where $S_{n+1},\cdots,S_{\kappa n}$ are obtained from $(m;S_2,\ldots,S_n)$ through Eq. (33).

The filling fraction $\nu=1/q$ Laughlin state $\Phi_{1/q}=\Pi_{i<j}(z_i-z_j)^q$ has a one-cluster form. Thus, $\Phi_{1/q}$ can also be viewed as an n-cluster polynomial for any positive n. When viewed as an n-cluster polynomial, the $\nu=1/q$ Laughlin state is described by

$$(m;S_2,\ldots,S_n) = \left(nq;q,\ldots,\frac{qn(n-1)}{2}\right).$$

Such a $\nu=1/q$ Laughlin state always appears as a solution of Eqs. (43) and (45) for any n.

B. Products of symmetric polynomials

Let $P(\{z_i^{(a)}\})$ and $P'(\{z_i^{(a)}\})$ be two derived symmetric polynomials of n-cluster form described by $(m;S_2,\ldots,S_n)$ and $(m';S_2',\ldots,S_n')$, respectively. Then, their product $\tilde{P}(\{z_i^{(a)}\})=P(\{z_i^{(a)}\})P'(\{z_i^{(a)}\})$ is also a symmetric polynomial of n-cluster form. $\tilde{P}(\{z_i^{(a)}\})$ is described by

$$(\tilde{m},\tilde{S}_2,\ldots,\tilde{S}_n) = (m+m';S_2+S_2',\ldots,S_n+S_n').$$

This is because the pattern of zeros of $\tilde{P}(\{z_i^{(a)}\})$ is related to the patterns of zeros of $P(\{z_i^{(a)}\})$ and $P'(\{z_i^{(a)}\})$ through

$$\tilde{D}_{ab} = D_{ab} + D_{ab}'.$$

Also, the relation between $(m;S_2,\ldots,S_n)$ and D_{ab} is linear [see Eqs. (10), (12), and (33)]. Therefore, if two $\boldsymbol{S}$ vectors, $\boldsymbol{S}$ and $\boldsymbol{S}'$, describe two existing symmetric polynomials, then their sum $\tilde{\boldsymbol{S}}=\boldsymbol{S}+\boldsymbol{S}'$ also describes an existing symmetric polynomial, whose fillings fractions are reciprocally additive.

Indeed, the solutions of Eqs. (43) and (45) have a structure that is consistent with the above result. We note that Eqs. (43) and (45) are linear in the $\boldsymbol{S}$ vector $\boldsymbol{S}=(m;S_2,\ldots,S_n)$. Thus, if $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are two solutions of Eqs. (43) and (45), then

$$\boldsymbol{S} = k_1\boldsymbol{S}_1 + k_2\boldsymbol{S}_2 \tag{55}$$

is also a solution for any non-negative integers k_1 and k_2. Therefore, we can divide the solutions of Eqs. (43) and (45) into two classes: primitive solutions and nonprimitive solutions. The primitive solutions are those that cannot be written as a sum of two other solutions. All solutions of Eqs. (43) and (45) are linear combinations of primitive solutions with non-negative integral coefficients.

As an application of the product rule, let us consider a symmetric polynomial of n-cluster form $\Phi(\{z_i\})$ which is described by $(m;S_2,\ldots,S_n)$. We can construct a new symmetric polynomial of n-cluster form from $\Phi(\{z_i\})$,

$$\tilde{\Phi}(\{z_i\}) = \Phi(\{z_i\})\prod_{i<j}(z_i-z_j)^q,$$

where q is even. The symmetric polynomial $\tilde{\Phi}(\{z_i\})$ is described by

$$(\tilde{m};\tilde{S}_2,\ldots,\tilde{S}_n) = \left(m+nq;S_2+q,\ldots,S_n+\frac{qn(n-1)}{2}\right).$$

VIII. SOME EXAMPLES

In this section, we will give some examples of symmetric polynomials described by the $\boldsymbol{S}$ vector $(m;S_2,\ldots,S_n)$ that satisfy Eqs. (43), (45), and (51).

A. *n*=1 cases

If $n=1$, the different patterns of zeros are characterized by an even integer m. We find $S_a=ma(a-1)/2$ and $D_{ab}=mab$. Each even m corresponds to a $\nu=1/m$ Laughlin state

$$\Phi_{1/m}(\{z_i\}) = \prod_{i<j}(z_i-z_j)^m.$$

We have introduced three equivalent ways to describe a pattern of zeros: the $\boldsymbol{S}$ vector $(m;S_2,\ldots,S_n)$, the $\boldsymbol{h}$ vector $(\frac{m}{n};h_1^{\text{sc}},\ldots,h_n^{\text{sc}})$, and the boson occupation number n_l: $(n_0,\ldots,n_{m-1})$. For the $\nu=1/m$ Laughlin state those data are given by

$$\Phi_{1/m} \quad : \quad \boldsymbol{S}=(m;),$$

$$\left(\frac{m}{n};h_1^{\text{sc}}\right) = (m;0),$$

$$(n_0,\ldots,n_{m-1}) = (1,0,\ldots,0).$$

B. *n*=2 cases

If $n=2$, the different patterns of zeros are characterized by two integers m,S_2. The following two sets of m, S_2 are the primitive solutions of Eqs. (43) and (45):

$$(m;S_2) = (1;0), \quad (m';S_2') = (4;2).$$

Let us discuss the solution $(m;S_2)=(1;0)$ in more detail. The corresponding boson occupation numbers are

$$(n_0,n_1,\ldots)=(2,2,2,\ldots),$$

where there are two bosons occupying each orbital. Let us check condition (47) for the $J=1/2$ case wherein there are only two orbitals. This leads to a state $|\{2,2\}\rangle$ with four bosons described by the wave function

$$\Phi_{\{2,2\}}=z_1z_2+z_1z_3+z_1z_4+z_2z_3+z_2z_4+z_3z_4.$$

On sphere the above wave function becomes (see Appendix A)

$$\Phi^{\text{sp}}_{\{2,2\}}=\mathcal{S}[v_1v_2u_3u_4],$$

where $\mathcal{S}$ is the symmetrization operator. Since $S_4=2$, we find that $J_4=4J-S_4=0$ and P_{4,J_4} is a projection into the subspace with vanishing total angular momentum. A direct calculation reveals that $P_{4,J_4}\Phi^{\text{sp}}_{\{2,2\}}=0$. Thus, $(m;S_2)=(1;0)$ does not satisfy condition (47) and does not correspond to any translation invariant symmetric polynomial.

Now let consider m,S_2 that satisfy a new condition (51) in addition to Eqs. (43) and (45). The following two sets of m,S_2 are the primitive solutions:

$$\Phi^2_{\frac{2}{2};Z_2}:\ (m;S_2)=(2;0),$$

$$\left(\frac{m}{n};h_1^{\text{sc}},\ldots,h_n^{\text{sc}}\right)=\left(\frac{2}{2};\frac{1}{2},0\right),$$

$$(n_0,\ldots,n_{m-1})=(2,0),$$

and

$$\Phi_{1/2}:\ (m;S_2)=(4;2),$$

$$\left(\frac{m}{n};h_1^{\text{sc}},\ldots,h_n^{\text{sc}}\right)=\left(\frac{4}{2};0,0\right),$$

$$(n_0,\ldots,n_{m-1})=(1,0,1,0).$$

Here, we also listed the corresponding $\boldsymbol{h}$ vector $\boldsymbol{h}=(\frac{m}{n};h_1^{\text{sc}},\cdots,h_n^{\text{sc}})$ and the boson occupation numbers $(n_0,\cdots,n_{m-1})$.

Let us discuss the solution $(m;S_2)=(2;0)$ in more details. We find $(S_1,S_2,S_3,S_4)=(0,0,2,4)$ and

$$\begin{pmatrix}D_{11} & D_{12}\\ D_{21} & D_{22}\end{pmatrix}=\begin{pmatrix}0 & 2\\ 2 & 4\end{pmatrix},$$

which means that we will have no zero if we bring two particles together and a second order zero if we bring a third particle to a two-particle cluster. Such a pattern of zeros describes the following translation invariant symmetric polynomial:

$$\Phi_{2/2;Z_2}(\{z_i\})=\mathcal{A}\left(\frac{1}{z_1-z_2}\frac{1}{z_3-z_4}\cdots\right)\prod_{i<j}(z_i-z_j) \qquad (56)$$

which is the filling fraction $\nu=1$ bosonic Pfaffian state.[14] Here, $\mathcal{A}$ is the antisymmetrization operator. The Pfaffian state can be written as a correlation of the following operator in a CFT:

$$V_e(z)=\psi(z)e^{i\phi(z)},$$

where $\psi(z)$ is the Majorana fermion operator in the Ising CFT (which is also the Z_2 parafermion CFT). The $h_1^{\text{sc}}=1/2$ in the $\boldsymbol{h}$ vector is the scaling dimension of ψ.

