Strong electron-phonon interaction in multiband superconductors

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We discuss the effects of anisotropy on superconducting critical temperature and order parameter in a strongly coupled regime. The multiband representation is used as a model for anisotropy. We show that strong-coupling effects in multiband superconductors lead to pair breaking due to interband coupling because soft phonon modes play the same role as static impurities. This effect makes the order parameters in different bands equal to each other and limits the upper bound on critical temperature.

I. INTRODUCTION

Effects of anisotropy on superconducting critical temperature and energy gap become of primary importance by approaching the strong-coupling regime when transition temperature T_c becomes of the order or even larger than the characteristic energy Ω of a boson mode, which mediate superconductivity. This issue received little attention up to now. In the weak-coupling limit, the effects of anisotropy were investigated shortly after the Bardeen–Cooper–Schrieffer (BCS) theory (see, e.g., Ref. [1]), and for multiband systems (Refs. [2] and [3]). Following the paper by Markowitz and Kadanoff,[4] different authors (references can be found in the review[5]) introduced the so-called separable interaction

$$V_{\mathbf{k}\mathbf{k}'} = (1 + a_{\mathbf{k}})V(1 + a_{\mathbf{k}'}), \qquad (1)$$

where $a_{\mathbf{k}}$ is an anisotropy parameter with the Fermi surface averaging $\langle a_{\mathbf{k}} \rangle$ being equal to zero. The result is the enhancement of the effective coupling constant

$$\lambda_{\mathrm{eff}} = \langle N(0)V_{\mathbf{k}\mathbf{k}'} \rangle = N(0)V(1 + \langle a_{\mathbf{k}}^2 \rangle) > N(0)V,$$

and corresponding rising of the T_c according to the standard BCS expression

$$T_c = 1.14\theta_D \exp(-1/\lambda_{\mathrm{eff}}), \qquad (2)$$

where θ_D is the phonon cutoff.

For multiband clean systems in the weak-coupling limit, the effective coupling constant in Eq. (2) is determined by the *maximum eigenvalue* of the matrix $\lambda_{\alpha\beta}$, where α and β are band indices. *Intraband* impurity scattering does not affect superconducting properties (Anderson's theorem) while the *interband* one averages out the order parameters Δ_α and λ_{eff} (and T_c), and corresponds to the average value

$$\langle \lambda \rangle = \frac{\sum_{\alpha\beta} N_\alpha(0)\lambda_{\alpha\beta}}{\sum_\alpha N_\alpha(0)}, \qquad (3)$$

(see, e.g., Refs. [6] and [7]). For positively defined matrix $\lambda_{\alpha\beta}$, the maximum eigenvalue is bigger than $\langle \lambda \rangle$, and we have the enhancement of T_c for multiband systems in comparison with the averaged value. This result does not depend on the sign of the nondiagonal matrix elements which determine the anisotropic contribution.[8]

Recent theoretical studies of superconductivity in the two-band superconductor MgB_2 (Ref. [9]), and calculations of covalent metals as the hypothetical hexagonal LiB and boron-doped diamond renewed the interest on the problem of an upper bound on superconducting critical temperature in strongly coupled anisotropic systems. Some estimates provide values of λ in anisotropic superconductors as large as four (Ref. [10]) or even 25 (Ref. [11]).

Let us first recall the result for the strong-coupling approach to isotropic systems. In the present paper we do not discuss effects of the strong electron–phonon interaction on phonon frequencies (see Ref. [12]). For the case of $\Omega \ll 2\pi T_c$ (which can occur for large λ), real phonons give the pair breaking contributions to the superconducting pairing, as well as to the quasiparticle renormalization. The largest terms corresponding to pair breaking and quasiparticle damping (see Appendix A) cancel each other out (Refs. [13] and [14]), and, as the result, one arrives at the following strong-coupling expression (see Ref. [15])

$$T_c = \mathrm{const}\sqrt{\lambda\Omega^2}, \qquad (4)$$

where in the simplest approximation $\mathrm{const} = (2\pi)^{-1} \simeq 0.15$ (numerical calculations give 0.1827). There are interpolation expressions connecting strong- and weak-coupling limits (see reviews[5,16,17]).

