

Emergence of h/e -period oscillations in the critical temperature of small superconducting rings threaded by magnetic flux

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As a function of the magnetic flux threading the object, the Little-Parks oscillation in the critical temperature of a large-radius, thin-walled superconducting ring or hollow cylinder has a period given by $h/2e$, due to the binding of electrons into Cooper pairs. On the other hand, the single-electron Aharonov-Bohm oscillation in the resistance or persistent current for a clean (i.e., ballistic) normal-state system, having the same topological structure, has a period given by h/e . A basic question is whether the Little-Parks oscillation changes its character, as the radius of the superconducting structure becomes smaller, and if it is even comparable to the zero-temperature coherence length. We supplement a physical argument that the h/e oscillations should also be exhibited with a microscopic analysis of this regime, formulated in terms of the Gor'kov approach to BCS theory. We see that, as the radius of the ring is made smaller, an oscillation in the critical temperature of period h/e emerges in addition to the usual Little-Parks $h/2e$ -period oscillation. We argue that, in the clean limit, there is a superconductor-normal transition at nonzero flux as the ring radius becomes sufficiently small and that the transition can be either continuous or discontinuous, depending on the radius and the external flux. In the dirty limit, we argue that the transition is rendered continuous, which results in continuous quantum phase transitions tuned by flux and radius.

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I. INTRODUCTION

The Little-Parks critical-temperature oscillations with magnetic flux of a large-radius thin-walled hollow cylindrical superconductor display a period $h/2e$.^{1,2} This oscillation period reflects the binding of electrons into Cooper pairs.³ Recent experiments by Liu *et al.*⁴ have probed the regime in which the diameter of the hollow cylinder is comparable to zero-temperature coherence of the superconductor. Their results confirmed the prediction by de Gennes⁵ of the destruction of superconductivity in small rings for a certain regime of the external flux. Liu *et al.* raised interesting and fundamental issues of what would happen if the circumference of the structure were to be smaller than the superconducting coherence length, as well as whether or not the Ginzburg-Landau approach would be valid in this regime. These issues make a microscopic treatment desirable.

On the other hand, the single-electron Aharonov-Bohm oscillations in the resistance or persistent current in a clean metallic ring have period h/e .⁶ This leads to a related fundamental issue: As the radius of the ring becomes smaller, how would the single-particle h/e period manifest itself in a Little-Parks type of experiment? Furthermore, does disorder affect the oscillation of critical temperature and the character of the transition between superconducting and normal states? And if so, how? These questions are not only of theoretical interest but are also likely to be addressed experimentally in view of recent progress in fabrication and experiments on small superconducting rings.^{7,8}

Recent work by Czajka *et al.*,⁹ involving the exact diagonalization of the Hubbard model for small numbers of sites

and the numerical solution of the Bogoliubov–de Gennes (BdG) equation, shows that impurities can play an important role if they are located such that pinned density waves are in phase with one another: charge-density-wave (CDW) order would then be enhanced and superconducting order reduced. They also found that the mean-field results obtained via the BdG equations were consistent with the exact diagonalization results, even in the small systems they studied. Very recently, a numerical study by Loder *et al.*,¹⁰ and analytical work by Juricic *et al.*¹¹ and by Barash¹² on clean d -wave superconducting loops have shown h/e -period oscillations in the supercurrent.

Ginzburg-Landau (GL) theory is valid near the superconducting transition as it is an expansion in powers of the superconducting order parameter. Therefore it may not be applicable at sufficiently low temperatures. Moreover, the GL approach cannot give a complete account of multiconnected geometries of small size as it is a description of the center-of-mass wave function of the Cooper pairs. For small rings, in addition to propagating around the circumference together, electrons in Cooper pairs can split apart and rejoin, and this process is not included in the GL description. In this paper, we study the oscillations of the critical temperature of s -wave superconducting rings theoretically via consideration of the microscopic BCS theory of superconductivity¹³ and analyzed using Gor'kov's approach.¹⁴ Our central purpose is to address the issues raised by Liu *et al.*⁴ and the ones mentioned above. Our focus is on the correction to the oscillations that is due to finiteness of the radius of the superconducting structure. We consider both the clean and dirty regimes, and in the latter regime we shall ignore any ten-

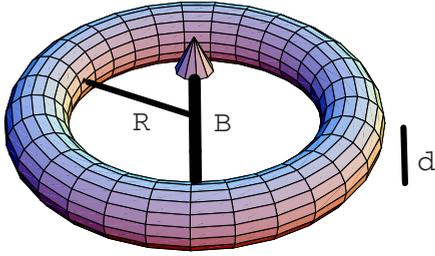


FIG. 1. (Color online) A ring with a magnetic flux threading through it. The radius of the ring is R and the thickness in the cross section is d .

dency toward CDW ordering by assuming that the significant configurations of the impurities are sufficiently random and that any pinned density waves are not in phase with one another. We shall see the emergence of an h/e oscillation in the critical temperature as the radius becomes smaller. We shall also see that the transition to the normal state can be either discontinuous or continuous in the clean limit but is always continuous in the dirty limit.

The origin of the $h/2e$ oscillations lies in the fact that a Cooper pair carries charge $2e$, which when circumnavigating the ring acquires a phase $e^{i2e\Phi/\hbar c}$, where Φ is the magnetic flux linking the ring. The period in flux is thus $h/2e$ (with c conveniently set to unity¹⁵). The emergence of h/e oscillations for small rings is due to the additional process in which electrons in a Cooper pair can, from time to time (so to speak), separate, propagate separately, and rejoin, with the two trajectories having a nonzero winding number relative to one another. This process, which can only occur for ring sizes comparable to or smaller than the Cooper-pair size, induces an oscillation with period h/e .

The organization of this Paper is as follows. In Sec. II we discuss the critical-temperature oscillations in the clean limit and in Sec. III we include the effect of disorder and discuss the dirty limit. We make some concluding remarks in Sec. IV. In Appendix A we derive the effective one-dimensional (1D) Gor'kov equations by averaging over the cross section (or thickness) of the ring. In Appendix B we provide supplementary details of the calculations that lead to the results in the main text. In Appendix C we provide two heuristic arguments for the emergence of the h/e period (1) by examining Cooper's problem on a ring and (2) by using an instanton approach. These two arguments lead to the physical picture described above and complement the Gor'kov Green function approach adopted in the main text.

II. CLEAN LIMIT

A. Gor'kov equations

We consider a ring (to be more precise, a torus) of radius R (and thickness d , namely, the diameter in the cross section, which is smaller than both R and the Cooper-pair size ξ_0) with a magnetic-flux density B threading the ring parallel to the ring axis (see Fig. 1). We can describe this field by the vector potential $\vec{A}(r) = (B/2)\hat{z} \times \vec{r}$. On the ring itself (the circumference of which is $L = 2\pi R$), the vector potential \vec{A} is

given by $\hat{\theta}\Phi/L$, where $\hat{\theta}$ is the azimuthal unit vector and Φ is the total flux enclosed by the ring, i.e., $\Phi = \int da \cdot B(\vec{r})$.

The normal and anomalous Green functions obey the Gor'kov equations^{3,14,16}

$$\left[i\hbar\omega_n - \frac{1}{2M} \left(-i\hbar\nabla - \frac{e\vec{A}}{c} \right)^2 + \mu \right] G(\vec{r}, \vec{r}'; \omega_n) + \Delta(\vec{r})F^\dagger(\vec{r}, \vec{r}'; \omega_n) = \hbar\delta(\vec{r} - \vec{r}'), \quad (1a)$$

$$\left[-i\hbar\omega_n - \frac{1}{2M} \left(i\hbar\nabla - \frac{e\vec{A}}{c} \right)^2 + \mu \right] F^\dagger(\vec{r}, \vec{r}'; \omega_n) - \Delta^*(\vec{r})G(\vec{r}, \vec{r}'; \omega_n) = 0, \quad (1b)$$

where M is the electron mass, e is the electron charge, \vec{r} and \vec{r}' are three-dimensional coordinates in the ring, and the order parameter Δ is defined self-consistently via

$$\Delta^*(\vec{r}) = \frac{V}{\beta} \sum_{\omega_n} F^\dagger(\vec{r}, \vec{r}'; \omega_n), \quad (2)$$

in which $\omega_n \equiv 2\pi T(n + 1/2)$ are Matsubara frequencies, $\beta \equiv 1/k_B T$, T is the temperature, and V is the BCS pairing strength.

We now invoke the narrowness of the ring to justify dropping all dependences on \vec{r} and \vec{r}' except those associated with the one-dimensional coordinates along the ring, x and x' . Owing to the physical periodicity of the ring, all functions of x and x' are periodic with period L (and the vector potential is a constant along the circumference of the ring). The Gor'kov equations then become one dimensional:¹⁷

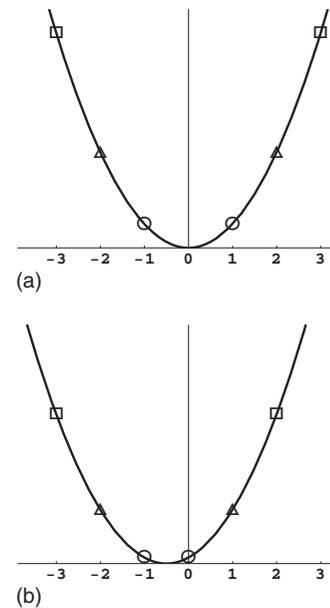


FIG. 2. Pairing of states (Ref. 3). (a) Upper panel: the external flux is $\Phi = 0$, the pairing is between $n_1 + n_2 = 0$. (b) Lower panel: $\Phi = h/2e$, the pairing is between $n_1 + n_2 = -1$. The pairing configuration changes from type (a) to type (b) at external flux $\Phi = h/4e$ (see Ref. 3).

$$\left[+i\hbar\omega_n - \frac{1}{2M} \left(i\hbar\partial_x + \frac{e\Phi}{cL} \right)^2 + \mu \right] G(x, x'; \omega_n) + \Delta(x) F^\dagger(x, x'; \omega_n) = \hbar \delta(x - x'), \quad (3a)$$

$$\left[-i\hbar\omega_n - \frac{1}{2M} \left(i\hbar\partial_x - \frac{e\Phi}{cL} \right)^2 + \mu \right] F^\dagger(x, x'; \omega_n) - \Delta^*(x) G(x, x'; \omega_n) = 0. \quad (3b)$$

We can expand G and F in Fourier series, as follows:

$$G(x_1, x_2; \omega_n) = \frac{1}{L} \sum_{n_1, n_2} g_{n_1, n_2}(\omega_n) e^{i(2\pi n_1/L)x_1 + i(2\pi n_2/L)x_2}, \quad (4a)$$

$$F^\dagger(x_1, x_2; \omega_n) = \frac{1}{L} \sum_{n_1, n_2} f_{n_1, n_2}^\dagger(\omega_n) e^{i(2\pi n_1/L)x_1 + i(2\pi n_2/L)x_2}, \quad (4b)$$

where n_1 and n_2 are integers labeling single-particle states. Due to the translational (to be more precise, rotational) invariance of the system, we assume that $G(x, x)$ has no dependence on x . This sets constraints on the nonzero Fourier com-

ponents for G : $n_1 + n_2 = 0$. Furthermore, we assume that $\Delta(x) = e^{i2\pi mx/L} \Delta_0$, and hence, $F^\dagger(x, x) \sim e^{-i2\pi mx/L} \Delta_0^*$. This sets constraints on the nonzero Fourier components for F^\dagger : $n_1 + n_2 = -m$. The meaning of this is that the pairing occurs between the single-particle states n_1 and $n_2 = -m - n_1$ ³ (see Fig. 2).