The other solution $(m;S_2)=(4;2)$ gives rise to the following D_{ab}:

$$\begin{pmatrix}D_{11} & D_{12}\\ D_{21} & D_{22}\end{pmatrix}=\begin{pmatrix}2 & 4\\ 4 & 8\end{pmatrix}.$$

It describes the symmetric polynomial

$$\Phi_{1/2}(\{z_i\})=\prod_{i<j}(z_i-z_j)^2, \qquad (57)$$

which is the filling fraction $\nu=1/2$ bosonic Laughlin state. The $\nu=1/2$ Laughlin state can be written as a correlation of the following operator in the Gaussian model [or $U(1)$ CFT]:

$$V_e(z)=e^{i\sqrt{2}\phi(z)}.$$

We note that the $\nu=1/2$ bosonic Laughlin state is characterized by a pattern of boson occupation numbers $(n_l)=(1,0,1,0,1,0,\ldots)$ and that the $\nu=1$ bosonic Pfaffian state is characterized by $(n_l)=(2,0,2,0,2,0,\ldots)$. Those patterns match the boson occupation distributions of the two states in the thin cylinder limit.[40–44] This appears to be a general result: the n_l that characterize a symmetric polynomial correspond to one of the boson occupation distribution of the same state in the thin cylinder limit. Or more precisely: the n_l that characterize a symmetric polynomial correspond to the boson occupation distribution of the same state in the thin sphere limit.[45]

The solution $(m;S_2)=(4;0)=2\times(2;0)$ gives rise to the following D_{ab}:

$$\begin{pmatrix}D_{11} & D_{12}\\ D_{21} & D_{22}\end{pmatrix}=\begin{pmatrix}0 & 4\\ 4 & 8\end{pmatrix}. \qquad (58)$$

It describes a symmetric polynomial which is the square of $\Phi_{2/2;Z_2}$,

$$\Phi_{Z_2Z_2}(\{z_i\})=\Phi^2_{2/2;Z_2}(\{z_i\})=\left[\mathcal{A}\left(\frac{1}{z_1-z_2}\frac{1}{z_3-z_4}\cdots\right)\right]^2\prod_{i<j}(z_i-z_j)^2. \qquad (59)$$

Let us consider another translation invariant symmetric polynomial

$$\Phi_{dw}(\{z_i\})=\mathcal{S}\left(\frac{1}{(z_1-z_2)^2}\frac{1}{(z_3-z_4)^2}\cdots\right)\prod_{i<j}(z_i-z_j)^2, \qquad (60)$$

where $\mathcal{S}$ is the symmetrization operator. We note that $\Phi_{dw}(\{z_i\})$ and $\Phi_{Z_2Z_2}(\{z_i\})$ have the same pattern of zeros given by Eq. (58). In the following, we would like to show that

$$\Phi_{dw}(\{z_i\})\propto\Phi_{Z_2Z_2}(\{z_i\}).$$

We first note that $\Phi_{Z_2Z_2}(\{z_i\})$ can be written as a correlation of the following operator in a CFT:

$$V_e(z) = \lambda_1(z)\lambda_2(z)e^{i\phi(z)},$$

where $\lambda_1(z)$ is the Majorana fermion operator in a Ising CFT and $\lambda_2(z)$ is the Majorana fermion operator in another Ising CFT. Thus, the operator $V_e(z)=\lambda_1(z)\lambda_2(z)e^{i\phi(z)}$ is an operator in Ising $\times$ Ising $\times$ U(1) CFT.

The state $\Phi_{dw}(\{z_i\})$ can also be written as a correlation of the following operator in a CFT:

$$V_e(z) = \partial_z\tilde{\phi}(z)e^{i\phi(z)},$$

where $\tilde{\phi}(z)$ is the field in a second $U(1)$ CFT. Thus, the operator $V_e(z)=\partial_z\phi(z)e^{i\phi(z)}$ is an operator in $\mathrm{U}(1)\times\tilde{\mathrm{U}}(1)$ CFT.

From the bosonization of the Ising $\times$ Ising CFT, one can show that the Ising $\times$ Ising CFT is equivalent to the $\tilde{\mathrm{U}}(1)$ CFT and $\lambda_1\lambda_2$ has the same N-body correlation functions as $\partial_z\tilde{\phi}(z)$. Thus, $\Phi_{dw}(\{z_i\})=\Phi_{Z_2Z_2}(\{z_i\})$. Numerical calculations have suggested that $\Phi_{dw}(\{z_i\})$ has gapless excitation and is unstable.

Next, let us consider the following two polynomials:

$$\Phi_{(Z_2)^q}(\{z_i\}) = \Phi^q_{2/2;Z_2}(\{z_i\}) = \left[\mathcal{A}\left(\frac{1}{z_1-z_2}\frac{1}{z_3-z_4}\cdots\right)\right]^q\prod_{i<j}(z_i-z_j)^q, \tag{61}$$

and

$$\Phi_{qw}(\{z_i\}) = \mathcal{S}_q\left(\frac{1}{(z_1-z_2)^q}\frac{1}{(z_3-z_4)^q}\cdots\right)\prod_{i<j}(z_i-z_j)^q, \tag{62}$$

where $\mathcal{S}_q=\mathcal{S}$ when q=even and $\mathcal{S}_q=\mathcal{A}$ when q=odd. The two symmetric polynomials have the same pattern of zeros D_{ab}. However, when $q>2$, the two polynomials are different. Those polynomials provide us examples that there can be more than one polynomial that have the same pattern of zeros.

C. $n=3$ cases

If $n=3$, the different patterns of zeros are characterized by three integers m,S_2,S_3. The following two sets of m,S_2,S_3 are the primitive solutions of Eqs. (43) and (45):

$$\Phi_{\frac{3}{2};Z_3}: (m;S_2,S_3) = (2;0,0),$$

$$\left(\frac{m}{n};h_1^{\mathrm{sc}},\ldots,h_n^{\mathrm{sc}}\right) = \left(\frac{2}{3};\frac{2}{3},\frac{2}{3},0\right),$$

$$(n_0,\ldots,n_{m-1}) = (3,0),$$

and

$$\Phi_{1/2}: (m;S_2,S_3) = (6;2,6),$$

$$\left(\frac{m}{n};h_1^{\mathrm{sc}},\ldots,h_n^{\mathrm{sc}}\right) = \left(\frac{6}{3};0,0,0\right),$$

$$(n_0,\ldots,n_{m-1}) = (1,0,1,0,1,0).$$

When n=odd, we find that the solutions of Eqs. (43) and (45) automatically satisfy Eq. (51).

From the $\boldsymbol{h}$ vector of the solution $(m;S_2,S_3)=(2;0,0)$, we find that the corresponding polynomial $\Phi_{\frac{3}{2};Z_3}$ describes the Z_3 Read–Rezayi parafermion state[19] since $h_1^{\mathrm{sc}}=h_2^{\mathrm{sc}}=2/3$ in the $\boldsymbol{h}$ vector match the scaling dimensions of the simple-current operators in the Z_3 parafermion CFT. Such a state has a filling fraction $\nu=3/2$. Here, we have been using $\Phi_{n/m;Z_n}$ to denote a Z_n parafermion state. We will follow such a convention for the rest of this paper. The second solution $(m;S_2,S_3)=(6;2,6)$ describes the $\nu=1/2$ Laughlin state.

D. $n=4$ cases

When $n=4$, the different patterns of zeros are characterized by four integers m,S_2,S_3,S_4. The primitive solutions of Eqs. (43) and (45) are given by the following three sets of m,S_2,S_3,S_4:

$$(m;S_2,S_3,S_4) = (1;0,0,0),(2;0,1,2),(8,2,6,12).$$

The solution $\boldsymbol{S}=(2;0,1,2)$ is the same as solution $\boldsymbol{S}=(1;0)$ for the $n=2$ case (i.e., the two solutions give rise to the same sequence $\{S_a\}$, $a=1,2,3,\ldots$). Such a solution does not satisfy Eq. (47), as shown in Sec. VIII B.

The solution $\boldsymbol{S}=(1;0,0,0)$ does not satisfy Eq. (47) either. Let us check condition (47) for the $J=1/2$ case wherein there are only two orbitals. This leads to a state $|\{4,4\}\rangle$ with eight bosons. On a sphere, such a state is given by

$$\Phi^{\mathrm{sp}}_{\{4,4\}} = \mathcal{S}[v_1v_2v_3v_4u_5u_6u_7u_8].$$

Since $S_8=4$, we find that $J_8=8J-S_8=0$ and P_{8,J_8} is a projection into the subspace with vanishing total angular momentum. Explicit calculation shows that the state $\Phi^{\mathrm{sp}}_{\{4,4\}}$ has a vanishing projection onto the $J_{\mathrm{tot}}=0$ subspace.