Moussa and Cohen[10] have imposed two possible upper bounds on a maximal critical temperature of multiband superconductors: the lower one is determined by the averaged coupling constant [Eq. (3)] while the upper one is governed by the maximal (positive) eigenvalue of the matrix for the first momentum of the Eliashberg functions $\alpha_{\alpha\beta}^2(\omega)F_{\alpha\beta}(\omega)$,

$$[\lambda\Omega^2]_{\alpha\beta} = M_{\alpha\beta}(1) = 2\int_0^\infty d\omega\,\omega\,\alpha_{\alpha\beta}^2(\omega)F_{\alpha\beta}(\omega) \qquad (5)$$

(for the Einstein spectrum, this value is equal to $\lambda_{\alpha\beta}\Omega^2$).

The purpose of this work is to analyze self-consistently the effects of anisotropy on the upper bound on T_c. We show that the low-frequency phonons play a role similar to *intraband* and *interband* static impurities. The latter can lead to the suppression of the anisotropy and, as a result, the upper bound on T_c is determined by the averaged coupling con-

stant. We consider in more detail the applications to the multiband systems.

II. GENERAL DESCRIPTION OF MULTIBAND SYSTEMS

The gap functions $\Delta_\alpha(\omega_n)$ can be calculated within an extension of the Eliashberg formalism to multiple bands;

$$\Delta_\alpha(\omega_n)Z_\alpha(\omega_n) = \pi T \sum_\beta \sum_{|\omega_m| \leq \omega_c} \frac{(\lambda_{\alpha\beta} - \mu^*_{\alpha\beta})\Delta_\beta(\omega_m)}{\sqrt{\omega_m^2 + \Delta_\beta^2(\omega_m)}}, \quad (6)$$

$$Z_\alpha(\omega_n) = 1 + \frac{\pi T}{\omega_n} \sum_\beta \sum_{\omega_m} \lambda_{\alpha\beta} \frac{\omega_m}{\sqrt{\omega_m^2 + \Delta_\beta^2(\omega_m)}}, \quad (7)$$

where

$$\lambda_{\alpha\beta} = 2 \int_0^\infty \omega \alpha^2_{\alpha\beta}(\omega)F_{\alpha\beta}(\omega)d\omega/[\omega^2 + (\omega_m - \omega_n)^2],$$

$Z_\alpha(\omega_n)$ are the Migdal renormalization functions, $\omega_n = \pi T(2n-1)$, standard Eliashberg functions define the superconducting properties, and thermodynamical properties are

$$\alpha^2_{\alpha\beta}(\omega)F_{\alpha\beta}(\omega) = \frac{1}{N_\alpha(0)} \sum_{\mathbf{k},\mathbf{k}',\nu} |g^{\alpha\beta,\nu}_{\mathbf{k},\mathbf{k}'}|^2 \delta(\varepsilon^\alpha_\mathbf{k})\delta(\varepsilon^\beta_{\mathbf{k}'})$$
$$\times \delta(\omega - \omega^\nu_{\mathbf{k}-\mathbf{k}'}), \quad (8)$$

where $\alpha, \beta = \{1, 2, \ldots\}$, $N_\alpha(0)$ is the partial density of states per spin at the Fermi energy, $g^{\alpha\beta}_{\mathbf{k},\mathbf{k}'}$ is the electron–phonon interaction (EPI) matrix element, $\mu^*_{\alpha\beta}$ is the renormalized Coulomb pseudopotential matrix element, and $\varepsilon^\alpha_\mathbf{k}$ and $\varepsilon^\beta_\mathbf{k}$ are the quasiparticle energies. Defining $\lambda_{\alpha\beta} = 2\int \omega^{-1}\alpha^2_{\alpha\beta}(\omega)F_{\alpha\beta}(\omega)d\omega$, we obtain the partial EPI constants. The values of $\Delta_\alpha(\omega_n)$ enter to the expression for the superconducting density of states[18]

$$N(\omega) = \sum_\alpha N_\alpha(0)\text{Re}\left[\left. \frac{\omega_n}{\sqrt{\omega_n^2 + \Delta_\alpha^2(\omega_n)}} \right|_{i\omega_n \to \omega + i\delta} \right]. \quad (9)$$