The Gor'kov equations can be expressed in terms of the Fourier components of G and F^\dagger as follows:

$$\left[+i\hbar\omega_n - \frac{\hbar^2}{2MR^2} (n_1 + m - \phi)^2 + \mu \right] g_{n_1+m, -n_1-m} + \Delta_0 f_{n_1, -n_1-m}^\dagger = \hbar, \quad (5a)$$

$$\left[-i\hbar\omega_n - \frac{\hbar^2}{2MR^2} (n_1 + \phi)^2 + \mu \right] f_{n_1, -n_1-m}^\dagger - \Delta_0^* g_{n_1+m, -n_1-m} = 0, \quad (5b)$$

where $\phi \equiv \Phi/(-hc/e) = \Phi/(hc/|e|)$. In the following we shall set $\hbar = 1$, $c = 1$, and $k_B = 1$, for the sake of convenience. However, we shall refer to the single-particle flux quantum hc/e as h/e .¹⁵ These equations can be solved explicitly, yielding

$$g_{n_1+m, -n_1-m} = \frac{-i\omega_n - \Omega(n_1 + \phi)^2 + \mu}{[i\omega_n - \Omega(n_1 + m - \phi)^2 + \mu][-i\omega_n - \Omega(n_1 + \phi)^2 + \mu] + |\Delta_0|^2}, \quad (6a)$$

$$f_{n_1, -n_1-m}^\dagger = \frac{\Delta_0^*}{[i\omega_n - \Omega(n_1 + m - \phi)^2 + \mu][-i\omega_n - \Omega(n_1 + \phi)^2 + \mu] + |\Delta_0|^2}, \quad (6b)$$

where, for the sake of convenience, we have introduced $\Omega \equiv 1/2MR^2$ (noting that we have set $\hbar = 1$ in $\hbar^2/2MR^2$). The self-consistency [Eq. (2)] then becomes

$$\Delta_0^* = \frac{VT}{L} \sum_{\omega_n} \sum_{n_1} f_{n_1, -n_1-m}^\dagger(\omega_n). \quad (7)$$

We mainly discuss the case in which the chemical potential μ is kept fixed, e.g., by contact with a particle reservoir, or else we assume that the variation of μ with temperature and flux is sufficiently weak to be negligible near the superconducting transition.

B. Critical temperature

To solve for $T_c(\phi)$, we set $\Delta_0 = 0$ in the self-consistency equation, thus obtaining

$$1 = \frac{VT}{L} \sum_{\omega_n} \sum_{n_1 \in \mathbb{Z}} \frac{1}{[i\omega_n - \Omega(n_1 + m - \phi)^2 + \mu][-i\omega_n - \Omega(n_1 + \phi)^2 + \mu]}. \quad (8)$$

It is important to note the underlying assumption that the

transition from superconducting to normal is associated with a vanishing order parameter, and hence, is a continuous transition. The consistency of this assumption needs to be checked once we obtain the solution. We shall see (below) that for sufficiently small radii the assumption is not valid and the transition is actually associated with nonvanishing order parameter, and hence, is discontinuous.

Next, we make use of the root of the Poisson summation formula (i.e., the Dirac comb and its Fourier series), $\sum_{n_1 \in \mathbb{Z}} \delta(x - n_1) = \sum_{k \in \mathbb{Z}} e^{i2\pi xk}$, to turn the summation over n_1 in Eq. (8) into one over the conjugate variable k :

$$\begin{aligned} & \sum_{n_1} \frac{1}{[i\omega_n - \Omega(n_1 + m - \phi)^2 + \mu][-i\omega_n - \Omega(n_1 + \phi)^2 + \mu]} \\ &= \sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} dx \\ & \quad \times \frac{e^{i2\pi xk}}{[i\omega_n - \Omega(x + m - \phi)^2 + \mu][-i\omega_n - \Omega(x + \phi)^2 + \mu]}. \end{aligned} \quad (9)$$

Instead of placing a cutoff on the energy, we follow Gor'kov¹⁴ and place the cutoff (which, in practice, is the

Debye frequency ω_D) on the Matsubara frequency.

1. Large-radius limit

For sufficiently large R , we can ignore the correction terms associated with finite radius (i.e., $k \neq 0$) and thus we obtain an equation relating the critical temperature at non-zero flux to that at zero flux. In the limits that the Debye frequency is much smaller than the chemical potential (i.e., $\omega_D/\mu \ll 1$) and that the chemical potential is much larger than the level spacing (i.e., $\sqrt{2MR^2\mu} \ll 1$), we obtain (see Appendix B 2 for the derivation)

$$\ln \left[\frac{T_c(\phi)}{T_c^0} \right] = \psi \left(\frac{1}{2} \right) - \text{Re} \psi \left[\frac{1}{2} - i \frac{x_m(\phi)}{2\pi T_c} \sqrt{\frac{2\mu}{MR^2}} \right], \quad (10)$$

where $T_c^0 \equiv T_c(0)$ is the critical temperature at zero flux for the same radius, $x_m(\phi) \equiv \phi - m/2$ [with m being chosen to minimize $(2\phi - m)^2$, i.e., the kinetic energy in the Ginzburg Landau picture], and $\psi(x)$ is the digamma function,

$$\psi(x) \equiv -\gamma + \sum_{k=0}^{\infty} \left(\frac{1}{k+1} - \frac{1}{k+x} \right), \quad (11)$$

where γ is the Euler constant.¹⁸

The zero-flux, zero-temperature order parameter Δ_0 is related to the corresponding critical temperature T_c^0 in the weak-coupling limit by $\Delta_0 = \Gamma T_c^0$, where $\Gamma = \pi/e^\gamma \approx 1.76$. We call the length $\xi_0 \equiv v_F/\pi\Delta_0$ the Cooper-pair size (where v_F is the Fermi velocity) so as to distinguish it from the zero-temperature Ginzburg-Landau coherence length $\xi(0)$. There are two further relevant energy scales: the chemical potential μ and the single-particle energy-level spacing Ω . The ratio $\Omega\mu/\Delta_0^2$ can be estimated as

$$\frac{\Omega\mu}{\Delta_0^2} \approx \left(\frac{\hbar^2}{2MR^2} \right) \left(\frac{Mv_F^2}{2} \right) / \left(\frac{\hbar v_F}{\pi\xi_0} \right)^2 = \left(\frac{\pi\xi_0}{2R} \right)^2, \quad (12)$$

which is set by on the ratio of the ring radius to the Cooper-pair size. This motivates us to define a measure ρ of the ratio of the radius to the Cooper-pair size via

$$\left(\frac{\pi}{2\rho} \right)^2 \equiv \frac{\Omega\mu}{\Delta_0^2}. \quad (13)$$

Then, defining $t \equiv T_c/T_c^0$, we can rewrite the equation for the critical temperature as

$$\ln t = \psi \left(\frac{1}{2} \right) - \text{Re} \psi \left[\frac{1}{2} - i \frac{x_m(\phi)\Gamma}{2\rho t} \right]. \quad (14)$$

An equation of this form was studied by Sarma,¹⁹ and by Maki and Tsuneto,²⁰ both in the context of the effect of a magnetic exchange field on superconductivity. The correspondence is that the role of the exchange field normalized to the zero-temperature gap (i.e., $\mu H/\pi\Delta_0$) is, in the present setting, played by the combination of the normalized inverse ring radius and the flux [i.e., $x_m(\phi)/\rho$]. In Refs. 19 and 20 it was found that, for large enough exchange field (here, small enough ring radius), Eq. (14) has multiple solutions, and that,

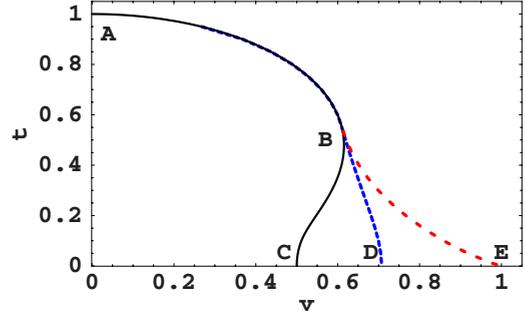


FIG. 3. (Color online) Phase diagram: reduced critical temperature $t(\phi, R) = T_c(\phi, R)/T_c(0, R)$ vs flux and radius $v = \pi|\phi - m/2|(\xi_0/R)$ in the clean limit, and in the limit that the finiteness corrections are ignored (i.e., bulk limit). In the case of the exchange-field effect, discussed in Refs. 19 and 20, the horizontal axis becomes $v = \mu_B H/\Delta_0$, where μ_B is Bohr magneton and H is the exchange field. The curve ABC is the solution to Eq. (14) while ABD is the curve representing the true equilibrium transition line. From A to B, the transition is continuous, whereas from B to D the transition is discontinuous, construction first made in Refs. 19 and 20 in the context of the exchange-field effect. The curve BE represents the metastability limit for “superheating,” whereas the curve BC represents the metastability limit for supercooling (Ref. 20).

moreover, the correct interpretation is that the transition between the normal and superconducting states becomes discontinuous (beyond certain value of the exchange field). By contrast, for small enough exchange field (here, large enough ring radius), the transition is continuous. By borrowing the results of Refs. 19 and 20, as summarized in Fig. 3, we have that the threshold at which the transition changes character between continuous and discontinuous occurs at $|x_m(\phi)|/\rho \approx 0.6/\pi$ (i.e., point B in Fig. 3). Furthermore, one has that for extremely small rings such that $|x_m(\phi)|/\rho > 1/\sqrt{2}\pi$ (i.e., point D), the system never becomes superconducting.²¹ Of course, these estimates would need to be modified if the finiteness of the radius were to be taken into account but we expect that the qualitative separation into discontinuous-transition and continuous-transition regimes would still hold.

For large-radius rings (i.e., $\rho \gg 1$), we can expand the second digamma function in Eq. (14) to second order and the logarithm to first order, thus obtaining

$$-(1-t) \approx \frac{1}{2} \psi'' \left(\frac{1}{2} \right) \left[\frac{x_m(\phi)\Gamma}{2\rho t} \right]^2 \approx -8.41 \left[\frac{x_m(\phi)\Gamma}{2\rho t} \right]^2. \quad (15)$$

Hence, we see that the fractional reduction in the critical temperature due to the flux is given by

$$\begin{aligned} (1-t) &\approx 8.41 \left[\frac{x_m(\phi)\Gamma}{2\rho t} \right]^2 \approx 8.41 \left[\frac{x_m(\phi)\Gamma}{2\rho} \right]^2 \\ &= 8.41 \left(\frac{\Gamma}{4\rho} \right)^2 (2\phi - m)^2. \end{aligned} \quad (16)$$

The integer m must be chosen such that $(2\phi - m)^2$ is minimum so as to obtain the most stable solution; hence, we recover the standard Little-Parks oscillation result for which

the period is $h/2e$. Equation (16) can be re-expressed as

$$\sqrt{\frac{1-t}{8.41}} \frac{4R}{\Gamma \xi_0} = \min_{m \in \mathbb{Z}} |2\phi - m|. \quad (17)$$

The left-hand side can be much smaller than unity if R is much smaller than ξ_0 but the value of the right-hand side (r.h.s) can range from 0 to $1/2$, depending on the value of external flux. This means that there can exist a range of fluxes for which no solution of t exists. This reflects the fact that superconductivity is destroyed over certain ranges of flux.⁵ To make a connection with the result by de Gennes,⁵ let us multiply Eq. (17) by 2π and take the cosine of both sides. We recover the de Gennes result for the transition temperature (for the case in which the length of the side arm in Ref. 5 is set to zero)

$$\cos\left(2\pi \frac{R}{\xi(t)}\right) \approx \cos\left(2\pi \frac{\Phi}{h/2e}\right), \quad (18)$$

where $\xi(t) \approx 0.74\Gamma \xi_0 / \sqrt{1-t}$ is the temperature-dependent Ginzburg-Landau coherence length in the clean limit. We note that, however, the Ginzburg-Landau approach is, strictly speaking, valid only near $t \approx 1$. Furthermore, the existence of discontinuous transitions is beyond the reach of the Ginzburg-Landau approach.