Thus, we consider the solutions of Eqs. (43), (45), and (51) to exclude those invalid cases. The primitive solutions of Eqs. (43), (45), and (51) are

$$\Phi_{4/2;Z_4}: (m;S_2,\ldots,S_n) = (2;0,0,0),$$

$$\left(\frac{m}{n};h_1^{\mathrm{sc}},\ldots,h_n^{\mathrm{sc}}\right) = \left(\frac{2}{4};\frac{3}{4},1,\frac{3}{4},0\right),$$

$$(n_0,\ldots,n_{m-1}) = (4,0); \tag{63}$$

$$\Phi_{2/2;Z_2}: (m;S_2,\ldots,S_n) = (4;0,2,4),$$

$$\left(\frac{m}{n};h_1^{\mathrm{sc}},\ldots,h_n^{\mathrm{sc}}\right) = \left(\frac{4}{4};\frac{1}{2},0,\frac{1}{2},0\right),$$

$$(n_0,\ldots,n_{m-1}) = (2,0,2,0); \tag{64}$$

$$\Phi_{1/2}: (m;S_2,\ldots,S_n) = (8;2,6,12),$$

$$\left(\frac{m}{n};h_1^{sc},\ldots,h_n^{sc}\right)=\left(\frac{8}{4};0,0,0,0\right),$$

$$(n_0,\ldots,n_{m-1})=(1,0,1,0,1,0,1,0).$$

Among those three primitive solutions, only $\Phi_{4/2;Z_4}$ is new. From the $\boldsymbol{h}$ vector in Eq. (63), we find that the solution $(m;S_2,S_3,S_4)=(2;0,0,0)$ describes the Z_4 parafermion state $\Phi_{4/2;Z_4}$ with $\nu=2$.

The solution $(m;S_2,S_3,S_4)=(4;0,2,4)$ is the same as the $\nu=1$ bosonic Pfaffian state $(m;S_2)=(2;0)$ discussed before. So the $\boldsymbol{h}$ vectors of the two solutions, $(\frac{m}{n};h_1^{sc},\ldots,h_4^{sc})=(\frac{4}{4};\frac{1}{2},0,\frac{1}{2},0)$ and $(\frac{m}{n};h_1^{sc},h_2^{sc})=(\frac{2}{2};\frac{1}{2},0)$, characterize the same state. In fact, the repeated $(\frac{1}{2},0)$ pattern in $(h_1^{sc},\ldots,h_4^{sc})$ implies that $(\frac{m}{n};h_1^{sc},\ldots,h_4^{sc})=(\frac{4}{4};\frac{1}{2},0,\frac{1}{2},0)$ can be reduced to $(\frac{m}{n};h_1^{sc},h_2^{sc})=(\frac{2}{2};\frac{1}{2},0)$. Also, the two solutions lead to the same pattern of boson occupation numbers: $(n_l)=(2,0,2,0,2,0,2,0,\ldots)$, again indicating that the two solutions describe the same state.

The solution $(m;S_2,S_3,S_4)=(8;2,6,12)$ describes the $\nu=1/2$ Laughlin state $\Phi_{1/2}$ characterized by $(\frac{m}{n};h_1^{sc},\ldots,h_4^{sc})=(\frac{8}{4};0,0,0,0)$. We have seen that $\boldsymbol{h}_{1/2}=(\frac{2}{1};0)$, $\boldsymbol{h}_{2/4}=(\frac{4}{2};0,0)$, $\boldsymbol{h}_{3/6}=(\frac{6}{3};0,0,0)$, and $\boldsymbol{h}_{4/8}=(\frac{8}{4};0,0,0,0)$ all describe the same $\nu=1/2$ Laughlin state $\Phi_{1/2}$. All the above solutions share the same pattern of boson occupation numbers: $(n_l)=(1,0,1,0,1,0,1,0,\ldots)$.

The product state $\Phi_{4/2;Z_4}\Phi_{2/2;Z_2}$ described by

$$\Phi_{4/2;Z_4}\Phi_{2/2;Z_2}:(m;S_2,\ldots,S_n)=(6;0,2,4),$$

$$\left(\frac{m}{n};h_1^{sc},\ldots,h_n^{sc}\right)=\left(\frac{6}{4};\frac{5}{4},1,\frac{5}{4},0\right),$$

$$(n_0,\ldots,n_{m-1})=(2,0,2,0,0,0)$$

is a possible stable $\nu=2/3$ FQH state. Note that the above $\boldsymbol{h}$ vector is the sum of the $\boldsymbol{h}$ vectors of the Z_2 and Z_4 parafermion states.

E. $n=5$ cases

When $n=5$, conditions (43) and (45) have the following three sets of primitive solutions:

$$\Phi_{5/2;Z_5}:(m;S_2,\ldots,S_n)=(2;0,0,0,0),$$

$$\left(\frac{m}{n};h_1^{sc},\ldots,h_n^{sc}\right)=\left(\frac{2}{5};\frac{4}{5},\frac{6}{5},\frac{6}{5},\frac{4}{5},0\right),$$

$$(n_0,\ldots,n_{m-1})=(5,0);$$

$$\Phi_{5/8;Z_5^{(2)}}:(m;S_2,\ldots,S_n)=(8;0,2,6,10),$$

$$\left(\frac{m}{n};h_1^{sc},\ldots,h_n^{sc}\right)=\left(\frac{8}{5};\frac{6}{5},\frac{4}{5},\frac{4}{5},\frac{6}{5},0\right),$$

$$(n_0,\ldots,n_{m-1})=(2,0,1,0,2,0,0,0);$$

$$\Phi_{1/2}:(m;S_2,\ldots,S_n)=(10;2,6,12,20),$$

$$\left(\frac{m}{n};h_1^{sc},\cdots,h_n^{sc}\right)=\left(\frac{10}{5};0,0,0,0,0\right),$$

$$(n_0,\ldots,n_{m-1})=(1,0,1,0,1,0,1,0,1,0).$$

All other solutions are linear combinations of the above three solutions.

$(\frac{m}{n};h_1^{sc},\ldots,h_5^{sc})=(\frac{2}{5};\frac{4}{5},\frac{6}{5},\frac{6}{5},\frac{4}{5},0)$ describes a Z_5 parafermion state $\Phi_{5/2;Z_5}$ studied by Read and Rezayi.[19] $(\frac{m}{n};h_1^{sc},\ldots,h_5^{sc})=(\frac{8}{5};\frac{6}{5},\frac{4}{5},\frac{4}{5},\frac{6}{5},0)$ describes a new parafermion state $\Phi_{5/8;Z_5^{(2)}}$ with $\nu=5/8$. The third state $(\frac{m}{n};h_1^{sc},\ldots,h_5^{sc})=(\frac{10}{5};0,0,0,0,0)$ describes the $\nu=1/2$ Laughlin state $\Phi_{1/2}$.

The Z_5 parafermion state $\Phi_{5/2;Z_5}$ can be expressed as a correlation of simple-current operators ψ_1 in the Z_5 parafermion CFT. ψ_1 has a scaling dimension of $h_1^{sc}=4/5$. The new parafermion state $\Phi_{5/8;Z_5^{(2)}}$ can be expressed as a correlation of simple-current operators ψ_2 in the Z_5 parafermion CFT. ψ_2 has a scaling dimension of $h_2^{sc}=6/5$. In general, the simple-current operator ϕ_l of a Z_n parafermion CFT has a scaling dimension

$$h_l^{sc}=\frac{l(n-l)}{n}.$$

Here, we have been using $\Phi_{n/m;Z_n^{(k)}}$ to denote a generalized Z_n parafermion state. We will follow such a convention in the rest of this paper.

F. $n=6$ cases

When $n=6$, conditions (43), (45), and (51) have the following four sets of primitive solutions:

$$\Phi_{6/2;Z_6}:(m;S_2,\ldots,S_n)=(2;0,0,0,0,0),$$

$$\left(\frac{m}{n};h_1^{sc},\ldots,h_n^{sc}\right)=\left(\frac{2}{6};\frac{5}{6},\frac{4}{3},\frac{3}{2},\frac{4}{3},\frac{5}{6},0\right),$$

$$(n_0,\ldots,n_{m-1})=(6,0).$$

$\Phi_{\frac{2}{2};Z_2}$, $\Phi_{3/2;Z_3}$, and $\Phi_{1/2}$. Three of the four primitive solutions have been discussed before and only one solution, $\Phi_{6/2;Z_6}$, is new. The $\Phi_{6/2;Z_6}$ state is the Z_6 parafermion state.[19] $\Phi_{2/2;Z_2}$ and $\Phi_{\frac{3}{2};Z_3}$ are the Z_2 and Z_3 parafermion states discussed before.