The Eliashberg functions satisfy the following symmetry relations

$$N_\alpha(0)\alpha^2_{\alpha\beta}(\omega)F_{\alpha\beta}(\omega) = N_\beta(0)\alpha^2_{\beta\alpha}(\omega)F_{\beta\alpha}(\omega). \quad (10)$$

For $T = T_c$, we have (neglecting the Coulomb pseudopotential)

$$\Delta_\alpha(\omega_n)Z_\alpha(\omega_n) = \pi T \sum_\beta \sum_{\omega_m} \frac{\lambda_{\alpha\beta}\Delta_\beta(\omega_m)}{|\omega_m|},$$

$$Z_\alpha(\omega_n) = 1 + \frac{\pi T}{\omega_n} \sum_\gamma \sum_{\omega_m} \lambda_{\alpha\gamma} \text{sign } \omega_m,$$

or

$$\Delta_\alpha(\omega_n)\left[1 + \frac{\pi T}{\omega_n} \sum_\gamma \sum_{\omega_m} \lambda_{\alpha\gamma} \text{sign } \omega_m \right] = \pi T \sum_\beta \sum_{\omega_m} \frac{\lambda_{\alpha\beta}\Delta_\beta(\omega_m)}{|\omega_m|}.$$

Finally

$$\tilde{\Delta}_\alpha(n)\rho(T_c) = \sum_\beta \sum_{n' \geq 1} B_{\alpha\beta}(n,n')\tilde{\Delta}_\beta(n'), \quad (11)$$

where for $n, n' \geq 1$, the matrix $B_{\alpha\beta}(n,n')$ has a form [we have used the symmetry of the gap function $\Delta_\alpha(\omega_n) = \Delta_\alpha(-\omega_n)$]

$$B_{\alpha\beta}(n,n') = \frac{\lambda_{\alpha\beta}(n-n') + \lambda_{\alpha\beta}(n+n'+1)}{\sqrt{(2n-1)(2n'-1)}} - \delta_{\alpha\beta}\delta_{nn'}S_\beta(n), \quad (12)$$

where

$$S_\beta(n) = \frac{1}{2n-1} \sum_\gamma \left[\lambda_{\beta\gamma}(0) + 2\sum_{m=1}^{n-1} \lambda_{\beta\gamma}(m) \right], \quad (13)$$

$\tilde{\Delta}_\alpha(n) = \Delta_\alpha(n)/\sqrt{2n-1}$, and $\rho(T_c)$ is the maximum eigenvalue of the matrix $B_{\alpha\beta}(n,n')$.

Here for the simplest Einstein spectrum with the frequency Ω, we have $\lambda_{\alpha\beta}(m) = \lambda_{\alpha\beta}\Omega^2/(\Omega^2 + 4\pi^2 T_c^2 m^2)$.

The value of T_c is determined by the equation

$$\rho(T_c) = 1. \quad (14)$$

III. STRONG COUPLING

The simplest way to estimate $\rho(T_c)$ for superstrong coupling is to put in Eq. (11) $n = n' = 1$. In this case, we have

$$\tilde{\Delta}_\alpha\left[\rho(T_c) + \sum_\gamma \lambda_{\alpha\gamma} \right] = \sum_\beta \lambda_{\alpha\beta}[\Omega^2/(2\pi T_c)^2 + 1]\tilde{\Delta}_\beta. \quad (15)$$

In the isotropic system $\lambda_{\alpha\beta} = \lambda \delta_{\alpha\beta}$, the last two terms in both sides of the equation cancel each other out and we have a standard expression for superstrong coupling (see Refs. 5 and 17),

$$T_{c,iso} = \frac{\Omega}{2\pi}\sqrt{\lambda}. \quad (16)$$

For the nondiagonal matrix $\lambda_{\alpha\beta}$, this cancellation does not occur and the large $\lambda_{\alpha\beta}$ terms play the role of pair breaking (see Appendix A).