2. Finite-radius correction

What is the correction to T_c that arises from the finiteness of the radius? By taking into account this correction, we arrive at the following equation obeyed by the critical temperature (see Appendix B 3 for the derivation):

$$\begin{aligned} \ln t(\phi) = & \psi\left(\frac{1}{2}\right) - \operatorname{Re} \psi\left[\frac{1}{2} - \frac{ix_m(\phi)\Gamma}{2\rho t(\phi)}\right] \\ & - 4 \sum_{k=1}^{\infty} \{\cos(2\pi k\bar{n}) e^{-2\pi k\rho/\Gamma} f_c(0, \rho, 1) \\ & - \operatorname{Re}[e^{i2\pi k\bar{n}} e^{-2\pi k\rho/\Gamma} f_c[x_m(\phi), \rho, t(\phi)] \cos 2\pi k\phi]\}, \end{aligned} \quad (19)$$

where we recall that $t(\phi) = T_c(\phi)/T_c(0)$, Γ is the BCS constant $\Gamma = \pi/e^\gamma \approx 1.76$. We have defined $\bar{n} \equiv \sqrt{2MR^2\mu}$, which is related to the number of Cooper pairs, and the function f_c is defined via the hypergeometric function ${}_2F_1(a, b; c; z)$:¹⁸

$$\begin{aligned} f_c[x_m(\phi), \rho, t] \equiv & {}_2F_1\left[\frac{1}{2} - \frac{ix_m(\phi)\Gamma}{2\rho t}, 1, \frac{3}{2}\right. \\ & \left. - \frac{ix_m(\phi)\Gamma}{2\rho t}, e^{-4\pi k\rho/\Gamma}\right] \Big/ \left[1 - \frac{ix_m(\phi)\Gamma}{\rho t}\right] \end{aligned} \quad (20)$$

In Figs. 4 and 5 we show the flux dependence of the critical temperature for two particular ring radii. For the larger radius case, the value of t at $\Phi = h/2e$ is essentially the same as that at $\Phi = h/e$ and 0. For the smaller radius case and for $\cos(2\pi\bar{n}) = 1$ (i.e., all the pair states are occupied at and below Fermi level), the amplitude at $\Phi = h/2e$ is reduced relatively to that at $\Phi = h/e$ and 0. Thus, as the radius becomes

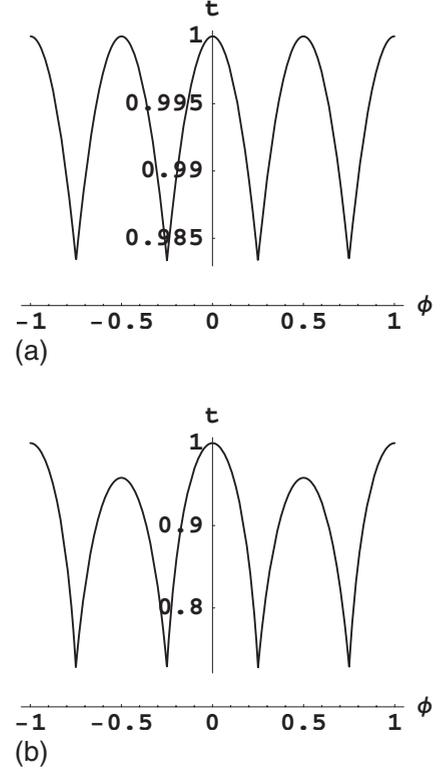


FIG. 4. Critical temperature (normalized to its zero-flux value) t vs flux $\phi = \Phi/(h/e)$ for $R = 5\xi_0$ (upper) and $R = 1.5\xi_0$ (lower) in the clean limit. The plots are made with $\cos(2\pi\bar{n}) = 1$ and with only the $k=1$ term in Eq. (19) retained.

small, we clearly see the emergence of the single-particle flux quantum period h/e . Also worth noticing is the occurrence of a second solution (with lower value) for the critical temperature at sufficiently small radii, as shown in Fig. 5. Compared to the higher T_c solutions for which our approximation of the T_c [Eq. (19)] by keeping only the first two correction terms yields rather precise values, the evaluation

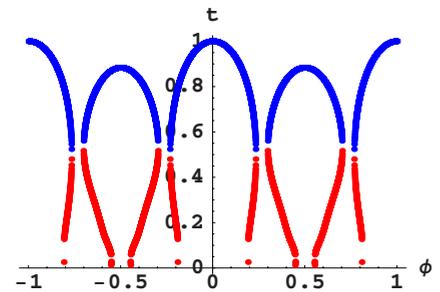


FIG. 5. (Color online) Reduced critical temperature $t(\phi) \equiv T_c(\phi)/T_c(0)$ (normalized to its zero-flux value) vs flux $\phi \equiv \Phi/(h/e)$ for $R = 1.25\xi_0$ with $\cos(2k\pi\bar{n}) = 1$ in the clean limit. The higher T_c branches (blue) are calculated by retaining only the first two terms in Eq. (19), whereas the lower T_c branches (red) are calculated by retaining as many as ten such terms. The plot shows multiple solutions to Eq. (19) and signifies the emergence of discontinuous transitions. The upper and lower branches of the solutions are expected to merge at certain values of ϕ ; the appearance of a gap between them is an artifact of our considering only a finite number of values of ϕ .

of T_c at these lower values of temperatures requires more terms (e.g., four or more) to be included in order to maintain the same accuracy.

For the case of large rings (i.e., $\rho \gg 1$), to determine the leading corrections, it is adequate to retain only the $k=1$ term and to set the correction $\sim ix_m(\phi)\Gamma/\rho t$ to zero, when compared to values of order unity, in the second f_c function. Thus, we arrive at the formula

$$\ln t(\phi) \approx \psi\left(\frac{1}{2}\right) - \operatorname{Re} \psi\left[\frac{1}{2} - i\frac{x_m(\phi)\Gamma}{2\rho t}\right] - 4 \cos(2\pi\bar{n})e^{-2\pi\rho/\Gamma}f(0,\rho,1)[1 - \cos 2\pi\phi]. \quad (21)$$

The change in the reduced critical temperature $t(\phi)$ is then approximately given by

$$1 - t(\phi) \approx 8.41 \left(\frac{\Gamma}{4\rho}\right)^2 (2\phi - m)^2 + 4 \cos(2\pi\bar{n}) \tanh^{-1}(e^{-2\pi\rho/\Gamma}) \times (1 - \cos 2\pi\phi), \quad (22)$$

where we have used the fact that ${}_2F_1(1/2, 1, 3/2, y^2) = \tanh^{-1}(y)/y$. Again, the integer m is to be chosen to minimize $(2\phi - m)^2$. We see that, in addition to the parabolic dependence on ϕ , there is a sinusoidal correction of period h/e . This is the emergence of the single-particle flux dependence. We also note that this correction is not universal in that it depends sensitively on the value of μ [and, moreover, the form of $\cos(2\pi\bar{n})$ results from the simple quadratic single-particle spectrum]. It can happen that the correction due to the finiteness of the radius actually increases the critical temperature, i.e., when $\cos(2\pi\bar{n}) < 0$.

As we have argued using the results of Sarma,¹⁹ and Maki and Tsuneto,²⁰ the occurrence of multiple solutions in Eq. (14) for $T_c(\phi)$, for certain ranges of $|x_m(\phi)|/\rho$, leads to a change from a continuous to a discontinuous superconducting-to-normal phase transition. Even with the corrections to T_c due to the finite-radius effect, as the radius of the ring decreases (to a value comparable to the coherence length), we observe that Eq. (19) still possesses multiple solutions for $T_c(\phi)$ for certain ranges of $|x_m(\phi)|/\rho$ (see Fig. 5). This implies that, somewhere in these ranges (of $|x_m(\phi)|/\rho$), there exists a change from continuous (at larger radius) to discontinuous (at smaller radius) superconducting-to-normal transition. This is shown schematically in Fig. 6. If the radius is sufficiently large, the transition is always continuous. If the radius is sufficiently small, the curve representing the discontinuous transition in one ‘‘dome’’ can intersect with that of the nearby dome [above the ‘‘void’’ region where no solution for T_c of Eq. (19) exists]. If this void region is large, the curve of the discontinuous transition can go to $t=0$ at certain value of ϕ without intersecting that from the nearby dome. However, calculations of $T_c(\phi)$ alone cannot determine the precise location of the change from continuous to discontinuous; considerations of free energies are necessary to settle this issue.

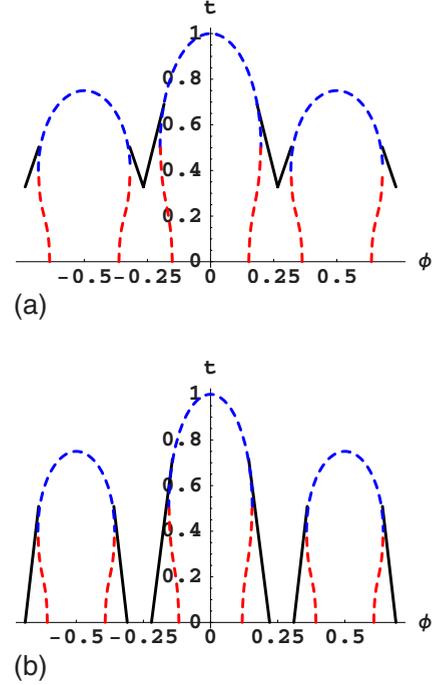


FIG. 6. (Color online) Schematic depiction of the reduced critical temperature $t(\phi) \equiv T_c(\phi)/T_c(0)$ (normalized to its zero-flux value) vs flux $\phi = \Phi/(h/e)$ for small-radius rings in the clean limit. As illustrated in Fig. 5 for sufficiently small radii, there are multiple solutions for T_c , as exemplified there by the higher (blue) branches and the lower (red) branches. Near $\phi=0$ and $\pm 1/2$, the upper branches are the equilibrium phase boundary and the transition to normal state is continuous. Away from these regions, the globally stable equilibrium phases must be sought by free-energy consideration, and the corresponding phase boundaries are indicated schematically by the solid lines and represent discontinuous transitions. Upper panel: For all flux values, there exists a superconducting state. Lower panel: For smaller radius, it can happen that there are flux values for which no superconducting state exists.

III. DISORDERED REGIME

How does disorder, in the form of potential scattering from fixed impurities, alter the physical picture that we have obtained so far? Does it affect the critical temperature? Does it change the order of the superconducting-to-normal transition? We now address these issues, assuming that the configuration of the impurities is sufficiently random that the tendency toward CDW formation is not enhanced.

A. Expansion of anomalous green function; disorder average

For the most part, we shall consider impurities that produce scalar potential scattering; *mutatis mutandis*, the effects of exchange and spin-orbit scattering can be straightforwardly included. To begin with, we include potential scattering via the $V(x)$ term in the Gor'kov equations,¹⁴ which now read (here we restore the constants \hbar and c)

$$\left[+ i\hbar\omega_n - \frac{1}{2M} \left(i\hbar\partial_x + \frac{e\Phi}{cL} \right)^2 - V(x) + \mu \right] G(x, x'; \omega_n) + \Delta(x)F^\dagger(x, x'; \omega_n) = \hbar\delta(x - x'), \quad (23a)$$

$$\left[-i\hbar\omega_n - \frac{1}{2M} \left(i\hbar\partial_x - \frac{e\Phi}{cL} \right)^2 - V(x) + \mu \right] F^\dagger(x, x'; \omega_n) - \Delta^*(x)G(x, x'; \omega_n) = 0, \quad (23b)$$

where $V(x)$ represents the potential from static impurities, i.e.,

$$V(x) = \sum_a u(x - x_a), \quad (24)$$

and x_a indicates the spatial location of a th impurity.

