

Reply to “Comment on ‘Renormalization group analysis and numerical simulation of propagation and localization of acoustic waves in heterogeneous media’ ”

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Based on extensive new numerical simulations, we show that Chu’s arguments and objections against our previous results are invalid. In addition, we explain the origin of the differences between our results and the previous ones, obtained based on a simple model of one-dimensional disordered materials.

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Chu believes that our numerical simulation of propagation of acoustic waves in the one-dimensional (1D) lattice model that we studied^{1,2} and the localization lengths that we computed for that model are “doubtful.” To prove his point, he refers to the results of some of the previous work^{3–5} on wave propagation in two-dimensional (2D) and three-dimensional (3D) media, such as Fig. 5(a) of Kirkpatrick,⁴ which is for 2D systems, and Fig. 1(a) of Magierski,⁵ which is for 3D media. In our work, however, we did not present any results for the localization length of 2D and 3D media and, therefore, the papers that Chu refers to have no direct relevance to our work on the 1D media that we studied.

More quantitatively, all the previous works^{3–5} used a common model based on wave scattering by hard spheres, implying immediately that in their model there is a specific length scale a —the sphere’s radius. Therefore, their result that the localization length increases for both the low and high frequencies can be understood in the following way:

(1) In the limit that the frequency ω vanishes or the wavelength ζ diverges, the acoustic wave cannot “see” the length scale a , as $\zeta/a \rightarrow \infty$, and thus, the localization length ξ diverges.

(2) In the opposite limit, $\omega \rightarrow \infty$ or $\zeta \rightarrow 0$, the acoustic waves cannot again see a and, therefore, ξ is large again, because in this case the acoustic waves propagate directly without any scattering, as their frequency is very large, and their wavelength is much smaller than a .

In the problem that we studied by using the renormalization group (RG) method, we considered a continuous system of scattering points or, in other words, the limit $a \rightarrow 0$, and in the low-frequency regime. This suggests that case (2) above has no relevance to our RG results whatsoever.

In the numerical simulation of the 1D media, we used the transfer-matrix (TM) method,^{6,7} with a lattice constant $\Delta = 1$. The average value of $\lambda = e(x)/\rho$ was $\lambda_0 = 10$, where $e(x)$ is the stiffness coefficient at x and ρ is the medium’s mean density. Therefore, only case (1) above might possibly be

relevant to our model. But, due to the different nature of the two models, we do not see any physical reason as to why the localization length in our 1D lattice system should increase at high frequencies. The TM method that we used is a rigorous numerical technique, used widely. It is known, of course, that the TM method can be numerically unstable,⁸ and Chu attempts to use this to cast doubts on our numerical results. However:

(i) The results shown in Fig. 8 of our paper² agree qualitatively with Figs. 13 and 14 of Ishii.⁸

(ii) As pointed out in the paper,² all of our results were tested by high-order finite-difference (FD) approximations [Eqs. (30) and (31) of our paper²]. Thus, Chu’s argument that the TM formulation is equivalent to a second-order FD approximation and, therefore, the results might not be accurate is not valid.

(iii) To strengthen his point, Chu also quotes from our paper² that, “The determination of the localization length remains a major numerical task,” in an apparent attempt to remind us that even we agree that this is a difficult problem and, therefore, we might somehow have made a mistake in our computations. However, that statement has been taken out of its context by Chu, as it was intended only for 2D and 3D systems for which we presented no results for the localization length in the paper,² not the 1D system that Chu discusses in his Comment.

Although, as stated above, the previous results^{3–5} for case (2) are all for 2D and 3D media, but suppose, for the sake of argument, that we accept Chu’s premise of the relevance of case (2) above to the 1D lattice system that we studied.² Then, in our simulations the group $\omega\Delta/\sqrt{\lambda_0}$ is, roughly speaking, the same as the coupling Ea/c in Kirkpatrick’s work,⁵ where E is the energy of the wave, c is the speed of light, and Δ is the lattice constant. Note that λ_0 in our model and simulation has the role of c in Kirkpatrick’s paper. He found that in 2D the minimum of the localization length ξ occurs around $Ea/c \approx 0.9$. Translating this number to an