Let us consider, for the sake of simplicity, the two-band system. The solution of Eq. (14) has a form

$$T_{c,2b} = \frac{\Omega}{2\pi}\sqrt{\frac{A + \sqrt{B^2 + 4CD}}{2C}}$$

with the eigenvector

$$\{\tilde{\Delta}_1, \tilde{\Delta}_2\} = \left\{ \frac{A' + \sqrt{B^2 + 4CD}}{2\lambda_{21}C}, 1 \right\}, \quad (17)$$

where $A = \lambda_{21}\lambda_{11} + \lambda_{11} + 2\lambda_{12}\lambda_{21} + \lambda_{12}\lambda_{22} + \lambda_{22}$, $A' = \lambda_{11}(1 + \lambda_{21}) - \lambda_{22}(1 + \lambda_{12})$ $B = \lambda_{21}\lambda_{11} + \lambda_{11} + 2\lambda_{12}\lambda_{21} + \lambda_{12}\lambda_{22} + \lambda_{22}$, $C = 1 + \lambda_{12} + \lambda_{21}$, and $D = \lambda_{12}\lambda_{21} - \lambda_{11}\lambda_{22}$.

In this case, in the order of $O(1/\lambda)$ (we suppose $\lambda_{11} \sim \lambda_{12} \sim \lambda_{21} \sim \lambda_{22} \sim \lambda \gg 1$), $\tilde{\Delta}_1 = \tilde{\Delta}_2$ and

$$T_{c,2b} \simeq \frac{\Omega}{2\pi}\sqrt{\langle\lambda\rangle}, \qquad (18)$$

where $\langle\lambda\rangle$ means averaging over both bands, and

$$\langle\lambda\rangle = \frac{(\lambda_{11}+\lambda_{12})N_1(0)+(\lambda_{22}+\lambda_{21})N_2(0)}{N_1(0)+N_2(0)},$$

and $N_\alpha(0)$ are the partial densities of states. In the case of multiple band system, we recover the general Eq. (3). For the non-Einstein spectrum, we have $\langle\lambda\rangle\Omega^2 \Rightarrow \langle[\lambda\Omega^2]_{\alpha\beta}\rangle = \langle M_{\alpha\beta}(1)\rangle = \langle 2\int_0^\infty d\omega\omega[\alpha^2(\omega)F(\omega)]_{\alpha\beta}\rangle$. This means that strong coupling leads to washing out the effects of anisotropy.

Similar statements were made in Refs. 19 and 20 where the authors have considered the momentum dependent interaction. In the former paper, the separable interaction similar to Eq. (1) $[\alpha^2(\omega)F(\omega)]_{\mathbf{pp'}}=\alpha^2(\omega)F(\omega)g(\mathbf{p})g(\mathbf{p'})$ with $\langle g(\mathbf{p})\rangle=1$ was used. They got the result that the expression for T_c in the superstrong limit reduces to the isotropic one while the "pairing potential" is proportional to $g(\mathbf{p})$. This contradicts the more general statement in the latter article where the *positive* (attractive) interactions $\alpha^2(\mathbf{k},\mathbf{k'},\omega)F(\mathbf{k},\mathbf{k'},\omega)$ for all $\mathbf{k},\mathbf{k'}$ was investigated and it was shown that the gap function becomes $\mathbf{k}$ independent, which leads to the isotropic expression for T_c. The detail inspection of the situation in Ref. 19 also shows that the real order parameter, which enters to the density of states [see Eq. (9)], is isotropic for large λ.

The exact behavior of T_c depends on the structure of the matrix $\alpha_{\alpha\beta}^2(\omega)F_{\alpha\beta}(\omega)$ but qualitative results remain. We have investigated numerically the evolution of T_c and eigenvectors Δ_α as functions of the coupling strength $\lambda_{\alpha\beta}$ for the model matrix of the Eliashberg functions

$$\alpha_{\alpha\beta}^2(\omega)F_{\alpha\beta}(\omega) = \alpha^2(\omega)F(\omega)\begin{pmatrix} 1 & 1/5 \\ 1/10 & 0 \end{pmatrix}. \qquad (19)$$

We suppose, for simplicity, $2N_1(0)=N_2(0)$ and $\lambda_{22}=0$ (i.e., no intrinsic superconductivity in the second band). The average EPI constant $\langle\lambda\rangle$ is equal to 0.467λ. Results for T_c are presented in Fig. 1. We see that for weak and intermediate couplings, there is an enhancement of T_c due to anisotropic effects, in comparison to the averaged value. For small EPI, the result coincide with the weak-coupling expression for $\lambda_{\text{eff}}=\lambda_{\text{max}}=1.02\lambda$, where λ_{max} is a maximal eigenvalue of the matrix [Eq. (19)]. This enhancement, however, vanishes for large values of λ when phonons lead to isotropization of the superconducting order parameter.