Following Gor'kov,¹⁴ we introduce a Green function $G^0(x, x'; \omega_n)$ that satisfies

$$\left[i\hbar\omega_n - \frac{1}{2M} \left(i\hbar\partial_x + \frac{e\Phi}{cL} \right)^2 - V(x) + \mu \right] G^0(x, x'; \omega_n) = \hbar \delta(x - x'). \quad (25)$$

We can then express F^\dagger exactly in terms of Δ^* and G^0 as

$$F^\dagger(x_1, x_2; \omega_n) = \frac{1}{\hbar} \int dx G^0(x, x_1; \omega_n) \Delta^*(x) G(x, x_2; -\omega_n). \quad (26)$$

Because we are only concerned with solving for T_c , near the transition, it is sufficient to keep G to zeroth order in Δ and replace G in Eq. (26) by G^0 ; thus we have

$$F^\dagger(x_1, x_2; \omega_n) \approx \frac{1}{\hbar} \int dx G^0(x, x_1; \omega_n) \Delta^*(x) G^0(x, x_2; -\omega_n). \quad (27)$$

We now consider the self-consistency [Eq. (2)] and average over the quenched disorder associated with the locations of the impurities:

$$\overline{\Delta^*(r)} = \frac{V}{\beta} \sum_{\omega_n} \int dx \overline{G^0(x, r; \omega_n) G^0(x, r; -\omega_n) \Delta^*(x)}, \quad (28)$$

where $\overline{\cdots}$ indicates disorder averaging. We note that, as explained by Gor'kov,¹⁴ the Green function G^0 oscillates on a much smaller length scale than Δ^* and, hence, the disorder average of Δ^* can be factorized. For convenience, we use \tilde{G} to denote the disorder average of G^0 , i.e., $\tilde{G}(x, r; \omega_n) \equiv \overline{G^0(x, r; \omega_n)}$, the translational invariance of which is restored, viz.,

$$\tilde{G}(x, r; \omega_n) = \frac{1}{L} \sum_{n_1} \tilde{G}(n_1; \omega_n) e^{i(2\pi n_1/L)(x-r)}. \quad (29)$$

To calculate the disorder average of the product of two Green functions, we introduce the kernel K ,¹⁴ defined via

$$\begin{aligned} & \overline{G^0(x, r; \omega_n) G^0(x, r'; -\omega_n)} \\ & \equiv \frac{1}{L^2} \sum_{n_1, n_2 \in \mathbb{Z}} K_{\omega_n}(n_1, n_2) e^{i(2\pi n_1/L)(x-r)} e^{i(2\pi n_2/L)(x-r')}. \end{aligned} \quad (30)$$

If we retain only the ladder diagrams (i.e., ignoring the crossed diagrams²²), we arrive at the result

$$K_{\omega}(n_1, n_2) = \tilde{G}_{\omega}(n_1) \tilde{G}_{-\omega}(n_2) \left[1 + n_{\text{imp}} \sum_q |u(q)|^2 K_{\omega}(n_1 - q, n_2 + q) \right], \quad (31)$$

where we have simplified the notation by dropping the subscript n on ω and moving ω from an argument to a subscript, and n_{imp} is the impurity concentration. Assuming that the order parameter retains the form $\overline{\Delta^*(r)} = \Delta_0^* e^{-i2\pi m r/L}$, the self-consistency equation can be reduced to

$$1 = \frac{V}{\beta L} \sum_{\omega} \sum_{n_1, n_2} \delta_{n_1+n_2, m} K_{\omega}(n_1, n_2). \quad (32)$$

B. Critical temperature

We leave the detailed calculation of the kernel to Appendix B 4 and simply quote here the resulting Eq. (B48) for the critical temperature in the disordered regime, i.e., $\tau_0 T_c \ll 1$:

$$\begin{aligned} \ln t = & \psi\left(\frac{1}{2}\right) - \psi\left[\frac{1}{2} + \frac{\Gamma l_e \xi_0 x_m^2(\phi)}{tR^2}\right] + \sum_{k=1}^{\infty} 4e^{-\pi k R/l_e} \cos(2\pi k \bar{n}) \\ & \times \{e^{-(2\pi k/\Gamma)(R/\xi_0)t} \cos(2\pi k \phi) f_d[x_m(\phi), R, t(\phi)] \\ & - e^{-(2\pi k/\Gamma)(R/\xi_0)t} f_d(0, R, 1)\}, \end{aligned} \quad (33)$$

where $l_e \equiv v_F \tau_0$ is the elastic mean-free path, τ_0 is the elastic scattering time, we recall that $\Gamma = \pi/e^\gamma$, and

$$\begin{aligned} & f_d[x_m(\phi), R, t(\phi)] \\ & \equiv {}_2F_1\left[\frac{1}{2} + \frac{\Gamma l_e \xi_0 x_m^2}{tR^2}, 1, \frac{3}{2} + \frac{\Gamma l_e \xi_0 x_m^2}{tR^2}, e^{-(4\pi k/\Gamma)(R/\xi_0)t}\right] \Big/ \left(1 + \frac{2\Gamma l_e \xi_0 x_m^2}{tR^2}\right). \end{aligned} \quad (34)$$

We remark that the argument in the second digamma function in Eq. (33) is real, in contrast with the clean case, for which it is complex [see Eq. (10)]. A consequence of this is that there is no longer a doublet of solutions for T_c . Moreover, the resulting single solution is consistent with the assumption that the order parameter becomes vanishingly small as the temperature approaches its critical value from the superconducting side. Therefore, the transition to the superconducting state is continuous.

To explore the consequences of Eq. (33), we first examine the large-radius (i.e., bulk) limit and show that we recover the de Gennes results for the case of rings. We then proceed to compare how the finiteness of the radius affects the oscillations of $T_c(\phi)$.

1. Large-radius limit

Ignoring correction due to the finiteness of the radius, we obtain the following equation for T_c of the bulk superconductor, i.e.,

$$\ln \left[\frac{T_c(\phi)}{T_c^0} \right] = \ln t(\phi) = \psi \left(\frac{1}{2} \right) - \psi \left[\frac{1}{2} + \frac{\Gamma l_e \xi_0 x_m^2(\phi)}{t(\phi) R^2} \right], \quad (35)$$

as we should. This is in agreement with the results of de Gennes⁵ for rings and those obtained by Lopatin *et al.*²³ for hollow cylinders. To see that the de Gennes results are recovered, we note that the critical value of $\Gamma l_e \xi_0 x_m^2(\phi)/R^2$, beyond which no superconducting solution for any $T > 0$ exists, is given by $\Gamma/4\pi$.²⁴ For a fixed radius, this defines the critical flux Φ_c , which defines the boundary between normal and superconducting states. We then determine that the critical flux $\phi_c \equiv \Phi_c/(h/e)$ satisfies

$$\frac{2R\sqrt{\pi}}{\sqrt{\xi_0 l_e}} = 2\pi \min_{m \in \mathbb{Z}} |2\phi_c - m|, \quad (36)$$

which gives the critical flux for a given radius, or vice versa, via

$$\cos \left(\frac{R}{\sqrt{\xi_0 l_e/2\sqrt{\pi}}} \right) = \cos \left(\frac{2\pi\Phi_c}{h/2e} \right). \quad (37)$$

When the flux dependence of the critical temperature is weak, as in the large-radius limit, $t \equiv 1 - T_c/T_c^0$ is close to unity and $\Gamma l_e \xi_0 x_m^2(\phi)/R^2 \ll 1$ so one can expand the logarithm in $(1-t)$, and the second digamma function around $1/2$ in Eq. (35) to obtain

$$(1-t) \approx \frac{\pi^2 \Gamma l_e \xi_0}{2 R^2} \left(\phi - \frac{m}{2} \right)^2. \quad (38)$$

As we did for Eq. (18) in the clean limit, we can express this equation as

$$\cos \left[2\pi \frac{R}{\xi(t)} \right] \approx \cos \left(2\pi \frac{\Phi}{h/2e} \right), \quad (39)$$

where $\xi(t) \approx 0.84\Gamma\sqrt{\xi_0 l_e} \sqrt{1-t}$ is the temperature-dependent Ginzburg-Landau coherence length in the dirty limit. Thus, we recover the de Gennes results⁵ but via a microscopic calculation. We note that, strictly speaking, the Ginzburg-Landau approach is only valid near $t \approx 1$ (see also discussion at the end of Sec. II B 1).

2. Finite-radius regime

Strictly speaking, the de Gennes results [Eq. (39)] are only valid in the large-radius limit, as is clearly seen from our microscopic derivation. In the regime in which the radius of the ring is comparable to the zero-temperature coherence length or the Cooper-pair size, we should take into account the effect of finite radius and use Eq. (33) instead of Eq. (35). In addition to the length scale defined by the Cooper-pair size ξ_0 , the behavior of T_c also depends on another length scale, viz., the mean-free path l_e . The correction terms (i.e., the terms in the summation) in Eq. (33) decay exponentially with R/l_e so that the series converges rather rapidly, even for $R \sim l_e$ or slightly smaller. In Fig. 7 we contrast the predictions of T_c in the large-radius limit [Eq. (35)] to those that include the finiteness corrections [Eq. (33)] for a small radius

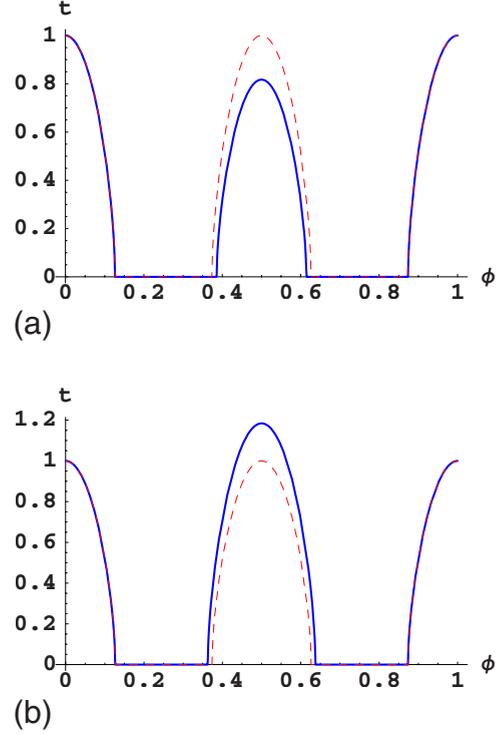


FIG. 7. (Color online) Critical temperature (normalized to its zero-flux value) for the disordered regime vs flux $\phi = \Phi/(h/e)$ for $R = 0.2\xi_0$ and the mean-free path $l_e = 0.2\xi_0$ (solid blue line), and for the same radius but $l_e \ll R$ (dashed red line). The plots are made with $\cos(2\pi k\bar{n}) = 1$ and $(-1)^k$ for the upper and lower panels, respectively. Moreover, only the $k = 0, 1$, and 2 terms in Eq. (33) have been retained as the convergence is rather good.

($R \sim 0.2\xi_0, l_e$). Figure 7 also shows that the $T_c(\phi)$ oscillation has a component of period h/e , in addition to the usual Little-Parks component of period $h/2e$. In Fig. 8 we plot the critical temperature vs radius for various flux values. We see that the deviation from the bulk result is more significant near $\phi = 1/2$, whereas the bulk result shows almost no deviation in the range $\phi = 0 - 0.25$.

The deviation of the critical temperature of small-radius rings from that of large-radius rings is, however, not universal as it depends on both the magnitude and sign of $\cos(2\pi\bar{n}) = \cos(2\pi\sqrt{2MR^2\mu})$. We have seen that when the sign of $\cos(2\pi\sqrt{2MR^2\mu})$ is positive, the corrections from the finiteness of the radius can cause a reduction in the critical temperature near the flux value $\Phi = h/2e$, compared to the bulk case. This corresponds to the case in which, at zero flux, all the electrons are paired up. However, when the sign of the cosine is negative, the critical temperature at $\Phi = h/2e$ can actually exceed its value at zero flux; this corresponds to the case in which, at $\Phi = h/2e$, all electrons are paired up, thus, the system is more stable there than at $\Phi = 0$. If the cosine term should happen to vanish accidentally, the oscillation at period h/e would disappear.