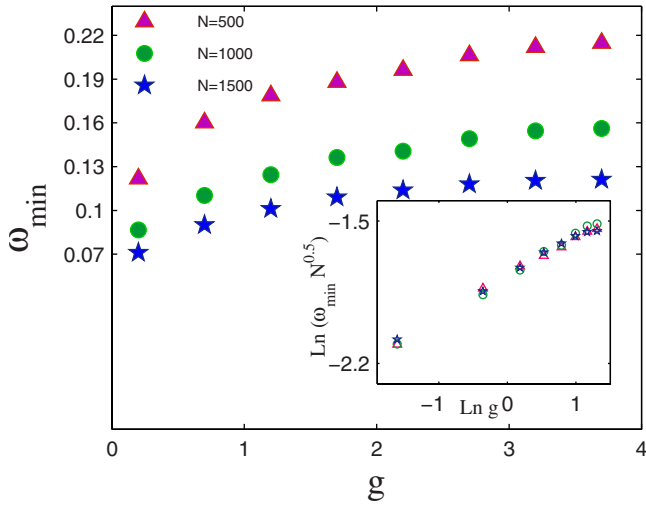


FIG. 1. (Color online) Dependence of the minimum allowed frequency of the system on the coupling constant g . The inset shows the collapse of the numerical results onto a single curve.

equivalent one for our 1D lattice model leads us to a minimum of ξ around $\omega \approx 2.84$ (i.e., $\omega\Delta/\sqrt{\lambda_0} \approx 0.9$, with $\sqrt{\lambda_0} \approx 3.16$ and $\Delta=1$), which is way outside the frequency range shown in Fig. 8 of our paper,² where we present the results only up to $\omega=0.1$. Therefore, even if we accept Chu’s premise, the results of Fig. 8 of our paper² are not “doubtful,” as they do not cover the frequency range in which the alleged missing minimum of ξ occurs. However, to further check the analysis and the numerical results presented previously,² we carried out additional TM calculations of the 1D lattice up to the maximum allowed frequency in the system.

First, using the direct diagonalization of the TM, we computed the dependence on the size N of the lattice and the coupling constant g of the minimum and maximum allowed frequencies that are present in the lattice. Here, N is the number of sites in the lattice, and the coupling constant g of the RG theory is defined by,^{1,2} $g=2D_0/\lambda_0^2$, with D_0 being the strength of the white—uncorrelated—part of the disorder. These computations are necessary, as the localization length ξ should be computed between the two frequencies.

In Figs. 1 and 2 the dependence on the coupling g of the minimum frequency ω_{\min} and maximum frequency ω_{\max} are presented. In these simulations the lattice constant was taken to be $\Delta=1$. The inset of Fig. 1 presents the scaling of the group $\omega_{\min}N^{0.5}$ vs the coupling constant g , indicating that all the data collapse onto a single curve. To obtain similar results for other values of the lattice constant, one should multiply the frequency ω in these figures by $1/\Delta$.

Second, to calculate the localization length ξ , we discretized the acoustic wave equation on a 1D lattice with a lattice constant Δ . Then, the governing equation for the wave component $\psi(x_i, \omega)$ at site i of the lattice was written in the TM form.² To ensure the stability of the numerical method, we checked, after multiplying the transfer matrices M times, the length of the resulting vector \mathbf{v} , normalized it again, and then continued with the new vector.^{6,7,9–11} The Lyapunov exponent γ , i.e., the inverse of the localization length, was then expressed in terms of the vector lengths d_α obtained after N normalization of \mathbf{v} . That is,

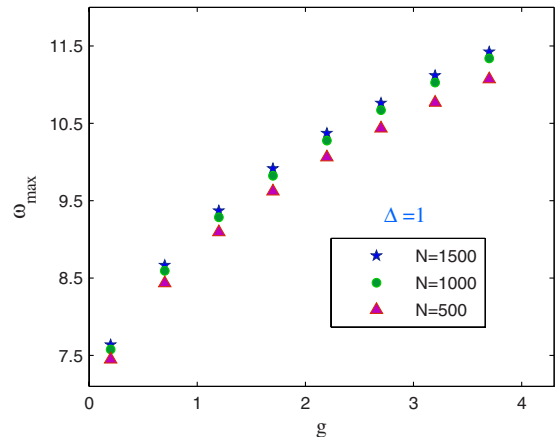


FIG. 2. (Color online) Same as in Fig. 1 but for the maximum allowed frequency.