We have to note that the result [Eq. (18)] is obtained under the condition of nonvanishing $\langle\lambda\rangle$ and in the Born approximation[21] for the spin-independent interaction.

Recently the model for the system with strong-coupling anisotropic interaction was considered in Ref. 22. It was supposed that the difference between the interaction in the quasiparticle channel $\langle\alpha^2(\mathbf{k},\mathbf{k'},\omega)F(\mathbf{k},\mathbf{k'},\omega)\rangle_{FS}$ and in the Cooper channel $\langle\Delta(\mathbf{k})\alpha^2(\mathbf{k},\mathbf{k'},\omega)F(\mathbf{k},\mathbf{k'},\omega)\Delta(\mathbf{k'})\rangle_{FS}/\langle\Delta^2(\mathbf{k})\rangle_{FS}$ is independent of the coupling strength. The above analyses (as well as Ref. 19 and 20) show that this difference vanishes
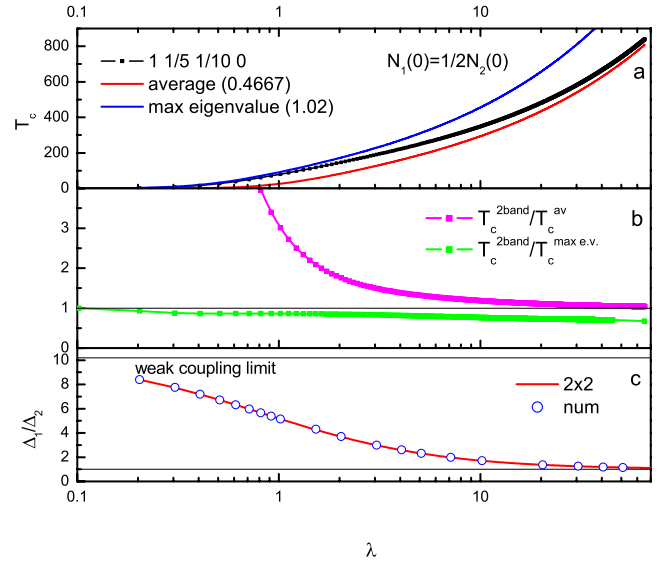

FIG. 1. (Color online) Critical temperature (a) and the gap ratio (c) in the two-band case as a function of intraband coupling constant λ in the first band. The panel (b) shows that T_c in the strongly coupled regime is determined by the average coupling constant. The panel (c) shows the ratio of the order parameters in the two bands. The gap function becomes isotropic in the strongly coupled regime. The numerically calculated ratio Δ_1/Δ_2 is very accurately described by the expression Eq. (17) in a broad range of λ.

for strong coupling. This removes unphysical results for T_c obtained in this limit in the mentioned paper.

In Appendix B, the sensitivity of T_c to different phonon modes is considered by calculating the variational derivatives. It is shown that the negative (divergent at small frequencies) contribution to the nondiagonal variational derivative of T_c vanishes in the strongly coupled regime.

IV. CONCLUSIONS

We have shown that strong-coupling effects in the multiband superconductors lead to the appearance of strong damping, which results from pair breaking due to interband coupling.

For systems with attractive interaction, this effect leads to averaging of order parameters in different bands. As a result, asymptotic behavior of T_c is described by the well-known single-band expression $T_c \propto \sqrt{\langle\lambda\Omega^2\rangle} = \sqrt{2\int_0^\infty d\omega\omega\langle\alpha^2(\omega)F(\omega)\rangle}$. This means that the upper bound on T_c in the superstrong coupling regime is determined by the averaged coupling constant while the higher upper bound corresponding to the maximal eigenvalue of the matrix $[\lambda\Omega^2]_{\alpha\beta}$ is never reached.