When the radius is not small, the change in $T_c(\phi)$, relative to $T_c(0)$, is small and we therefore need only retain terms in Eq. (33) to order $k = 1$, and also may replace t on the r.h.s. by 1, as we did to obtain Eq. (38). Thus we obtain

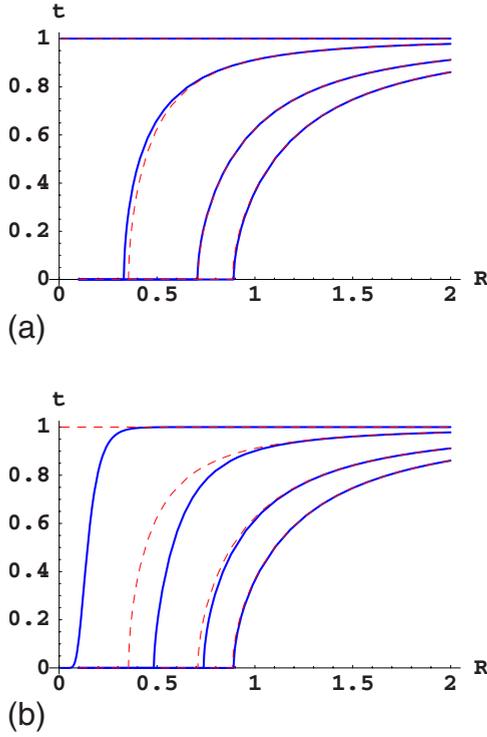


FIG. 8. (Color online) Phase boundaries between normal state (to the left of the curves) and the superconducting state in the temperature [$t(\phi, R) = T_c(\phi, R)/T_c(\phi=0, R)$, normalized to its zero-flux value] vs radius (R , in unit of ξ_0) plane. Upper panel: $\phi = \Phi/(h/e) = 0, 0.1, 0.2, 0.25$ (from the top down). The solid blue boundaries take into account the finite-radius corrections, whereas the dashed red boundaries are solutions to the bulk equation [Eq. (35)]. Lower panel: $\phi = \Phi/(h/e) = 0.5, 0.40, 0.30, 0.25$ (from the top down). In all plots, we have chosen $l_e = 0.2\xi_0$ and $\cos(2\pi k\bar{n}) = 1$, and have kept terms up to $k=2$ in Eq. (33). Note the significant deviations associated with the finite-radius corrections near $\phi=0.5$.

$$(1-t) \approx \frac{\pi^2 \Gamma l_e \xi_0}{2 R^2} \left(\phi - \frac{m}{2} \right)^2 + 4e^{-\pi R/l_e} \cos(2\pi\bar{n}) \tanh^{-1}(e^{-(2\pi\Gamma)(R/l_e)}) (1 - \cos 2\pi\phi). \quad (40)$$

Hence, an oscillation of period h/e is clearly seen to emerge, and whether the critical temperature reduces or increases (e.g., at $\Phi = h/2e$), compared to the bulk Little-Parks value, is seen to depend on the sign of the cosine term. For the case where all electrons are paired up at zero flux, the critical temperature is lower at $\Phi = h/2e$ than at $\Phi = 0$.

IV. CONCLUDING REMARKS

We have considered the oscillations in the critical temperature of a superconducting ring of finite radius in the presence of a threading magnetic flux. We have found that, as the radius of the ring is (parametrically) reduced, an oscillation in the critical temperature of period of h/e emerges, in addition to the usual Little-Parks dependence (the period of which is $h/2e$). Our results provide corrections, due to the

finiteness of the ring radius, to the results that de Gennes obtained for a flux-threaded ring.⁵ We have argued that in the clean limit there is a superconductor-normal transition, as the ring radius becomes sufficiently small at nonzero flux, and that the transition can be either continuous or discontinuous, depending on the radius and/or flux. In the disordered regime, we have argued that the transition is rendered continuous, which results in a quantum critical point tuned by flux and radius.

One may wonder how the system behaves as it goes from clean to dirty limit. At which point does the existence of multiple solutions in the critical temperature disappear? By analyzing Eq. (B39), we obtain that, ignoring the finiteness corrections, double solutions disappear when the disorder is such that $l_e/\xi_0 \lesssim 1.73$.

One question we should also address is the thickness d of the ring cross section. It causes an orbital pair-breaking effect. For the purpose of estimating this, we can use the result from a wire with same thickness in a field perpendicular to the wire axis (for the calculation, see, e.g., Ref. 2). The fractional decrease in critical temperature, when it is small, can be estimated to be (not including the oscillation by flux)

$$1-t(\phi) \approx \frac{\pi^2 \Gamma}{24} \phi^2 \frac{\xi_0 l_e^2 d^2}{l_e R^4}, \quad (41)$$

which causes a quadratic decrease in the critical temperature on top of the oscillations discussed in the present paper. If we take the same values shown in Fig. 7, i.e., $R \approx l_e \approx 0.2\xi_0$, for $d \approx 0.2R$, the decrease is about 3.6% at $\phi=1/2$ and about 14.4% at $\phi=1$. The largest decrease in the critical temperature shown in Fig. 7 is about 20%, and the h/e component is not swamped by the pair-breaking effect and can still be observed.

Although we have obtained the emergence of the h/e oscillation and its amplitude by the microscopic calculations, the physics behind these effects can be illustrated via heuristic arguments associated with the corresponding Cooper problem on a ring together with a path-integral based instanton tunneling approach. We briefly discuss these points in Appendix C.

The nonuniversal factor $\cos(2\pi\bar{n}) = \cos(2\pi\sqrt{2MR^2\mu})$ determines whether the finiteness of the ring radius leads to a decrease or an increase in the critical temperature near flux $\Phi = h/2e$. Although this factor is model dependent (i.e., dependent on the form of the single-particle spectrum) for the quadratic spectrum we consider here, $\sqrt{2MR^2\mu}$ roughly counts the number of electrons divided by four as there are two spin species, and positive and negative (angular) momenta. If we restrict ourselves to the case in which all electrons are paired, then $\bar{n} = \sqrt{2MR^2\mu} \approx (N_{\text{pair}} - 1)/2$, because for $\mu=0$ two electrons can still occupy the $n=0$ state. When all the levels at μ and below are filled at zero flux, $\cos(2\pi\sqrt{2MR^2\mu}) \approx 1$. When there is one pair fewer or more (or equivalently, when all pairs are occupied at $\Phi = h/2e$), $\cos(2\pi\sqrt{2MR^2\mu}) \approx -1$. This seems to result in an even-odd effect not from the number of electrons²⁵ but from the number of Cooper pairs. However, whether this even-odd effect holds in general requires further investigation.

Notes added in proof. We would like to point out that the issue of flux-dependent supercurrents for s -wave rings was also studied by Zhu and Wang in Ref. 28. In addition to Refs. 10–12, there is a very recent work on the same issue in d -wave superconductors in Ref. 29.

Throughout our paper, we have essentially assumed that the switching of pairing configuration (see Fig. 2) occurs at flux values being an odd integer multiple of $h/4e$ near superconducting-normal transition. This is consistent with our numerical results that by allowing the switching to be varied, the largest possible critical temperature is obtained when the above assumption is obeyed. In a very recent paper by Vakaryuk in Ref. 30, the author concludes that the switching can be flux dependent at zero temperature. If the two results are to be consistent, we are led to the conclusion that the switching is also temperature dependent. However, this requires further investigation.

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APPENDIX A: AVERAGED GOR'KOV EQUATIONS OVER RADIAL DIRECTION

Consider a ring on a plane with inner and outer radii a and A , respectively. Assume that the thickness $(A-a)$ is much smaller than the mean radius $(A+a)/2$ and the zero-temperature coherence length $\xi(0)$. We reduce the two-dimensional Gor'kov equations into effective one-dimensional equations by defining the averaged Green function,

$$\tilde{G}(\theta, \theta') \equiv \frac{2}{A^2 - a^2} \int_a^A d\rho \rho \int_a^A d\rho' \rho' G(\rho, \theta, \rho', \theta'; \omega), \quad (\text{A1})$$

and similarly for \tilde{F}^\dagger . Under this averaging, the two-dimensional delta function becomes one-dimensional,

$$\sum_{k \in \mathbb{Z}} \int_{-\infty}^{\infty} dx \frac{e^{-i\pi mk} e^{i2\pi xk}}{\left[i\omega_n - \frac{1}{2MR^2}(x-x_0)^2 + \mu \right] \left[-i\omega_n - \frac{1}{2MR^2}(x+x_0)^2 + \mu \right]}, \quad (\text{B1})$$

where $x_0 \equiv \phi - m/2$. The integral over x can be performed using contour integration. For example, for $k \geq 0$ and $\omega_n > 0$, we can close the contour in the upper plane, and evalu-

$$\frac{2}{A^2 - a^2} \int_a^A d\rho \rho \int_a^A d\rho' \rho' \delta(\vec{r} - \vec{r}') = \delta(\theta - \theta'). \quad (\text{A2})$$

In the Laplacian operator, there is a term $\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} G(\rho, \rho')$. When performing the above average, we get for this term

$$\begin{aligned} & \frac{2}{A^2 - a^2} \int_a^A d\rho \rho \int_a^A d\rho' \rho' \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} G(\rho, \theta, \rho', \theta'; \omega) \\ &= \frac{2}{A^2 - a^2} \int_a^A d\rho' \rho' \rho \left. \frac{\partial}{\partial \rho} G \right|_{\rho=a}^{\rho=A}, \end{aligned} \quad (\text{A3})$$

which, under the condition of no current flowing radially through the ring boundary, gives zero identically. Similarly, if we regard the anomalous Green function as playing the role of the Ginzburg-Landau order parameter in the theory of superconductivity, no flow of supercurrent radially gives the average to be zero for F^\dagger . Furthermore, under the limit that the average of the product of two functions can be approximated by the product of two averaged functions, we obtain a set of reduced one-dimensional Gor'kov equations with the coordinate variables being azimuthal angles θ and θ' , and the radii being set to the fixed value of the average radius. After appropriately renormalizing the Green functions by the inverse of the average radius (thus making the dimension consistent), we arrive at Eq. (3a) and (3b). We remark that even if the ring has the geometry of a torus, the same averaged 1D equations will result provided that the cross section is relatively small, compared with the ring area and the coherence length.

APPENDIX B: CRITICAL TEMPERATURE EQUATIONS: SUPPLEMENTARY DETAILS

In this appendix, we supplement the details leading to the critical-temperature equations in both clean limit and the disordered regime.