$$\gamma = \frac{1}{MN} \sum_{\alpha=1}^N \ln(d_\alpha). \quad (1)$$

Moreover, the error in estimating γ is given by

$$\frac{\Delta\gamma}{\gamma} = \frac{1}{\sqrt{N}} \frac{\sqrt{\langle(\ln d_\alpha)^2\rangle - \langle\ln d_\alpha\rangle^2}}{\langle\ln d_\alpha\rangle}, \quad (2)$$

where the brackets indicate an averaging over the sequence of $\{d_\alpha\}$. According to the Oseledec theorem,¹² γ is a self-averaged quantity and, therefore, the error of its estimates should approach zero with increasing N as $1/\sqrt{N}$.

The results for the frequency dependence of the localization length ξ in a lattice of size $N=500$ are presented in Fig. 3. The three values of the coupling constants g for which the results of Fig. 3 were computed were selected to be below, at, and above the critical value of the coupling constant, $g_c \approx 0.72$, at which the system (of size $N=500$) undergoes² a phase transition from localized to delocalized states. As Fig. 3 indicates, the delocalized modes have seemingly

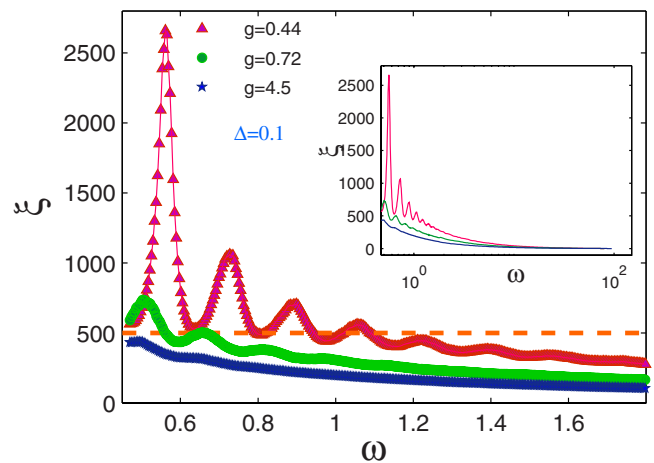


FIG. 3. (Color online) Frequency dependence of the localization length ξ in a 1D lattice of size $N=500$ for three values of the coupling constant g . Inset: the same results but up to the maximum frequency.

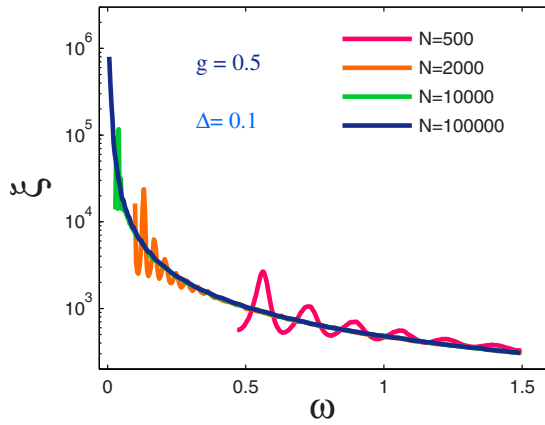


FIG. 4. (Color online) Frequency dependence of the localization length ξ of a 1D lattice of size N , up to maximum frequency, with a lattice constant, $\Delta=0.1$. The extended modes exhibit oscillatory behavior. For larger N , the oscillations become much smaller, hence indicating that they are an artifact of finite value of N .

oscillatory behavior. If the coupling constant g increases, all the modes will be localized, and the oscillatory behavior disappears.

At a fixed g the oscillatory behavior is, in fact, an artifact of the lattice’s finite size rather than being indicative of any physical effect. To demonstrate this, we carried out the TM simulations for $g < g_c$ and several lattice sizes N , with a lattice constant, $\Delta=0.1$. The results, presented in Fig. 4, confirm our assertion. As N increases, the amplitude of the oscillation decreases significantly, hence indicating that they are purely an artifact of the finite size of the lattice.