ACKNOWLEDGMENTS

APPENDIX A

We extend the results of Ref. 14 for effects of low-frequency intermediate boson modes ($\Omega \lesssim 2\pi T_c$) on the criti-

cal temperature of the multiband superconductors. On the real frequency axis, the equations for the complex order parameter $\Delta_\alpha(\omega)$ and the renormalization function $Z_\alpha(\omega)$ have forms (neglecting the Coulomb contribution)

$$Z_\alpha(\omega)\Delta_\alpha(\omega) = \sum_\beta \int_{-\infty}^{\infty} dz K_{\alpha\beta}(z',\omega) \mathrm{Re}\frac{\Delta_\beta(z')}{z'}, \quad (A1)$$

$$[1 - Z_\alpha(\omega)]\omega = -\sum_\beta \int_{-\infty}^{\infty} dz K_{\alpha\beta}(z',\omega), \quad (A2)$$

where $K_{\alpha\beta}(z',\omega)$ is a kernel of the interelectron interaction via intermediate bosons with the spectral function $\alpha_{\alpha\beta}^2(\Omega)F_{\alpha\beta}(\Omega)$,

$$K_{\alpha\beta}(z',\omega) = \frac{1}{2}\int_0^\infty d\Omega\, \alpha_{\alpha\beta}^2(\Omega)F_{\alpha\beta}(\Omega)$$
$$\times \left[\frac{\tanh\frac{z'}{2T_c} + \coth\frac{\Omega}{2T_c}}{z' + \Omega - \omega - i\delta} - \{\Omega \rightarrow -\Omega\} \right].$$

Now let us separate the functions $\alpha_{\alpha\beta}^2(\Omega)F_{\alpha\beta}(\Omega)$ on to low-energy part $[\alpha_{\alpha\beta}^2(\Omega)F_{\alpha\beta}(\Omega)]^<$ and the high-energy one

$$\alpha_{\alpha\beta}^2(\Omega)F_{\alpha\beta}(\Omega) = [\alpha_{\alpha\beta}^2(\Omega)F_{\alpha\beta}(\Omega)]^< \Theta(2\pi T_c - \Omega)$$
$$+ [\alpha_{\alpha\beta}^2(\Omega)F_{\alpha\beta}(\Omega)]^> \Theta(\Omega - 2\pi T_c).$$

The same procedure can be done for the kernel $K_{\alpha\beta}(z',\omega)$,

$$K_{\alpha\beta}(z',\omega) = K_{\alpha\beta}^<(z',\omega) + K_{\alpha\beta}^>(z',\omega). \quad (A3)$$

In the first term on the right-hand side of Eq. (A3), we can neglect the frequency Ω in the denominator. In this case,

$$K_{\alpha\beta}^<(z',\omega) = \frac{\Gamma_{\alpha\beta}^<}{\pi}\frac{1}{z' - \omega - i\delta}, \quad (A4)$$

where

$$\Gamma_{\alpha\beta}^< = \pi\int_0^\infty d\Omega [\alpha_{\alpha\beta}^2(\Omega)F_{\alpha\beta}(\Omega)]^< \coth(\Omega/2T_c) \simeq 2\pi\lambda_{\alpha\beta}^< T_c$$
$$(A5)$$

is the matrix of the electron scattering on the low-energy excitations. Now we use the dispersion relation for the order parameter $\Delta_\beta(\omega)$,

$$i\frac{\Delta_\beta(\omega)}{\omega} = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dz'}{\omega - z' + i\delta}\mathrm{Re}\frac{\Delta_\beta(z')}{z'}, \quad (A6)$$

which is a consequence of the dispersion relation for the electron Green function in the Nambu representation. Combining Eqs. (A1) and (A2) with Eqs. (A3)–(A6), we get

$$\Delta_\alpha(\omega)\left[1 + \sum_\beta \frac{i\Gamma_{\alpha\beta}^<}{\omega} + \sum_\beta \int_{-\infty}^{\infty} dz K_{\alpha\beta}^>(z',\omega) \right]$$
$$= \sum_\beta \frac{i\Gamma_{\alpha\beta}^<}{\omega}\Delta_\beta(\omega) + \sum_\beta \int_{-\infty}^{\infty} dz K_{\alpha\beta}^>(z',\omega)\mathrm{Re}\frac{\Delta_\beta(z')}{z'}.$$