Using $\Phi_{2/2;Z_2}$, $\Phi_{3/2;Z_3}$, and $\Phi_{6/2;Z_6}$, we can construct some interesting and possibly stable composite states:

$$\Phi_{3/2;Z_3}\Phi_{6/2;Z_6}:(m;S_2,\ldots,S_n)=(6;0,0,2,4,6),$$

$$\left(\frac{m}{n};h_1^{sc},\ldots,h_n^{sc}\right)=\left(\frac{6}{6};\frac{3}{2},2,\frac{3}{2},2,\frac{3}{2},0\right),$$

$$(n_0,\ldots,n_{m-1})=(3,0,3,0,0,0);$$

$$\Phi_{2/2;Z_2}\Phi_{6/2;Z_6}:(m;S_2,\ldots,S_n)=(8;0,2,4,8,12),$$

$$\left(\frac{m}{n};h_1^{sc},\ldots,h_n^{sc}\right)=\left(\frac{8}{6};\frac{4}{3},\frac{4}{3},2,\frac{4}{3},\frac{4}{3},0\right),$$

$$(n_0,\ldots,n_{m-1})=(2,0,2,0,2,0,0,0);$$

$$\Phi_{2/2;Z_2}\Phi_{3/2;Z_3}:(m;S_2,\ldots,S_n)=(10;0,2,6,12,18),$$

$$\left(\frac{m}{n};h_1^{sc},\ldots,h_n^{sc}\right)=\left(\frac{10}{6};\frac{7}{6},\frac{2}{3},\frac{1}{2},\frac{2}{3},\frac{7}{6},0\right),$$

$$(n_0,\ldots,n_{m-1})=(2,0,1,0,1,0,2,0,0,0);$$

$$\Phi_{2/2;Z_2}\Phi_{3/2;Z_3}\Phi_{6/2;Z_6}:(m;S_2,\ldots,S_n)=(12;0,2,6,12,18),$$

$$\left(\frac{m}{n};h_1^{sc},\ldots,h_n^{sc}\right)=\left(\frac{12}{6};2,2,2,2,2,0\right),$$

$$(n_0,\ldots,n_{m-1})=(2,0,1,0,1,0,2,0,0,0,0,0).$$

The filling fractions of those states are given by $\nu=n/m$.

G. $n=7$ cases

When $n=7$, conditions (43) and (45) have the following five sets of primitive solutions:

$$\Phi_{7/2;Z_7}:(m;S_2,\ldots,S_n)=(2;0,0,0,0,0,0),$$

$$\left(\frac{m}{n};h_1^{sc},\ldots,h_n^{sc}\right)=\left(\frac{2}{7};\frac{6}{7},\frac{10}{7},\frac{12}{7},\frac{12}{7},\frac{10}{7},\frac{6}{7},0\right),$$

$$(n_0,\ldots,n_{m-1})=(7,0);$$

$$\Phi_{7/8;Z_7^{(2)}}:(m;S_2,\ldots,S_n)=(8;0,0,2,6,10,14),$$

$$\left(\frac{m}{n};h_1^{sc},\ldots,h_n^{sc}\right)=\left(\frac{8}{7};\frac{10}{7},\frac{12}{7},\frac{6}{7},\frac{6}{7},\frac{12}{7},\frac{10}{7},0\right),$$

$$(n_0,\ldots,n_{m-1})=(3,0,1,0,3,0,0,0);$$

$$\Phi_{7/18;Z_7^{(3)}}:(m;S_2,\ldots,S_n)=(18;0,4,10,18,30,42),$$

$$\left(\frac{m}{n};h_1^{sc},\ldots,h_n^{sc}\right)=\left(\frac{18}{7};\frac{12}{7},\frac{6}{7},\frac{10}{7},\frac{10}{7},\frac{6}{7},\frac{12}{7},0\right),$$

$$(n_0,\ldots,n_{m-1})=(2,0,0,0,0,1,0,0,0,2,0,0,0,0,0);$$

$$\Phi_{7/14;C_7}:(m;S_2,\ldots,S_n)=(14;0,2,6,12,20,28),$$

$$\left(\frac{m}{n};h_1^{sc},\ldots,h_n^{sc}\right)=\left(\frac{14}{7};2,2,2,2,2,2,0\right),$$

$$(n_0,\ldots,n_{m-1})=(2,0,1,0,1,0,1,0,2,0,0,0,0,0); \quad (65)$$

and $\Phi_{1/2}$.

$\Phi_{7/2;Z_7}$ is the Z_7 parafermion state which can be expressed as a correlation of simple-current operators ψ_1 in the Z_7 parafermion CFT. ψ_1 has a scaling dimension of $h_1^{sc}=6/7$. $\Phi_{7/8;Z_7^{(2)}}$ and $\Phi_{7/18;Z_7^{(3)}}$ are two new Z_7 parafermion states. $\Phi_{7/8;Z_7^{(2)}}$ can be expressed as a correlation of simple-current operators ψ_2 while $\Phi_{7/18;Z_7^{(3)}}$ can be expressed as a correlation of simple-current operators ψ_3 in the Z_7 parafermion CFT. ψ_2 has a scaling dimension of $h_2^{sc}=10/7$ and ψ_3 has a scaling dimension of $h_3^{sc}=12/7$.

Let us discuss the state $\Phi_{7/14;C_7}$ in more detail. $\Phi_{7/14;C_7}$ has the form

$$\Phi_{7/14;C_7}(\{z_i\})=G_{C_7}(\{z_i\})\prod(z_i-z_j)^2,$$

where the pattern of the zeros (or poles) for $G_{C_7}(\{z_i\})$ is given by [see Eq. (25)]

$$(d_{ab})=\begin{pmatrix}-2&-2&-2&-2&-2&-4&0\\-2&-2&-2&-2&-4&-2&0\\-2&-2&-2&-4&-2&-2&0\\-2&-2&-4&-2&-2&-2&0\\-2&-4&-2&-2&-2&-2&0\\-4&-2&-2&-2&-2&-2&0\\0&0&0&0&0&0&0\end{pmatrix},$$

where $a,b=1,\ldots,7$. It implies that there is a second order pole as an a cluster approaches a b cluster if $a+b<7$ and a fourth order pole as an a cluster approaches a b cluster if $a+b=7$. Such a pattern of poles is reproduced by

$$G_{C_7}(\{z_i\})=\mathcal{S}[f_{C_7}^2(z_1,\ldots,z_7)f_{C_7}^2(z_8,\ldots,z_{14})\cdots],$$

where

$$f_{C_n}(z_1,\ldots,z_n)=\frac{1}{z_1-z_2}\frac{1}{z_2-z_3}\cdots\frac{1}{z_{n-1}-z_n}\frac{1}{z_n-z_1}.$$

To confirm such a result, we note that for $a=2,\ldots,7$, the minimal total powers of a variables in $f_{C_7}^2$ are s_a with $(s_2,\ldots,s_7)=(-2,-4,-6,-8,-10,-14)$ [see Eq. (29)]. The minimal total powers of a variables in $\Pi(z_i-z_j)^2$ are $\tilde{s}_a$ with $(\tilde{s}_2,\ldots,\tilde{s}_7)=(2,6,12,20,30,42)$. Thus, the minimal total powers of a variables in $\Phi_{7/14;C_7}$ are given by $S_a=s_a+\tilde{s}_a$: $(S_2,\ldots,S_7)=(0,2,6,12,20,28)$, which agrees with Eq. (65).

The function $f_{C_7}(z_1,\ldots,z_7)$ can be represented graphically as in Fig. 3. In such a graphic representation, the maximum total order of the poles of a variables is the maximum number of lines that connect a dots. Note that the maximum total order of the poles of a variables is the negative of the minimal total power of zeros of a variables.

In fact, for any n, we have a state $\Phi_{n/2n;C_n}$ described by $(h_1^{sc},\ldots,h_{n-1}^{sc},h_n^{sc})=(2,\ldots,2,0)$. The explicit wave function is given by

$$\Phi_{n/2n;C_n}=\prod_{i<j}(z_i-z_j)^2\mathcal{S}[f_{C_n}^2(z_1,\ldots,z_n)f_{C_n}^2(z_{n+1},\ldots,z_{2n})\cdots]$$

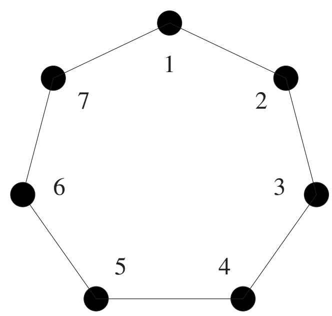

FIG. 3. The graph that represents f_{C_7}. Each line between the ith dot and the jth dot represent a factor $1/(z_i-z_j)$.

H. $n=8$ cases

When $n=8$, conditions (43), (45), and (51) have the following six sets of primitive solutions:

$$\Phi_{8/2;Z_8}: (m;S_2,\ldots,S_n)=(2;0,0,0,0,0,0,0),$$

$$\left(\frac{m}{n};h_1^{\text{sc}},\ldots,h_n^{\text{sc}}\right)=\left(\frac{2}{8};\frac{7}{8},\frac{3}{2},\frac{15}{8},2,\frac{15}{8},\frac{3}{2},\frac{7}{8},0\right),$$

$$(n_0,\ldots,n_{m-1})=(8,0);$$

$$\Phi_{8/18;Z_8^{(3)}}: (m;S_2,\ldots,S_n)=(18;0,2,8,14,24,36,48),$$

$$\left(\frac{m}{n};h_1^{\text{sc}},\ldots,h_n^{\text{sc}}\right)=\left(\frac{18}{8};\frac{15}{8},\frac{3}{2},\frac{7}{8},2,\frac{7}{8},\frac{3}{2},\frac{15}{8},0\right),$$

$$(n_0,\ldots,n_{m-1})=(2,0,1,0,0,0,2,0,0,0,1,0,2,0,0,0,0,0);$$

$$\Phi_{8/8;C_8/Z_2}: (m;S_2,\ldots,S_n)=(8;0,0,2,4,8,12,16),$$

$$\left(\frac{m}{n};h_1^{\text{sc}},\ldots,h_n^{\text{sc}}\right)=\left(\frac{8}{8};\frac{3}{2},2,\frac{3}{2},2,\frac{3}{2},2,\frac{3}{2},0\right),$$

$$(n_0,\cdots,n_{m-1})=(3,0,2,0,3,0,0,0);$$

$\Phi_{\frac{4}{2};Z_4}$; $\Phi_{2/2;Z_2}$; and $\Phi_{1/2}$.