Third, to address the main point of Chu’s Comment, we computed the frequency dependence of the localization length ξ for several values of the lattice constant Δ , up to the maximum frequency ω_{max} , in order to check whether there is any increase in the localization length at high frequencies. The idea is that, by shrinking the lattice constant, the lattice simulation results might approach those obtained previously with the continuous model.³⁻⁵ The results, shown in Fig. 5, confirm, however, our previous results.² Note that for each lattice constant Δ the localization length is not computed beyond the maximum allowed frequency. We conclude, therefore, that none of Chu’s objections to our results is valid.

Since our previous results² are confirmed by those presented above, the question is: What is the source of the discrepancy between our lattice-based simulation results and those obtained previously.³⁻⁵ We suspected that the difference might be due to the existence of short-range correlations in the disorder, the significance of which was emphasized in our recent paper,⁷ as well as the different geometries used in our simulations (a lattice) and the previous ones (spheres). Thus, we carried out a series of simulation with a simple model, which not only contains short-range correlated disorder, but also mimics the geometry used in the previous works.³⁻⁵ We considered white-noise disorder in the lattice and repeated the value of $\lambda(x_i)$ at each site x_i twice. That is, if the lattice structure is (A, B, C, D, \dots) , where the symbols indicate the value of λ at the sites, we constructed another

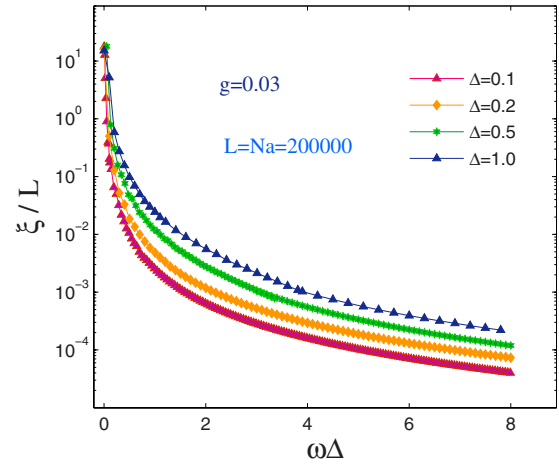


FIG. 5. (Color online) Frequency dependence of the localization length ξ in a 1D lattice of size N and lattice constant Δ , where $L = N\Delta = \text{const}$.

lattice, twice as long, with the structure $(A, A, B, B, C, C, D, D, \dots)$. Thus, each pair represents a 1D “sphere,” mimicking the previous geometry,³⁻⁵ but also introducing short-range correlations into the model. We then carried out extensive TM simulations using the new model and computed the frequency dependence of the localization length ξ . The results, shown in Fig. 6, are drastically different from those presented above and in our previous papers.^{1,2} They indicate that there is a minimum in the value of the localization length. Therefore, the minimum is *not* an artifact of the finite lattice constant, rather, it is due to the specific geometry and the existence of short-range correlations in the system. Since such features did not exist in our original simulations, and those in Figs. 1–5 above, it is not surprising that they did not exhibit the same maximum (or minimum).

Summarizing, as our new simulation results presented above indicate, our previous results^{1,2} are valid, not “doubtful,” as Chu claims. The origin of the discrepancy between our lattice-based results and the previous results³⁻⁵ is due to the different geometry of the two systems, as well as the existence of short-range correlations.

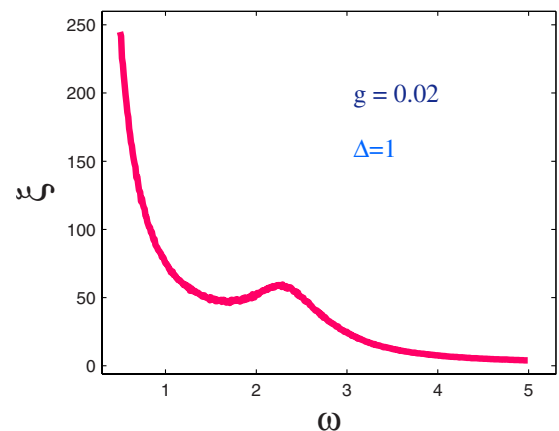


FIG. 6. (Color online) Frequency dependence of the localization length ξ in a 1D lattice of 1D “spheres” with short-range correlations.

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