We see that the low-frequency excitations play a role of in-

traband and interband static impurities. Intraband $\Gamma_{\alpha\alpha}^<$ ones drop out from the above equation (so-called Anderson's theorem). It is interesting to note that the famous cancellation of the largest terms proportional to $\lambda^<$ (see, e.g., Ref. 5) comes not from the strong renormalization of the quasiparticle energy (Re Z) but from the damping $\Gamma^< \sim \pi\lambda^< T$.

APPENDIX B

In Ref. 23, the sensitivity of T_c to different phonon modes was considered by calculating the variational derivatives $\delta T_c/\delta[\alpha^2(\Omega)F(\Omega)]_{\alpha\beta}$. For the diagonal elements ($\alpha = \beta$), the result for small $\Omega(\Omega \ll 2\pi T_c)$ coincides with the one obtained by Bergmann and Rainer,[24] $\delta T_c/\delta[\alpha^2(\Omega)F(\Omega)] \sim \Omega$, for the isotropic single-band system. This corresponds to the enhancement of T_c by adding low-frequency phonons (bosons).

In the multiband case, the interband derivative has the following form

$$\frac{\delta T_c}{\delta[\alpha^2(\Omega)F(\Omega)]_{\alpha \neq \beta}} \sim \frac{N_\alpha(0)}{\Omega}\sum_{n\geq 1}\frac{\Delta_\alpha(\omega_n)[\Delta_\beta(\omega_n) - \Delta_\alpha(\omega_n)]}{\omega_n^2}$$
$$+ O(\Omega),$$

and $\delta T_c/\delta[\alpha^2(\Omega)F(\Omega)]_{12}$ and $\delta T_c/\delta[\alpha^2(\Omega)F(\Omega)]_{21}$ have different signs. This contradicts to the symmetry relation [Eq. (10)]. If we change the function $[\alpha^2(\Omega)F(\Omega)]_{12}$, the counterpart $[\alpha^2(\Omega)F(\Omega)]_{21}$ has to be changed automatically. Only the symmetrized *off-diagonal* combination $\delta T_c/\delta\alpha^2(\Omega)F(\Omega)_{\mathrm{o.d.}}$ has physical meaning. Here

$$\alpha^2(\Omega)F(\Omega)_{\mathrm{o.d.}}$$
$$= \frac{N_1(0)[\alpha^2(\Omega)F(\Omega)]_{12} + N_2(0)[\alpha^2(\Omega)F(\Omega)]_{21}}{N_1(0) + N_2(0)}.$$

As a result, we obtain

$$\frac{\delta T_c}{\delta\alpha^2(\Omega)F(\Omega)_{\mathrm{o.d.}}} \sim -\frac{N_\alpha(0)}{\Omega}\sum_{n\geq 1}\frac{[\Delta_2(\omega_n) - \Delta_1(\omega_n)]^2}{\omega_n^2} + O(\Omega).$$
$$(B1)$$

In contrast to the single-band case (see Ref. 17), the off-diagonal derivative has different behavior in the weak-coupling and strong-coupling regimes. For the former case, one can suppose $\Delta_\alpha(\omega_n) = \Delta_\alpha(\Theta - |\omega_n|)$ and $\delta T_c/\delta\alpha^2(\Omega)F(\Omega)_{\mathrm{o.d.}} \sim -\frac{(\Delta_2 - \Delta_1)^2}{\Omega}$. This means that the addition of nondiagonal interaction with low-frequency phonons leads to strong suppression of the critical temperature in weak-coupling anisotropic superconductors. This result was obtained in Ref. 25 for the anisotropic separable interaction. In the strong-coupling limit, as it was shown above, $\Delta_1 \Rightarrow \Delta_2$, then the first term in Eq. (B1) vanishes and $\delta T_c/\delta\alpha^2(\Omega)F(\Omega)_{\mathrm{o.d.}} \sim \Omega > 0$, similar to the intraband contribution. This result can be directly obtained from the Eq. (18).

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