$\Phi_{8/2;Z_8}$ is the Z_8 parafermion state which can be expressed as a correlation of simple-current operators ψ_1 in the Z_8 parafermion CFT. ψ_1 has a scaling dimension of $h_1^{\text{sc}}=7/8$. $\Phi_{8/18;Z_8^{(3)}}$ is a new Z_8 parafermion state. $\Phi_{8/18;Z_8^{(3)}}$ can be expressed as a correlation of simple-current operators ψ_3 in the Z_8 parafermion CFT. ψ_3 has a scaling dimension of $h_3^{\text{sc}}=15/8$.

Let us discuss the state $\Phi_{8/8;C_8/Z_2}$ in more details. We note that the $\boldsymbol{h}$ vector for the $\Phi_{8/8;C_8/Z_2}$ state is the difference of the $\boldsymbol{h}$ vectors of the $\Phi_{8/2;C_8}$ state and the $\Phi_{2/2;Z_2}$ state:

$$\left(\frac{3}{2},2,\frac{3}{2},2,\frac{3}{2},2,\frac{3}{2},0\right)=(2,2,2,2,2,2,2,0)-\left(\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0,\frac{1}{2},0\right).$$

$\Phi_{8/8;C_8/Z_2}$ has the form

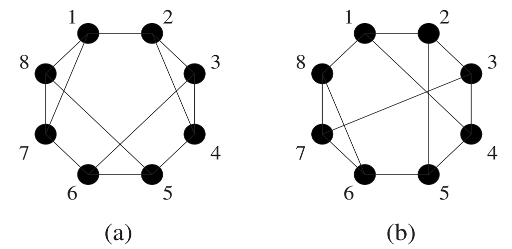

FIG. 4. The graphs that represent f_{C_8/Z_2}. Each line between the ith dot and the jth dot represents a factor $1/(z_i-z_j)$. After the antisymmetrization, f_{C_8/Z_2} from graph (a) gives rise to a nonzero antisymmetric function, while f_{C_8/Z_2} from graph (b) gives rise to vanishing antisymmetric function.

$$\Phi_{8/8;C_8/Z_2}(\{z_i\})=G_{C_8/Z_2}(\{z_i\})\prod(z_i-z_j)$$

where the minimal total power of a variables in $G_{C_8/Z_2}(\{z_i\})$ is given by s_a with $(s_2,\ldots,s_8)=(-1,-3,-4,-6,-7,-9,-12)$. We find that

$$G_{C_8/Z_2}(\{z_i\})=\mathcal{A}[f_{C_8/Z_2}(z_1,\ldots,z_8)f_{C_8/Z_2}(z_9,\ldots,z_{16})\cdots]$$

where $\mathcal{A}$ is the antisymmetrization operator and the function $f_{C_8/Z_2}(z_1,\ldots,z_8)$ is represented by Fig. 4(a).

We would like to mention that a state simpler than $\Phi_{8/8;C_8/Z_2}$ is $\Phi_{4/4;C_4/Z_2}$ that has the same pattern of zeros as a composite state of Z_4 parafermion state $\Phi_{4/2;Z_4}$:

$$\Phi_{4/4;C_4/Z_2}\approx\Phi_{4/2;Z_4}\Phi_{4/2;Z_4}.$$

Here, $\approx$ means to have the same pattern of zeros. $\Phi_{4/4;C_4/Z_2}$ has the form

$$\Phi_{4/4;C_4/Z_2}(\{z_i\})=G_{C_4/Z_2}(\{z_i\})\prod(z_i-z_j),$$

where the minimal total power of a variables in $G_{C_4/Z_2}(\{z_i\})$ is given by s_a with $(s_2,\ldots,s_4)=(-1,-3,-6)$. We find that

$$G_{C_4/Z_2}(\{z_i\})=\mathcal{A}[f_{C_4/Z_2}(z_1,\ldots,z_4)f_{C_4/Z_2}(z_5,\ldots,z_8)\cdots]$$

where the function $f_{C_4/Z_2}(z_1,\ldots,z_4)$ is represented by Fig. 5. In fact, $f_{C_4/Z_2}(z_1,\ldots,z_4)=\prod_{1\le i<j\le 4}\frac{1}{z_i-z_j}$ is the only function whose total order of poles for two-, three-, and four-particle clusters are given by 1, 3, and 6, respectively. Such a state is studied recently by Yu.[46]

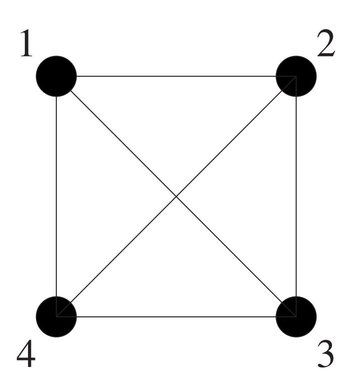

FIG. 5. The graph that represents f_{C_4/Z_2}. Each line between the ith dot and the jth dot represent a factor $1/(z_i-z_j)$.

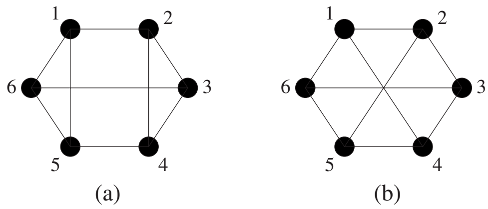

FIG. 6. The graphs that represent f_{C_6/Z_2}. Each line between the ith dot and the jth dot represents a factor $1/(z_i-z_j)$. After the antisymmetrization, f_{C_6/Z_2} from graph (a) gives rise to a nonzero antisymmetric function, while f_{C_6/Z_2} from graph (b) vanishes.

Another interesting state is $\Phi_{6/6;C_6/Z_2}$ which has the same pattern of zeros as a composite state of Z_3 and Z_6 parafermion states:

$$\Phi_{6/6;C_6/Z_2} \approx \Phi_{3/2;Z_3}\Phi_{6/2;Z_6}.$$

$\Phi_{6/6;C_6/Z_2}$ has the form

$$\Phi_{6/6;C_6/Z_2}(\{z_i\}) = G_{C_6/Z_2}(\{z_i\}) \prod (z_i - z_j),$$

where the minimal total power of a variables in $G_{C_6/Z_2}(\{z_i\})$ is given by s_a with $(s_2,\ldots,s_6)=(-1,-3,-4,-6,-9)$. We find that

$$G_{C_6/Z_2}(\{z_i\}) = \mathcal{A}[f_{C_6/Z_2}(z_1,\ldots,z_6)f_{C_6/Z_2}(z_7,\ldots,z_{12})\cdots]$$

where the function $f_{C_6/Z_2}(z_1,\ldots,z_6)$ is represented by Fig. 6(a).

I. $n=9$ cases

When $n=9$, conditions (43), (45), and (51) have the following six sets of primitive solutions:

$$\Phi_{9/2;Z_9}: (m;S_2,\ldots,S_n) = (2;0,0,0,0,0,0,0,0),$$

$$\left(\frac{m}{n};h_1^{\rm sc},\ldots,h_n^{\rm sc}\right) = \left(\frac{2}{9};\frac{8}{9},\frac{14}{9},2,\frac{20}{9},\frac{20}{9},2,\frac{14}{9},\frac{8}{9},0\right),$$

$$(n_0,\ldots,n_{m-1}) = (9,0);$$

$$\Phi_{9/8;Z_9^{(2)}}: (m;S_2,\ldots,S_n) = (8;0,0,0,2,6,10,14,18),$$

$$\left(\frac{m}{n};h_1^{\rm sc},\ldots,h_n^{\rm sc}\right) = \left(\frac{8}{9};\frac{14}{9},\frac{20}{9},2,\frac{8}{9},\frac{8}{9},2,\frac{20}{9},\frac{14}{9},0\right),$$

$$(n_0,\ldots,n_{m-1}) = (4,0,1,0,4,0,0,0);$$

$$\Phi_{9/32;Z_9^{(4)}}: (m;S_2,\ldots,S_n) = (32;0,6,14,26,42,60,84,108),$$

$$\left(\frac{m}{n};h_1^{\rm sc},\ldots,h_n^{\rm sc}\right) = \left(\frac{32}{9};\frac{20}{9},\frac{8}{9},2,\frac{14}{9},\frac{14}{9},2,\frac{8}{9},\frac{20}{9},0\right),$$

$$(n_0,\ldots,n_{m-1}) = (2,0,0,0,0,0,1,0,1,0,0,\; 0,1,0,0,0,1,0,1,0,0,0,0,0,0,2,0,0,0,0,0,0,0);$$

$$\Phi_{9/12;C_9/Z_3}: (m;S_2,\ldots,S_n) = (12;0,2,4,8,14,20,28,36),$$

$$\left(\frac{m}{n};h_1^{\rm sc},\ldots,h_n^{\rm sc}\right) = \left(\frac{12}{9};\frac{4}{3},\frac{4}{3},2,\frac{4}{3},\frac{4}{3},2,\frac{4}{3},\frac{4}{3},0\right),$$

$$(n_0,\ldots,n_{m-1}) = (2,0,2,0,1,0,2,0,2,0,0,0);$$

$\Phi_{3/2;Z_3}$; and $\Phi_{1/2}$.