1. Clean limit

Continuing from Eq. (9), we make a shift in x : $x \rightarrow x - m/2$, under which the integral becomes

ate the residues at $x = x_0 + \sqrt{2MR^2(\mu + i\omega_n)}$ and $x = -x_0 - \sqrt{2MR^2(\mu - i\omega_n)}$. We end up with (noting the cancellation of the factor $e^{-i\pi mk}$)

$$\begin{aligned}
& \frac{\pi e^{i2\pi k\phi + i2\pi k\sqrt{2MR^2(\mu + i\omega_n)}}}{\sqrt{\frac{2}{MR^2}(\mu + i\omega_n) \left[\omega_n + \frac{ix_0^2}{MR^2} - ix_0\sqrt{\frac{2}{MR^2}(\mu + i\omega_n)} \right]}} \\
& + \frac{\pi e^{-i2\pi k\phi - i2\pi k\sqrt{2MR^2(\mu - i\omega_n)}}}{\sqrt{\frac{2}{MR^2}(\mu - i\omega_n) \left[\omega_n - \frac{ix_0^2}{MR^2} + ix_0\sqrt{\frac{2}{MR^2}(\mu - i\omega_n)} \right]}}. \quad (\text{B2})
\end{aligned}$$

Including the contributions from the three other cases, we arrive at the self-consistency equation:

$$\begin{aligned}
1 = & \frac{|V|T}{2\pi R} \text{Re} \sum_{\omega_n > 0}^{\omega_D} \\
& \times \left\{ \frac{4\pi}{\sqrt{\frac{2}{MR^2}(\mu + i\omega_n) \left[\omega_n + \frac{ix_0^2}{MR^2} - ix_0\sqrt{\frac{2}{MR^2}(\mu + i\omega_n)} \right]}} \right. \\
& \left. + \sum_{k=1}^{\infty} \frac{8\pi e^{i2\pi k\sqrt{2MR^2(\mu + i\omega_n)}} \cos 2\pi k\phi}{\sqrt{\frac{2}{MR^2}(\mu + i\omega_n) \left[\omega_n + \frac{ix_0^2}{MR^2} - ix_0\sqrt{\frac{2}{MR^2}(\mu + i\omega_n)} \right]}} \right\}, \quad (\text{B3})
\end{aligned}$$

where we have put the Debye frequency ω_D as the upper cutoff in the Matsubara sum. This equation can be rewritten as

$$\begin{aligned}
1 = & \sqrt{\frac{2M}{\mu}} |V|T \text{Re} \sum_{\omega_n > 0}^{\omega_D} \\
& \times \left\{ \frac{1}{\sqrt{1 + i\frac{\omega_n}{\mu} \left[\omega_n + \frac{ix_0^2}{MR^2} - ix_0\sqrt{\frac{2}{MR^2}(\mu + i\omega_n)} \right]}} \right. \\
& \left. + \sum_{k=1}^{\infty} \frac{2e^{i2\pi k\sqrt{2MR^2(\mu + i\omega_n)}} \cos 2\pi k\phi}{\sqrt{1 + i\frac{\omega_n}{\mu} \left[\omega_n + \frac{ix_0^2}{MR^2} - ix_0\sqrt{\frac{2}{MR^2}(\mu + i\omega_n)} \right]}} \right\}. \quad (\text{B4})
\end{aligned}$$

Furthermore, for typical temperatures we have $\sqrt{\mu + i\omega_n} \approx \sqrt{\mu}(1 + i\omega_n/2\mu)$. The second term causes the contribution of the $k \neq 0$ terms to be exponentially small for large R , i.e., there is a factor,

$$e^{-2\pi kR\sqrt{2M}\omega_n/2\mu}. \quad (\text{B5})$$

2. Large-radius limit

For sufficiently large R , we can ignore the $k \neq 0$ correction terms, in which case, we have

$$\begin{aligned}
1 = & \sqrt{\frac{2M}{\mu}} |V|T \text{Re} \sum_{\omega_n > 0}^{\omega_D} \\
& \times \left\{ \frac{1}{\sqrt{1 + i\frac{\omega_n}{\mu} \left[\omega_n + \frac{ix_0^2}{MR^2} - ix_0\sqrt{\frac{2}{MR^2}(\mu + i\omega_n)} \right]}} \right\}. \quad (\text{B6})
\end{aligned}$$

The third term in the denominator is usually much larger than the second term, as the chemical potential μ is much larger than the level spacing $1/2MR^2$, i.e., $\sqrt{2MR^2}\mu \gg 1$. For

typical values of ω_D and μ , we have $\omega_D/\mu \ll 1$. Hence, we have

$$1 \approx \sqrt{\frac{2M}{\mu}} |V|T \text{Re} \sum_{\omega_n > 0}^{\omega_D} \frac{1}{\omega_n - ix_0\sqrt{\frac{2\mu}{MR^2}}} \quad (\text{B7})$$

$$= \sqrt{\frac{2M}{\mu}} \frac{|V|}{2\pi} \text{Re} \sum_{n=0}^{\omega_D/2\pi T} \frac{1}{\left(n + \frac{1}{2}\right) - \frac{ix_0}{2\pi T}\sqrt{\frac{2\mu}{MR^2}}}. \quad (\text{B8})$$

The solution for T to this equation gives the critical temperature T_c . Denoting by T_c^0 the critical temperature in the absence of flux (so that $x_0=0$), we have the corresponding equation

$$1 \approx \sqrt{\frac{2M}{\mu}} \frac{|V|}{2\pi} \sum_{n=0}^{\omega_D/2\pi T_c^0} \frac{1}{n + \frac{1}{2}}. \quad (\text{B9})$$

Taking the difference of Eqs. (B7) and (B9), we have

$$0 = \text{Re} \sum_{n=0}^{\omega_D/2\pi T_c(\phi)} \frac{1}{\left(n + \frac{1}{2}\right) - \frac{ix_0}{2\pi T_c(\phi)}\sqrt{\frac{2\mu}{MR^2}}} - \sum_{n=0}^{\omega_D/2\pi T_c^0} \frac{1}{n + \frac{1}{2}}. \quad (\text{B10})$$

If we extend both upper limits to infinity, we should compensate by the difference (assuming $\omega_D/T \gg 1$), i.e.,

$$\sum_{[\omega_D/2\pi T_c(\phi)+1]^{(\omega_D/2\pi T_c^0)+1}} \frac{1}{n + \frac{1}{2}} \approx \ln \left[\frac{T_c(\phi)}{T_c^0} \right]. \quad (\text{B11})$$

Therefore, we arrive at

$$0 = \text{Re} \sum_{n=0}^{\infty} \left[\frac{1}{\left(n + \frac{1}{2}\right) - \frac{ix_0}{2\pi T_c(\phi)}\sqrt{\frac{2\mu}{MR^2}}} - \frac{1}{n + \frac{1}{2}} \right] + \ln \left[\frac{T_c(\phi)}{T_c^0} \right]. \quad (\text{B12})$$

In terms of the digamma function $\psi(x)$ in Eq. (11), we have an implicit formula for $T_c(\phi)$:

$$\ln \frac{T_c(\phi)}{T_c^0} = \psi \left(\frac{1}{2} \right) - \text{Re} \psi \left[\frac{1}{2} - \frac{ix_0}{2\pi T_c(\phi)} \sqrt{\frac{2\mu}{MR^2}} \right]. \quad (\text{B13})$$

Using the quantities $\mu \approx v_F^2/2M$, $\xi_0 = v_F/\pi\Delta_0$, and $\rho \approx R/\xi_0$, and defining $t(\phi) \equiv T_c(\phi)/T_c^0$, we can rewrite Eq. (B13) as

$$\ln t(\phi) = \psi \left(\frac{1}{2} \right) - \text{Re} \psi \left[\frac{1}{2} - \frac{ix_0\Gamma}{2\rho t(\phi)} \right]. \quad (\text{B14})$$

3. Finiteness correction

What is the correction to $T_c(\phi)$ due to the finiteness of the radius? To address this question we need to take into account the $k \neq 0$ corrections to Eq. (B4). If we take $\omega_D/\mu \ll 1$ and $\sqrt{\mu + i\omega_n} \approx \sqrt{\mu}(1 + i\omega_n/2\mu)$, the self-consistency [Eq. (B4)] can be approximated as

$$1 \approx \sqrt{\frac{2M}{\mu}} \frac{|V|}{2\pi} \operatorname{Re} \sum_{n=0}^{\omega_D/2\pi T} \frac{1}{\left(n + \frac{1}{2}\right) - \frac{ix_0\Gamma T_c^0}{2\rho T}} \left[1 + \sum_{k=1}^{\infty} 2e^{i2\pi k\sqrt{2MR^2(\mu+i\omega_n)}} \cos 2\pi k\phi \right] \quad (\text{B15})$$

$$\approx \sqrt{\frac{2M}{\mu}} \frac{|V|}{2\pi} \operatorname{Re} \sum_{n=0}^{\omega_D/2\pi T} \frac{1}{\left(n + \frac{1}{2}\right) - \frac{ix_0\Gamma T_c^0}{2\rho T}} \left[1 + \sum_{k=1}^{\infty} 2e^{i2\pi k\sqrt{2MR^2\mu - 4k(n+1/2)(\pi\rho T)/(\Gamma T_c^0)}} \cos 2\pi k\phi \right]. \quad (\text{B16})$$

Taking the difference between this equation and the version corresponding to $\phi=0$, and using the trick for converting the cutoff at the Debye frequency in a logarithm [see Eqs. (B11)–(B13)], we arrive at

$$\begin{aligned} \ln t(\phi) = & \psi\left(\frac{1}{2}\right) - \operatorname{Re} \psi\left(\frac{1}{2} - \frac{ix_0\Gamma}{2\rho t}\right) - 4 \sum_{k=1}^{\infty} \left\{ \cos(2\pi k\sqrt{2MR^2\mu}) e^{-2\pi k\rho/\Gamma} \operatorname{HF}\left[\frac{1}{2}, 1, \frac{3}{2}, e^{-4\pi k\rho/\Gamma}\right] \right. \\ & \left. - \operatorname{Re} \left\{ e^{i2\pi k\sqrt{2MR^2\mu}} e^{-2\pi k\rho/\Gamma} \operatorname{HF}\left[\frac{1}{2} - \frac{ix_0\Gamma}{2\rho t(\phi)}, 1, \frac{3}{2} - \frac{ix_0\Gamma}{2\rho t}, e^{-4\pi k\rho/\Gamma}\right] \cos 2\pi k\phi \left(1 - \frac{ix_0\Gamma}{\rho t}\right) \right\} \right\}, \end{aligned} \quad (\text{B17})$$

where the ϕ dependence of t on r.h.s. is suppressed. We have extended the upper limit of the sum over n to infinity for the exponentially decaying terms (which introduces a negligible small error) and we have used the formula

$$\sum_{n=0}^{\infty} \frac{e^{-b(n+1/2)}}{n + \frac{1}{2} + a} = 2e^{-b/2} \operatorname{HF}\left[\frac{1}{2} + a, 1, \frac{3}{2} + a, e^{-b}\right] / (1 + 2a), \quad (\text{B18})$$

where $\operatorname{HF}[a, b, c, z]$ is the hypergeometric function ${}_2F_1(a, b; c; z)$.

4. Disordered regime

In this subsection, we calculate $\tilde{G}_\omega(n)$ and K_ω , and obtain the r.h.s. of the self-consistency [Eq. (31)]. For simplicity, we assume that the potential $u(r)$ is short ranged so that its Fourier transform $u(q)$ can be treated as a constant, essentially independent of momentum transfer q .

The one-particle self-energy can be obtained from summing one-particle-irreducible diagrams (and ignoring the crossed diagrams) and thus we obtain

$$\Sigma(\omega) = \sum_{n_1} \frac{n_{\text{imp}}|u|^2}{i\omega - \frac{1}{2MR^2}(n_1 - \phi)^2 + \mu} \approx -\frac{i}{2\tau(\omega)} \operatorname{sgn}(\omega), \quad (\text{B19})$$

the real part of the self-energy has been ignored and we shall call $\tau(\omega)$ the frequency-dependent scattering time. Under the condition that the Debye frequency is much smaller than the chemical potential, i.e., $\omega_D/\mu \ll 1$, and hence for the range of ω s that are relevant to superconductivity (i.e., $|\omega| < \omega_D$), we obtain

$$\begin{aligned} \frac{1}{2\tau(\omega)} \approx & \frac{1}{2\tau_0} \left\{ 1 + \sum_{k>0} 2 \cos(2\pi k\phi) \right. \\ & \left. \times \cos[2\pi k\sqrt{2MR^2\mu}] e^{-2\pi k\sqrt{MR^2/2\mu}|\omega|} \right\}, \end{aligned} \quad (\text{B20})$$

where

$$\frac{1}{2\tau_0} \equiv \frac{2\pi n_{\text{imp}}|u|^2}{\sqrt{2\mu/MR^2}}. \quad (\text{B21})$$

This form of $\tau(\omega)$ is an approximant because it is calculated with G^0 rather than \tilde{G} . To improve the approximation we then use the exact disordered Green function \tilde{G} to calculate the same self-energy diagrams again (see, e.g., Ref. 26), thus obtaining a self-consistency condition for $\tau(\omega)$ (that can be solved iteratively):

$$\begin{aligned} \frac{1}{2\tau(\omega)} = & \frac{1}{2\tau_0} \left\{ 1 + \sum_{k>0} 2 \cos(2\pi k\phi) \right. \\ & \left. \times \cos[2\pi k\sqrt{2MR^2\mu}] e^{-2\pi k\sqrt{MR^2/2\mu}|\omega|(1+1/2\tau(\omega)|\omega|)} \right\}. \end{aligned} \quad (\text{B22})$$

As the $k>0$ terms are exponentially small, we can approximate $\tau(\omega)$ on the right-hand side by τ_0 to arrive at

$$\begin{aligned} \frac{1}{2\tau(\omega)} \approx & \frac{1}{2\tau_0} \left\{ 1 + \sum_{k>0} 2 \cos(2\pi k\phi) \right. \\ & \left. \times \cos[2\pi k\sqrt{2MR^2\mu}] e^{-2\pi k\sqrt{MR^2/2\mu}|\omega|(1+1/2\tau_0|\omega|)} \right\}. \end{aligned} \quad (\text{B23})$$

Returning to the kernel K , we rewrite Eq. (31), following Gor'kov,¹⁴ as

$$K_\omega(n_1, n_2) = G_\omega(n_1) G_{-\omega}(n_2) [1 + \mathcal{L}_\omega(n_1 + n_2)], \quad (\text{B24})$$

where we have conveniently dropped the $\tilde{}$ sign in \tilde{G} , and

$$\mathcal{L}_\omega(n_1 + n_2) \equiv n_{\text{imp}} \sum_q |u(q)|^2 K_\omega(n_1 - q, n_2 + q), \quad (\text{B25})$$

and the argument $n_1 + n_2$ in \mathcal{L} reflects the conservation of total (azimuthal) momentum of the two incoming (and outgoing) electrons in the ladder diagrams after disorder averaging. For the purpose of evaluating self-consistency [Eq.

(32), our goal is to obtain K_ω for $n_2 = -n_1 + m$. By eliminating K from Eqs. (B24) and (B25), we have an equation for \mathcal{L}_ω :

$$\mathcal{L}_\omega(m) = n_{\text{imp}} \sum_{n'_1} |u|^2 G_\omega(n'_1) G_{-\omega}(-n'_1 + m) [1 + \mathcal{L}_\omega(m)], \quad (\text{B26})$$

which gives

$$\mathcal{L}_\omega(m) = \frac{A_\omega(m)}{1 - A_\omega(m)}, \quad (\text{B27})$$

where

$$A_\omega(m) \equiv n_{\text{imp}} \sum_{n'_1} |u|^2 G_\omega(n'_1) G_{-\omega}(-n'_1 + m). \quad (\text{B28})$$

The self-consistency [Eq. (32)] then becomes

$$1 = \frac{V}{\beta L} \sum_\omega \frac{1}{n_{\text{imp}} |u|^2} \frac{A_\omega(m)}{1 - A_\omega(m)} = \frac{V}{\beta L} \frac{4\pi\tau_0}{\sqrt{2\mu/MR^2}} \sum_\omega \frac{A_\omega(m)}{1 - A_\omega(m)}, \quad (\text{B29})$$

where we have used Eq. (B21) in the second equality. As with the evaluation of Eq. (B1) in the clean limit (see Appendix B), we obtain A_ω as

$$A_\omega = A_\omega^0 + A_\omega^{k>0} = \frac{1}{2\tau} \text{Re} \left[\frac{1}{|\omega|\eta - iX_0} \right] + \sum_{k>0} \frac{e^{-2\pi k \sqrt{MR^2/2\mu} |\omega| \eta}}{\tau} \cos(2\pi k \sqrt{2MR^2\mu}) \text{Re} \left[\frac{e^{i2\pi k \phi}}{|\omega|\eta - iX_0} \right], \quad (\text{B30})$$

where $X_0 \equiv (\phi - m/2) \sqrt{\frac{2\mu}{MR^2}}$, $\eta \equiv (1 + 2\tau|\omega|)/(2\tau|\omega|)$, and τ is a shorthand for $\tau(\omega)$.

To illustrate the corrections arising from the finiteness of the radius, we rewrite A_ω as

$$A_\omega = \frac{(1 + 2\tau|\omega|)(1 + a_\omega) - 2\tau b_\omega X_0}{(1 + 2\tau|\omega|)^2 + (2\tau X_0)^2}, \quad (\text{B31})$$

where $a_\omega (\ll 1)$ and $b_\omega (\ll 1)$ are

$$a_\omega \equiv \sum_{k>0} 2e^{-2\pi k \sqrt{MR^2/2\mu} |\omega| \eta} \cos(2\pi k \sqrt{2MR^2\mu}) \cos(2\pi k \phi), \quad (\text{B32a})$$

$$b_\omega \equiv \sum_{k>0} 2e^{-2\pi k \sqrt{MR^2/2\mu} |\omega| \eta} \cos(2\pi k \sqrt{2MR^2\mu}) \sin(2\pi k \phi). \quad (\text{B32b})$$

As the correction to $1/\tau(\omega)$ is exponentially small [see Eq. (B23)], in the exponentials, η can be approximated by

$$\eta_0 \equiv 1 + \frac{1}{2\tau_0|\omega|}. \quad (\text{B33})$$

On the other hand, for $1/\tau(\omega)$, not in the exponentials, we shall approximate it by retaining the leading correction [see Eqs. (B23)] via

$$\frac{1}{2\tau(\omega)} \approx \frac{1}{2\tau_0} (1 + a_\omega). \quad (\text{B34})$$

The quotient $A_\omega/(1 - A_\omega)$ then becomes

$$\frac{A_\omega}{1 - A_\omega} = \frac{1 + \left[a_\omega - \frac{2\tau b_\omega X_0}{1 + 2\tau|\omega|} \right]}{2\tau|\omega| + \frac{1}{1 + 2\tau|\omega|} (2\tau X_0)^2 - \left[a_\omega - \frac{2\tau b_\omega X_0}{1 + 2\tau|\omega|} \right]} \approx \frac{1 + \left[a_\omega - \frac{2\tau b_\omega X_0}{1 + 2\tau|\omega|} \right]}{2\tau|\omega| + \frac{1}{1 + 2\tau|\omega|} (2\tau X_0)^2}. \quad (\text{B35})$$

We remark that, from this equation, one can obtain the equation for the critical temperature for arbitrary mean-free path l_e .

a. Large-radius limit and arbitrary disorder

To illustrate the remark made above, we consider large-radius limit in which we ignore corrections due to the finite radius. In this limit Eq. (B35) becomes

$$\frac{A_\omega}{1 - A_\omega} = \frac{1}{2\tau_0|\omega| + \frac{1}{1 + 2\tau_0|\omega|} (2\tau_0 X_0)^2}. \quad (\text{B36})$$

Substituting this into Eq. (B29) we have

$$1 = \sqrt{\frac{2M}{\mu}} \frac{|V|}{2\pi} \text{Re} \sum_{n=0}^{\omega_D/2\pi T_c} \frac{(n + \frac{1}{2}) + \frac{1}{4\pi\tau_0 T_c}}{(n + \frac{1}{2}) \left[(n + \frac{1}{2}) + \frac{1}{4\pi\tau_0 T_c} \right] + \left(\frac{X_0}{2\pi T_c} \right)^2}. \quad (\text{B37})$$

In the clean limit $\tau_0 T_c \gg 1$, we see that this reduces to Eq. (B8), which we obtained in the absence of disorder. However, here τ_0 is arbitrary. Subtracting Eq. (B37) from the corresponding equation at $\phi=0$ (i.e., $X_0=0$) and using the trick to get logarithm of the ratio of the critical temperatures, as we did in Eqs. (B11)–(B13), we have

$$\ln \left[\frac{T_c(\phi)}{T_c^0} \right] = \sum_{n=0}^{\infty} \left\{ \frac{(n + \frac{1}{2}) + \frac{1}{4\pi\tau_0 T_c}}{(n + \frac{1}{2}) \left[(n + \frac{1}{2}) + \frac{1}{4\pi\tau_0 T_c} \right] + \left(\frac{X_0}{2\pi T_c} \right)^2} - \frac{1}{n + \frac{1}{2}} \right\}. \quad (\text{B38})$$

By making the partial fractions of the first term in the summation and by using the definition of the digamma function [Eq. (11)], we arrive at

$$\ln \left[\frac{T_c(\phi)}{T_c^0} \right] = \psi \left(\frac{1}{2} \right) - \frac{1}{\sqrt{\alpha^2 - \chi^2}} \times \left[\frac{-\alpha + \sqrt{\alpha^2 - \chi^2}}{2} \psi \left(\frac{1 + \alpha + \sqrt{\alpha^2 - \chi^2}}{2} \right) + \frac{\alpha + \sqrt{\alpha^2 - \chi^2}}{2} \psi \left(\frac{1 + \alpha - \sqrt{\alpha^2 - \chi^2}}{2} \right) \right], \quad (\text{B39})$$

where, for convenience, we have defined $\alpha \equiv 1/4\pi\tau_0 T_c(\phi)$ and $\chi \equiv X_0/\pi T_c(\phi)$. This is the equation for the critical tem-

perature at arbitrary disorder for large radii of rings.

b. Finite-radius corrections and strong disordered regime

Now we return to the corrections due to the finiteness of the radius. In the strong disordered limit (i.e., $\tau_0\omega \ll 1$), Eq. (B35) becomes

$$\frac{A_\omega}{1-A_\omega} \approx \frac{1}{2\tau_0[\omega + 2\tau_0 X_0^2]} + \frac{a_\omega - 2\tau_0 b_\omega X_0}{2\tau_0[\omega + 2\tau_0 X_0^2]}. \quad (\text{B40})$$

This is further approximated, using Eq. (B34) as

$$\frac{A_\omega}{1-A_\omega} \approx \frac{1}{2\tau_0[\omega + 2\tau_0 X_0^2]} + \frac{2a_\omega - 2\tau_0 b_\omega X_0}{2\tau_0[\omega + 2\tau_0 X_0^2]}, \quad (\text{B41})$$

where we have used Eq. (B34). The first term on the r.h.s. represents the bulk term and is the Little-Parks term in the dirty limit. The second term takes into account the finite radius and contains flux dependence in period h/e . The equa-

tion for the critical temperature will then contain the digamma function and hypergeometric function with real arguments, in contrast with the clean limit. This makes the transition to normal state continuous, as the solution for the critical temperature is unique and the assumption of vanishing order parameter used in the linearized self-consistency condition is valid. Therefore, there can be quantum phase transitions tuned by flux and/or radius.