$\Phi_{9/2;Z_9}$ is the old Z_9 parafermion state. $\Phi_{9/8;Z_9^{(2)}}$ and $\Phi_{9/32;Z_9^{(4)}}$ are new Z_9 parafermion states, which can be expressed as a correlation of simple-current operators ψ_2 and ψ_4 in the Z_9 parafermion CFT, respectively. We also note that the $\boldsymbol{h}$ vector for the $\Phi_{9/12;C_9/Z_3}$ state is the difference of the $\boldsymbol{h}$ vectors of the $\Phi_{9/18;C_9}$ state and the $\Phi_{3/2;Z_3}$ state:

$$\left(\frac{4}{3},\frac{4}{3},2,\frac{4}{3},\frac{4}{3},2,\frac{4}{3},\frac{4}{3},0\right) = (2,2,2,2,2,2,2,2,0) - \left(\frac{2}{3},\frac{2}{3},0,\frac{2}{3},\frac{2}{3},0,\frac{2}{3},\frac{2}{3},0\right).$$

However, we do not know if a symmetric polynomial described by the $\boldsymbol{h}$ vector $(\frac{12}{9};\frac{4}{3},\frac{4}{3},2,\frac{4}{3},\frac{4}{3},2,\frac{4}{3},\frac{4}{3},0)$ really exists or not.

IX. DISCUSSION

In this paper, we use a local condition—the pattern of zeros–to classify symmetric polynomials of infinity variables. We find that symmetric polynomials of n-cluster form [see Eqs. (23) and (24)] can be labeled by a set in integers $(m;S_2,\ldots,S_n)$. Those integers must satisfy conditions (43), (45), and (47).

Using the symmetric polynomials labeled by n and $(m;S_2,\ldots,S_n)$, we have constructed a large class of simple FQH states. The constructed FQH states contain both the Laughlin states and non-Abelian states, such as the Read–Rezayi parafermion states, the new generalized parafermion states, and some other new non-Abelian states. Although the constructed FQH states are for bosonic electrons, the bosonic FQH states and the fermionic FQH states have a simple one-to-one correspondence:

$$\Phi_{\rm fermion} = \Phi_{\rm boson}\prod_{i<j}(z_i - z_j).$$

We can easily obtain fermionic FQH states from the corresponding bosonic ones.

We have seen that the ground state wave functions of different Abelian and non-Abelian fraction quantum Hall states can be characterized by patterns of zeros $\{S_a\}$. One may wonder: can we use the data $\{S_a\}$ to calculate various topological properties of the corresponding fraction quantum Hall state? In Ref. 45, we will show that many topological properties can indeed be calculated from $\{S_a\}$, such as the number of possible quasiparticle types and their quantum numbers.

However, $\{S_a\}$ cannot describe all FQH states. More complicated "multicomponent" FQH states, such as the $\nu=2/5$

Abelian FQH state, are not included in our construction. This suggests that certain non-Abelian states, such as the parafermion states, belong to the same class as the simple one-component Laughlin states. Thus, our result can be viewed as a classification of "one-component" FQH states, although the precise meaning of "one-component" remains to be clarified. String-net condensation and the associated tensor category theory[26] provide a fairly complete classification of nonchiral topological orders in two spatial dimensions. We hope the framework introduced in this paper be a step toward a classification of chiral topological orders in two spatial dimensions.

ACKNOWLEDGMENTS

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APPENDIX A: ELECTRONS ON A SPHERE

To test if the FQH Hamiltonian has an energy gap or not, we need to put the FQH state on a compact space to avoid the gapless edge excitations which are always present.[47] In this appendix, we will discuss how to put a FQH state on a sphere.[31] We assume that there is a uniform magnetic field on a sphere with a total N_ϕ flux quanta. The wave function of one electron in the first Landau level has the form[31]

$$\Psi(\theta,\varphi) = \sum_{m=0}^{N_\phi} c_m v^{N_\phi - m} u^m, \tag{A1}$$

where $u=\cos(\theta/2)e^{i\varphi/2}$ and $v=\sin(\theta/2)e^{-i\varphi/2}$.[48]

We can also use a complex number $z=u/v$ to parameterize the points on the sphere. In terms of z, the wave function becomes

$$\Psi(\theta,\varphi) = e^{-iN_\phi\varphi/2}\sin^{N_\phi}(\theta/2)\sum_{m=0}^{N_\phi} c_m z^m = \frac{e^{-iN_\phi\varphi/2}}{(1+|z|^2)^{N_\phi/2}}\sum_{m=0}^{N_\phi} c_m z^m. \tag{A2}$$

We see that we can use a polynomial $\Phi(z)$ to describe the wave function of one electron in the first Landau level:

$$\Phi(z) = \sum_{m=0}^{N_\phi} c_m z^m. \tag{A3}$$

Here, the power of z is equal or less than N_ϕ. Equations (A1) and (A2) allow us to go back and forth between the spinor representation $\Psi(u,v)$ and the polynomial representation $\Phi(z)$ of the states on the sphere.

Since polynomials (A3) represent wave functions on a sphere, hence they form a representation of $SU(2)$ [or $O(3)$] rotation of the sphere. The dimension of the representation is $N_\phi+1$. Such a representation is said to carry an angular momentum

$$J = \frac{N_\phi}{2}.$$

The $SU(2)$ Lie algebra is generated by

$$L^z = z\partial_z - J, \quad L^- = \partial_z, \quad L^+ = -z^2\partial_z + 2Jz,$$

which satisfy

$$[L^z, L^\pm] = L^\pm, \quad [L^+, L^-] = 2L^z.$$

Those operators act within the space formed by the polynomials of form (A3). The inner product in the space of the polynomials is defined through the inner product of the wave functions $\Psi_1(\theta,\varphi)=\frac{e^{-iN_\phi\varphi/2}}{(1+|z|^2)^{N_\phi/2}}\Phi_1(z)$ and $\Psi_2(\theta,\varphi)=\frac{e^{-iN_\phi\varphi/2}}{(1+|z|^2)^{N_\phi/2}}\Phi_2(z)$:

$$\langle\Phi_2|\Phi_1\rangle = \int \sin(\theta)d\theta d\varphi \Psi_2^*(\theta,\varphi)\Psi_1(\theta,\varphi) = \int 4\cos(\theta/2)\sin^3(\theta/2)d\frac{\cos(\theta/2)}{\sin(\theta/2)}d\varphi\Psi_2^*\Psi_1 = \int \frac{4d^2z}{(1+|z|^2)^2}\frac{1}{(1+|z|^2)^{N_\phi}}\Phi_2^*(z)\Phi_1(z).$$

Now let us consider a polynomial of two variables $\Phi(z_1,z_2)$ where the highest power for z_1 is $2J_1$ and the highest power for z_2 is $2J_2$ (here, $2J_1$ and $2J_2$ are integers). $\Phi(z_1,z_2)$ can also be viewed as a representation of $SU(2)$, wherein the generators of the $SU(2)$ Lie algebra are given by

$$L^z = L_1^z + L_2^z, \quad L^\pm = L_1^\pm + L_2^\pm,$$

$$L_1^z = z_1\partial_{z_1} - J_1, \quad L_1^- = \partial_{z_1}, \quad L_1^+ = -z_1^2\partial_{z_1} + 2J_1 z_1,$$

$$L_2^z = z_2\partial_{z_2} - J_2, \quad L_2^- = \partial_{z_2}, \quad L_2^+ = -z_2^2\partial_{z_2} + 2J_2 z_2.$$

$\Phi(z_1,z_2)$ is not an irreducible representation of $SU(2)$. It can be decomposed as $\oplus_{J=|J_1-J_2|}^{J_1+J_2}\mathcal{H}_J$ where $\mathcal{H}_J$ is an angular-momentum J representation of $SU(2)$. We may say that z_1 has angular momentum J_1 and z_2 has an angular momentum J_2. Thus, the angular momenta of $\Phi(z_1,z_2)$ are those obtained by combining the angular momentum J_1 and the angular momentum J_2.

What are the states in the space $\mathcal{H}_J$? Let $\Phi_{J,m}$, where $m=-J,-J+1,\ldots,J$, be the polynomials in the $\mathcal{H}_J$ space such that $\Phi_{J,m}$ is the eigenstate of L^z with eigenvalue m. Let us also introduce $z_\pm=z_1\pm z_2$. We see that

$$L^- = \partial_{z_1} + \partial_{z_2} = 2\partial_{z_+},$$

$$L^+ = -\frac{1}{2}(z_+^2 + z_-^2)\partial_{z_+} - z_+z_-\partial_{z_-} + 2J_1z_1 + 2J_2z_2,$$

$$L^z = z_+\partial_{z_+} + z_-\partial_{z_-} - J_1 - J_2.$$

From $L^-\Phi_{J,-J}=0$ and $L^z\Phi_{J,-J}=-J\Phi_{J,-J}$, we find that

$$\Phi_{J,-J} \propto z_-^{J_1+J_2-J} = (z_1 - z_2)^{J_1+J_2-J}. \tag{A4}$$

$\Phi_{J,m}$'s are generated from $\Phi_{J,-J}$ by applying L^+'s. Since L^+ never reduce the power of z_-, $\Phi_{J,m}$ thus has the form

$z_-^{J_1+J_2-J}f(z_1,z_2)=(z_1-z_2)^{J_1+J_2-J}f(z_1,z_2)$. This reveals a close relation between the order of zeros and the angular momentum: *the polynomial* $\Phi(z_1,z_2)$ with angular momentum J must have an order J_1+J_2-J zero as $z_1\to z_2$.

On the other hand, let $\Phi_D(z_1,z_2)$ be a polynomial that has a Dth order zero as $z_1\to z_2$, i.e., Φ_D has the form $(z_1-z_2)^D f(z_1,z_2)$. We note that the actions of $L^\pm$ and L^z do not decrease the power of zero as $z_1\to z_2$. Therefore, the action of $L^\pm$ and L^z can never change Φ_D to $\Phi_{J,-J}$ if $J>J_1+J_2-D$. Thus, $\Phi_D(z_1,z_2)$ can only contain angular momenta $J\le J_1+J_2-D$ which lead to zeros of order D or more.

We can let $z_1\to z_2$ in $\Phi_D(z_1,z_2)$ and obtain $F(z_2)$ as

$$\Phi_D(z_1,z_2) = (z_1-z_2)^D F(z_2) + O[(z_1-z_2)^{D+1}].$$

The minimum L^z eigenvalue for $(z_1-z_2)^D F(z_2)$ is $-J_1-J_2+D$ which corresponds to $F(z_2)=1$. After the $SU(2)$ rotations, we generate other polynomials $F(z_2)$ from $F(z_2)=1$. Those $F(z_2)$ polynomials form an angular momentum J_1+J_2-D representation of $SU(2)$. Thus, *fusing* z_1 and z_2 together produces a variable with an angular momentum J_1+J_2-D, where J_1 is the angular momentum of z_1, J_2 is the angular momentum of z_2 and D is the power of the zeros as $z_1\to z_2$.

APPENDIX B: THE RELATION TO CONFORMAL FIELD THEORY

It was pointed out that the symmetric polynomial $\Phi(z_1,\ldots,z_N)$ that describes a FQH state can be written as an N-point correlation function of a certain operator V_e in a CFT,[14,36,37]

$$\Phi(\{z_i\}) = \lim_{z_\infty\to\infty} z_\infty^{2h_N}\left\langle V(z_\infty)\prod_i V_e(z_i)\right\rangle. \quad \text{(B1)}$$

To find the corresponding CFT of a symmetric polynomial Φ described by a pattern of zeros $\{S_a\}$, we will first calculate the scaling dimension of the operator V_e from S_a.

1. Spin of the type-*a* particles

Due to the relation between the scaling dimension and the intrinsic spin, we will first calculate the intrinsic spin of the electron and the a-electron clusters. From Eq. (8), we see that the angular momentum J_a of the type-a particle has the form $J_a=aJ-S_a$, where S_a is an intrinsic property of the type-a particle and is independent of the magnetic flux $N_\phi=2J$ through the sphere [see Eq. (33)]. S_a depend only on the pattern of the zeros and is called the orbital spin of the type-a particle.[29,49,50]

As discussed in Ref. 50, the orbital spin contains two contributions $S_a=S_a^{\text{sv}}+h_a$. S_a^{sv} comes from the spin vector and h_a is the intrinsic spin. The intrinsic spin is related to the statistics of the particle through the spin-statistics theorem. The statistics in turn is related to the scaling dimension (for example, bosons always have integral scaling dimensions).

To separate the two contributions, we need to identify the contribution from the spin vector. This can be achieved by noting that the spin vector contribution is proportional to a: $S_a^{\text{sv}}=Ca$. The key is to find the proportional coefficient C.

For this purpose, let us consider S_{nN_c} in Eq. (38). S_{nN_c} is the orbital spin for the bound state of N_c type-n particles. We know that the type-n particles form the Laughlin state (20). For a bound state of N_c type-n particles, its orbital spin S_{nN_c} contains a term linear in N_c which is the contribution from the spin vector and a term quadratic in N_c which is the intrinsic spin. From Eq. (38), we see that the contribution from the spin vector to S_{nN_c} is

$$S_{nN_c}^{\text{sv}} = N_c\left(S_n - \frac{mn}{2}\right).$$

After replacing N_c by a/n, we identify the contribution from the spin vector to S_a:

$$S_a^{\text{sv}} = a\left(\frac{S_n}{n} - \frac{m}{2}\right). \quad \text{(B2)}$$

Thus, the intrinsic spin is

$$h_a = S_a - a\left(\frac{S_n}{n} - \frac{m}{2}\right). \quad \text{(B3)}$$

h_a is also the scaling dimension of the operator $(V_e)^a$.

2. Symmetric polynomial as a correlation in a conformal field theory

The electron operator $V_e(z)$ in the CFT expression of Φ [Eq. (B1)] has the form

$$V_e(z) = \psi_1(z)e^{i\phi(z)/\sqrt{\nu}},$$

where $e^{i\phi/\sqrt{\nu}}$ is the vertex operator in a Gaussian model. The vertex operator has a scaling dimension $\frac{1}{2\nu}$. ψ_1 is a simple-current operator;[14,35,37] i.e, ψ_1 satisfies the following fusion relation:

$$\psi_a\psi_b = \psi_{a+b}, \quad \psi_a \equiv (\psi_1)^a.$$

Such an Abelian fusion rule is closely related to the unique-fusion condition discussed in Sec. III A. If $\Phi(z_1,\ldots,z_N)$ has an n-cluster form, ψ_1 satisfies

$$\psi_n = (\psi_1)^n \sim 1.$$

$\Phi(z_1,\ldots,z_N)$ can be decomposed according to Eq. (21). The correlation of the Gaussian part $e^{i\phi(z)/\sqrt{\nu}}$ produces the Φ_ν part of Φ and the correlation of the simple-current part ψ_1 produces the G part of Φ.

The intrinsic spin h_a is actually the scaling dimension of the ath power of the electron operator, $V_a\equiv(V_e)^a$. The scaling dimension of the Gaussian part $e^{ia\phi(z)/\sqrt{\nu}}$ is $\nu^{-1}\frac{a^2}{2}=\frac{a^2m}{2n}$ and

$$h_a^{\text{sc}} = h_a - \frac{a^2m}{2n} \quad \text{(B4)}$$

is the scaling dimension of the simple-current operator ψ_a.

We can obtain the scaling dimension h_a of the operator V_a more directly without using the concept of spin vector and orbital spin. First, we note that the derived polynomial $P(\{z_i^{(a)}\})$ can be expressed as a correlation of $V_a(z_i^{(a)})$'s,

$$P(\{z_i^{(a)}\}) = \lim_{z_\infty \to \infty} z_\infty^{2h_N} \left\langle V(z_\infty) \prod_{i,a} V_a(z_i^{(a)}) \right\rangle.$$

The operator-product expansion of V_a's determines the powers of zeros in the correlation function $P(\{z_i^{(a)}\})$. The (D_{ab})th order zero as $z_i^{(a)} \to z_i^{(b)}$ implies that

$$V_a(z_1)V_b(z_2) \sim (z_1-z_2)^{D_{ab}} V_{a+b} + O[(z_1-z_2)^{D_{ab}+1}].$$

Thus, the scaling dimension of V_a satisfies

$$h_{a+b} = h_a + h_b + D_{ab}. \tag{B5}$$

From Eq. (10), we see that h_a and S_a satisfy the same equation (except the $S_1=0$ condition). Thus,

$$h_a = S_a - Ca$$

for a certain constant C. Since $\psi_{nk} \equiv (\psi_1)^{nk} = 1$ and $V_{kn} = e^{ink\phi/\sqrt{\nu}}$, h_{nk} contains only a term that is quadratic in k. From Eq. (38), we see that

$$S_{nk} = kS_n + \frac{mnk(k-1)}{2}. \tag{B6}$$

Thus, we must choose C to cancel the linear k term in Eq. (B6) to obtain h_{nk}. We find that $C = \frac{S_n}{n} - \frac{m}{2}$ and

$$h_a = S_a - \frac{aS_n}{n} + \frac{am}{2}. \tag{B7}$$

This allows us to obtain the scaling dimension of the simple current ψ_a which is given by h_a^{sc} in Eq. (34). We can also express S_a in terms of h_a^{sc} [see Eq. (35)].

We note that CFT requires that $h_a = 0 \bmod 1$ since V_a are bosonic operators. This requires

$$C = \frac{S_n}{n} - \frac{m}{2} = 0 \mod 1. \tag{B8}$$

Such a condition is satisfied if we impose condition (51).