As $\tau_0 X_0 \ll 1$ in the strong-disorder limit, the ratio $A_\omega/(1-A_\omega)$ becomes (by ignoring the b term)

$$\frac{A_\omega}{1-A_\omega} \approx \frac{1}{2\tau_0[\omega + 2\tau_0 X_0^2]} + \frac{2a_\omega}{2\tau_0[\omega + 2\tau_0 X_0^2]}. \quad (\text{B42})$$

From this we can obtain an equation for the critical temperature T_c in the presence of flux Φ , as we have done in the clean limit (see Appendix B 3). By inserting Eq. (B42) into the self-consistency [Eq. (B29)], we obtain

$$1 = \sqrt{\frac{2M}{\mu}} \frac{V}{2\pi} \sum_{n=0}^{\omega_D/2\pi T_c} \frac{1 + \sum_{k>0} 2e^{-2\pi k \sqrt{MR^2/2\mu} [1/2\tau_0 + 2\pi T_c(n+1/2)]} \cos(2\pi k \sqrt{2MR^2\mu}) \cos 2\pi k \phi}{n + \frac{1}{2} + \frac{\tau_0 X_0^2}{\pi T_c}}. \quad (\text{B43})$$

Subtracting from this the corresponding $\phi=0$ equation, i.e.,

$$1 = \sqrt{\frac{2M}{\mu}} \frac{V}{2\pi} \sum_{n=0}^{\omega_D/2\pi T_c^0} \frac{1 + \sum_{k>0} 2e^{-2\pi k \sqrt{MR^2/2\mu} [1/2\tau_0 + 2\pi T_c^0(n+1/2)]} \cos(2\pi k \sqrt{2MR^2\mu})}{n + \frac{1}{2}}, \quad (\text{B44})$$

we obtain the following implicit equation for the critical temperature:

$$0 = \sum_{n=0}^{\infty} \left(\frac{1}{n + \frac{1}{2}} - \frac{1}{n + \frac{1}{2} + \frac{\tau_0 X_0^2}{\pi T_c}} \right) - \sum_{n=\omega_D/2\pi T_c^0+1}^{\omega_D/2\pi T_c} \frac{1}{n} + \sum_{k>0} 2e^{-2\pi k \sqrt{MR^2/2\mu} (1/2\tau_0)} \cos(2\pi k \sqrt{2MR^2\mu}) \times \sum_{n=0}^{\infty} \left(\frac{e^{-2\pi k \sqrt{MR^2/2\mu} (2\pi T_c^0)(n+1/2)}}{n + \frac{1}{2}} - \frac{e^{-2\pi k \sqrt{MR^2/2\mu} (2\pi T_c)(n+1/2)} \cos 2\pi k \phi}{n + \frac{1}{2} + \frac{\tau_0 X_0^2}{\pi T_c}} \right), \quad (\text{B45})$$

where we have used the fact that when $n \approx \omega_D/2\pi T_c^0$ or higher, $1/(n + \frac{1}{2} + x) \approx 1/n$, the sum of the series being approximated is a logarithm, and we have extended the upper limit of n for the correction terms to infinity. By using formula (B18) for the hypergeometric function, we finally arrive at

$$\ln\left(\frac{T_c}{T_c^0}\right) = \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \frac{\tau_0 X_0^2}{\pi T_c}\right) + \sum_{k>0} 4e^{-2\pi k \sqrt{MR^2/2\mu} (1/2\tau_0)} \cos(2\pi k \sqrt{2MR^2\mu}) \times \left[e^{-2\pi k \sqrt{MR^2/2\mu} \pi T_c} \cos 2\pi k \phi \text{HF}\left(\frac{1}{2} + \frac{\tau_0 X_0^2}{\pi T_c}, 1, \frac{3}{2}\right) + \frac{\tau_0 X_0^2}{\pi T_c} e^{-2\pi k \sqrt{MR^2/2\mu} (2\pi T_c)} \right] / \left[\left(1 + \frac{2\tau_0 X_0^2}{\pi T_c}\right) - e^{-2\pi k \sqrt{MR^2/2\mu} (\pi T_c^0)} \text{HF}\left(\frac{1}{2}, 1, \frac{3}{2}, e^{-2\pi k \sqrt{MR^2/2\mu} (2\pi T_c^0)}\right) \right], \quad (\text{B46})$$

where we recall that $X_0 \equiv (\phi - m/2) \sqrt{\frac{2\mu}{MR^2}}$. If we use $\mu = Mv_F^2/2$, $l_e = v_F \tau_0$, and $\xi_0 = v_F / \pi \Delta_0$ (with $\Delta_0 = \Gamma T_c^0$), we have

$$\ln t = \psi\left(\frac{1}{2}\right) - \psi\left[\frac{1}{2} + \frac{\Gamma l_e \xi_0}{tR^2} \left(\phi - \frac{m}{2}\right)^2\right] + \sum_{k>0} 4e^{-\pi k R/l_e} \cos(2\pi k \sqrt{2MR^2\mu}) \times \left[e^{-(2\pi k/\Gamma)(R/\xi_0)t} \cos 2\pi k \phi \text{HF}\left[\frac{1}{2} + \frac{\Gamma l_e \xi_0}{tR^2} \left(\phi - \frac{m}{2}\right)^2, 1, \frac{3}{2} + \frac{\Gamma l_e \xi_0}{tR^2} \left(\phi - \frac{m}{2}\right)^2, e^{-(4\pi k/\Gamma)(R/\xi_0)t}\right] \times \left[1 + \frac{2\Gamma l_e \xi_0}{tR^2} \left(\phi - \frac{m}{2}\right)^2 \right]^{-1} - e^{-(2\pi k/\Gamma)(R/\xi_0)t} \text{HF}\left(\frac{1}{2}, 1, \frac{3}{2}, e^{-(4\pi k/\Gamma)(R/\xi_0)t}\right) \right]. \quad (\text{B47})$$

We can rewrite this equation using the relations $\mu = Mv_F^2/2$, $\Delta_0 = v_F/\pi\xi_0$, and $l_e = v_F\tau_0$, and thus arrive at

$$\ln t = \psi\left(\frac{1}{2}\right) - \psi\left[\frac{1}{2} + \frac{\Gamma l_e \xi_0}{tR^2} \left(\phi - \frac{m}{2}\right)^2\right] + \sum_{k=1}^{\infty} 4e^{-\pi k R/l_e} \cos(2\pi k \sqrt{2MR^2\mu}) \times \left[e^{-(2\pi k/\Gamma)(R/\xi_0)t} \cos 2\pi k \phi \frac{\text{HF}\left[\frac{1}{2} + \frac{\Gamma l_e \xi_0}{tR^2} \left(\phi - \frac{m}{2}\right)^2, 1, \frac{3}{2} + \frac{\Gamma l_e \xi_0}{tR^2} \left(\phi - \frac{m}{2}\right)^2, e^{-(4\pi k/\Gamma)(R/\xi_0)t}\right]}{1 + \frac{2\Gamma l_e \xi_0}{tR^2} \left(\phi - \frac{m}{2}\right)^2} - e^{-(2\pi k/\Gamma)(R/\xi_0)t} \text{HF}\left(\frac{1}{2}, 1, \frac{3}{2}, e^{-(4\pi k/\Gamma)(R/\xi_0)t}\right) \right]. \quad (\text{B48})$$

c. Finite-radius and arbitrary disorder

It is also possible to include corrections from the finiteness of the radius and derive the equation for the critical temperature, as we did in the previous section. The results will contain the r.h.s. of Eq. (B39) as the zeroth order, as well as corrections due to the finiteness of the radius in terms of hypergeometric functions. However, the formulas are too cumbersome and we do not list them here.

APPENDIX C: HEURISTIC ARGUMENT TO THE EMERGENCE OF h/e PERIOD OSCILLATIONS

1. Cooper problem on a ring

Let us consider Cooper's problem on a ring with a flux Φ threading through it. The orbital wave function can be written in the same form as in Eq. (4a) and (4b). Assuming that $\psi(x, x) \sim e^{i2\pi m x/L}$, and that the electron-electron interaction is factorizable, i.e., $V_{n,n'} = \Lambda U_n U_{n'}^*$, we can write down the time-independent Schrödinger equation in terms of the wavefunction amplitude a_n as

$$2\epsilon_n(\phi)a_n + \sum_{|n'|>n_F} V_{n,n'}a_{n'} = Ea_n, \quad (\text{C1})$$

where

$$2\epsilon_n(\phi) = \frac{1}{2MR^2}[(n+\phi)^2 + (m+n-\phi)^2], \quad (\text{C2})$$

and hence we arrive at

$$\frac{1}{\Lambda} = \sum_n \frac{1}{E - 2\epsilon_n(\phi)}. \quad (\text{C3})$$

The integer m has to be chosen to minimize the ground-state energy. Thus, we see that the solution is periodic in ϕ with period 1 (or in Φ with period h/e , namely the single-particle

flux quantum) and that the Little-Parks period $h/2e$ is exact only in the large- R limit, in which case the summation over n can be replaced by an integral, and hence, the period of ϕ is $1/2$. This illustrates the role of the single-particle oscillation. However, the amplitudes of the single-particle and Little-Parks oscillations have to be evaluated within a microscopic theory, as has been done in the main text.

2. Flux oscillation: an instanton approach

The instanton picture provides us another heuristic view of the emergence of h/e -period oscillations. The argument presented here is intended just to give some intuition. We refer to Rajaraman²⁷ for a pedagogical review of instanton techniques. A single instanton tunneling from 0 through a potential to 2π (which is identified as 0) on a circle gives an amplitude

$$\lim_{\tau \rightarrow \infty} \langle 2\pi | e^{-H\tau} | 0 \rangle_{(1,0)} = e^{-S_0} J K \tau \omega^{1/2} e^{-\omega\tau^2}, \quad (\text{C4})$$

where S_0 is the classical Euclidean action (in the absence of flux threading through the circle), J is a Jacobian factor, K is a constant independent of τ as $\tau \rightarrow \infty$, and ω is the harmonic-oscillator frequency near the bottom of the trapping potential. Now, if there is a flux Φ threading the ring, the amplitude will also acquire an additional phase factor $e^{i2\pi\phi}$, where $\phi \equiv \Phi/\Phi_0$ and $\Phi_0 = h/e$ if the particle carries charge e . For a charge $2e$ particle, the phase factor would be $e^{i4\pi\phi}$.

We consider a ring size that is large enough that the electrons in a Cooper pair generally traverse the ring together and rarely split so as to wind separately around the ring. The ground-state energy will give a qualitative estimate of the critical temperature of the associated superconductor. The total contribution of the amplitude for the instanton associated with the Cooper pair is then

$$\lim_{\tau \rightarrow \infty} \langle 2\pi | e^{-H\tau} | 0 \rangle = \omega^{1/2} e^{-\omega\tau/2} \sum_{n_1, n_2} \frac{1}{n_1! n_2!} \times (JK\tau e^{-S_0})^{n_1+n_2} e^{i4\pi(n_1-n_2)\phi}, \quad (\text{C5})$$

which gives

$$\lim_{\tau \rightarrow \infty} \langle 2\pi | e^{-H\tau} | 0 \rangle = \omega^{1/2} e^{-\omega\tau/2} \exp(2JK\tau e^{-S_0} \cos 4\pi\phi) \sim \lim_{\tau \rightarrow \infty} \langle 2\pi | E_0 \rangle \langle E_0 | 0 \rangle e^{-E_0\tau}. \quad (\text{C6})$$

This gives a ground-state energy of $E_0 = \frac{\omega}{2} - 2JK e^{-S_0} \cos 4\pi\phi$, which reveals the period $h/2e$, the Little-Parks period in critical temperature. The fact that this gives a dependence on flux being sinusoidal rather than quadratic results from the lack of accounting of the other many-body electrons and the existence of a condensate.

Now, occasionally (in the sense of contributing Feynman paths), the electrons in a Cooper pair separate and circumnavigate the ring (relative to one another) before reassociating. The contribution of such processes to the ground-state energy can be estimated via the instantons and anti-

instantons of such events associated with them. A single instanton involving one electron going from 0 to 2π has the amplitude

$$\lim_{\tau \rightarrow \infty} \langle 2\pi | e^{-H\tau} | 0 \rangle_{(1,0)} = -e^{-S_{0;e}} J_e K_e \tau \omega_e^{1/2} e^{-\omega_e\tau/2} e^{i2\pi\phi}, \quad (\text{C7})$$

where the subscript e indicates a single electron rather than a Cooper pair and the additional minus sign comes from the exchange of the two electrons. Summing all the instanton and anti-instanton processes, we arrive at the contribution to the ground-state energy from the two electrons

$$E_{0;e} = \omega_e + 4J_e K_e e^{-S_{0;e}} \cos 2\pi\phi, \quad (\text{C8})$$

which results in the emergence of a h/e contribution to the period of oscillation to critical temperature. We remark that it is owing to the separation of the length scales, and hence, time scales that we can separate the Cooper-pair and single-electron contributions. The amplitudes of the two oscillations are related to the respective actions and, for large radius, the amplitude corresponding to single-particle oscillation is expected to be small, due to the binding resulting from the attractive interparticle interaction.

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the radial direction and ignoring the fluctuations (see Appendix A). In our discussions we require that the mean average level spacing $\hbar v_F/d$ in the transverse direction is much greater than the superconducting gap Δ_0 , so that only the lowest level in the transverse direction is occupied. This leads to the condition that $d \ll \xi_0$, where ξ_0 is the Cooper-pair size $\xi_0 = \hbar v_F / \pi \Delta_0$.

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