Let us introduce $V_{-a} \equiv V_a^\dagger$ where we have assumed $\psi_a^\dagger = \psi_{n-a}$. Consider

$$G(z_1, \ldots, z_k) = \langle V_{a_1}(z_1) \cdots V_{a_k}(z_k) \rangle$$
$$= P(z_1, \ldots, z_k) \prod_{i<j} (z_i - z_j)^{D_{a_i,a_j}},$$

where $D_{a,b} = h_{a+b} - h_a - h_b$, P is a polynomial of z_i's, and $\Sigma_i a_i = 0$. The part $\Pi_{i<j}(z_i - z_j)^{D_{a,b}}$ reproduces the poles/zeros of the correlation function as $z_i - z_j \to 0$. As $z_1 \to \infty$, $\Pi_{i<j}(z_i - z_j)^{D_{a,b}}$ behaves as $z_1^{\Sigma_{i=2}^k D_{a_1,a_i}}$. As a CFT correlation function $G(z_1, \ldots, z_k)$ should behave as $1/z_1^{2h_{a_1}}$ in $z_1 \to \infty$ limit. Thus, the maximal power of z_1 in $P(z_1, \ldots, z_k)$ must be

$$\gamma_1 = -\sum_{i=2}^{k} D_{a_1,a_i} - 2h_{a_1},$$

$$= (k-3)h_{a_1} + \sum_{i=2}^{k} h_{a_i} - \sum_{i=2}^{k} h_{a_1+a_i},$$

where $a_1 = -\Sigma_{i=2}^k a_i$. When $k=4$, we have

$$\gamma_1 = h_{a_2+a_3+a_4} + h_{a_2} + h_{a_3} + h_{a_4} - h_{a_2+a_3} - h_{a_3+a_4} - h_{a_3+a_2},$$

where we have used $h_{-a} = h_a$. The requirement that $\gamma_1 \geq 0$ is the third condition in Eq. (45).

3. Generalized vertex algebra

To understand the CFT representation of the symmetric polynomial more deeply, let us consider generalized vertex algebra.[51] The CFT formed by the simple-current operators ψ_a is a special case of a generalized vertex algebra.

Consider operators $A(z)$, $B(w)$, etc., which form an operator-product-expansion algebra

$$A(z)B(w) = \frac{1}{(z-w)^{\alpha_{AB}}} \{[AB]_{\alpha_{AB}}(w) + (z-w)[AB]_{\alpha_{AB}-1}(w)$$
$$+ (z-w)^2 [AB]_{\alpha_{AB}-2}(w) + \cdots\}, \tag{B9}$$

and

$$(z-w)^{\alpha_{AB}} A(z)B(w) = \mu_{AB}(w-z)^{\alpha_{AB}} B(w)A(z), \tag{B10}$$

where μ_{AB} is a phase factor. Here, $(z-w)^{\alpha_{AB}} \equiv |z-w|^{\alpha_{AB}} e^{\alpha_{AB} i\theta_{zw}}$, where $(z-w) = |z-w| e^{i\theta_{zw}}$ and $-\pi < \theta_{zw} \leq \pi$. The operator product in Eq. (B9) is assumed to be radially ordered: $A(z)B(w) \to R[A(z)B(w)]$, where

$$(z-w)^{\alpha_{AB}} R[A(z)B(w)]$$
$$= \begin{cases} (z-w)^{\alpha_{AB}} A(z)B(w) & |z| > |w| \\ \mu_{AB}(w-z)^{\alpha_{AB}} B(w)A(z) & |w| > |z|. \end{cases}$$

We see that commutation relation (B10) ensures that the correlation functions of $A(z)$ and $B(w)$ are smooth functions. Let h_A, h_B and $h_{[AB]_{\alpha_{AB}}}$ be the scaling dimensions of A, B, and $[AB]_{\alpha_{AB}}$, then

$$\alpha_{AB} = h_A + h_B - h_{[AB]_{\alpha_{AB}}}.$$

The self-consistency of the vertex algebra requires α_{AB}'s to satisfy[51]

$$\alpha_{AB} + \alpha_{AC} - \alpha_{AD} = 0 \text{ mod } 1, \tag{B11}$$

and μ_{AB}'s to satisfy

$$\mu_{AB}\mu_{AC} = \mu_{AD}(-)^{\alpha_{AB}+\alpha_{AC}-\alpha_{AD}}, \tag{B12}$$

where $D = [BC]_{\alpha_{BC}}$.

The fusion rule of the simple-current operators $\psi_a \psi_b = \psi_{a+b}$ requires that those operators form the following vertex operator algebra:

$$\psi_a(z)\psi_b(w) = \frac{c_{ab}[\psi_{a+b}(w) + O(z-w)]}{(z-w)^{h_a^{\text{sc}} + h_b^{\text{sc}} - h_{a+b}^{\text{sc}}}},$$

$$\psi_a(z)\psi_{-a}(w) = \frac{1 + \frac{2h_a^{\text{sc}}}{c}(z-w)^2 T(w) + O[(z-w)^3]}{(z-w)^{2h_a^{\text{sc}}}},$$

where $\psi_{-a} \equiv \psi_{n-a}$ and $\psi_{a+n} \equiv \psi_a$.

We see that the above algebra of simple currents ψ_a is a special case of generalized vertex algebra. We have

$$\alpha_{ab}=h^{sc}_a+h^{sc}_b-h^{sc}_{a+b}.$$

Condition (B11) becomes

$$\Delta_3(a,b,c)=0 \bmod 1, \tag{B13}$$

where we have used Eqs. (49) and (46). From Eq. (B12), we see that if $\Delta_3(a,b,c)=$odd for certain choices of a,b,c, then μ_{ab} cannot be trivial (i.e., $\mu_{ab}=1$). When $\Delta_3(a,b,c)=$even for all a,b,c, then μ_{ab} can be a trivial solution $\mu_{ab}=1$. Thus condition (51) has a special meaning in CFT.

4. Conditions on the *h* vectors

Due to the one-to-one correspondence between the $\boldsymbol{S}$ vectors and the $\boldsymbol{h}$ vectors [see Eqs. (34) and (35)], we can translate conditions (43) and (45) on the $\boldsymbol{S}$ vectors to some conditions on the $\boldsymbol{h}$ vectors.

Note that an $\boldsymbol{h}$ vector is specified by n, m, and $h^{sc}_1,\cdots,h^{sc}_{n-1}$. We extend $h^{sc}_1,\ldots,h^{sc}_{n-1}$ to h^{sc}_a for any integer a by requiring

$$h^{sc}_0=0,\quad h^{sc}_a=h^{sc}_{a+n}.$$

Conditions (43) become

$$S_a=h^{sc}_a-ah^{sc}_1+\frac{a(a-1)m}{2n}=\text{non-negative integer},$$

$$m>0,\quad mn=\text{even},$$

$$2nh^{sc}_1+m=0 \bmod n. \tag{B14}$$

From $2nh^{sc}_1+m=0 \bmod n$, we see that $2nh^{sc}_1$ is an integer. From $2nh^{sc}_a-a(2nh^{sc}_1)+a(a-1)m=$even integer, we see that $2nh^{sc}_a$ are always integers. Also, $2nh^{sc}_{2a}$ are always even integers and $2nh^{sc}_{2a+1}$ are either all even or all odd. Since $h^{sc}_n=0$, thus when $n=$odd $2nh^{sc}_a$ are all even. Only when $n=$even can $2nh^{sc}_{2a+1}$ either be all even or all odd.

To generate sets of h^{sc}_a that satisfy the above conditions, we will use Eq. (34). Setting $a=1$ in Eq. (34), we get $2nh^{sc}_1=m(n-1)-2S_n$. We see that when $n=$odd, $2nh^{sc}_1$ $=$even. When $n=$even, $2nh^{sc}_1=$even when $m=$even and $2nh^{sc}_1=$odd when $m=$odd. We also see that $S_n\le m(n-1)/2$ which implies that $S_a\le m(n-1)/2$ for $a=2,3,\ldots,n$.

Conditions (45) become

$$nh^{sc}_{2a}-2anh^{sc}_1+ma(2a-1)=0 \bmod 2n,$$

$$h^{sc}_{a+b}-h^{sc}_a-h^{sc}_b\ge-\frac{abm}{n},$$

$$h^{sc}_{a+b+c}-h^{sc}_{a+b}-h^{sc}_{b+c}-h^{sc}_{a+c}+h^{sc}_a+h^{sc}_b+h^{sc}_c\ge 0. \tag{B15}$$

Condition (51) becomes

$$h^{sc}_{a+b+c}-h^{sc}_{a+b}-h^{sc}_{b+c}-h^{sc}_{a+c}+h^{sc}_a+h^{sc}_b+h^{sc}_c=\text{even}. \tag{B16}$$

The first condition in Eq. (B14) and the second condition in Eq. (B15) imply that

$$h^{sc}_{a+b}-h^{sc}_a-h^{sc}_b+\frac{abm}{n}\ge 0,$$

$$h^{sc}_{a+b}-h^{sc}_a-h^{sc}_b+\frac{abm}{n}=0 \bmod 1, \tag{B17}$$

which are part of defining conditions of parafermion CFT if $m=2$. Thus, the pattern of zeros of symmetric polynomial may have a natural relation to parafermion CFT.

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